Continuous Measurements of Quantum Phase

V.P.Belavkin
Mathematics Department University of Nottingham, UK

C.Bendjaballah
Ecole Superieure d'Electricite, Gif sur Yvette, France

30 May 1990
Published in: *Quantum Opt.* **6**, (1994) 169–186

Abstract

The problem of phase measurement for a quantum oscillator using the quantum nondemolition principle (NDP) which makes the selfadjoint operator and wave packet reduction postulate unnecessary. It is shown that this approach is an appropriate one, not only in order to introduce in a proper way the unsharp measurements of the quantum phase, but also to describe the relevant quantum jumps under such measurements. Our analysis suggests that in contrary to other quantum measurement problems (e.g. the position and the momentum of a particle), the operator to represent the phase of a quantum field cannot be separated from the quantum measurement process. Moreover, it allows us to describe the time continuous process of quantum phase jumps and also to study the stochastic dynamics of spontaneous localization for the oscillator under phase measurement.

Introduction

The problem of a phase operator, obeying the laws of quantum mechanics, has motivated several physicists since Dirac's unsatisfactory assumption of the existence of a Hermitian phase operator conjugate to the number operator. Numerous studies have been devoted to this subject and recent analysis in connection with the study of squeezed states of light, has renewed the interest in the quantum phase operator measurements.

The first point we may wonder what the quantum phase operator, discussed in [**?**], is about, specifically, how is it related to the quantum phase observable introduced in [1]. We may also consider to find which quantum phase distribution among those considered [**?**] corresponds to the observation of phase of a quantized light field. Which scheme of quantum phase measurements is optimal with respect to a given criterion say for optical communication? Another question of interest in relation with the quantum phase jumps, is to describe the state of the quantum oscilator after its phase has been measured.

Among the dynamical problems not yet solved for quantum phase there are following questions: Is there an interaction quantum mechanical model to reproduce the quantum phase jumps and the relevant phase probability distributions? How to describe the sequential and time continuous quantum phase measurements? What is happenning with the quantum oscillator in the process of the phase measurements? Is there a general physical principle or reduction equation which plays the role of the projection postulate for the timed measurements of such unusual observable as the quantum phase?

All these questions are essential from the theoretical point of view and deserve to be studied from a practical point of view. The aim of this paper is to show how they can be solved in the developed framework of the unsharp nondemolition measurement.

Although it seems that there is no Hermitian operator that directly corresponds to the phase of quantum electromagnetic field, it is known that measurements of the phase (actually phase difference or phase jump) in quantum optics do not involve such fundamental problems. There are several experiments in which the quantum phase can be measured indirectly. Furthermore, in order to apply techniques such as heterodyning, to use squeezed light or the phase shift keying modulation in optical space communications, it is important to compare the different models of indirect phase measurements and to determine which one is the most sharp and most adapted to quantum optical communications, for instance.

The difficulty of defining an appropriate quantum phase operator is not the reflection of the impossibility of quantum phase measurements but is simply related with the asence of a dynamically conjugated operator to the photon number which would generate the shifts of this quantum number.

Indeed, the elementary shift $n \mapsto n+1$ of $n = 0, 1, 2, \ldots$ cannot be inverted on $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$ and hence it does not generate a unitary group but only the semigroup $|n\rangle \mapsto |n+s\rangle$, $s \in \mathbf{Z}_+$ which does not posses a selfadjoint generator in the Hilbert space $\mathcal{H}$ of photon states. This situation would occur for the momentum operator of a quantum particle if its coordinate operator $\widehat{x}$ had its domain of variation restricted, say to the positive half–line $\mathbf{R}_+$. The translation generator $\widehat{p} = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x}$ of one parameter shift semigroup of the isometries $\mathrm{e}^{-\frac{i}{\hbar} s\widehat{p}}$, $s \in \mathbf{R}_+$ would not be a selfadjoint operator in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}_+)$, but it does not mean the absence of a possibility to measure the momentum of a quantum particle in $\mathbf{R}_+$.

This means that a proper representation of quantum phase must be made in terms of the quantum measurement involving a nonunitary scattering matrix for the isometric shifts of quantum states due to object–meter interaction rather than in terms of selfadjoint generators. This can be realized within the quantum theory of counting nondemolition measurements [3] defining both a probability distribution of the measurement and collapsed states after the measurement by generalized reduction transformations. Notice that due to the continuity of the phase spectrum $[0, 2\pi[$, the collapsed states couldnot be found by application of von Neumann projection postulate even if the phase operator had existed in the usual sense as a momentum operator $\widehat{p}$ with such spectrum for a quantum particle in the discrete space $\mathbf{Z}$.

Before showing how to solve this problem for the quantum phase, let us illustrate the nondemolition measurement scheme for a simple case of Hermitian operator $\widehat{p}$ generating the cyclic unitary shift

$$\mathrm{e}^{i\widehat{p}}|0\rangle = |s\rangle, \quad \mathrm{e}^{i\widehat{p}}|n\rangle = |n-1\rangle, \quad n = 1, \ldots, s$$

in the $s+1$–dimensional space $\mathcal{H} = \{\sum_{n=0}^{s} c_n |n\rangle\}$ where $|n\rangle$ are the eigenvectors for the values $n = 0, 1, \ldots, s$ of the number operator.

The operator $\widehat{p}$ considered in [**?**] as a discrete model of oscillator phase has the normalized eigenvectors $\eta_\vartheta = (s+1)^{-1/2} \sum_{n=0}^{s} \mathrm{e}^{in\vartheta}|n\rangle$ corresponding to the eigenvalues $\vartheta = \frac{2l\pi}{s+1}$, $l = 0, \ldots, s$.

The nondemolition measurement of $\widehat{p}$ is described by the pointer observable $\widehat{q}$ that is the coordinate operator of a meter acting in a copy $\mathcal{E} = \{\sum_{n=-k}^{s-k} a_n |n\rangle\}$ of $\mathcal{H}$ exactly as the operator $\widehat{p}$ acts in $\mathcal{H}$:

$$\mathrm{e}^{i\widehat{q}}|-k\rangle = |s-k\rangle, \quad \mathrm{e}^{i\widehat{q}}|n\rangle = |n-1\rangle, \quad n = 1-k, \ldots, s-k.$$

Here k is any integer which one can take as $k > s$ in order to have have the positivity of the pointer momentum $M : |n\rangle \mapsto -n|n|rangle$ in$\mathcal{E}$, generating the shift $\mathrm{e}^{i\vartheta M}\zeta_\lambda = \zeta_{\lambda-\vartheta}$ (modulo 2π) of the eigenvectors

$$\zeta_\lambda = (s+1)^{-\frac{1}{2}} \sum_{n=-k}^{s-k} \mathrm{e}^{in\lambda}|n\rangle, \quad \lambda = 0, \varepsilon, \ldots, s\varepsilon$$

for the discrete coordinate $\widehat{q}$ with $\varepsilon = \frac{2\pi}{s+1}$. The interaction of the quantum oscillator and the meter, initially independent, is described by the scattering operator $S = \mathrm{e}^{-i\widehat{p\otimes M}}$ in the product space $\mathcal{H} \otimes \mathcal{E}$. The unitary operator S shifts the initial value λ_0 of the pointer $\widehat{q}$ to the value $\lambda = \vartheta + \lambda_0$ (modulo 2π) corresponding to the eigenvalue ϑ of the phase $\widehat{p}$ by the Heisenberg transformation $S^\dagger (\mathbf{1} \otimes \widehat{q}) S = \widehat{p} \otimes \mathbf{1} + \mathbf{1} \otimes \widehat{q}$ (modulo 2π). Hence, if the initial state of the meter is fixed as the eigenvector $\zeta_0 = (s+1)^{-1/2} \sum_{n=0}^{s} |n\rangle$ corresponding to $\lambda_0 = 0$ and λ is the reading of the pointer $\widehat{q}$ after the interaction, then we will definitly have $\vartheta = \lambda$. This means that the quantum state of the oscillator after such measurement is given by the eigenvector η_λ of $\widehat{p}$ for any initial state vector $\eta \in \mathcal{H}$of the oscilator. It can be obtained [**?**] by the normalization $P_\lambda\eta/\|P_\lambda\eta\|$ of the orthogonal projection

$$P_\lambda\eta = \frac{1}{s+1} \sum_{n=-k}^{s-k} \mathrm{e}^{i(\widehat{p}-\lambda)n}\eta = \frac{1}{s+1} \sum_{m=0}^{s} c_m \mathrm{e}^{-im\lambda} \sum_{n=0}^{s} \mathrm{e}^{in\lambda}|n\rangle = \eta_\lambda\langle\eta_\lambda|\eta\rangle$$

given by the partial matrix element $P_\lambda = \langle\zeta_\lambda|S|\zeta_0\rangle$ of the scattering operator for any $\eta = \sum_{n=0}^{s} c_n|n\rangle$. Thus, the projection postulate appears as a conjecture of the possibility of choosing an eigenvector of $\widehat{q}$ as the initial state ζ_0 of the meter and of the filtering by ζ_λ after the interaction accordingly to the pointer reading λ.

