

# QUANTUM DIFFUSION, MEASUREMENT AND FILTERING

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ABSTRACT. A brief presentation of the basic concepts in quantum probability theory is given in comparison to the classical one. The notion of quantum white noise, its explicit representation in Fock space, and necessary results of noncommutative stochastic analysis and integration are outlined.

Algebraic differential equations that unify the quantum non Markovian diffusion with continuous non demolition observation are derived. A stochastic equation of quantum diffusion filtering generalising the classical Markov filtering equation to the quantum flows over arbitrary \*-algebra is obtained.

A Gaussian quantum diffusion with one dimensional continuous observation is considered. The a posteriori quantum state diffusion in this case is reduced to a linear quantum stochastic filter equation of Kalman-Bucy type and to the operator Riccati equation for quantum correlations. An example of continuous nondemolition observation of the coordinate of a free quantum particle is considered, describing a continuous collapse to the stationary solution of the linear quantum filtering problem found in the paper.

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## 1. INTRODUCTION

Beginning in the mid seventies, modern probability theory has, (along with traditional subjects, such as dynamical systems with random perturbations), been also concerned with fundamentally new stochastic objects — quantum dynamic systems with an inherently probabilistic nature. Mathematically, the concept of quantum probability arises not because of the lack of information for a complete description of the object, the instability of chaotic motion or the inaccuracy of measurement

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but is due to the noncommutativity of the algebra of random variables which are represented by the operators in the Hilbert space. As quantum probability theory is an intrinsically stochastic theory, it cannot be stated within the framework of the Kolmogorov axioms [1], which assume the fundamentally deterministic description of the classical systems under the given point states  $\omega \in \Omega$ . It is based on different axioms[2], [3] namely, the Neumann axioms, whose greater generality can be demonstrated even in the case of a finite number of alternative elementary events  $\omega = 1, \dots, n$ .

Let us illustrate for this simple case how the classical probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  can be represented as a special case of the quantum space that is defined by the triple  $(\mathbf{H}, \mathcal{A}, \mathbf{E})$ . Here  $\mathbf{H}$  is a (finite-dimensional) complex space of column-vectors  $h = [\eta^i]$ ,  $\eta^i \in \mathbf{C}$  with scalar product  $(g|h) = \sum \bar{\zeta}^i p_i \eta^i \equiv g^* h$  for  $g = [\zeta^i] \in \mathbf{H}$  defined by the weights (probabilities)  $p_i > 0$ ,  $\mathcal{A}$  is an associative, but not necessarily commutative matrix algebra  $X = [\xi_k^i]$  closed under the involution  $X \mapsto X^*$  defined by the Hermitian conjugation

$$(X^* g|h) = (g|Xh), [\xi_k^i]^* = [p_i^{-1} \bar{\xi}_i^k p_k],$$

and with matrix  $I = [\delta_k^i]$  as the identity  $I \in \mathcal{A}$ , and  $\mathbf{E}[X] = (e|Xe)$  is a positive normalized functional ( $\mathbf{E}[X^*X] \geq 0$ ,  $\mathbf{E}[I] = 1$ ) of expectation of noncommuting variables  $X$  defined by a fixed unit vector  $e \in \mathbf{H}$ , where  $\|e\|^2 := (e|e) = 1$ .

Classical random variables  $x : \Omega \rightarrow \mathbf{C}$  can also be described by the multiplication operators  $X = \hat{x}$ ,  $(\hat{x}h)(\omega) = x(\omega)h(\omega)$  in the complex Hilbert space  $\mathbf{H} = L^2(\Omega, \mathcal{F}, \mathbf{P})$  of  $\mathcal{F}$ -measurable  $\mathbf{P}$ -square-integrable functions  $h : \Omega \rightarrow \mathbf{C}$ ,

$$\|h\|^2 = \int |h(\omega)|^2 \mathbf{P}(d\omega) = (h|h) < \infty.$$

Their expectations  $\mathbf{E}[\hat{x}] = \int x(\omega) \mathbf{P}(d\omega)$  are defined as  $(e|\hat{x}e)$  by the unit function  $e(\omega) \equiv 1$  which is normalized with respect to any probability measure  $\mathbf{P}$ . Thus, however, only commutative operator algebras  $\mathcal{A}$  are obtained, whose elements are given in the finite case of  $\Omega = \{1, \dots, n\}$  by all the diagonal matrices  $\hat{x} = [\xi(i)\delta_k^i]$  with the commutative product, corresponding to the pointwise multiplication of the functions  $x(\omega) = \xi(i)$ , where  $\omega = i$ .

Conversely, any quantum probability space  $(\mathbf{H}, \mathcal{A}, \mathbf{E})$  can be reduced to the classical one  $(\Omega, \mathcal{F}, \mathbf{P})$  only in the case of the commutativity of the algebra  $\mathcal{A}$ ; in the finite-dimensional case this is realised by simultaneous reduction of the commuting matrices  $X \in \mathcal{A}$  to diagonal form  $[\xi(i)\delta_k^i]$ . The probabilities  $p_i$  of the elementary events  $\omega = i$  in the diagonal representation are defined by the restriction  $p_i = \mathbf{E}[P_i]$  of the functional  $\mathbf{E}[X] = \sum \xi(i)p_i$  on projective matrices  $P_j = [\xi_j(i)\delta_i^k]$ ,  $\xi_j(i) = \delta_j^i$ .

In this article a quantum analog of diffusion and the problem of its continuous measurement and stochastic filtering, that gives the solution of the Zeno paradox [4] (as a result of establishing an a posteriori stationary state), are considered within the framework of the noncommutative algebraic approach. A derivation of the stochastic equation is given for a nonnormalized a posteriori quantum state, which is obtained in [5] by renormalising the basic equation of nonlinear quantum filtering [6]. The solution of the equation has been found for the case of linear quantum diffusion of canonical commutation relations, obtained previously for the quantum Gaussian case by means of linear Markov filtering methods in [7], [8].

In presenting the second (basic) section, we deliberately avoided the questions concerning the sufficient conditions for the dense definition of the unbounded infinitesimal generators that guarantee the uniqueness of solutions of quantum stochastic and operator equations; this is beyond the scope of this article. We only point out that in the first and second sections these questions are not relevant (see [9]) for the Markovian case with complete pre-Hilbert domain  $D$  in the initial Hilbert space  $H$ , corresponding to boundedness of the operators  $L$  and  $H$  in  $D = H$ . Moreover, a solution exists for an unbounded algebra  $\mathcal{A}$  of canonical commutation relations, which is considered in the third section, in the framework of a quantum calculus of kernels for the operators  $L, H \in \mathcal{A}$  in the initial Fock scale  $\{F_\xi | \xi > 1\}$  [10] if their inductive limit  $\cup F_\xi$  is chosen as  $D$ . Besides, the explicitly solvable model of this section with linear unbounded generators  $L$  and  $H$ , does not require the estimates obtained in these scales.

For completeness the notation and explicit methods of quantum stochastic integration and the proof of their estimates in Fock scale [5] are briefly presented in the Appendix . The comprehensive statement of the author's general approach, outlined above, and the estimates for the integrals can be found in [10], [11]. The earlier results on quantum stochastic calculus in the framework of Hudson and Parthasarathy approach [12], are reviewed in [13].

The approach presented generalizes the results for purely quantum diffusion in [14] to the case of an arbitrary initial algebra  $\mathcal{A}$ . This enables a unified description of quantum and classical diffusion, their observation and filtering as special algebraic cases. In the sections 3 and 4, a one-dimensional variant of an infinite-dimensional quantum Gaussian filtering [14] is presented as well as an example of observation of a coordinate of a free quantum Brownian particle; this was analysed earlier in [16] by the method of solving the a posteriori Schrödinger equation [15].

## 2. QUANTUM DIFFUSION AND NONDEMOLITION MEASUREMENT

**1.1. Basic Notation.** Let  $H$  be a complex Hilbert space and  $D \subseteq H$  be a dense subspace defined as an inductive limit (see appendix 1) of some scale  $\{H_\xi | \xi > 1\}$  in the space  $H$ . Let the initial algebra  $\mathcal{A}$  of noncommutative random variables describing a 'quantum object' at the initial moment  $t = 0$  be represented by an involutive subalgebra  $\mathcal{A} \subseteq \mathcal{B}(D)$  of linear operators  $X : D \rightarrow D$ ,  $X^* \in \mathcal{A}$ , having (an inductively) continuous conjugate  $X^* : D \rightarrow D$ ,  $(X^*\chi|\psi) = (\chi|X\psi)$  with respect to the scalar product in  $H$ , with an identity operator  $I \in \mathcal{A}$ .

Let us denote by  $\mathcal{H} = H \otimes \mathcal{F}$  the tensor product  $H$  and the Fock space  $\mathcal{F} = \Gamma(\mathcal{K})$  over the Hilbert space  $\mathcal{K} = L^2(\mathbf{R}_+)$  of a 'quantum noise'  $\hat{w}_t(g)$ ,  $g \in \mathcal{K}$ , and let the pre-Hilbert space  $\mathcal{D}$  be an inductive limit of the Hilbert scale  $\mathcal{H}_\xi = H_\xi \otimes \mathcal{F}_\xi$ ,  $\xi > 1$ , where  $\{\mathcal{F}_\xi\}$  is the natural Fock scale (see appendix 2) over  $\mathcal{K}$ . We shall consider the quantum noise as a set  $\{\hat{w}_t(g) | g \in L^2(\mathbf{R}_+)\}$  of Brownian motions  $t \mapsto \hat{w}_t(g)$ , represented in  $\mathcal{F}$  by self-adjoint operators

$$\hat{w}_t(g) = \int_0^t (g(r) d\hat{a}_r^* + \bar{g}(r) d\hat{a}_r) \equiv \hat{a}_t^*(g) + \hat{a}_t(g^*),$$

with a Gaussian state on the algebra generated by them, which is induced by the vacuum function  $\delta_0 \in \mathcal{F}$ . Here  $\{\hat{a}_r, \hat{a}_r^* | r \in \mathbf{R}_+\}$  are canonical operators of creation  $\hat{a}_r^*$  and annihilation  $\hat{a}_r$  in  $\mathcal{F}$  (see Appendix 3) called quantum stochastic integrators, and  $g^*(t) = \bar{g}(t)$ . Note that each operator function  $t \mapsto \hat{w}_t(g)$  that has commutative

values  $[\widehat{w}_s(g), \widehat{w}_t(g)] = 0$  is equivalent to a classical Brownian motion with intensity  $|g(t)|^2$ , with respect to the vacuum vector  $e = \delta_\emptyset$ . This follows from the formula

$$e^{i\widehat{w}(g)} = e^{i\widehat{a}^*(g)} e^{-\frac{1}{2}\|g\|^2} e^{i\widehat{a}(g^*)}$$

and  $e^{\widehat{a}(f)}\delta_\emptyset = \delta_\emptyset$  for any  $f \in \mathcal{K}$ , due to which the quantum characteristic function

$$\mathbf{E}[e^{i\widehat{w}_t(g)}] = (\delta_\emptyset | e^{i\widehat{w}_t(g)} \delta_\emptyset)$$

coincides with the classical Gaussian characteristic function

$$\int \exp\left\{i \int_0^t g(r) dw_r\right\} \mathbf{P}(d\omega) = \exp\left\{-\frac{1}{2} \int_0^t |g(r)|^2 dr\right\}$$

of the standard Wiener process  $w_t$ . However, the different Brownian motions  $\widehat{w}_t(f)$  and  $\widehat{w}_t(g)$  with  $f^*g \neq g^*f$  do not have any classical representation on a single probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  because of noncommutativity (see A3 in Appendix 3):

$$[\widehat{w}_t(f), \widehat{w}_t(g)] = \int_0^t \left( \bar{f}(r)g(r) - \bar{g}(r)f(r) \right) dr.$$

**Definition 1.** Let  $\{\mathcal{A}_t | t \in \mathbf{R}_+\}$  be an increasing set of involutive subalgebras  $\mathcal{A}_t \subseteq \mathcal{A}_s$ ,  $t \leq s$  of the operators  $X_t \in \mathcal{B}(\mathcal{D})$  generated by operators  $X \in \mathcal{A}$ ,  $\widehat{w}_t(g)$ ,  $g \in \mathcal{K}$ , such that

$$X_t \in \mathcal{A}_t \Leftrightarrow [X_t, Y] = 0, \quad \forall Y \in \mathcal{B}(\mathcal{D}) : [X_0, Y] = 0 = [W_t(g), Y],$$

where the operators  $X_0 \in \mathcal{A}_0$ ,  $W_t(g)$ ,  $g \in L^2(0, t]$  are assumed to act in  $\mathcal{H} = \mathbb{H} \times \mathcal{F}$  as  $X_0 \otimes \hat{1}$  and  $I \otimes \widehat{w}_t(g)$ . A measurable operator function  $F(t) : \mathcal{D} \rightarrow \mathcal{D}$  is called adapted if  $F(t) \in \mathcal{A}_t$  for almost all  $t \in \mathbf{R}_+$ .