In this paper, most of the questions of fundamental interest will be considered. We will try to find an answer which is as complete as possible. More specifically, we show how the non–demolition principle of the quantum measurements theory [**?**], free of a particular type of reduction postulate, helps to solve these problems, exactly in the same way as they have been solved for the ordinary quantum observables like the position of a quantum particle. As a result we obtain a stochastic wave reduction equation for the state vector of the quantum oscillator under the time continuous phase observation. In contrast to the stochastic diffusive models of continuous reduction and spontaneous localizations [**?**, 4, 8], the continuous phase reduction equation is governed on the individual level by not the Wiener but the Poisson process which can even have singular intensity distribution on the phase space $[0, 2\pi[$.

1 The quantum phase operators

Let us consider a quantum system having a purely discrete energy spectrum, a quantum harmonic oscillator for instance, with the countable family of eigenvectors $|n\rangle$ $(n = 0, 1, \ldots)$. We represent the state vectors η of such a system in the Hilbert space $\mathcal{H}$ of the linear combinations $\sum_{n=0}^{\infty} c_n |n\rangle$, where $\{c_n\}$ are complex coefficients such that $\|\eta\|^2 = \sum_{n=0}^{\infty} |c_n|^2 < \infty$, by the $*$–analytic functions $\eta(z^*) = \sum_{n=0}^{\infty} c_n z^n$ in the unit disc $|z| < 1$, similarly to [**?**, **?**]. Each function is completely defined by the boundary values $h(\vartheta) = \lim_{r\uparrow 1} \eta(r\mathrm{e}^{i\vartheta})$ and the scalar product $\langle \eta|\zeta\rangle = \int_0^{2\pi} \eta(\mathrm{e}^{i\vartheta})^* \zeta(\mathrm{e}^{i\vartheta}) \frac{\mathrm{d}\vartheta}{2\pi}$ is considered as the correlation $(h|f) = \langle h^* f\rangle$ of $h(\vartheta)^*$ with another such a function $f(\vartheta) = \lim_{r\uparrow 1} \zeta(r\mathrm{e}^{i\vartheta})$ respectively to the shift invariant (mod. 2π) probability measure $\frac{\mathrm{d}\vartheta}{2\pi}$. The z^*–transformation $\{c_n\} \mapsto \eta(z)$ constitutes a phase representation via the natural embedding $\eta(z) \mapsto \eta(\mathrm{e}^{i\vartheta}) = \sum_{n=0}^{\infty} c_n \mathrm{e}^{-in\vartheta}$ of space $\mathcal{H}$ into space $\mathcal{E}$ of all periodic functions $f(\vartheta)$ modulo 2π with $\langle |f|^2\rangle = \frac{1}{2\pi} \int_0^{2\pi} |f(\vartheta)|^2 \mathrm{d}\vartheta < \infty$. A periodic wave function $f(\vartheta) = \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{-in\vartheta}$ represents a vector $\zeta \in \mathcal{H}$ only if it is analytic in the complex domain $\vartheta \in \mathbf{C}$, $\Im\vartheta < 0$ such that $\zeta(z^*) = f(i \ln z)$, $\forall\, |z| \leq 1$ is its z^*–transformation. In this enlarged space $\mathcal{E}$, there exists a self–adjoint phase operator $\widehat{\varphi}$ defined by the multiplication $[\widehat{\varphi} f](\vartheta) = \varphi(\vartheta) f(\vartheta)$ for $\varphi(\vartheta) = \vartheta - 2\pi \lfloor \frac{\vartheta}{2\pi} \rfloor$ with the generalized eigenvectors $|\vartheta\rangle$ of the spectral decomposition, $\widehat{\varphi} = \frac{1}{2\pi} \int_0^{2\pi} |\vartheta\rangle\ \vartheta\ \langle\vartheta| \mathrm{d}\vartheta$, where $\lfloor \mathbf{x} \rfloor$ is the integer part of $x \in \mathbf{R}$ and $\langle\vartheta|$ is the evaluating functional: $f \mapsto f(\vartheta)$ defined for any periodic function $f(\vartheta)$.

This operator does not leave the subspace $\mathcal{H}$ invariant due to the non–analyticity of $\varphi(\vartheta)$ at $\Im\vartheta < 0$ but the unitary operator $\mathrm{e}^{-i\widehat{\varphi}}$ functions as an isometry $[Z^\dagger \eta](z) = z^* \eta(z)$ in $\mathcal{H}$ because $[\mathrm{e}^{-i\widehat{\varphi}} h](\vartheta) = \mathrm{e}^{-i\vartheta} h(\vartheta)$ is analytic and $\mathrm{e}^{-i\widehat{\varphi}} |n\rangle = |n+1\rangle$ for all n.

Note that the operator $Z^\dagger$ is not adjoint to the inverse operator $\hat{z} = \mathrm{e}^{i\hat{\varphi}}$ which maps $\mathcal{H}$ onto the bigger subspace $\mathcal{H} \oplus \mathbf{C} \subset \mathcal{E}$ but to the non–unitary operator $Z = E\hat{z}$ given by the orthoprojector E in $\mathcal{E}$ onto $\mathcal{H}$, and known as the exponential phase operator $[Z\eta](z) = \frac{1}{z^*} (\eta(z) - \eta(0))$ [**?**]. The latter operator is not normal because the commutator $[Z^\dagger, Z] = |0\rangle\langle 0|$ is not zero, but it has the overcomplete family $\{|z)||z|<1\}$ of non–orthogonal left eigenvectors in $\mathcal{H}$

$$|z)=\sum_{n=0}^{\infty} z^n|n\rangle \quad : \quad Z|z)=|z)z \tag{1.1}$$

describing the unsharp phase vectors $E|\vartheta\rangle$ as the limits $|\mathrm{e}^{i\vartheta})=\lim_{r\uparrow 1}|r\mathrm{e}^{i\vartheta})$.

The corresponding positive operator–valued measure $\Pi=\{\Pi_\Delta\}$, defined for any Borel subset $\Delta\subseteq[0,2\pi[$ as

$$\Pi_\Delta=\frac{1}{2\pi}\int_\Delta|\mathrm{e}^{i\vartheta})(\mathrm{e}^{i\vartheta}|\mathrm{d}\vartheta\,:\,\eta\mapsto\frac{1}{2\pi}\int_\Delta|\mathrm{e}^{i\vartheta})h(\vartheta)\mathrm{d}\vartheta$$

with $(\mathrm{e}^{i\vartheta}|\eta=h(\vartheta)=\langle\vartheta|h$ for any $\eta\in\mathcal{H}$, is normalized to the identity operator $I=\Pi_{[0,2\pi[}$ in $\mathcal{H}$, and describes the probabilities $\Pr(\vartheta\in\Delta)=\langle\eta|\Pi_\Delta|\eta\rangle$ of an unsharp measurement of the oscillator phase in an initial state $\eta\in\mathcal{H}$.

This measurement gives the best results among all the unsharp measurements of the phase parameter of a quantum coherent state under the maximum likelihood criterion, as was first shown in [1] and also in [9, **?**, **?**]. However, the nonorthogonal vectors $|\mathrm{e}^{i\vartheta})=E|\vartheta\rangle$ are unnormalizable because $(\mathrm{e}^{i\vartheta}|\mathrm{e}^{i\vartheta})=\lim_{r\uparrow 1}(1-r^2)^{-1}\to\infty$. Hence they are not in the Hilbert space and cannot be regarded as the vectors of *a posteriori states*, i.e. the states after the phase measurements, or even proportional to such vectors.

A family $\{\chi_\vartheta:\vartheta\in[0,2\pi[\}$ of normalized vectors $\chi_\vartheta\in\mathcal{H},\|\chi_\vartheta\|=1$ is called *a posteriori state vectors* if there exists a probability amplitude $c(\vartheta)$, $\int_0^{2\pi}|c(\vartheta)|^2\mathrm{d}\vartheta/2\pi=1$ such that the vector–function $\chi(\vartheta)=c(\vartheta)\chi_\vartheta$ reproduces the state-vector of the combined system "ocilator plus phase meter" in the pointer representation after the interaction.

According to the vector superposition principle, there must be a complete family

$$\{G(\vartheta):\vartheta\in[0,2\pi[,\quad\int_0^{2\pi}G(\vartheta)^\dagger G(\vartheta)\mathrm{d}\vartheta=2\pi I\}$$

of linear transformations $G(\vartheta)$ in $\mathcal{H}$ giving the vector–function $\chi(\vartheta)=G(\vartheta)\eta$.

We say that the reduction transformations $G(\vartheta))$ satisfy the compatibility condition with respect to the phase vectors if they commute with these vectors in the sense that each $(z|$ intertwines them with c-numbers $g_z(\vartheta)$:

$$(z|G(\vartheta)=g_z(\vartheta)(z|,\,\forall\,\vartheta\in[0,2\pi[,|z|<1 \tag{1.2}$$

Now we can prove that the ideal phase measurement demolish the quantum system because there is no way to get the posterior state vectors using the projection or any other reduction postulate $G(\vartheta) : \eta \mapsto \chi(\vartheta)$, which would be compatible with such measurement.

Indeed, the normalized posterior state vectors $\chi_\vartheta = \chi(\vartheta)/c(\vartheta)$are uniquely defined up to $Arg\ c(\vartheta)$ by the square root $|c(\vartheta)|$ of the probability density

$$|c(\vartheta)|^2 = \frac{1}{2\pi i}\int_{|z|=1} |g_z(\vartheta)|^2|\eta(z)|^2\mathrm{d}z \tag{1.3}$$

for the quantum phase measurement respectively to the measure $\frac{\mathrm{d}\vartheta}{2\pi}$.

As follows from (1.2), the reduction transformations $G(\vartheta)$ must be diagonal

$$[G(\vartheta)\eta](z) = (z|G(\vartheta)\eta = g_z(\vartheta)\eta(z)\,, \int_0^{2\pi} |g_z(\vartheta)|^2\mathrm{d}\vartheta = 2\pi$$

in the phase representation, corresponding to $|z| \uparrow 1$. However, as one can easily see from (1.3), this condition is in contradiction with the reproducibility condition $|c(\vartheta)|^2 = |h(\vartheta)|^2$ of the probability densities $|h(\vartheta)|^2$ corresponding to the best unsharp phase measurement $(\mathrm{e}^{i\vartheta}|\eta = h(\vartheta) = \eta(\mathrm{e}^{i\vartheta})$.

This is because there is no square integrable complex function $\vartheta \mapsto g_z(\vartheta)$ on $[0, 2\pi[$, with $|g_z(\vartheta)|^2 = \frac{1}{2\pi}\delta(\vartheta - \mathrm{Arg}\,z)$ for $|z| = 1$ and hence (1.3) does not hold for all $\eta \in \mathcal{H}$ and $|c(\vartheta)| = |\eta(\mathrm{e}^{i\vartheta})|$.

This difficulty which makes all attempts to study a sequential process of such demolition measurements meaningless, is not exceptional for the phase. It is inseparable from any quantum sharp measurement of usual observables like a position with a continuous spectrum. In order to obtain a reduction transformation for the sequential or even time continuous phase measurements satisfying condition (1.2), it is necessary to consider a scheme with indirect phase observation or a discretization of the phase spectrum as was proposed in [**?**].

According to the NDP theory [**?**], we must consider another quantum system to describe an apparatus with a given pointer observable whose spectral values $\lambda \in [0, 2\pi[$ are first assumed as discrete ones $\lambda \in \{0, \varepsilon, \ldots, s\varepsilon\} \equiv \Lambda_s$, where $s \geq k$ and $\varepsilon = \frac{2\pi}{s+1}$, $k \geq 0$ is any integer. The minimal $(s + 1)$–dimensional Hilbert space $\mathcal{E}_s$ for this apparatus consists of the set of all discrete complex functions $f : \Lambda_s \to \mathbf{C}$ which can be considered as the restrictions on $\Lambda_s \subset [0, 2\pi[$ of the trigonometric functions $f(\lambda) = \sum_{n=-k}^{s-k} a_n \mathrm{e}^{-i\lambda n}$.

This constitutes the isometric embedding

$$\mathcal{E}_s \subset \mathcal{E}, \quad \int_0^{2\pi} |f(\lambda)|^2 \mathrm{d}\lambda = \sum_{l=0}^{s} |f(\varepsilon l)|^2 \varepsilon$$

into the Hilbert space $\mathcal{E} = L^2(\Lambda)$ of all square–integrable functions on $\Lambda = [0, 2\pi[$ given by the interpolation formula $f(\lambda) = \sum_{l=0}^{s} \delta_l^\varepsilon(\lambda) f(\varepsilon l)\varepsilon$, where $\delta_l^\varepsilon(\lambda)$ stands for $\frac{1}{2\pi}\sum_{n=-k}^{s-k} \mathrm{e}^{in(\varepsilon l - \lambda)}$.

We assume that at the initial time, the oscillator and the apparatus are independent and the state of the apparatus is fixed $g \in \mathcal{K}$, $\|g\| = 1$, in the $(k+1)$–dimensional subspace $\mathcal{K} = \mathcal{E}_k$ of the trigonometric polynomials $g(\lambda) = \sum_{m=0}^{k} b_m \mathrm{e}^{im\lambda}$. The compound system at this stage is described by the product state vectors $\eta \otimes g$, in $\mathcal{H} \otimes \mathcal{K}$ so that their joint wave function $\psi(z, \lambda)$ in the (z, λ)–representation $(|z| \le 1, \lambda \in \Lambda)$, is initially written as $\psi_0(z, \lambda) = \eta(z) g(\lambda)$. At the instant of the measurement, the oscillator and the apparatus are coupled by the scattering operator $S = \exp(-i\widehat{\varphi} \otimes M)$ where M is the angular momentum operator $-i\frac{d}{d\lambda}$ in $\mathcal{E}$, having the simple finite spectrum $\{0, 1, \ldots, k\}$ on the invariant subspace $\mathcal{K} \subset \mathcal{E}$, because $M|n\rangle = -n|n\rangle$, $\forall\, n$. Thus defined in $\mathcal{E} \otimes \mathcal{E}$, the operator S is also unitary in the invariant Hilbert subspace $\mathcal{E} \otimes \mathcal{K}$ and it is isometric on $\mathcal{H} \otimes \mathcal{K} \subset \mathcal{E} \otimes \mathcal{K}$ as follows from the phase–momentum compatibility property

$$(\mathrm{e}^{i\vartheta}| \otimes \langle n| S = (\mathrm{e}^{i\vartheta}| \mathrm{e}^{i\widehat{\varphi} n} \otimes \langle n| = \mathrm{e}^{in\vartheta} (\mathrm{e}^{i\vartheta}| \otimes \langle n| \tag{1.4}$$

for $n = 0, -1, \ldots, -k$ as a result of the positivity of the spectrum $m = -n$ of $M = \sum_{n \le 0} |n\rangle m \langle n|$ in $\mathcal{K}$ and the property $(\mathrm{e}^{i\vartheta}| \mathrm{e}^{i\widehat{\varphi}} = \mathrm{e}^{i\vartheta} (\mathrm{e}^{i\vartheta}|$ (1.1). Note that although the unitary scattering $S_t = 1_t S$, where $1_t = 0$ if $t \le 0$ and $1_t = 1$ if $t > 0$, admits the formal Hamiltonian $\hbar\widehat{\varphi} \otimes \delta(t) M$ in the extended space $\mathcal{E} \otimes \mathcal{K}$, the correspondent isometry $Z^{\dagger M} = S|(\mathcal{H} \otimes \mathcal{K})$ has no Hamiltonian as being adjoint to the partial isometry $Z^M = \sum_{n \le 0} |n\rangle Z^{-n} \langle n|$.

Let us denote by $\{|\lambda\rangle : \lambda \in \Lambda_s\}$, $|\lambda\rangle = \sum_{n=-k}^{s-k} \mathrm{e}^{i\lambda n} |n\rangle$, $\lambda \in \Lambda_s$, the orthogonal family of eigenvectors for the diagonal operator $[\hat{q} f](\lambda) = \lambda f(\lambda)$ in $\mathcal{E}_s$ of the discrete angle $\lambda \in \Lambda_s$, considered in [?] as an approximate finite dimensional phase operator. The partial matrix elements

$$G(\lambda) = \langle \lambda | S g = g(\lambda - \hat{\varphi}), \quad \lambda \in \Lambda_s \tag{1.5}$$

for a given $g \in \mathcal{K}$, define the linear reduction transformations $\mathcal{H} \to \mathcal{H}$,

$$[G(\lambda)\eta](z) = g(\lambda - i \ln z^*)\, \eta(z) \tag{1.6}$$

satisfying the nondemolition condition (1.2) with the trigonometric polynomial function $g(\lambda) = g(\lambda - \vartheta)$ for all $z = \mathrm{e}^{i\vartheta}$, $\lambda \in \Lambda_s$. If the initial state vector g of the meter was localized at the point $\lambda = 0$, then the posterior state vectors $\chi_\lambda = \frac{G(\lambda)\eta}{c(\lambda)}$ in the phase representation $x_\lambda(\vartheta) = \lim_{r\uparrow 1} \chi_\lambda(r\mathrm{e}^{i\vartheta})$ are given by the localization

$$x_\lambda(\vartheta) = \frac{g(\lambda - \vartheta)h(\vartheta)}{c(\lambda)} \tag{1.7}$$

of the initial wave function $h(\vartheta) = \lim_{r\uparrow 1} \eta(r\mathrm{e}^{i\vartheta})$ around the result of the measurement $\lambda \in \Lambda_s$ with $c(\lambda) = \|G(\lambda)\eta\|$. This result also holds for the phase measurement with the continuous spectrum $\Lambda = [0, 2\pi[$, because the reduced description of the non–demolition measurement in terms of the transformations (1.5), only depends on the initial state-vector g of the $k+1$ dimensional space $\mathcal{K}$ and does not depend on the dimensionality $s+1$ of the space $\mathcal{E}_s$ so that the continuous limit $s \to \infty$ appears straightforward.