We shall consider here only quantum stochastic integrals of the form

$$(2.1) \quad i_0^t(F, D) = \int_0^t (F(r) dA_r + D(r) dA_r^*),$$

where  $A_r = I \otimes \widehat{a}_r$ ,  $A_r^* = I \otimes \widehat{a}_r^*$  and  $F, D$  are locally square-integrable (see Appendix 4) together with adjoint  $F^*, D^*$  adapted operator-functions  $\mathbf{R}_+ \rightarrow \mathcal{A}_t$ . Note that on the exponential vectors, described by the product-functions  $h(\tau) = k^{\otimes}(\tau) \psi$ , where  $\psi \in \mathcal{D}$  and  $k \in L^2(\mathbf{R}_+)$ , the integrals (2.1) are weakly defined as the usual operator integrals

$$(h | i_0^t(F, D) h) = \int_0^t \left( h | [F(r)k(r) + D(r)\bar{k}(r)] h \right) dr.$$

This gives in particular,  $\mathbf{E}[i_0^t(F, D)] = 0$  for  $\mathbf{E}[X] = (e|Xe)$ , where  $e(\tau) = \delta_\emptyset(\tau) \psi$  is the exponential vector, corresponding to  $k = 0$ . Such operator integrals on the exponential domain were constructed by Hudson and Parthasarathy [12] for the case of bounded  $F(t), D(t)$ .

For the adapted integrals (2.1), the quantum Ito formula [10]–[12] can be obtained as in the case of (A.5) (see Appendix 5), corresponding to  $X_t(t) = 0$  for almost all  $t$ . This formula defines (see Appendix 5) the pointwise multiplication

$$X(t)Y(t) = X(0)Y(0) + \int_0^t d(XY)(r)$$

of the adapted operator-functions

$$X(t) = X(0) + \int_0^t dX(r), \quad Y(t) = Y(0) + \int_0^t dY(r)$$

in terms of the product  $dX(t)dY(t) = D(t)^*F(t)dt$  of their stochastic differentials  $dX = D^*dA + DdA^*$ ,  $dY = F^*dA + FdA^*$ :

$$(2.2) \quad \begin{aligned} d(XY) = & dXY + XdY + dXdY = D^*Fdt + (D^*Y + XF^*)dA \\ & +(DY + XF)dA^*. \end{aligned}$$

The classical Ito formula for the stochastic integrals

$$I_0^t(f, \omega) = \int_0^t f(r, \omega) dw_r$$

with respect to the standard Wiener process  $w = \{w_t | t \in \mathbf{R}_+\}$  can be obtained from (2.2) by the Segal one-to-one transformation  $\omega : \widehat{w}_t \mapsto w_t$ , where  $\widehat{w}_t = \widehat{a}_t + \widehat{a}_t^*$ . The latter represents the adapted operator integrals  $\iota_0^t(\widehat{f}, \widehat{f}) = \int_0^t \widehat{f}(r) d\widehat{w}_r$  for the non-anticipated functionals  $\widehat{f}(t) = f(t, \widehat{w})$  of commuting selfadjoint operators  $\widehat{w} = \{\widehat{w}_t\}$  with

$$\|\widehat{f}\delta_\emptyset\|_t^2 := \int_0^t \|\widehat{f}(r)\delta_\emptyset\|^2 < \infty$$

in the form of the Ito integrals  $\omega[\iota_0^t(\widehat{f}, \widehat{f})] = I_0^t(f, \omega)$ , so that

$$\|\iota_0^t(\widehat{f}, \widehat{f})\delta_\emptyset\|^2 = \|\widehat{f}\delta_\emptyset\|_t^2 = \int \|\iota_0^t(f, \omega)\|^2 \mathbf{P}(d\omega),$$

where  $\mathbf{P}$  is the standard Wiener probability measure.

**1.2. Quantum diffusion.** Quantum stochastic evolution in the open system  $\{\mathcal{A}_t\}$  is described by an adapted family  $\{\iota(t) | t \in \mathbf{R}_+\}$  of  $*$ -representations  $\iota(t) : X \mapsto X(t)$  of the initial algebra  $\mathcal{A} \subseteq \mathcal{B}(\mathcal{D})$  into  $\mathcal{B}(\mathcal{D})$ , i.e. of linear maps  $\mathcal{A} \rightarrow \mathcal{A}_t$ , with the properties:

$$\iota(t, X^*X) = \iota(t, X)^*\iota(t, X), \quad \iota(t, I) = I_0 := I \otimes \hat{1}.$$

It is called diffusion motion if the operator-valued functions  $t \mapsto X(t)$  have the quantum stochastic differentials of the form

$$(2.3) \quad dX(t) + C(t)dt = D^*(t)dA_t + D(t)dA_t^*.$$

Here  $C(t) = \gamma(t, X)$  is an adapted operator-valued function, locally integrable ( $p = 1$ ) for every  $X \in \mathcal{A}$  defined by the linear maps  $\gamma(t) : \mathcal{A} \rightarrow \mathcal{A}_t$ ,  $t \in \mathbf{R}_+$ .  $D^*(t) = \delta^*(t, X)$ ,  $D(t) = \delta(t, X)$  are adapted operator-valued functions, locally square-integrable for each  $X \in \mathcal{A}$ , defined by the linear maps  $\delta^*(t)$ ,  $\delta(t) : \mathcal{A} \rightarrow \mathcal{A}_t$ .

Define the output process  $Y = \{Y(t) | t \in \mathbf{R}_+\}$ , which is subject to measurement and described by a commutative family of (essentially) selfadjoint operators  $Y(t) = Y(t)^*$  on  $\mathcal{D}$  with the initial condition  $Y(0) = 0$  and stochastic differentials

$$(2.4) \quad dY(t) = G(t)dt + F^*(t)dA_t + F(t)dA_t^*.$$

Here  $G(t) = G(t)^* \in \mathcal{A}_t$  is essentially self-adjoint and locally-integrable ( $p = 1$ ).  $F(t) \in \mathcal{A}_t$  is locally square-integrable together with its conjugate:  $F^*(t) = F(t)^*$ ;  $G(t)$  and  $F(t)$  are adapted operator-valued functions of  $t \in \mathbf{R}_+$ .

Unlike the classical case, not every involutive subalgebra  $\mathcal{B}$  of  $\mathcal{A}$ , but only a central one  $\mathcal{B} \subset \mathcal{A} \cap \mathcal{A}'$ , defines the conditional expectations  $\mathbf{E}[X|\mathcal{B}]$  for any state vector  $e \in \mathbf{H}$  as the positive projections  $\mathcal{A} \rightarrow \mathcal{B}$  which are compatible with  $\mathbf{E}[X] = (e|Xe)$  such that  $\mathbf{E}[\mathbf{E}[X|\mathcal{B}]] = \mathbf{E}[X]$  for all  $X \in \mathcal{A}$ . Hence, not every stochastic process described by the equation (2.4), can be considered as an output process for quantum diffusion, defined by equation (2.3), but only that for which

the posterior expectations of  $X(t)$  with respect to the observation  $Y(s)$ ,  $s \leq t$ , exist.

**Definition 2.** A process  $Y(t)$  is called *causal, or nondemolition* with respect to the process  $X(t)$  if

$$(2.5) \quad [X(t), Y(s)] := X(t)Y(s) - Y(s)X(t) = 0$$

for all  $t \geq s$ ,  $s \in \mathbf{R}_+$ .

The nondemolition condition together with the self-nondemolition of  $Y$ , i.e. with the commutativity  $[Y(t), Y(s)] = 0$ ,  $\forall t, s$ , is necessary and sufficient [11] for the existence of the conditional expectations  $\hat{\pi}(t, X) = \mathbf{E}[X(t)|\mathcal{B}_t]$  for the operators  $X(t) = \iota(t, X)$ , with respect to the  $*$ -algebras

$$\mathcal{B}_t = \{Y \in \mathcal{B}(\mathcal{D}) \mid [X, Y] = 0, \quad \forall X \in \mathcal{B}(\mathcal{D}) : [X, Y(s)] = 0, \quad \forall s \leq t\}$$

generated by the family  $\{Y(s)|s \leq t\}$  and for every initial vector-function  $e \in \mathcal{D}$ ,  $\|e\| = 1$ . If the process  $Y$  with  $Y(0) = 0$  is nondemolition with respect to the coefficients  $C, D^*, D$  of the equation (2.3), then it is nondemolition with respect to the solution  $X$ , corresponding to any initial  $X(0) \in \mathcal{A}$ . This and other sufficient conditions of the next proposition obviously follow from the integral representation

$$X(t) = X - \int_0^t (C(r) dr - D^*(r) dA_r - D(r) dA_r^*).$$

**Proposition 1.** The integrals  $X(t)$  of (2.3) with  $X \in \mathcal{A}$  are  $*$ -representations  $\iota(t) : X \mapsto X(t)$  iff the linear maps

$$\gamma(t) : X \mapsto C(t), \quad \delta^*(t) : X \mapsto D^*(t), \quad \delta(t) : X \mapsto D(t)$$

satisfy the following differential conditions

- (i)  $\gamma(t, X^*) = \gamma(t, X)^*$ ,  $\delta(t, X^*) = \delta^*(t, X)^*$ ,  $\forall X \in \mathcal{A}$ ,
- (ii)  $\gamma(t, X^*X) = \iota(t, X)^*\gamma(t, X) + \gamma(t, X)^*\iota(t, X) - \delta(t, X)^*\delta(t, X)$ ,  
 $\delta(t, X^*X) = \iota(t, X)^*\delta(t, X) + \delta(t, X)^*\iota(t, X) - \delta^*(t, X^*X)^*$ ,
- (iii)  $\gamma(t, I) = 0$ ,  $\delta(t, I) = 0 = \delta^*(t, I)$ ,  $\forall t \in \mathbf{R}_+$ .