Note that the periodic function $g(\lambda) = \sum_{m=0}^{k} b_m \mathrm{e}^{i\lambda m}$, representing the initial state vector $g \in \mathcal{K}$ on the measurement scale Λ_s, can be sharply localized at a point, say $\lambda = 0$ if and only if $s = k$ and $k < \infty$. The unsharply localized $g(\lambda)$ defines the $(\mathrm{mod}\, 2\pi)$ additive noise $\lambda_0 \in \Lambda_s$ to the quantum phase ϑ under the nondemolition measurement $\lambda = \vartheta + \lambda_0$ in the sense that the probability density (1.3) is given by the convolution

$$|c(\lambda)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |g\,(\lambda - \vartheta)\,|^2 |h\,(\vartheta)\,|^2 \mathrm{d}\vartheta \tag{1.8}$$

of the periodic distributions $|g\,(\lambda)\,|^2$ and $|h\,(\vartheta)\,|^2$.

As an example, let us consider an initial state vector $g \in \mathcal{E}_s$, $(k = s)$, specified by the following limiting cases: a): $s < \infty$, $r \to 1$ and b): $s \to \infty$, $r < 1$ of the geometric sequence $b_m = \sqrt{\frac{1-r^2}{1-r^{2(s+1)}}} r^m$, $m = 0, \ldots, s$. The reduction transformations (1.5) in the phase representation, are defined by

$$g(\lambda - \vartheta) = \sqrt{\frac{1-r^2}{1-r^{2(s+1)}} \frac{1-r_{\vartheta-\lambda}^{s+1}}{1-r_{\vartheta-\lambda}}}, \quad r_\vartheta = r\mathrm{e}^{-i\vartheta}$$

The conditional probability density $|g(\lambda - \vartheta)|^2$ is given by

$$|g(\lambda)|^2 = \frac{1-r^2}{1-2r\cos\lambda + r^2} \frac{1-2r^{s+1}\cos(s+1)\lambda + r^{2(s+1)}}{1-r^{2(s+1)}} \tag{1.9}$$

which is localized on Λ_s at the point $\lambda = 0$ if $r < 1$, with $\sum_{\lambda\in\Lambda_s} |g(\lambda)|^2 = s+1$. The characteristic sequence $c_s^\vartheta(m) = \frac{1}{s+1}\sum_{\lambda\in\Lambda_s} |g(\lambda-\vartheta)|^2 \mathrm{e}^{-i\lambda m}$, $(0 \le m \le s)$ of this distribution, has the form of the convolution

$$(b^\vartheta * b^\vartheta)_m = \sum_{k+n=m}^{\mathrm{mod}\,(s+1)} (b_{-n}^\vartheta)^* b_k^\vartheta$$

of $b_n^\vartheta = b_n \mathrm{e}^{-i\vartheta n}$ and $(b_{-n}^\vartheta)^* = b_{-n}^{*-\vartheta}$, so that

$$\begin{aligned} c_s^\vartheta(m) &= \mathrm{e}^{-i\vartheta m}\sum_{n=0}^{s-m} b_n^* b_{n+m} + \mathrm{e}^{i\vartheta(s+1-m)}\sum_{n=0}^{m-1} b_{n+s+1-m}^* b_n \\ &= \frac{1-r^2}{1-r^{2(s+1)}} \sum_{k+n=m}^{\mathrm{mod}(s+1)} r_{-\vartheta}^{-n} r_\vartheta^k \\ &= \frac{1-r^{2(s+1-m)}}{1-r^{2(s+1)}} r_\vartheta^m + \frac{1-r^{2m}}{1-r^{2(s+1)}} r_{-\vartheta}^{s+1-m} \end{aligned} \tag{1.10}$$

Since $c_s^\vartheta(m) = \mathrm{e}^{-i\vartheta m}$ for $r = 1$ and $\vartheta \in \Lambda_s$, the conditional distribution $|g(\lambda-\vartheta)|^2$ is sharply localized on Λ_s at $\vartheta = \lambda$ only in the limit case a) and it is almost sharply localized in case b) if $\varepsilon = 1 - r \ll 1$. This is because $c^\vartheta(m) = \lim_{s\to\infty} c_s^\vartheta(m) = r_\vartheta^m \simeq (1 - m\varepsilon)\mathrm{e}^{-i\vartheta m}$ for all $m = 0, 1, \ldots$.

2 The sequential phase measurements

Now let us fix a countable set $\tau = \{t_1, t_2, \ldots\}$ of time instants $\{t_1 < t_2 < \ldots\}$ for the nondemolition phase measurements which are described as previously in terms of the independent meters at the different $t \in \tau$. We shall suppose that $\tau_t = \tau \cap [0, t[$ is a finite subsequence for each $t < \infty$, i.e. $|\tau_t| < \infty$, where $|\tau|$ is the cardinality $n = 0, 1, \ldots$ of $\tau = \{t_1, , \ldots, t_n\}$ with $n = 0$ corresponding to the empty sequence $\tau = \emptyset$. Let us denote by $\tau(\cdot)$: $\lambda \mapsto \tau(\lambda)$ a partition

$$\tau = \sum_{\lambda\in\Lambda_s} \tau(\lambda) = \cup_{\lambda\in\Lambda_s}\tau(\lambda), \quad \tau(\lambda)\cap\tau(\lambda') = \emptyset, \quad \forall\, \lambda' \ne \lambda$$

into the subsets $\tau(\lambda) = \{t_n \in \tau | \lambda_n = \lambda\}$, corresponding to the fixed values $\lambda \in \Lambda_s$ in a sequence $(\lambda_1, \lambda_2, \ldots)$ of the measurement data $\lambda_n \in \Lambda_s$ at the given time instants t_n and by $n_t(\lambda) = |\tau_t(\lambda)|$ the corresponding cardinalities of $\tau_t(\lambda) = \tau(\lambda) \cap [0, t[$, defining a number distribution

$$n_t(\cdot) : \lambda \mapsto n_t(\lambda), \quad \sum_{\lambda \in \Lambda_s} n_t(\lambda) = n_t$$

on the finite subset $\Lambda_s \subset [0, 2\pi[$ for each $t < \infty$. The partitions $\tau(\cdot)$ describe the graphs $\upsilon = \{y_1, y_2, \ldots\} \subset \tau \times \Lambda_s, y_n = (t_n, \lambda_n)$ of the maps $y : t_n \in \tau \mapsto \lambda_n \in \Lambda_s$, giving a measurement result $y(t) = \lambda$ at a time instant $t \in \mathbf{R}_+$ if and only if $\mathrm{d}_t n(\lambda) = 1$, where $\mathrm{d}_t n(\lambda) = n_{t+\mathrm{d}t}(\lambda) - n_t(\lambda)$, $0 < \mathrm{d}t < t_{n+1} - t_n$ for each $t \in [t_n, t_{n+1}[$.

In these terms one can define the stochastic propagator $T(t, \upsilon) : \mathcal{H} \to \mathcal{H}$, corresponding to a random sequence $\upsilon_t = (y_1, y_2, \ldots, y_n)$ of the events $y \in \tau_t \times \Lambda_s$ for the sequential phase measurements up to a time instant $t \in [t_n, t_{n+1}[$ as a chronological product

$$T(t, \upsilon) = U(t - t_n) G(\upsilon_t), \quad G(\upsilon_t) = G(y_n) \ldots G(y_1) \tag{2.1}$$

Here $U(t) : \mathcal{H} \to \mathcal{H}$ is the unitary time–evolution operator of the quantum harmonic oscillator

$$U(t) = \sum_{n=0}^{\infty} |n\rangle \mathrm{e}^{-i\omega n t} \langle n| = \mathrm{e}^{-i\omega N t} \tag{2.2}$$

where N is the photon number operator. The reduction transformations (1.5) are taken in the Heisenberg picture

$$G(t, \lambda) = U(t)^{\dagger} G(\lambda) U(t) = g\left(\lambda I - i \ln Z(t)^{\dagger}\right) \tag{2.3}$$

where we have $Z(t) = U(t)^{\dagger} Z U(t) = \mathrm{e}^{-i\omega t} Z$ as follows from property (1.1) and also $U(t)|z) = |z\mathrm{e}^{-i\omega t})$. Such oscillating operators $G(t, \lambda) = G(\lambda + \omega t)$ commute for the different values of $(t, \lambda) \in \mathbf{R}_+ \times \Lambda_s$ due to the commutativity of the operators $G(\lambda) = g\left(\lambda I - i \ln Z^{\dagger}\right)$ at different $\lambda \in \Lambda_s$.

The reduced state vector $\chi(t, \upsilon_t) = T(t, \upsilon)\eta$ is not normalized for a fixed sequence of measurement events $y = (t, \lambda)$, but it is normalized in the mean square sense

$$\langle \|\chi(t)\|^2 \rangle_o = \int_{\Upsilon_t} \|G(\upsilon_t)\eta\|^2 \mathrm{P}_o(\mathrm{d}\upsilon_t) = 1, \quad \forall \eta : \|\eta\| = 1 \tag{2.4}$$

with respect to $\mathrm{P}_0(\mathrm{d}\upsilon_t) = \pi_0(\mathrm{d}\tau_t)(s+1)^{-n_t}$ which is the probability measure of $\mathrm{d}\upsilon_t = \mathrm{d}\tau_t \times \{\lambda_1, \ldots, \lambda_n\}$. The measure P_0 is defined on $\Upsilon_t = \sum_{n\geq 0} \Gamma_t(n) \times \Lambda_s^n$ by any probability measure $\pi_0(\mathrm{d}\tau)$ on the space $\Gamma_t = \bigcup_{n\geq 0} \Gamma_t(n)$, the union of the spaces $\Gamma_t(n)$ of all finite $\tau = \{0 \leq t_1 < \ldots < t_n < t\}, n = 0, 1, \ldots$ with $\tau = \emptyset$ for $n = 0$, and $\tau = t$ for $n = 1$.

Indeed, the integral over Υ_t can be written, by definition, as an integral over Γ_t

$$\int_{\Upsilon_t} \|G(\upsilon_t)\eta\|^2 \mathrm{P}_o(\mathrm{d}\upsilon_t) = \\ \int_{\Gamma_t} \frac{1}{(s+1)^{n_t}} \sum_{\lambda\in\Lambda_s} \|G(t_{n_t}, \lambda_{n_t}) \ldots G(t_1, \lambda_1)\eta\|^2 \pi_o(\mathrm{d}\tau_t) \tag{2.5}$$

and

$$\sum_{\lambda\in\Lambda_s} \|G(t_n, \lambda_n) \ldots G(t_1, \lambda_1)\eta\|^2 = (s+1)^n \|\eta\|^2$$

as

$$\|G(t_n, \lambda_n) \ldots G(t_1, \lambda_1)\eta\|^2 = \\ \frac{1}{2\pi} \int_0^{2\pi} \Big| g(\lambda_n - \vartheta_n) \ldots g(\lambda_1 - \vartheta_1) h(\vartheta) \Big|^2 \mathrm{d}\vartheta\,, \tag{2.6}$$

where $\vartheta_n = \vartheta - \omega t_n$. One can easily show [5] that the map $\chi(t) : \upsilon \mapsto \chi(t, \upsilon)$ as a $\mathcal{H}$–valued stochastic processes, satisfies the filtering wave equation

$$\mathrm{d}\chi(t) + i\omega N\chi(t)\mathrm{d}t = \sum_{\lambda\in\Lambda_s} \Big[G(\lambda) - I\Big]\chi(t)\,\mathrm{d}_t n(\lambda) \tag{2.7}$$

in the Ito sense: $\mathrm{d}\chi(t) = \chi(t + \mathrm{d}t) - \chi(t)$, with the independent increments $\mathrm{d}_t n(\lambda) \in \{0, 1\}$, according to the following multiplication table:

$$\mathrm{d}_t n(\lambda)\mathrm{d}_t n(\lambda') = \mathrm{d}_t n(\lambda)\delta_{\lambda\lambda'}, \quad \delta_{\lambda\lambda'} = \begin{cases} 1\,, & \lambda' = \lambda \\ 0\,, & \lambda' \neq \lambda \end{cases} \tag{2.8}$$

Thus, the stochastic differential equation (2.7) for an initial state vector $\chi(0) = \eta$ defines uniquely the statistics of the sequential phase measurements by the probability density $\|\chi(t, \upsilon)\|^2$ of the *output* measure

$$\mathrm{P}(\mathrm{d}\upsilon_t) = \|G(\upsilon_t)\eta\|^2 \mathrm{P}_o(\mathrm{d}\upsilon_t)\,, \quad \upsilon_t \in \Upsilon_t \tag{2.9}$$

restricted to any finite $t \in \mathbf{R}_+$ with respect to a given input probability measure P_o of $\upsilon \in \mathrm{d}\upsilon_t$. It also defines the *a posteriori* state vector $\chi_\upsilon(t) = \frac{\chi(t,\upsilon)}{\|\chi(t,\upsilon)\|}$ if the probability density of the observation of the sequence υ is not equal to 0.

To study a specific example, let us fix the stationary Poisson probability measure $\pi_o(\mathrm{d}\tau)$ given on the set $\Gamma = \Gamma_t \times \Gamma_{[t}$ of all sequences $\tau = \left(\tau_t, \tau_{[t}\right)$ with finite $\tau_t \subset [0,t[$ and infinite $\tau_t \subset [t,\infty[$ for any $t < \infty$, by the marginal distributions

$$\pi_o(\mathrm{d}\tau) = \nu^{n_t} \exp(-\nu t)\mathrm{d}\tau_t, \quad \mathrm{d}\tau_t = \prod_{n=1}^{n_t} \mathrm{d}t_n \tag{2.10}$$

with an intensity $\nu > 0$ and $\mathrm{d}\tau_t = 1$ if $n_t = 0$. The Lebesque measure $\mathrm{d}\tau$ on Γ corresponds to the exponential integral

$$\int_\Gamma f(\tau)\mathrm{d}\tau = \sum_{n=0}^{\infty} \underset{0\le t_1 \dots < t_n < \infty}{\int \dots \int} f(t_1,\dots,t_n)\mathrm{d}t_1 \dots \mathrm{d}t_n$$

taking the value

$$\int_{\Gamma_t} f(\tau)\mathrm{d}\tau = \exp\left(\int_0^\infty f(t)\mathrm{d}t\right)$$

if $f(\tau) = \prod_{t\in\tau} f(t)$ for all $\tau \in \Gamma$.

The output probability measure (2.9) on the space $\Upsilon = \Gamma \times \Lambda^\infty$ of all infinite sequences $\upsilon = (\upsilon_1, \upsilon_2, \dots)$ with finite $\upsilon_t = (y_n)_{n\le n_t} \in \Upsilon_t$ is completely defined by the limit $t' \to \infty$ of the characteristic functional

$$\varphi_s(t', l) = \int_{\Upsilon_{t'}} \exp\Big(-i \int_0^{t'} \sum_{\lambda\in\Lambda_s} l(t,\lambda)\mathrm{d}_t n(\lambda)\Big)\mathrm{P}(\mathrm{d}\upsilon)$$

as the expectation value of $f(\upsilon) = \prod_{y\in\upsilon} \mathrm{e}^{-il(y)}$, given by

$$\varphi_s(t', l) = \int_\Upsilon f(\upsilon)\mathrm{P}(\mathrm{d}\upsilon) = \frac{1}{2\pi}\int_0^{2\pi} \varphi_s^\vartheta(t', l)\,|h(\vartheta)|^2\mathrm{d}\vartheta$$

Therefore

$$\varphi_s^\vartheta(t', i\ln f) \quad = \quad \int_{\Upsilon_{t'}} \prod_{y\in\upsilon} \Big(f(y)|g_\vartheta(y)|^2\Big)\mathrm{P}_o(\mathrm{d}\upsilon)$$

$$
\begin{aligned}
&= \int_{\Gamma_{t'}} \prod_{t\in\tau} \left(\frac{1}{s+1} \sum_{\lambda\in\Lambda_s} f(t,\lambda) |g_\vartheta(t_n,\lambda)|^2 \right) \pi_o(\mathrm{d}\tau) \\
&= \mathrm{e}^{-\nu t'} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_0^{t'} \frac{\nu}{s+1} \sum_{\lambda\in\Lambda_s} f(t,\lambda) |g_\vartheta(t,\lambda)|^2 \pi_o(\mathrm{d}t) \right)^n \\
&= \exp\left(\frac{\nu}{s+1} \int_0^{t'} \sum_{\lambda\in\Lambda_s} [f(t,\lambda)-1] |g_\vartheta(t,\lambda)|^2 \mathrm{d}t \right)
\end{aligned}
$$

where we considered that $g_\vartheta(t,\lambda) = g(\lambda+\omega t-\vartheta)$.

These functions are normalized so that $\sum_{\lambda\in\Lambda_s} |g_\vartheta(t,\lambda)|^2 = s+1$, $\forall\, \vartheta \in [0,2\pi[$. In particular, one obtains the characteristic sequence

$$
\mathrm{C}_s(t,m) = \int_{\Upsilon_t} \mathrm{e}^{-im\Theta_t(\upsilon)} \mathrm{P}(\mathrm{d}\upsilon)
$$

of the second quantized phase $\Theta_t(\upsilon) = \sum_{\lambda\in\Lambda_s} \lambda n_t(\lambda)$ at time instant t. It is given as the mean value of $\mathrm{C}_s^\vartheta(t',m) = \varphi_s^\vartheta(t',l)$ at $l(t,\lambda) = m\lambda$,

$$
\mathrm{C}_s(t,m) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(\int_0^{t'} [\mathrm{c}_s^\vartheta(t,m)-1]\mathrm{d}t \right) |h(\vartheta)|^2 \mathrm{d}\vartheta
$$

where $\mathrm{c}_s^\vartheta(t,m) = \mathrm{c}_s^{\vartheta(t)}(m)$ is the characteristic sequence of the distribution on $|g(\lambda+\omega t-\vartheta)|^2$, having form (1.10) in the case of the geometric sequence b_m given by (1.9).