The process  $X(t)$  satisfies the condition (2.5) iff the stochastic derivations  $C(t), D^*(t), D(t)$  also satisfy the condition (2.5) as  $X(t)$  with respect to the nondemolition process  $Y(t)$  for all  $X \in \mathcal{A}$ , and the derivatives  $G, F^*, F$  in (2.4) satisfy the differential nondemolition conditions

$$(2.6) \quad \begin{aligned} [X(t), F^*(t)] = 0 &= [F(t), X(t)], \quad \forall t \in \mathbf{R}_+, \\ D^*(t)F(t) - F^*(t)D(t) &= [G(t), X(t)]. \end{aligned}$$

PROOF. The stochastic differentials  $dX(t) = X(t + dt) - X(t)$  of the linear  $*$ -maps  $\iota(t) : X \mapsto X(t)$  are defined by the linear  $*$ -maps  $\gamma(t)$ ,  $\delta^*(t)$ ,  $\delta(t)$  by virtue of linear independence of the fundamental differentials  $dt$ ,  $dA_t$  and  $dA_t^*$ . The conditions (ii) are found by applying the Ito formula (2.2) to  $X(t)^*X(t)$ :

$$\begin{aligned} d(X(t)^*X(t)) &= dX(t)^*dX(t) + dX(t)^*X(t) + X(t)^*dX(t) \\ &= [\delta(t, X)^*\delta(t, X) - \gamma(t, X)^*\iota(t, X) - \iota(t, X)^*\gamma(t, X)] dt \\ &\quad + d\iota_0^t(\delta(X)^*\iota(X) + \iota(X)^*\delta^*(X), \delta(X^*)\iota(X) + \iota(X)^*\delta(X)). \end{aligned}$$

By equating the stochastic derivatives of this differential and

$$d\iota(t, X^*X) = d\iota_0^t(\delta^*(X^*X), \delta(X^*X)) - \gamma(X^*X) dt,$$

we obtain that  $\delta^*$  and  $\delta$  are the derivations of the algebra  $\mathcal{A}$ , and  $-\gamma$  has the positive-definite dissipator

$$\iota(X)^*\gamma(X) + \gamma(X)^*\iota(X) - \gamma(X^*X) = \delta(X)^*\delta(X).$$

The condition (iii) follows from  $d\iota(t, I) = 0$  because of the independence of  $\gamma$ ,  $\delta^*$ ,  $\delta$ .

If  $Y(t)$  is a nondemolition process for  $X(t)$ , then

$$[dX(t), Y(s)] = [X(t+dt), Y(s)] - [X(t), Y(s)] = 0$$

with  $t \geq s$ ; hence the nondemolition for  $C$ ,  $D^*$ ,  $D$ :

$$[C(t), Y(s)] = 0, \quad [D^*(t), Y(s)] = 0, \quad [D(t), Y(s)] = 0, \quad \forall t \geq s,$$

follows by commutativity of  $Y(s)$  with the independent differentials  $dt$ ,  $dA_t$  and  $dA_t^*$ . Applying equation (2.2) to the differential of the commutator  $[X(t), Y(t)] = 0$  we obtain (taking into account the equality  $[dX(t), Y(t)] = 0$ ):

$$\begin{aligned} d[X(t), Y(t)] &= [dX(t), dY(t)] + [dX(t), dY(t)] + [X(t), dY(t)] \\ &= \left( D^*(t)F(t) - F^*(t)D(t) + [X(t), G(t)] \right) dt \\ &\quad + d i_0^t([X, F^*], [X, F]) = 0, \end{aligned}$$

which yields the differential self-nondemolition conditions (2.6). Hence, all conditions of the proposition are necessary.

**1.3. The Markov case.** The quantum diffusion (2.3) with coefficients

$$C(t) = \iota(t, C_t), \quad D^*(t) = \iota(t, D_t^*), \quad D(t) = \iota(t, D_t),$$

corresponds to the Markov stochastic evolution (in strong sense). Here  $C_t, D_t^*, D_t \in \mathcal{A}$  are defined by the structural maps

$$\gamma_t : X \mapsto C_t, \quad \delta_t^* : X \mapsto D_t^*, \quad \delta_t : X \mapsto D_t,$$

for which the conditions (i)–(iii) indicated above were obtained by Hudson and Evans in [14]. The self-nondemolition conditions (2.6) give the restrictions for the coefficients  $G$  and  $F^*, F$  in this case. We shall restrict ourselves to consideration of the standard case  $F(t) = I_0 = F^*(t)$  of the indirect measurement

$$(2.7) \quad Y(t) = \int_0^t \iota(t, G_r) dr + I \otimes \widehat{w}_t$$

of the diffusion of a square-integrable initial process defined locally by  $G_t \in \mathcal{A}$  over the standard Wiener process  $w_t$  represented in  $\mathcal{H}$  by the operators  $I \otimes \widehat{w}_t = A_t + A_t^*$ . It is not hard to prove [5] that  $Y(r) = V_t(I \otimes \widehat{w}_r) V_t^*$ ,  $\forall t > r \in \mathbf{R}_+$  by the uniqueness of the stochastic operator equation

$$dV_t + \frac{1}{8} G(t) V_t dt = i G(t) V_t d\widehat{w}_t, \quad V_0 = I,$$

where

$$G(t) = \iota(t, G_t), \quad \widehat{w}_t = \frac{i}{2} (\widehat{a}_t^* - \widehat{a}_t) = \frac{1}{2} \widehat{w}_t(i).$$

The above implies the local unitary equivalence of the processes  $A_t + A_t^*$  and  $Y(t)$ , which is always the case for the locally norm-square-integrable operator-functions  $G_t : \mathbb{H} \rightarrow \mathbb{H}$  [9].

**Corollary 1.** In the case under consideration the condition (2.6) completely defines the structure of the inner derivation  $X \mapsto [X, G_t]$  for  $\delta_t - \delta_t^*$ :

$$(2.8) \quad \delta_t(X) = \frac{1}{2} [X, G_t] - \alpha_t(X),$$

where  $\alpha_t(X^*) = \alpha_t(X)^*$ ,  $\forall X \in \mathcal{A}$  is a  $*$ -derivation of the algebra  $\mathcal{A}$ . The conditions (i)–(iii) here also define the structure of the maps  $\gamma_t : \mathcal{A} \rightarrow \mathcal{A}$  in the form

$$(2.9) \quad \begin{aligned} \gamma_t(X) &= \frac{1}{2} \lambda_t(X) - \beta_t(X), \\ \lambda_t(X) &= \frac{1}{4} [G_t, [G_t, X]] + G_t \alpha_t(X) + \alpha_t(X) G_t - \alpha_t^2(X), \end{aligned}$$

where  $\alpha_t^2(X) = \alpha_t(\alpha_t(X))$ ,  $\beta_t : \mathcal{A} \rightarrow \mathcal{A}$  is some  $*$ -derivation

$$\beta_t(X^*X) = X^* \beta_t(X) + \beta_t(X^*) X, \quad \forall X \in \mathcal{A}.$$

PROOF. By taking into account the differentiation property

$$[G_t, X^*X] = X^*[G_t, X] + [G_t, X^*] X,$$

and similarly for  $\alpha_t(X^*X)$  we obtain

$$\begin{aligned} \lambda_t(X^*X) &= \frac{1}{4} [G_t, X^*[G_t, X] + [G_t, X^*]X] \\ &\quad + G_t(X^* \alpha_t(X) + \alpha_t(X^*) X) + (X^* \alpha_t(X) + \alpha_t(X^*) X) G_t \\ &\quad - (X^* \alpha_t^2(X) + 2\alpha_t(X)^* \alpha_t(X) + \alpha_t^2(X)^* X) \\ &= X^* \lambda_t(X) + \lambda_t(X)^* X - \frac{1}{2} [X, G_t]^* [X, G_t] \\ &\quad + [X, G_t]^* \alpha_t(X) + \alpha_t(X)^* [X, G_t] - 2\alpha_t(X)^* \alpha_t(X) \\ &= X^* \lambda_t(X) + \lambda_t(X)^* X - 2\delta_t(X)^* \delta_t(X). \end{aligned}$$

Hence  $\gamma_t^0 = \frac{1}{2} \lambda_t$  possesses the property (ii) of the map  $\gamma_t$ :

$$\gamma_t^0(X^*X) = X^* \gamma_t^0(X) + \gamma_t^0(X)^* X - \delta_t(X)^* \delta_t(X),$$

and the  $*$ -property  $\gamma_t^0(X^*) = \gamma_t^0(X)^*$ , as does  $\gamma_t$  in (iii). From here the result that  $\beta_t = \gamma_t^0 - \gamma_t$  is a  $*$ -derivation follows.

The maps  $\gamma_t$  are called generators for the Lindblad equation

$$d\mu_0^t/dt + \mu_0^t \circ \gamma_t = 0,$$

where  $\mu \circ \gamma(X) = \mu(\gamma(X))$ , which is an algebraic analog of the Kolmogorov equation. This is satisfied by the operators  $X_0^t = \mu_0^t(X)$ ,  $X \in \mathcal{A}$  of the conditional expectation  $\mu_0^t(X) = (\delta_\emptyset | X(t) \delta_\emptyset)$ , with respect to the vacuum function  $\delta_\emptyset \in \mathcal{F}$  defined by

$$X_0^t \psi = [\iota(t, X) h] (\emptyset), \quad \forall h = \psi \otimes \delta_\emptyset, \quad \psi \in \mathcal{D}.$$

The differential conditions obtained for Markov diffusion are necessary for the existence of a unique solution  $X(t)$  of equation (2.3) for all  $X(0) = X \in \mathcal{A}$  and so, for the Lindblad equation. They are sufficient in the case [14] of constant  $\alpha_t$ ,  $\beta_t$  and  $G_t$  and the boundedness of the algebra  $\mathcal{A}$  (for instance, with  $\mathcal{D} = \mathcal{H}$ ). Moreover, as was proved in [10], the maps  $\iota(t) : X \mapsto X(t)$  are representations of  $\mathcal{A}$  in  $\mathcal{A}_t$ , satisfying the condition of nondemolition (2.5) for all  $X \in \mathcal{A}$ . This is also true under significantly more general conditions of local  $p$ -integrability over the norm of the operators  $G_t : \mathcal{H} \rightarrow \mathcal{H}$  (with  $p = 2$ ) and the maps  $\alpha_t$  ( $p = 2$ ) and  $\beta_t$  ( $p = 1$ ) from  $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$  to  $\mathcal{A}$ .

In the case of inner derivations

$$\alpha_t(X) = i[S_t, X], \quad \beta_t(X) = i\left[H_t + \frac{1}{4}(S_t G_t + G_t S_t), X\right],$$

(as is the case for the von-Neumann algebra  $\mathcal{A}$ ), a quantum Markov diffusion is defined by structure maps of the type

$$(2.10) \quad \begin{aligned} \delta_t(X) &= [X, L_t], & \delta_t^*(X) &= [L_t^*, X], \\ \gamma_t(X) &= \frac{1}{2}(L_t^*[L_t, X] + [X, L_t^*]L_t) + i[X, H_t], \end{aligned}$$

where

$$L_t = \frac{1}{2}G_t + iS_t, \quad L_t^* = \frac{1}{2}G_t - iS_t, \quad H_t = H_t^* \in \mathcal{A}.$$

### 3. QUANTUM DIFFUSION AND FILTERING.

**2.1. The a posteriori dynamics.** The quantum diffusion (2.3) under nondemolition measurement (2.4), is described by the classical random variables  $x_t(\omega) = \langle X(t) \rangle_t(\omega)$  of the conditional expectations  $\langle X(t) \rangle_t = \mathbf{E}[X(t)|\mathcal{B}_t]$ , for the operators  $X(t) = \iota(t, X)$ , on the trajectories  $\omega \in \Omega$  of the process  $Y(t)$ , with respect to initial vector-valued function  $e_0 = \psi_0 \otimes \delta_\emptyset$ ,  $\psi_0 \in \mathbf{D}$ . As was established for the first time in [5], [6], the random process  $x_t : \Omega \rightarrow \mathbf{C}$ , considered as a stochastic map  $X(t) \mapsto x_t(\omega)$ , satisfies the Ito filtering equation for quantum diffusion

$$(3.1) \quad d\langle \iota(t, X) \rangle_t + \langle \gamma(t, X) \rangle_t dt = \langle \kappa(t, X) - \langle G(t) \rangle_t \iota(t, X) \rangle_t d\tilde{Y}$$

with respect to the stochastic map  $\langle \cdot \rangle_t : X(t) \mapsto x_t$ . In this equation  $\tilde{Y}(t) = Y(t) - \int_0^t \langle G(r) \rangle_r dr$  is an innovation martingale for the observed process (2.4), and  $\kappa(t) : \mathcal{A} \rightarrow \mathcal{B}(\mathbf{D})$  is a linear  $*$ -map, which in the case of  $F(t) = I_0 = F^*(t)$  is of the particularly simple form:

$$(3.2) \quad \kappa(t, X) = \frac{1}{2}(G(t)X(t) + X(t)G(t)) - \alpha(t, X),$$

where  $\alpha(t, X)$  is defined by the  $*$ -derivation  $\delta(t) + \delta^*(t) = -2\alpha(t)$ . Equation (3.1), derived in [5] by means of the martingale methods of quantum nonlinear filtering, extends the basic equation (8.10), [18] of the optimal diffusion filtering to the case of the noncommutative operator algebras  $\mathcal{A}$ . By complete analogy with the classical case the quantum Ito formula was used with the innovation process and the representation theorem [11], which requires the conditional expectations  $x_t(\omega)$  to exist. This requirement is met by the self-nondemolition condition (2.5) for all  $s \leq t$  which is trivially satisfied in the commutative case  $s > t$ .

In the quantum Markov case of the indirect measurement (2.7) the conditional expectation  $x_t(\omega) = \pi_t(X, \omega)$  of the operators  $X(t)$  can be found as in the classical case by solving an autonomous stochastic equation for the a posteriori state  $\hat{\pi}_t(X) = \langle \iota(t, X) \rangle_t$ . The latter is defined on the trajectories  $\omega$  as a linear stochastic positive normalized map  $\omega(\hat{\pi}_t(X)) = x_t(\omega)$  of the algebra  $\mathcal{A}$  into  $\mathbf{C}$  satisfying the condition

$$\int x_t(\omega) y(\omega) \mathbf{P}_0^t(d\omega) = (e_0 | X(t) Y e_0), \quad \forall X \in \mathcal{A}.$$

In this equation,  $Y \in \mathcal{B}_t$  is any bounded operator in the algebra of the observed  $\mathcal{B}_t$ ,  $y(\omega) = \omega(\hat{y})$  is the Segal transformation of the operator  $\hat{y}$  in  $\mathcal{F}$  that corresponds to the unitary-equivalent operator  $I \otimes \hat{y} = V_t^* Y V_t$ , and  $\mathbf{P}_0^t$  is an induced (by the unitary

transformation) probability measure on the trajectories  $\omega|_{[0,t]} := \{w_r | r \in [0,t]\}$ , restricted to the interval  $[0,t]$  with respect to the initial vector-state

$$\int y(\omega) \mathbf{P}_0^t(d\omega) = (V_t^*(\psi_0 \otimes \delta_\emptyset) \mid (I \otimes \widehat{y}) V_t^*(\psi_0 \otimes \delta_\emptyset)).$$

**2.2. The filtering equation.** Let us sketch the essentials in the derivation of a stochastic Markov quantum filtering equation, obtained for the general output process in [9]. First, we shall prove that the vacuum conditional expectation  $\mu_g^t(X) = (\delta_\emptyset | \pi_g(t, X) \delta_\emptyset)$  of the product  $\pi_g(t, X) = \iota(t, X) e_g(t)$ , where

$$(3.3) \quad e_g(t) = \exp \left\{ \int_0^t g(r) dY(r) - \frac{1}{2} g(r)^2 dr \right\},$$

satisfies the linear evolution equation

$$(3.4) \quad \frac{d}{dt} \mu_g^t(X) + \mu_g^t \circ \gamma_t(X) = \mu_g^t(G_t \cdot X - \alpha_t(X)) g(t),$$

where  $\mu_g^0(X) = X$ , and  $G \cdot X = (GX + XG)/2$ . Let us assume the uniqueness of this solution, which is always true for locally  $p$ -integrable bounded (over the norm) maps  $G_t$ ,  $\alpha_t$  ( $p = 2$ ) and  $\beta_t$  ( $p = 1$ ). We shall prove that  $\mu_g^t(X)$  is the mathematical expectation of the product  $\widehat{\mu}_g^t(X) = \widehat{\mu}^t(X) \widehat{e}_g^t$ , of the stochastic operators  $\mu^t(X, \omega) = \omega[\widehat{\mu}^t(X)]$  (which satisfy a quantum filtering equation) with the exponentials

$$e_g^t(\omega) = \exp \int_0^t \left[ g(r) dw_r - \frac{1}{2} g(r)^2 \right] dr = \omega(\widehat{e}_g^t),$$

(which are defined with respect to the trajectories  $\omega : t \mapsto w_t$  of the standard Wiener process  $w_t$ ,  $t \in \mathbf{R}_+$ ). The above means that the output process  $Y(r)$ , restricted by any  $t \in \mathbf{R}_+$ , is absolutely continuous with respect to the standard restricted process  $w^t = \{w_r | r < t\}$ . This follows from the unitary equivalence  $Y(r) = V_t Y_r V_t^*$ ,  $\forall r < t$  and  $Y_r = I \otimes \widehat{w}_r$ ,  $\forall t \in \mathbf{R}_+$ , representing the output up to a time  $t$  with respect to the initial vector-function  $e_0 = \psi_0 \otimes \delta_\emptyset$  and  $e_0^t = V_t^* e_0$  correspondingly. The probability density  $\rho_0^t(\omega) = \mathbf{P}_0^t(d\omega) / \mathbf{P}(d\omega)$ , for the measurement of the trajectory  $\{w_r | r < t\}$ , of the process  $Y$  on the interval  $[0,t]$ , is defined with respect to the standard Wiener probability measure  $\mathbf{P}(d\omega)$  by the formula  $\rho_0^t(\omega) = \varphi_0^t(I, \omega)$ . Here  $\varphi_0^t(\omega) = \varphi_0 \circ \mu^t(\omega)$  is the stochastic functional  $\varphi_0^t(X, \omega) = (\psi_0 | \mu^t(X, \omega) \psi_0)$ , which corresponds to the initial state  $\varphi_0(X) = (\psi_0 | X \psi_0)$  on the algebra  $\mathcal{A}$ . Finally, we deduce a nonlinear equation for the a posteriori state  $\pi_t(\omega)$  using the ordinary Ito formula and the normalization of the stochastic functional  $\varphi_0^t(\omega)$ .

**Theorem 1.** *Let the equation (3.4) has the unique solution  $X_g^t = \mu_g^t(X)$ , corresponding to the initial condition  $\mu_g^0(X) = X$  for each  $X \in \mathcal{A}$  and  $g \in \mathcal{K}$ . Then it coincides with the vacuum expectation  $X_g^t = (\delta_\emptyset | \pi_g(t, X) \delta_\emptyset)$ , where  $\pi_g(t, X) = \iota(t, X) e_g(t)$  and is defined by the Wiener average*

$$(3.5) \quad X_g^t = \int_0^t X^t(\omega) \exp \left\{ \int_0^t g(r) dw_r - \frac{1}{2} g(r)^2 dr \right\} \mathbf{P}(d\omega)$$

over the continuous trajectories  $\omega \in \Omega$ . Here  $X^t(\omega) = \omega[\widehat{\mu}^t]$  is the Segal transformation of the solution  $\widehat{X}^t = \widehat{\mu}^t(X)$  to the operator filtering equation

$$(3.6) \quad d\widehat{\mu}^t(X) + \widehat{\mu}^t \circ \gamma_t(X) dt = \widehat{\mu}^t(G_t \cdot X - \alpha_t(X)) d\widehat{w}_t$$

with initial condition  $\widehat{\mu}^0(X) = X$ . In this case the linear stochastic equation (3.6) also has a unique solution in the Ito sense, which defines almost everywhere ( $\rho_0^t(\omega) \neq 0$ ) for each  $\psi_0 \in \mathbb{H}$  the a posteriori state

$$\pi_t(X, \omega) = \frac{(\psi_0 | X^t(\omega) \psi_0)}{\rho_0^t(\omega)},$$

where the probability density  $\rho_0^t(\omega) = (\psi_0 | I^t(\omega) \psi_0)$  is given by the positive operator  $I^t(\omega) = \mu^t(I, \omega)$ , satisfying the martingale property

$$\int I^t(\omega) \mathbf{P}(d\omega | w^r) = I^r(\omega), \forall r < t.$$

PROOF. First we find a quantum stochastic equation for  $X_g(t) = \pi_g(t, X)$  using the Ito formula

$$de_g(t) = g(t) e_g(t) dY(t), \quad e_g(0) = 1,$$

where

$$dY(t) = G(t) dt + F(t)^* dA_t + dA_t^* F(t).$$

We obtain according to (2.2)

$$\begin{aligned} d(X(t) e_g(t)) &= dX(t) d e_g(t) + dX(t) e_g(t) + X(t) d e_g(t) \\ &= (g D^* F - C + g X G)(t) e_g(t) dt \\ &\quad + d\iota_0^t(D^* + X F^* g D + X F g) e_g(t) \\ &= \pi_g\left(t, (G_t X + X G_t) \frac{g(t)}{2} - \alpha_t(X) g(t) - \gamma_t(X)\right) dt \\ &\quad + \pi_g(t, \delta_t^*(X) + g(t) X) dA_t + \pi_g(t, \delta_t(X) + g(t) X) dA_t^*, \end{aligned}$$

where the explicit form (2.8) has been used for

$$D^* = \delta^*(X), \quad D = \delta(X), \quad C = \gamma(X), \quad GX + D = G \cdot X - \alpha(X)$$

with  $F = I_0 = F^*$ .

Taking into account the martingale property of the quantum stochastic integral (2.1) with respect to the vacuum-vector  $\delta_0 \in \mathcal{F}$ , we find the equation (3.4) for the operator

$$\mu_g^t(X) : \psi \mapsto \mu_g^t(X) \psi = [\pi_g(t, X)h](\emptyset)$$

in  $\mathbb{H}$  is implied by the action of  $\pi_g(t, X) = X(t) e_g(t)$  on  $h = \psi \otimes \delta_0$ :

$$d\mu_g^t(X) = \mu_g^t((G_t \cdot X - \alpha_t(X) g(t) - \gamma_t(X)) dt), \quad \forall X \in \mathcal{A},$$

where  $\gamma_t : \mathcal{A} \rightarrow \mathcal{A}$  is defined in the form (2.10).