3 The spontaneous phase localization

First of all, let us seek the general filtering wave equation for a sequential unsharp nondemolition measurements of the quantum phase. More precisely, we want to modify equation (2.7) for the case of continuous observation of the stochastic process only

$$
\Phi_t(\upsilon) = \sum_{n=1}^{n_t} \lambda_n - 2\pi \left\lfloor \frac{1}{2\pi} \sum_{n=1}^{n_t} \lambda_n \right\rfloor = \varphi\left(\sum_{n=1}^{n_t} \lambda_n \right) \tag{3.1}
$$

It is given by the random sums $\sum_{n=1}^{n_t} \lambda_n = \sum_{\lambda\in\Lambda_s} \lambda n_t(\lambda)$ (mod.2π) of the angular positions $\lambda_n \in \Lambda_s$ of the independent pointers at the random instants $t_n \in \mathbf{R}_+$ with the Poisson distribution (2.10).

This is a stochastic Λ_s–valued process having the spontaneous jumps $\lambda_n \neq 0$ at the time instants $t_n \in \sum_{\lambda\in\Lambda_s^\circ}\tau(\lambda)$, where $\tau(\lambda)=\{t_n\in\tau|\lambda_n=\lambda\}$ and the domain $\Lambda_s^\circ=\{\varepsilon,2\varepsilon,\ldots,s\varepsilon\}$, $\left(\varepsilon=\frac{2\pi}{s+1}\right)$ does not contain 0. When there is no jump of $\Phi_t(\upsilon)$, $t\notin\sum_{\lambda\in\Lambda_s^\circ}\tau(\lambda)$, there is no reduction of the state vector, and $\mathrm{d}\chi_t$ must be proportional to the infinitesimal increment $\mathrm{d}t$. The general filtering equation (2.7) now takes the following linear form [6]

$$\mathrm{d}\chi(t)+K\chi(t)\mathrm{d}dt=\sum_{\lambda\in\Lambda_s}L(\lambda)\chi(t)\mathrm{d}_t n(\lambda) \tag{3.2}$$

where K and $L(\lambda)$ are the operators in $\mathcal{H}$, such that $K-K^\dagger=2i\omega N$ and $L(0)=0$, due to the continuity of $\chi(t)$ at $t\in\tau(0)$. The self–adjoint part of the operator K can be derived from the condition $\langle\mathrm{d}\|\chi(t)\|^2\rangle_0=0$ of the mean square normalization (2.4) by application of the Ito formula

$$\begin{aligned}\mathrm{d}\|\chi\|^2 &= 2\Re(\chi|\mathrm{d}\chi)+\|\mathrm{d}\chi\|^2\\ &= -2\Re(\chi|K\chi)\mathrm{d}t+\sum_{\lambda\in\Lambda_s}\Big[2\Re(\chi|L(\lambda)\chi)+\|L(\lambda)\chi\|^2\Big]\mathrm{d}_t n(\lambda)\end{aligned}$$

From the conditions that the increments $\mathrm{d}_t n(\lambda)$ are independent and that the mean value is such that $\langle\mathrm{d}_t n(\lambda)\rangle=\nu(\lambda)\mathrm{d}t$, it is derived

$$K+K^\dagger=\sum_{\lambda\in\Lambda_s}\Big[L(\lambda)+L(\lambda)^\dagger L(\lambda)+L(\lambda)^\dagger\Big]\nu(\lambda) \tag{3.3}$$

where $\nu(\lambda)\geq 0$, is a given intensity distribution $\nu=\sum_{\lambda\in\Lambda_s}\nu(\lambda)$ of the Poisson process $n_t=\sum_{\lambda\in\Lambda_s}n_t(\lambda)$.

This completely defines the modification of the Eq.(2.7) if $L(\lambda)$ is taken as $G(\lambda)-G(0)$ such that the summation in (3.2) is important only over $\lambda\in\Lambda_s^\circ$. The stochastic propagator $T(t,\upsilon):\eta\mapsto\chi(t,\upsilon)$ for this equation is given as in (2.1) by the nonunitary time evolution operator $U(t)=\mathrm{e}^{-Kt}$ and

$$G(t,\lambda)=\mathrm{e}^{Kt}\Big[L(\lambda)+I\Big]\mathrm{e}^{-Kt},\quad G(t,0)=I \tag{3.4}$$

It is defined only on the pairs $\upsilon_n\in\upsilon$ with $\lambda_n\neq 0$ and normalized w.r.t. the Poisson measure $\mathrm{P}_o(\mathrm{d}\upsilon_t)=\exp\Big((\nu(0)-\nu)t\Big)\prod_{n=1}^{n_t-n_t(0)}\Big[\nu(\lambda_n)\mathrm{d}t_n\Big]$ on the space Υ° of all sequences $\upsilon=\{\upsilon_1,\upsilon_2,\ldots\}$, $\upsilon_n\in\mathbf{R}_+\times\Lambda_s^\circ$ with $\lambda\neq 0$.

Let us now examine the limit at $s\to\infty$ corresponding to the continuous scale of the phase meter. First, we assume that the intensity $\nu(\lambda)=\nu_s(\lambda)$ of the jumps per degree of freedom, is inversely proportional to their number $s+1$, $\nu_s(\lambda) = \frac{2\pi}{s+1}\dot{\nu}(\lambda)$. Therefore one can obtain the limit $s \to \infty$

$$\mathrm{d}\chi(t) + i\omega N\chi(t)\mathrm{d}t = \int_0^{2\pi} \Big[G(\lambda) - I\Big]\chi(t)\mathrm{d}_t n(\mathrm{d}\lambda) \tag{3.5}$$

of the stochastic wave equation (2.7) and in the same way, the continuous limit of the modified equation (3.2). It is controlled by the stochastic Poisson measure $n_t(\Delta) = |\upsilon \cap ([0,t[\times\Delta| = \int_\Delta \mathrm{d}_\lambda n_t$, counting the number of events $(t_n, \lambda_n) \in \upsilon$ on the time interval $[0,t[$ in a phase domain $\Delta \subseteq [0,2\pi[$ or $(\Delta \subseteq]0,2\pi[)$ where $\mathrm{d}_\lambda n_t = n_{t,\lambda+\mathrm{d}\lambda} - n_{t,\lambda} \equiv n_t(\mathrm{d}\lambda)$, $n_{t,\lambda} = n_t\left([0,\lambda[\right)$. The increments $\mathrm{d}_t n(\Delta) = n_{t+\mathrm{d}t}(\Delta) - n_t(\Delta)$ satisfy the multiplication table

$$\mathrm{d}_t n(\Delta)\mathrm{d}_t n(\Delta') = \mathrm{d}_t n(\Delta \cap \Delta') \tag{3.6}$$

that is a generalization of the discrete table (2.8).

The mean value $\langle n_t(\Delta)\rangle = t\nu(\Delta)$ of such a process is given by an intensity measure $\nu(\Delta) \geq 0$, characterizing the operator $K+K^\dagger$ in (3.2) as the integral

$$K + K^\dagger = \int_0^{2\pi} \Big[L(\lambda) + L(\lambda)^\dagger L(\lambda) + L(\lambda)^\dagger\Big]\nu(\mathrm{d}\lambda) \tag{3.7}$$

respectively to $\nu(\mathrm{d}\lambda) = \dot{\nu}(\lambda)\mathrm{d}\lambda = \mathrm{d}_\lambda\nu$ where $\mathrm{d}_\lambda\nu = \nu_{\lambda+\mathrm{d}\lambda} - \nu_\lambda$, $\dot{\nu}(\lambda) = \frac{\mathrm{d}_\lambda\nu}{\mathrm{d}\lambda}$. Note that due to $\nu(0) = \frac{\dot{\nu}(0)}{s+1} \to 0$ as $s \to \infty$, the operator–valued function $L(0)$ cannot necessarily be equal to 0. The stochastic propagator (2.1) resolving the equation (3.5) is then normalized as in (2.4) respectively to the probability measure $\mathrm{P}_o(\mathrm{d}\upsilon_t)$ on $\Upsilon_t = \sum_{n\geq 0}\Gamma_t(n) \times \Lambda_s^n$, $\quad \Lambda = [0,2\pi[$ having the density $\mathrm{e}^{-\nu t}\prod_{n=1}^{n_t}\dot{\nu}(\lambda_n)$, $\nu = \int \dot{\nu}(\lambda)\mathrm{d}\lambda$ with respect to $\mathrm{d}\upsilon_t = \prod_{n=1}^{n_t}(\mathrm{d}\lambda_n \mathrm{d}t_n)$. Thus the continuous limit of the modified equation considered in (3.5) in the case $L(\lambda) = G(\lambda) - I$ and $\dot{\nu}(\lambda) = \frac{\nu}{2\pi}$ as $K + K^\dagger = 0$ in this case.