Now, if  $X^t(\omega) = \mu^t(X, \omega)$  satisfies the stochastic equation

$$dX^t(\omega) + C^t(\omega) dt = D^t(\omega) d\widehat{w}_t, \quad X^0(\omega) = X,$$

we can derive a differential for

$$\widehat{\mu}_g^t(X) = \widehat{\mu}^t(X) \widehat{e}_g^t, \quad \widehat{e}_g^t = \exp \left\{ \int_0^t \left( g(r) d\widehat{w}_r - \frac{1}{2} g(r)^2 dr \right) \right\},$$

by means of the Ito formula of classical stochastic calculus.

$$dX^t(\omega) + C^t(\omega) dt = D^t(\omega) d\widehat{w}_t, \quad X^0(\omega) = X.$$

Using  $d\hat{e}_g^t = g(t)\hat{e}_g^t d\hat{w}_t$  we have

$$\begin{aligned} d(\hat{X}^t \hat{e}_g^t) &= d\hat{X}^t d\hat{e}_g^t + d\hat{X}^t \hat{e}_g^t + \hat{X}^t d\hat{e}_g^t \\ &= (g(t)\hat{D}^t - \hat{C}^t) \hat{e}_g^t dt + (\hat{D}^t + \hat{X}^t g(t)) \hat{e}_g^t d\hat{w}_t, \end{aligned}$$

what can be written in the form of the stochastic equation

$$dX_g^t(\omega) + (C_g^t(\omega) - D_g^t(\omega)g(t))dt = (D_g^t(\omega) + X_g^t(\omega)g(t))dw_t$$

for  $X_g^t(\omega) = \omega(\hat{X}^t \hat{e}_g^t)$ . Hence, the mathematical expectation (3.5) of  $X_g^t(\omega) = X^t(\omega)e_g^t(\omega)$  with respect to the Gaussian measure  $\mathbf{P}$  of the standard Wiener process  $w$  satisfies the equation

$$d\mu_g^t(X) = (D_g^t g(t) - C_g^t) dt, \quad \mu_g^0(X) = X.$$

Comparison of this equation with equation (3.4) gives the coefficients

$$C_g^t = \int C^t(\omega) e_g^t(\omega) \mathbf{P}(d\omega), \quad D_g^t = \int D^t(\omega) e_g^t(\omega) \mathbf{P}(d\omega)$$

in the form:

$$C_g^t = \mu_g^t(\gamma_t(X)), \quad D_g^t = \mu_g^t(G_t \cdot X - \alpha_t(X)).$$

Consequently  $C^t(\omega) = \omega(\hat{C}^t)$ ,  $D^t(\omega) = \omega(\hat{D}^t)$  are the coefficients

$$\hat{C}^t = \hat{\mu}^t(\gamma_t(X)), \quad \hat{D}^t = \hat{\mu}^t(G_t \cdot X - \alpha_t(X)),$$

that define an equation for  $X^t(\omega) = \mu^t(X, \omega)$  in the form of (3.6). The solution  $I^t(\omega) = \mu^t(I, \omega)$  to this equation for the initial condition  $X = I$  defines a positive operator-valued diffusive process  $I^t(\omega) = \omega(\hat{\mu}^t(I))$  which satisfies to the martingale equation  $dI^t(\omega) = G^t(\omega) dw_t$  with the initial condition  $I^0(\omega) = I$ , where  $G^t(\omega) = \mu^t(G, \omega)$  and the properties  $\alpha_t(I) = 0 = \gamma_t(I)$  are substituted into (3.6). Thus the Theorem 1 is proved.

**2.3. The classical case.** The remark that follows provides an explanation why equation (3.6) is a noncommutative analog of the Zakai filtering equation.

**Remark 1.** Let  $\mathcal{A}$  be a commutative algebra equivalent to the space  $C^\infty(\mathbf{R}^d)$  of infinitely-differentiable functions  $x : \mathbf{R}^d \rightarrow \mathbf{C}$  with the pointwise product, and the involution  $x^*(z) = \bar{x}(z)$ . Then equation (3.6) is an operator representation of the Zakai equation

$$(3.7) \quad d\mu_{z_0}^t + \frac{1}{2} \Delta_t \mu_{z_0}^t dt = (g_t + \nabla_t) \mu_{z_0}^t dw_t, \quad \mu_{z_0}^0 = \delta_{z_0},$$

for the nonnormalized a posteriori distribution  $\mu_{z_0}^t(dz, \omega)$  of the Markov diffusion process  $z(t)$  described by the stochastic equation

$$(3.8) \quad dz + c_t(z) dt = a_t(z) dv_t, \quad z(0) = z_0,$$

with the indirect measurement

$$dy(t) = g_t(z(t))dt + dw_t$$

defined by the standard Wiener process  $w_t = -v_t$ . Here

$$\begin{aligned} \int x(z) \nabla_t \mu(dz) &= - \int \sum_{k=1}^d a^k(z) x'_k(z) \mu(dz), \\ \int x(z) \Delta_t \mu(dz) &= +2 \int \sum_{k=1}^d c_t^k(z) x'_k(z) \mu(dz) \\ &\quad - \int \sum_{k,l=1}^d a_t^k(z) a_t^l(z) x''_{kl}(z) \mu(dz), \end{aligned}$$

with  $x'_k = \partial_k x$ ,  $x''_{kl} = \partial_k \partial_l x$ . The integral  $\int \mu_{z_0}^t(dz, \omega) = \rho_{z_0}^t(\omega)$  defines the probability density of the output process  $y(r, \omega)$  on the interval  $0 \leq r < t$  with respect to the Wiener distribution  $\mathbf{P}(d\omega)$  with given initial state  $z_0 \in \mathbf{R}^d$ ,  $\delta_{z_0}(dz) = 1$  with  $z_0 \in dz$ ,  $\delta_{z_0}(dz) = 0$ , and  $z_0 \notin dz$ .

Indeed, in the case of the commutative algebra  $\mathcal{A} \simeq C^\infty(\mathbf{R}^d)$ ,  $G_t$  is the multiplication operator by the given function  $g_t(z)$  of the state  $z \in \mathbf{R}^d$  of the Markov process  $z(t)$ . This process has the generator  $\gamma_t(X)(z) = [\Gamma_t x](z)$ , defined by the diffusion operator  $\Gamma_t$  on the measurable functions  $x : z \mapsto x(z)$ ,  $x \in C^\infty(\mathbf{R}^d)$ . The indirect measurement of  $g_t(z)$  is given by the output process

$$y(t) = \int_0^t g_r(z(r)) dr + w_t.$$

Since  $[G_t, X] = 0$ ,  $\forall X \in \mathcal{A}$ ,  $\delta_t = -\alpha_t$  is a real derivation

$$\alpha_t(X)(z) = a_t(z) \partial x(z) := a_t^k(z) \partial_k x(z), \quad \partial_k = \frac{\partial}{\partial z^k},$$

and  $\gamma_t = G_t \cdot \alpha_t - \beta_t - \frac{1}{2} \alpha_t^2$ , where  $G_t \cdot \alpha_t(X)(z) = g_t(z) \alpha_t(X)(z)$ , the operation

$$(3.9) \quad \Gamma_t = (a_t g - b_t) \partial - \frac{1}{2} (a_t \partial)^2 = c_t^k \partial_k - \frac{1}{2} a_t^k a_t^l \partial_k \partial_l$$

is a standard generator of the diffusion process  $z(t)$ , with  $c_t^k = a_t^l (g_t \delta_l^k - \frac{1}{2} \partial_l a_t^k) - b_t^k$ .

Note that the noise  $v_t = -w_t$  in the classical system appeared essentially the same as in the observation channel because it was represented in the Fock space of the Wiener process  $w_t$ . In order to represent a classical stochastic system in the same way with the noise  $v_t \neq 0$  which is independent of  $w_t$ , it is necessary to start from the Fock space  $\mathcal{F} = \Gamma(\mathcal{K})$  over  $\mathcal{K} = \mathbf{C}^m \otimes L^2(\mathbf{R}_+)$  with multiplicity  $m \geq 2$ , as is the case in [14].

**2.4. The a posteriori equation.** In the general case the filtering equation (3.6) defines the nonnormalized a posteriori state  $\widehat{\varphi}_0^t = \varphi_0 \circ \widehat{\mu}_0^t$ , which is the vector-state  $\widehat{\varphi}_0^t(X) = (\widehat{\psi}_0^t | X \widehat{\psi}_0^t)$  for all  $\varphi_0(X) = (\psi_0 | X \psi_0)$  in the case of inner derivations (2.10). The normalized a posteriori state  $\widehat{\pi}_t(X) = \widehat{\varphi}_0^t(X) / \widehat{\varphi}_0^t(I)$  satisfies the (nonlinear) a posteriori equation

$$(3.10) \quad d\widehat{\pi}_t(X) + \widehat{\pi}_t \circ \gamma_t(X) dt = \widehat{\pi}_t(\kappa_t(X) - \widehat{\pi}_t(G_t) X) d\widetilde{w}_t,$$

with initial condition  $\widehat{\pi}_0(X) = \varphi_0(X)$ , where

$$\kappa_t(X) = G_t \cdot X - \alpha_t(X), \quad d\widetilde{w}_t = dw_t - \widehat{\pi}_t(G_t) dt.$$

The nonlinear stochastic equation (3.10) is the Markov case of the general a posteriori diffusive equation (3.1) with innovating martingale  $d\widetilde{Y}$  represented by  $d\widetilde{w}$ .