The operators $G(\lambda) = g(\lambda I - \hat{\varphi})$ are defined in the continuous phase limit by a meter wave function $g(\lambda) = \sum_{m=0}^\infty b_m \mathrm{e}^{im\lambda}$ in the infinite dimensional space $\mathcal{K} \sim \mathcal{H}$.

The output probability measure (2.9) of the continuous phase measurement is now described by the characteristic functional

$$\Phi_s^\vartheta(t', l) = \int_{\Upsilon_{t'}} \exp\Big(-i\int_0^{t'}\int_0^{2\pi} f(t,\lambda)\mathrm{d}_t n(\mathrm{d}\lambda)\Big)\mathrm{P}(\mathrm{d}\upsilon_t) \tag{3.8}$$

of the form $\Phi_s(t',l) = \frac{1}{2\pi}\int\limits_0^{2\pi} \Phi_s^\vartheta(t',l)|h(\vartheta)|^2$, where

$$\Phi_s^\vartheta(t', i\ln f) = \exp\Big(\frac{\nu}{2\pi}\int\limits_0^{t'}\int\limits_0^{2\pi}\Big(f(t,\lambda)-1\Big)|g_\vartheta(t,\lambda)|^2\mathrm{d}\lambda\mathrm{d}t\Big) \tag{3.9}$$

with $g_\vartheta(t,\lambda) = g(\lambda+\omega t-\vartheta)$ normalized so that $\int |g_\vartheta(t,\lambda)|^2\mathrm{d}\lambda = 2\pi$.

In particular, the characteristic sequence of the second-quantized continuous phase $\Theta_t(v) = \int_0^{2\pi}\lambda n_t(\mathrm{d}\lambda)$ at time t, is given by the mean value of

$$C^\vartheta(t',m) = \exp\Big(\nu\int\limits_0^{t'}[c^\vartheta(t,m)-1]\mathrm{d}t\Big) \tag{3.10}$$

respectively to the initial phase distribution $|h(\vartheta)|^2\mathrm{d}\vartheta/2\pi$. Here, $c^\vartheta(t,m) = \mathrm{e}^{im(\omega t-\vartheta)}c(m)$ is the characteristic sequence

$$c(m) = \frac{1}{2\pi}\int_0^{2\pi}|g(\lambda)|^2\mathrm{e}^{-i\lambda m}\mathrm{d}\lambda,$$

that is geometric $c(m) = r^{|m|}$ in the case

$$|g(\lambda)|^2 = \frac{1-r^2}{1-2r\cos(\lambda)+r^2}, \quad -1<r<1$$

corresponding to $b_m = \sqrt{1-r^2}r^m \quad m = 0,1,2,\ldots$.

4 The singular limit of quantum phase

Now let us consider a singular measurement limit $s\to\infty$ for the spontaneous phase (3.1) with continuous jumps $\lambda_n \in [0,2\pi[$, when the intensity of the spontaneous jumps per degree of freedom is fixed as $\frac{1}{s+1}\sum_{\lambda\in\Lambda_s}\nu(\lambda) = 1$, i.e. $\nu = s+1\to\infty$. Suppose also that $l_s(m) = \sum_{\lambda\in\Lambda_s}[\mathrm{e}^{-im\lambda}-1]\nu(\lambda)$ has a finite limit $l(m)$ at $s\to\infty$, $\forall\, m = 0,\pm1,\pm2,\ldots$ with the following properties $l(0) = 0, \quad l(-n) = l(n)^*$,

$$\sum_n c_n = 0 \Rightarrow \sum_n c_n^* l(m-n)c_m \geq 0, \quad \forall\, c_n\in\mathbf{C}. \tag{4.1}$$

Therefore the normalized distribution $\nu_s(\lambda) = \frac{\nu(\lambda)}{s+1}$ on Λ_s in the limit that $k \to \infty$, $s = 2k$, is concentrated at $\lambda = 0$, $\nu(\lambda) \sim 2\pi\delta^\varepsilon(\lambda)$, where

$$\delta^\varepsilon(\lambda) = \frac{1}{2\pi}\sum_{m=-k}^{k} \mathrm{e}^{im\lambda} = \frac{\cos((k+1)\lambda) - \cos(k\lambda)}{2\pi(\cos(\lambda) - 1)} = \begin{cases} \frac{1}{\varepsilon}, & \lambda = 0 \\ 0, & \lambda \in \Lambda_s^\circ \end{cases} \tag{4.2}$$

is the Dirac function $\delta(\lambda)$ on $[0, 2\pi[$ at $\varepsilon = \frac{2\pi}{2k+1} \to 0$.

However, the probability of the jumps $\lambda_n \neq 0$ of $\Phi_t(\upsilon)$ does not tend to 0 because they have a non zero intensity function $\dot{\nu}(\lambda) = \lim_{k\to\infty} \frac{1}{\varepsilon}\nu(\lambda)$, coinciding on $]0, 2\pi[$ with the distribution

$$\dot{\gamma}(\lambda) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} l(m)\mathrm{e}^{im\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\gamma_\lambda \tag{4.3}$$

where $\gamma_\lambda = \gamma_\lambda^+ + \gamma_\lambda^-$, with $\gamma_\lambda^+ = \sum_{m=1}^{\infty} \frac{1}{2im\pi} l(m)\mathrm{e}^{im\vartheta} = \gamma_\lambda^{-*}$ is defined as the formal limit at $s = 2k \to \infty$ of $\gamma\left([0, \lambda[\right) = \sum_{\vartheta\in\Lambda_s}^{\vartheta<\lambda} \gamma(\vartheta)$

$$\gamma(\vartheta) = \frac{1}{2k+1}\sum_{m=-k}^{k} l_s(m)\mathrm{e}^{im\vartheta} = \nu(\vartheta) - 2\pi\delta^\varepsilon(\vartheta)\,. \tag{4.4}$$

The function $\dot{\gamma}(\lambda)$ is the mean value $\langle \dot{\upsilon}_t(\lambda)\rangle = t\dot{\gamma}(\lambda)$ of the distribution–valued process

$$\dot{\upsilon}_t(\lambda) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \lambda_t(m)\mathrm{e}^{im\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\upsilon_{t,\lambda} \tag{4.5}$$

coinciding with $\dot{n}_t(\lambda) = \lim_{k\to\infty} \frac{1}{\varepsilon} n_t(\lambda)$ at $\lambda \neq 0$. It is given by the complex stochastic amplitudes $\lambda_t(n)^* = \lambda_t(-n)$, $\lambda_t(0) = 0$ with the mean values $\langle \lambda_t(m)\rangle_0 = tl(m)$ and the Ito multiplication table:

$$\sum_{n,m} c_n^* \mathrm{d}\lambda_t^*(n)\mathrm{d}\lambda_t(m) c_m = \sum_{n,m} c_n^* \mathrm{d}\lambda_t(m-n) c_m \tag{4.6}$$

for the linear combinations $\sum_m \lambda_t(m) c_m$ with the complex coefficients $\sum_m c_m = 0$.

At the limit $k \to 0$, the stochastic phase (3.1) can be formally defined in terms of the amplitudes $\lambda_t(m)$ as $\varphi(Y_t)$, where

$$Y_t = \int_0^{2\pi} \lambda\dot{\upsilon}_t(\lambda)\mathrm{d}\lambda = \lim_{k\to\infty} \frac{1}{2k+1}\sum_{m=-k}^{k} \frac{1}{im}\lambda_t(m) \tag{4.7}$$

Moreover, since $\int_0^{2\pi} \dot{v}_t(\lambda)\mathrm{d}\lambda = 0$, one easily shows

$$\int_0^{2\pi} \lambda \dot{n}_t(\lambda)\mathrm{d}\lambda = \int_0^{2\pi} \lambda \dot{v}_t(\lambda)\mathrm{d}\lambda = -\int_0^{2\pi} v_{t,\lambda}\mathrm{d}\lambda$$

where $v_{t,\lambda} = v_{t,\lambda}^- + v_{t,\lambda}^+$ and $v_{t,\lambda}^+ = \sum_{m=1}^\infty \frac{1}{2im\pi}\lambda_t(m)\mathrm{e}^{im\lambda} = {v_{t,\lambda}^-}^*$ is given by the formal limit $s = 2k \to \infty$ of $v_t([0,\lambda[) = \sum_{\vartheta\in\Lambda_s}^{\vartheta<\lambda} v_t(\vartheta)$, where

$$v_t(\vartheta) = \frac{1}{2k+1}\sum_{m=-k}^{k} \lambda_{s,t}(m)\mathrm{e}^{im\vartheta}, \quad \lambda_{s,t}(m) = \sum_{\lambda\in\Lambda_s} (\mathrm{e}^{-im\lambda} - 1)n_t(\lambda)\,. \quad (4.8)$$

Hence the integral $Y_t = -\int_0^{2\pi} v_{t,\lambda}\mathrm{d}\lambda$ is defined as the formal limit

$$-\int_0^{2\pi} v_{t,\lambda}\mathrm{d}\lambda := \lim_{s\to\infty}\left(Y_t^- + Y_t^+\right) = -\lim_{s\to\infty}\sum_{\lambda\in\Lambda_s^\circ} v_t([0,\lambda[)\varepsilon\,, \quad (4.9)$$

where

$$Y_t^+ = \frac{1}{2k+1}\sum_{m=1}^{k}\frac{\varepsilon}{\mathrm{e}^{im\varepsilon}-1}\lambda_{s,t}(m) = {Y_t^-}^*$$

since $v_t([0,\lambda[) = \frac{1}{2k+1}\sum_{m=-k}^k \frac{\mathrm{e}^{im\lambda}}{\mathrm{e}^{im\varepsilon}-1}\lambda_{s,t}(m)$.