It can be deduced from the linear one (3.6) for the nonnormalized state  $\widehat{\varphi}_t(X) = \varphi_0(\widehat{\mu}_0^t(X))$  by applying the classical Ito formula to the product  $\widehat{\varphi}_t(X) = \widehat{\rho}_0^t \widehat{\pi}_t(X)$  and noting that the positive martingale  $\widehat{\rho}_0^t = \widehat{\varphi}_0^t(I)$  has the stochastic differential  $d\widehat{\rho}_0^t = \widehat{\varphi}_0^t(G_t)d\widehat{w}_t$ . Indeed,

$$\begin{aligned} d\widehat{\varphi}_0^t(X) &= \widehat{\rho}_0^t d\widehat{\pi}_t(X) + d\widehat{\rho}_0^t \widehat{\pi}_t(X) + \widehat{\rho}_0^t d\widehat{\pi}_t(X) \\ &= \widehat{\varphi}_0^t(G_t) \widehat{\pi}_t \circ \widetilde{\kappa}_t(X) dt + (\widehat{\varphi}_0^t(G_t) \widehat{\pi}_t(X) + \widehat{\rho}_0^t \widehat{\pi}_t \circ \widetilde{\kappa}_t(X)) d\widehat{w}_t \\ &\quad - \widehat{\rho}_0^t (\widehat{\pi}_t \circ \gamma_t(X) + \widehat{\pi}_t \circ \widetilde{\kappa}_t(X) \widehat{\pi}_t(G_t)) dt \\ &= \varphi_0^t(G_t \cdot X - \alpha_t(X)) d\widehat{w}_t - \widehat{\varphi}_0^t \circ \gamma_t(X) dt, \end{aligned}$$

where  $\widetilde{\kappa}_t(X) = \kappa_t(X) - \widehat{\pi}_t(G_t)X$ . Note that in the deduction of the equation the following relation was used:

$$\widehat{\rho}_0^t \widehat{\pi}_t \circ \widetilde{\kappa}_t(X) \widehat{\pi}_t(G_t) = \widehat{\pi}_t \circ \widetilde{\kappa}_t(X) \widehat{\varphi}_0^t(G_t).$$

#### 4. LINEAR QUANTUM DIFFUSION WITH OBSERVATION

Let  $\Xi$  be a symplectic  $\sharp$ -space, i.e. a complex space with the involution

$$\eta \in \Xi \mapsto \eta^\sharp, \quad \eta^{\sharp\sharp} = \eta, \quad \left( \sum \lambda_i \eta_i \right)^\sharp = \sum \lambda_i^* \eta_i^\sharp, \quad \forall \lambda_i \in \mathbf{C},$$

and skew-symmetric bilinear  $\sharp$ -form:  $s : \Xi \times \Xi \rightarrow \mathbf{C}$ ,

$$s(\eta, \eta^\sharp) = -s(\eta^\sharp, \eta), \quad s(\eta^\sharp, \eta)^* = s(\eta, \eta^\sharp).$$

We denote by  $\text{Re } \Xi$  the real space of the  $\sharp$ -invariant vectors  $\eta = \eta^\sharp \in \Xi$ , and assume that  $\text{Re } \Xi$  is a Hilbert space with respect to the scalar product  $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle$ , satisfying the inequality

$$\xi^2 \eta^2 - \langle \xi, \eta \rangle^2 \geq \frac{1}{4} s(\xi, \eta)^2, \quad \forall \xi, \eta \in \text{Re } \Xi,$$

where  $\xi^2 = \langle \xi, \xi \rangle$ ,  $\eta^2 = \langle \eta, \eta \rangle$ .

A linear map  $R : \Xi \rightarrow \mathcal{B}(D)$ , satisfying the  $\sharp$ -property  $R(\eta)^* = R(\eta^\sharp)$  defines an operator representation of the canonical commutation relations if

$$(4.1) \quad [R(\eta), R(\eta^\sharp)] = R(\eta) R(\eta^\sharp) - R(\eta^\sharp) R(\eta) = \frac{1}{i} s(\eta, \eta^\sharp) I.$$

on a complex pre-Hilbert space  $D$ . It is called Gaussian with respect to a normalised vector  $\psi_0 \in D$ ,  $\|\psi_0\|^2 = (\psi_0 | \psi_0) = 1$  if

$$(4.2) \quad (\psi_0 | e^{R(i\xi)} \psi_0) = e^{\vartheta_0(i\xi) - \xi^2/2} \equiv \theta_0(\xi), \quad \xi \in \text{Re } \Xi.$$

Here  $\vartheta_0(\eta) = \langle \eta, \vartheta_0 \rangle$  is the linear continuous  $\sharp$ -functional  $\vartheta_0(\eta)^* = \vartheta_0(\eta^\sharp)$  of the mathematical expectation  $(\psi_0 | R(\eta) \psi_0) = \vartheta_0(\eta)$ , that is defined by some  $\vartheta_0 \in \text{Re } \Xi$  by means of the complexified bilinear form

$$\langle \xi + i\eta, \vartheta_0 \rangle = \langle \xi, \vartheta_0 \rangle + i\langle \eta, \vartheta_0 \rangle$$

on  $\Xi$ . The product  $\langle \xi, \eta \rangle$  corresponds to the symmetric covariance

$$\text{Re} (R(\xi) \psi_0 | R(\eta) \psi_0) - \vartheta_0(\xi) \vartheta_0(\eta) = \langle \xi, \eta \rangle, \quad \forall \xi, \eta \in \text{Re } \Xi.$$

The exponents  $e^{R(i\xi)} = X(\xi) = e^{iR(\xi)}$  are defined as unitary operators, which form the Weyl family  $\{X(\xi) | \xi \in \text{Re } \Xi\}$ ,

$$X(\xi) X(\eta) = e^{is(\xi, \eta)} X(\xi + \eta), \quad \forall \xi, \eta \in \text{Re } \Xi,$$

with the self-adjoint generators

$$R(\xi) \psi = -i \frac{d}{d\lambda} X(\lambda \xi) \psi|_{\lambda=0}, \quad \xi \in \text{Re } \Xi.$$

Such a representation can be realised in the Fock space  $H = F$  over the completion  $K$  of the (quotient) space  $\Xi$  with respect to the (semi) positive definite scalar product

$$(\xi | \eta) = \langle \eta, \xi^\sharp \rangle + \frac{i}{2} s(\eta, \xi^\sharp), \quad \forall \xi, \eta \in \Xi.$$

Indeed, the space  $F$  can be defined as the completion of the (quotient) span  $D$  of the exponential vectors  $\{\eta^\otimes | \eta \in \Xi\}$  with respect to the scalar product

$$(\xi^\otimes | \eta^\otimes) = \sum_{n=0}^{\infty} \frac{1}{n!} (\xi | \eta)^n = \exp(\xi | \eta).$$

Let  $A^* : \Xi \rightarrow \mathcal{B}(D)$  be a linear map, defining the creation operators in  $D$  by the adjoints  $A^*(\eta) = A(\eta^\sharp)^*$  to the annihilation operators

$$A(\xi) : \eta^\otimes \mapsto (\xi | \eta) \eta^\otimes, \quad \forall \xi \in \text{Re } \Xi.$$

From the canonical commutation relations

$$[A(\eta^\sharp), A^*(\eta)] = (\eta | \eta) I \geq 0, \quad \forall \eta \in \Xi,$$

one can obtain the relations (4.1) for the linear combinations

$$R(\eta) = \vartheta_0(\eta) I + 2\Re A(\eta), \quad 2\Re A(\eta) = A^*(\eta) + A(\eta).$$

This defines the Gaussian representation  $\xi \mapsto R(\xi)$  with respect to the vacuum vector  $\psi_0 = 0^\otimes$  in  $F$ , so that  $A(\eta) \psi_0 = 0$ ,  $\forall \eta$ . The Weyl operators  $X(\xi)$  are defined in  $F$  as

$$(4.3) \quad X(\xi) = \theta_0(\xi) e^{A^*(i\xi)} e^{A(i\xi)}, \quad \forall \xi \in \text{Re } \Xi.$$

We obtain the representation (4.2):  $\theta_0(\xi) = (\psi_0 | X(\xi) \psi_0)$  if we take into account the fact that  $e^{A(\eta)} \psi_0 = \psi_0$ .

Let us denote by  $\mathbf{j} : \eta \mapsto \mathbf{j}\eta (= \eta)$  a canonical bounded map from the Hilbert space  $\Xi$  with respect to the norm

$$|\eta| = \langle \eta^\sharp, \eta \rangle^{1/2} = \sqrt{(\text{Re } \eta)^2 + (\text{Im } \eta)^2}, \quad \text{Im } \eta^\sharp = \text{Re } i\eta,$$

into the pre-Hilbert space  $\Xi$  with respect to the (semi) norm  $\|\eta\| = (\eta | \eta)^{1/2}$ ,

$$\begin{aligned} \|\eta\|^2 &= (\eta | \eta) = |\eta|^2 + \frac{i}{2} s(\eta, \eta^\sharp) = |\eta|^2 + s(\text{Re } \eta, \text{Im } \eta) \\ &\leq |\eta|^2 + |s(\text{Re } \eta, \text{Im } \eta)| \leq |\eta|^2 + 2|\text{Re } \eta| |\text{Im } \eta| \leq 2|\eta|^2. \end{aligned}$$

Then we can write  $(\xi | \eta) = \langle \xi^\sharp, \mathbf{g}\eta \rangle$ , where  $\mathbf{g} = \mathbf{1} - \frac{i}{2}\mathbf{s}$ , so that  $\mathbf{g}\eta$  is the complex bounded functional  $\vartheta(\xi) = (\xi^\sharp | \eta) = \langle \xi, \mathbf{g}\eta \rangle$  on  $\text{Re } \Xi$  which together with  $\vartheta^\sharp(\xi) = \langle \xi^\sharp, \mathbf{g}\eta \rangle^* = (\eta | \xi)$  defines the Hermitian functional

$$2 \text{Re } \vartheta = \vartheta + \vartheta^\sharp = 2 \text{Re } \eta + \mathbf{s} \text{Im } \eta,$$

where  $\mathbf{s} : \text{Re } \Xi \rightarrow \text{Re } \Xi$  is a skew-symmetric operator  $\langle \xi, \mathbf{s}\eta \rangle = s(\eta, \xi)$ ,  $|\mathbf{s}\eta| \leq 2|\eta|$ .

Let us consider a quantum diffusion for the operators  $R(t, \eta) = \pi(t, R(\eta))$  with continuous indirect sequential observation of the operators

$$G_t = L_t + L_t^* = R(\zeta_t + \zeta_t^\sharp).$$

Here  $L_t = R(\zeta_t)$ ,  $L_t^* = R(\zeta_t^\sharp)$  are defined by the weak locally square-integrable families  $\{\zeta_t\}$ ,  $\{\zeta_t^\sharp\}$  of the elements  $\zeta_t$ ,  $\zeta_t^\sharp \in \Xi$ ,  $t \in \mathbf{R}_+$  so that

$$\int_0^t \varepsilon_r(\vartheta^\sharp, \vartheta) dr < \infty, \forall \vartheta \in \Xi, t \in \mathbf{R}_+,$$

where

$$(4.4) \quad \varepsilon_t(\vartheta^\sharp, \vartheta) = \vartheta^\sharp(\zeta_t) \vartheta(\zeta_t^\sharp) = |\langle \vartheta^\sharp, \zeta_t \rangle|^2.$$

Taking into account the fact that any derivation of the Weyl algebra  $\mathcal{A}$  is internal, we consider only the structural maps (2.10), given by the operators  $L_t, L_t^*$  and by a Hamiltonian  $H_t$  in the initial Fock space  $\mathbf{H} = \mathbf{F}$ . Moreover, we shall assume that the Hamiltonian is obtained by the normal ordering  $H_t =: h_t(R) :$  of a quadratic form  $h_t(R)$  of the operators  $A + A^*$  such that

$$(\psi_\eta | H_t \psi_\eta) = v_t(\vartheta) + \frac{1}{2} \omega_t(\vartheta, \vartheta), \quad \vartheta = \vartheta_0 + 2 \operatorname{Re}(\mathbf{g}\eta).$$

In this equation  $\psi_\eta = \exp\{-\frac{1}{2}(\eta|\eta) + A^*(\eta)\} \psi_0$  are the normalised exponential vectors

$$\psi_\eta = e^{-\|\eta\|^2/2} \eta^\otimes, \quad \|\psi_\eta\|^2 = (\psi_\eta | \psi_\eta) = 1, \quad \forall \eta \in \Xi,$$

generating  $\mathbf{H} = \mathbf{F}$  as the completion of the linear envelope  $\{\psi_\eta\}$ .  $\{v_t | t \in \mathbf{R}_+\}$  is a locally integrable family of the linear  $\sharp$ -forms  $v_t : \Xi \rightarrow \mathbf{C}$ , and  $\{\omega_t | t \in \mathbf{R}_+\}$  is a locally integrable family of real symmetric forms

$$\omega_t(\vartheta^\sharp, \vartheta) = \omega_t(\operatorname{Re} \vartheta, \operatorname{Re} \vartheta) + \omega_t(\operatorname{Im} \vartheta, \operatorname{Im} \vartheta) = \omega_t(\vartheta, \vartheta^\sharp).$$

We shall consider the forms  $v_t, \omega_t$  to be continuous, so that

$$v_t(\vartheta) = \langle v_t, \vartheta \rangle, \quad \omega_t(\vartheta', \vartheta) = \vartheta'(\omega_t \vartheta) = \langle \vartheta', \omega_t \vartheta \rangle, \quad \forall \vartheta, \vartheta' \in \Xi,$$

where  $v_t \in \operatorname{Re} \Xi$  and  $\omega_t$  is a symmetric operator which is bounded with respect to the norm in  $\Xi$ .