Notice that $\lambda_{s,t}(0) = 0$ and the stochastic amplitudes $\lambda_{s,t}(m)$, $m = \pm1, \pm2, \ldots, \pm k$ have the mean values $\langle\lambda_{s,t}(m)\rangle = tl_s(m)$ and satisfy the multiplication table (4.6) mod $(2k+1)$ as their limits $\lambda_t(m) = \lim_{k\to\infty}\lambda_{s,t}(m)$. The right hand side of the continuous reduction equation (3.2) can be written in the phase representation $x(t,\vartheta) = \langle\vartheta|\chi(t) = \chi(t,\mathrm{e}^{i\vartheta})$ as

$$\int_0^{2\pi}\langle\vartheta|L(\lambda)\chi(t)\mathrm{d}_t\dot{v}(\lambda)\mathrm{d}\lambda = \sum_{m=0}^{\infty} b_m\mathrm{e}^{-im\vartheta}x(t,\vartheta)\mathrm{d}_t\lambda(m)$$

where $\mathrm{d}_t\lambda(m) = \lambda_{t+\mathrm{d}t}(m) - \lambda_t(m)$, the sum being finite for the admissible wave functions $g(\lambda) = \sum_{m=0}^\infty b_m\mathrm{e}^{im\lambda}$ of the meters ($b_m = 0$ for $m > k$).

In addition to the examples described in sections 1 and 2, let us consider the number distribution $\nu(\lambda)$ given on Λ_s by the density (1.9) with the value of r given by $r = 1 - \frac{\alpha}{s+1}$, $\alpha > 0$, as $\nu(\lambda) = |g(\lambda)|^2$. One can easily find that

$$l_s(m) = (s+1)\,(c_s(m) - 1) \quad \to \quad l(m) = -\beta|m|,$$

where $c_s(m)$ is given by (1.10) with $\vartheta = 0$ and $\beta = \alpha\frac{\mathrm{e}^\alpha - 1}{\mathrm{e}^\alpha + 1}$. The corresponding distribution (4.3) is defined as the formal series

$$\dot{\gamma}(\lambda) = \frac{\beta}{i\pi}\sum_{m=1}^{\infty} m\sin(\lambda m) = \frac{\beta}{i}\frac{\mathrm{d}}{\mathrm{d}\lambda}[\delta^-(\lambda) - \delta^+(\lambda)] \tag{4.10}$$

where $\delta^+(\lambda) = \frac{1}{2\pi}\sum_{m=1}^{\infty}\mathrm{e}^{im\lambda} = \delta^-(\lambda)^*$. This distribution coisides with the positive one $\dot{\nu}(\lambda) = \beta/2\pi(1-\cos\lambda)$ on the trigonometric polynomials

$$f(\lambda) = g(\lambda) - g(0) = \sum_{m=-k}^{k} c_m \mathrm{e}^{im\lambda} = \sum_{m=-k}^{k} b_m(\mathrm{e}^{im\lambda} - 1)$$

given with $c_m = b_m, m \neq 0, c_0 = -\sum_{m\neq 0} b_m$. This is derived from

$$\int_0^{2\pi} |f(\lambda)|^2\dot{\gamma}(\lambda)\mathrm{d}\lambda = 2\beta\sum_{m,n=-k}^{k} b_n^*|m\diamond n|b_m = \frac{1}{2\pi}\int_0^{2\pi}|f(\lambda)|^2\frac{\beta}{1-\cos\lambda}\mathrm{d}\lambda \geq 0,$$

where $2|m\diamond n| = |m| + |n| - |m-n| = \frac{1}{2\pi}\int_0^{2\pi}(\mathrm{e}^{-in\lambda} - 1)(\mathrm{e}^{im\lambda} - 1)\frac{\mathrm{d}\lambda}{1-\cos\lambda}$ because $\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos m\lambda - 1}{1-\cos\lambda}\mathrm{d}\lambda = |m|$.

Hence operator (3.6) defines the Hermitian form $\langle\chi|(K+K^\dagger)|\chi\rangle$ for the phase reduction equation (3.2) as the finite sum

$$\frac{2\pi}{\beta}\int_0^{2\pi}\langle\chi|\left(L(\lambda) + L(\lambda)^\dagger L(\lambda) + L(\lambda)^\dagger\right)|\chi\rangle\dot{\gamma}(\lambda)\mathrm{d}\lambda =$$
$$\int_0^{2\pi}|x(\vartheta)|^2\Big(\sum_{m,n=1}^{k} 2\,(m\wedge n)\, b_m^* b_n \mathrm{e}^{i(m-n)\vartheta} - \sum_{m=1}^{k} m\left(b_m\mathrm{e}^{-im\vartheta} + b_m^*\mathrm{e}^{im\vartheta}\right)\Big)\mathrm{d}\vartheta$$

for any initial trigonometric wave function $g(\lambda)$ of the meter with admissible Fourier coefficients ($b_m = 0$ if $m < 0$). Note that the distribution

$$\gamma(\lambda - \vartheta) = |g(\lambda-\vartheta)|^2 - 2\pi\delta^\varepsilon(\lambda - \vartheta)$$

defines at the limit $s\to\infty$ an infinite dimensional Minkowski metrics in the space $\mathcal{E} = \bigcup_{s<\infty}\mathcal{E}_s$ of all trigonometric polynomials $f(\lambda)$, giving an example of the Ito algebra [2] which is particularly important for the phase measurements with the singular input Poisson noise, inducing the phase spontaneous localizations.

Conclusion

It is demonstrated that the quantum phase operator of a discrete (or continuous) spectrum can be treated in the conventional Hilbert space as a conditional expectation (projection) of a proper observable of the phase meter without any limiting procedure. It is shown that under the NDP, the problems mentioned in [**?**] in the phase measurements, namely the convergence for $s \to \infty$ of the operators, commutators, states, eigenvalues,... defined in $\mathcal{E}$ to the usual ones in $\mathcal{H}$, the normalization of the phase states, and the restriction to the "physical" states, are simply solved by the proper choice of the measurement apparatus and are not relevant to the quantum oscillator itself.

It is shown, that by using a NDP approach, the time continuous process of the phase observations can be completely described. Moreover, this principle allows us to treat the sequential unsharp measurements of quantum phase in terms of non–projective reduction transformations. The time continuous measurements of the quantum oscillator phase are described as a stochastic process of a spontaneous localization of the phase pointer. The unsharp localization of the quantum phase under the observation at random instants is caused by the proper interaction of the oscillator with the meter.

This quantum jump model shows once again that there is no universal individual-trajectory reduction dynamics but the stochastic reduction equation derived here for photon-phase countings has another type than in the diffusive case [**?**, **?**]. Because of the boundness of the phase spectrum, the quantum phase jumps can not even be approximized by a diffuion prrocess, but there is a singular Poisson limit in this case. But as in the diffusive case the phase reduction equation depends on the quantum mechanical model of spontaneous interaction only via the chosen phase operator, the initial state–vector of the meter, and the statistics of the input counting process, inducing the photon emission.

These two only possible types of stochastic dynamics constitute the real limitation for a form of spontaneous reduction equations.

References

[1] Belavkin V. P. Hypothesis testing for quantum optical signals. *Radio*

Eng Physics, 21(1):95–104, 1976.

[2] Belavkin V. P. Chaotic states and stochastic integrations in quantum systems. *Usp. Math Nauk (Russian Math Surveys)*, (1), 1992.

[3] Belavkin V.P. A continuous counting observation and posterior quantum dynamics. *J Phys A Math Gen*, (22):L 1109–L 1114, 1989.

[4] Belavkin V.P. A new wave equation for a continuous non-demolition measurement. *Phys Letters A*, 140(78):355–358, 1989.

[5] Belavkin V.P. A stochastic posterior Schrödinger equation for counting non-demolition measurement. *Letters in Math Phys*, (20):85–89, 1990.

[6] Belavkin V.P. and A Barchielli. Measurements continuous in time and posteriori states in quantum mechanics. *J. Phys. A. Math. Gen.*, 24:1495–1514, 1991.

[7] Belavkin V.P. and Staszewski P. Nondemolition observation of a free quantum particle. *Phys. Rev. A.*, 45(3):1347–1356, 1992.

[8] Ghirardi G.C., Pearle P., and Rimini A. Markov processes in Hilbert space and continuous spontaneous localization of systems of indentical particles. *Phys. Rev. A.*, 42:78–89, 1990.

[9] Helstrom C.W. *Quantum Detection and Estimation theory.* Academic Press, New York, 1976.