**Proposition 2** Under the assumptions made above about the linearity of  $L_t, L_t^*$  in  $\{R(\eta)\}$  and quadraticity of  $H_t$  the equation (2.3) for  $X(t) = R(t, \eta)$ ,  $\eta \in \Xi$  with  $D = [X, L]$ ,  $D^* = [L^*, X]$  and  $C$  defined by the generator (2.10), is linear and autonomous with respect to the family  $\{R(\eta) | \eta \in \Xi\}$ ,

$$(4.5) \quad dR(t, \eta) + R(t, i\kappa_t \mathbf{s}\eta) dt = d\widehat{v}_t(\eta) + Iv_t(\mathbf{s}\eta) dt,$$

where  $\kappa_t$  is a complex bounded operator in  $\Xi$  and

$$d\widehat{v}_t(\eta) = s(\eta, \operatorname{Im} \zeta_t) d\widehat{w}_t + s(\eta, 2 \operatorname{Re} \zeta_t) d\widehat{u}_t = \frac{1}{2} d\widehat{w}_t(s(\zeta, i\eta)),$$

where  $\widehat{u}_t = \Im A_t^*$ .

Indeed, for the operators  $H_t$  with the quadratic Wick symbols  $h_t(\vartheta) = v_t(\vartheta) + \frac{1}{2} \omega_t(\vartheta, \vartheta)$  the following commutation relations hold:

$$i[H_t, R(\eta)] = v_t(\mathbf{s}\eta) I + R(\omega_t \mathbf{s}\eta), \quad \forall \eta \in \Xi.$$

In addition, the equations (4.1) also give

$$i[R(\eta), L_t] = s(\eta, \zeta_t) I, \quad i[L_t^*, R(\eta)] = s(\zeta_t^\sharp, \eta) I$$

for  $L_t = R(\zeta_t)$ ,  $L_t^* = R(\zeta_t^\sharp)$ . By substituting this into (2.10) we obtain  $C_t = R(i\kappa_t \mathbf{s}\eta) - v_t(\mathbf{s}\eta) I$ , where

$$\kappa_t = \frac{1}{2} \gamma_t + i\omega_t, \quad \gamma_t = \varepsilon_t - \varepsilon_t^\sharp.$$

Here  $\varepsilon_t$  is an operator in  $\Xi$ , that defines the positive form (3.4)

$$\varepsilon_t(\vartheta^\sharp, \vartheta) = \langle \vartheta^\sharp, \varepsilon_t \vartheta \rangle, \quad \langle \vartheta^\sharp, \varepsilon_t^\sharp \vartheta \rangle = \varepsilon_t(\vartheta, \vartheta^\sharp), \quad \forall \vartheta \in \Xi.$$

In the right-hand side of the equation (2.3) we obtain the quantum noise

$$\begin{aligned} D_t^* dA_t + D_t dA_t^* &= [R(\zeta_t^\sharp), R(\eta)] dA_t + [R(\eta), R(\zeta_t)] dA_t^* \\ (4.6) \qquad \qquad \qquad &= i(s(\eta, \zeta_t^*) dA_t - s(\eta, \zeta_t) dA_t^*) = d\widehat{v}_t(\eta). \end{aligned}$$

The following theorem establishes the existence and uniqueness of the solution of equation (4.5) together with the integral of the locally square-integrable real function  $g(t)$  with respect to the stochastic differentials

$$(4.7) \qquad \qquad \qquad dY(t) = R(t, \zeta_t + \zeta_t^\sharp) dt + dW_t,$$

where  $W_t = A_t + A_t^*$  is the Fock representation of the Wiener process — an error of measurement  $L_t + L_t^*$  in Fock space  $\mathcal{F} = \Gamma(\mathcal{K})$ .

**Theorem 2.** *Let the equations (4.5), (4.6) for the linear diffusion be defined by the functions  $v_t, \omega_t$  and  $\varepsilon_t$ , which are locally integrable with respect to the norm in  $\Xi$ , such that*

$$|v_t|_t^{(1)} = \int_0^t |v_r| dr < \infty, \quad |\kappa|_t^{(1)} = \int_0^t |\kappa_r| dr < \infty, \quad \forall t,$$

where  $|v_t| = \sqrt{v_t^2}$ ,  $|\kappa_t| = \sup\{\kappa_t(\vartheta', \vartheta) \mid |\vartheta'|, |\vartheta| \leq 1\}$ , and

$$|\zeta + \zeta^\sharp|_t^{(2)} = \left( \int_0^t (2 \operatorname{Re} \zeta_r)^2 dr \right)^{1/2} < \infty, \quad \forall t \in \mathbf{R}_+.$$

Then they have a unique solution defined in the Hilbert space  $\mathbf{F} \otimes \mathcal{F}$  as

$$(4.8) \qquad R(t, \eta) + \int_0^t g(r) dY(r) = \int_0^t s(v_r, \eta_r) dr + R(\eta(t)) + i_0^t(f^*, f),$$

by the quantum stochastic integral  $i_0^t(f^*, f) = a(f_t^*) + a^*(f_t)$  with

$$a(f_t^*) = \int_0^t f^*(r, \eta) dA_r, \quad a^*(f_t) = \int_0^t f(r, \eta) dA_r^*.$$

Here  $\eta(t) = \phi_0^{(g)}(t, \eta) = \eta_0$ ,  $\eta_r = \phi_r^{(g)}(t, \eta)$ ,  $r \in [0, t[$  is the solution of the backward conjugate equation

$$(4.9) \qquad \qquad \qquad -\dot{\eta}_r + i\kappa_r s \eta_r = g(r) (\zeta_r + \zeta_r^\sharp),$$

with the boundary condition  $\eta_t = \eta$ . The complex functions  $f_t, f_t^*$  of  $r \in \mathbf{R}_+$  and  $\eta \in \Xi$  equal zero at  $r > t$  and are defined for  $r \leq t$  by the real function  $g(r)$  as

$$f^*(r, \eta) = g(r) + is(\phi_r^{(g)}(t, \eta), \zeta_r^\sharp) = f(r, \eta^\sharp)^*.$$

PROOF First, we write the weak solution of the equation (4.9) in the standard Duhamel form

$$\eta_r = \phi(t) \eta + \int_r^t g(s) \phi_r(s) (\zeta_s + \zeta_s^\sharp) ds,$$

where  $\phi_r(t) \eta = \phi_r^{(0)}(t, \eta)$  is the solution of the equation (4.9) with zero right-hand side  $g = 0$ . The resolving operator  $\phi_r(t)$  exists as the chronologically ordered

exponential

$$\phi_r(t) = \sum_{n=0}^{\infty} \int \cdots \int_{r \leq t_1 < \cdots < t_n < t} \kappa_{t_1} \mathbf{s} \cdots \kappa_{t_n} \mathbf{s} dt_1 \cdots dt_n.$$

This comes from the estimate of the norm

$$\begin{aligned} |\phi_r(t)| &= \sup \left\{ |\phi_r(t) \eta| : |\eta| < 1 \right\} \\ &\leq \sum_{n=0}^{\infty} |\mathbf{s}|^n \int \cdots \int_{0 \leq t_1 < \cdots < t_n < t} |\kappa_{t_1}| \cdots |\kappa_{t_n}| dt_1 \cdots dt_n \\ &\leq \exp \left\{ \int_0^t |2\kappa_r| dr \right\}, \end{aligned}$$

which is finite as  $|\mathbf{s}| \leq 2$ . Hence, the linear form  $\langle \eta_r, \vartheta \rangle$  is uniquely defined on  $\Xi \ni \vartheta$  for every  $r \in [0, t)$  as a bounded functional with the estimate

$$\begin{aligned} |\langle \eta_r, \vartheta \rangle| &\leq |\langle \phi_r(t), \vartheta \rangle| + \int_r^t |g(s)| |\langle \phi_r(s), 2 \operatorname{Re} \zeta_s, \vartheta \rangle| ds \\ &\leq |\eta| |\phi_r^\top(t) \vartheta| + |g|_t^{(2)} |\zeta + \zeta^\sharp|_t^{(2)} \left( \int_r^t |\phi_r^\top(s) \vartheta|^2 ds \right)^{1/2} \\ &\leq \left( |\eta| + |g|_t^{(2)} |\zeta + \zeta^\sharp|_t^{(2)} \sqrt{t-r} \right) |\vartheta| \exp \{ 2 |\kappa|_t^{(1)} \}. \end{aligned}$$

Now we can integrate the left-hand side of the equation (4.8) by parts, taking into account (4.6) and (4.9):

$$\begin{aligned} R(t, \eta) &+ \int_0^t g(r) (R(r, 2 \operatorname{Re} \zeta_r) dr + 2\Re dA_r) \\ &= R(t, \eta) + \int_0^t (2\Re \{g(r) dA_r\} - R(r, \dot{\eta}_r - i\kappa_r \mathbf{s} \eta_r) dr) \\ &= R(0, \eta_0) + \int_0^t (2\Re \{g(r) dA_r\} + dR(r, \eta_r) + R(r, i\kappa_r \mathbf{s} \eta_r) dr) \\ &= R(\phi_0^{(g)}(t, \eta)) + \int_0^t (2\Re \{g(r) + is(\eta_r, \zeta_r^\sharp) dA_r\} + v_r(\mathbf{s} \eta_r) dr). \end{aligned}$$

Here  $dR(r, \eta_r)$  is the quantum stochastic differential  $dR(r, \eta)|_{\eta=\eta_r}$ , satisfying the equation (4.5) for  $t = r$ . This proves Theorem 2.

**Remark 2** The solution  $R(t, \eta)$  of the equation (4.5) given by the integral (4.8) for  $g = 0$  preserves the commutation relations (4.1) and satisfies the nondemolition principle

$$[R(t, \eta), Y_g(t)] = 0, \quad \forall \eta \in \Xi, \quad g \in L^2(\mathbf{R}_+),$$

with respect to the commutative (self-nondemolition) processes

$$Y_g(t) = \int_0^t g(r) dY(r), \quad g \in L^2(\mathbf{R}_+).$$

Indeed, by using the quantum Ito formula we can obtain

$$\begin{aligned} [dR(t, \eta), dR(t, \eta^\sharp)] &= [d\hat{v}_t(\eta), d\hat{v}_t(\eta^\sharp)] = \gamma_t(\mathbf{s}\eta, \mathbf{s}\eta^\sharp) dt, \\ [dR(t, \eta), dY_g(t)] &= [d\hat{v}_t(\eta), d\hat{w}_t g(t)] = is(\eta, \zeta + \zeta^\sharp) g(t) dt. \end{aligned}$$

Hence, if  $[R(t, \eta), R(t, \eta^\sharp)] = (1/i)s(\eta, \eta^\sharp)I$ , then

$$\begin{aligned} [dR(t, \eta), R(t, \eta^\sharp)] &= \kappa_t^\top(\mathbf{s}\eta, \mathbf{s}\eta^\sharp)dt, \\ [R(t, \eta), dR(t, \eta^\sharp)] &= -\kappa_t(\mathbf{s}\eta, \mathbf{s}\eta^\sharp)dt \end{aligned}$$

and

$$\begin{aligned} d[R(t, \eta), R(t, \eta^\sharp)] &= \kappa_t^\top - \kappa_t + \gamma_t(\mathbf{s}\eta, \mathbf{s}\eta^\sharp) dt = 0, \\ d[R(t, \eta), Y_g(t)] &= [dR(t, \eta), Y_g(t)] + [R(t, \eta), dY_g(t)] + [dR(t, \eta), Y_g(t)] \\ &= \left\{ [R(t, \eta), R(t, \zeta_t + \zeta_t^\sharp)] + is(\eta, \zeta_t + \zeta_t^\sharp)I \right\} g(t) dt = 0, \end{aligned}$$

if  $[R(t, \eta), Y_g(t)] = 0$  and, consequently,  $[dR(t, \eta), Y_g(t)] = 0$ .

**Example.** Let us consider the simplest nontrivial case of the space  $\Xi = \mathbf{C}^2$  of column vectors  $\eta = \begin{bmatrix} \eta_p \\ \eta_q \end{bmatrix}$ ,  $\eta_p, \eta_q \in \mathbf{C}$  with the involution  $\eta \mapsto \eta^\sharp$  (a complex conjugation  $\eta^\sharp = \begin{bmatrix} \eta_p^* \\ \eta_q^* \end{bmatrix}$ ) and nondegenerate symplectic form

$$s(\eta, \eta^\sharp) = 2(\eta_p \eta_q^* - \eta_q \eta_p^*).$$

This corresponds to the canonical commutation relations  $[P, Q] = (2/i)I$  for the operators momentum  $P$  and coordinate  $Q$  for the one-dimensional quantum particle. They are defined in the Fock space over  $\mathbf{K} = \mathbf{C}$  as (4.1) for  $R(\eta) = \eta_p P + \eta_q Q$  due to the degeneracy of the semi positive definite scalar product

$$(\xi | \eta) = \eta_p \xi_p^* + \eta_q \xi_q^* + \frac{i}{2} s(\eta, \xi^\sharp) = (\xi_p^* + i\xi_q^*)(\eta_p - i\eta_q).$$

This Fock representation in  $\mathbf{F} = \Gamma(\mathbf{C})$  is associated with the Gaussian state (4.2) (which corresponds to the standard scalar product  $\langle \xi^\sharp, \eta \rangle = \xi_p^* \xi_p + \xi_q^* \xi_q$ ) and is equivalent to the Shrödinger representation in  $\mathbf{H} = L^2(\mathbf{R})$ . In this representation  $Q_0 = x$  and  $P_0 = \frac{2}{i} \frac{d}{dx}$ , and

$$\psi_0(x) = \frac{1}{(2\pi)^{1/4}} \exp \left\{ -\frac{1}{4}(x - q)^2 + \frac{i}{2} qx \right\},$$

where  $q = (\psi_0 | Q \psi_0) = \vartheta_q$  and  $p = (\psi_0 | P \psi_0) = \vartheta_p$ .

Suppose that this particle exhibits free quantum Brownian motion, i.e.  $H = (1/(2m))P^2$  is its Hamiltonian, where  $m > 0$  is the mass of the particle, so that

$$\begin{aligned} P(t) &= P \otimes \hat{1} + \hat{v}_t, & Q(t) &= Q \otimes \hat{1} + \frac{1}{m} \int_0^t P(r) dr, \\ \hat{v}_t &= i\sqrt{\lambda}(A_t^* - A_t) = I \otimes 2\sqrt{\lambda}\Im A_t^* \end{aligned}$$

is a realisation in the Fock space  $\mathbf{F} \otimes \mathcal{F}$ , where  $\mathcal{F} = \Gamma(L^2(\mathbf{R}^+))$ , of the Wiener process with intensity  $2\lambda$  with respect to the vacuum-vector  $\delta_\emptyset \in \mathcal{F}$ . This corresponds to the quantum stochastic equation (4.5) with the parameters  $v_t = 0$ ,

$$\omega_t = \begin{pmatrix} 1/2m & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{s}_t = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad \zeta_t = \frac{\sqrt{\lambda}}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \zeta_t^\sharp.$$

In this case the equation (4.6) describes an indirect observation of the coordinate of the particle, realised by measuring the increment of the commutative process

$$Y(t) = \int_0^t \sqrt{\lambda} Q(r) dr + W_t, \quad Y(0) = 0,$$

with precision  $\lambda > 0$ , where  $W_t = A_t + A_t^*$  is the standard stochastic error of the measurement represented by another realisation in  $F \otimes \mathcal{F}$  of the standard Wiener process with respect to  $\delta_\emptyset$ . Note, that by the noncommutativity

$$[\widehat{v}(s), \widehat{w}(r)] = 2i\sqrt{\lambda} \min\{s, r\} I \otimes \hat{1}$$

it is impossible to represent these quantum processes together as Wiener classical processes on the same probability space, although each of them separately allows such a representation. In addition, the nondemolition condition of the pair  $(P, Q)$  with respect to the observation  $Y$  can be verified directly by calculating the commutators

$$[P(t), Y(s)] = 0, \quad [Q(t), Y(s)] = 0, \quad \forall s \leq t.$$

(One can check, that for  $s > t$  the above do not commute)

### 5. MARKOVIAN FILTERING OF GAUSSIAN QUANTUM PROCESS

The solution (4.8), obtained for the linear quantum diffusion equation (4.5) with continuous observation (4.6), enables us to find the Weyl operators (4.3) in the Heisenberg picture  $X(t, \xi) = \iota(t, X(\xi))$ . To this end let us represent the product

$$X_g(t, \xi) = \exp \left\{ \int_0^t \left( g(r) dY(r) - \frac{1}{2} g(r)^2 dr \right) \right\} X(t, \xi)$$

as the exponent of the operator (4.8):

$$\begin{aligned} X_g(t, \xi) &= \exp \left\{ R(t, i\xi) + \int_0^t \left( g(r) dY(r) - \frac{1}{2} g(r)^2 dr \right) \right\} \\ &= \exp \left\{ \int_0^t \left( s(v_t, \phi_r^{(g)}(t)) - \frac{1}{2} g(r)^2 \right) dr \right. \\ &\quad \left. + R(\phi_0^{(g)}(t)) + \iota_0^t(f^*, f) \right\} (i\xi). \end{aligned}$$

Taking into account that

$$\exp \{ \iota_0^t(f^*, f) \} = e^{|f_t|^2/2} e^{a^*(f_t)} e^{a(f_t^*)},$$

we can obtain

$$X_g(t, \xi) = e^{I_g^t(\xi)} e^{a^*(f_t(i\xi))} X \left( \frac{1}{i} \phi_0^{(g)}(t, i\xi) \right) e^{a(f_t(i\xi))},$$

where  $f_t(r, i\xi) = g(r) - is(\eta_r, \zeta_t)$ ,  $f_t^*(r, i\xi) = g(r) + is(\eta_r, \zeta_t^\#)$  for  $\eta_t = i\xi$ . The integral over the trajectories of the equation (4.9)

$$I_g^t(\xi) = \int_0^t s(v_r, \phi_r^{(g)}(t, i\xi)) dr + \frac{|f_t|^2(i\xi) - g_t^2}{2},$$

where

$$g_t^2 = \int_0^t g(r)^2 dr, \quad |f_t|^2(i\xi) = \int_0^t f^*(r, i\xi) f(r, i\xi) dr,$$

can be written using (4.4) in the following form

$$(5.1) \quad I_g^t(\xi) = \int_0^t \left\{ s(v_r + \text{Im} \zeta_r^\# g(r), \eta_r) + \frac{1}{2} \varepsilon_r(\mathbf{s}\eta_r, \mathbf{s}\eta_r) \right\} dr.$$

Given that  $e^{a(f^*)} \delta_\emptyset = 0$  we deduce the following result.

**Proposition 3** The mathematical expectation  $(\psi \otimes \delta_\emptyset | X_g(t, \xi) \psi \otimes \delta_\emptyset)$  with respect to the state vector  $h = \psi \otimes \delta_\emptyset$  for arbitrary  $\psi \in \mathbf{F}$  can be written in the form  $\langle \psi | \chi_g^t(\xi) \psi \rangle$ , where the expectation  $\chi_g^t(\xi)$  of the operator  $X_g(t, \xi)$  with respect to the vacuum vector  $\delta_\emptyset \in \mathcal{F}$  has the form

$$(5.2) \quad \chi_g^t(\xi) = \exp \left\{ I_g^t(\xi) + R(\phi_0^{(g)}(t, i\xi)) \right\}.$$

The function  $t \mapsto \chi_g^t(\xi)$  satisfies the operator equation

$$(5.3) \quad \begin{aligned} i \frac{d}{dt} \chi_g^t(\xi) &+ \left\{ s(v_t, \xi) + \frac{i}{2} \varepsilon_t(\mathbf{s}\xi, \mathbf{s}\xi) \right\} \chi_g^t(\xi) \\ &= \left\{ \langle \boldsymbol{\kappa}_t \mathbf{s}\xi, \partial \rangle + 2 \operatorname{Im} \langle \zeta_t g(t), i\partial + \frac{1}{2} \mathbf{s}\xi \rangle \right\} \chi_g^t(\xi), \end{aligned}$$

in the partial (functional) derivatives of the first order

$$\langle \zeta, \partial \rangle X(\xi) = (i/2)(X(\xi)R(\zeta) + R(\zeta)X(\xi)).$$

Indeed, it is not difficult to find, by means of direct substitution, that the characteristic operator-function (5.2) satisfies the equation (5.3). However, this can be skipped, as far as this equation with the commutation relation

$$[R(\zeta), X(\xi)] = s(\zeta, \xi) X(\xi), \quad \forall \zeta, \xi \in \Xi,$$

is concerned, it can be written in the form of the equation (3.4), that was derived before for the vacuum expectation  $\mu_g^t(X)$  of any operator of the form  $X_g(t) = e_g(t)X(t)$ :

$$\frac{d}{dt} \mu_g^t(\xi) + \mu_g^t(\gamma_t(X(\xi))) = g(t) \mu_g^t(X(\xi) R(\zeta_t) + R(\zeta_t^\sharp) X(\xi)),$$

where

$$\gamma_t(X) = i[X, v_t(R) + \frac{1}{2} \omega_t(R, R)] + \frac{1}{2} (R(\zeta_t^\sharp)[R(\zeta_t), X] + [X, R(\zeta_t^\sharp)]R(\zeta_t)).$$

In particular, for  $g = 0$  this equation defines the Markovian map  $\mu_0^t : \mathcal{A} \rightarrow \mathcal{A}$  for the quantum diffusion  $R(t, \eta)$ ; this map also has the characteristic operator-valued function  $\chi_0^t(\xi) = \mu_0^t(X(\xi))$  in the form (5.2) which is defined by the integral  $I_0^t(\xi)$  and  $\phi_r^{(0)}(t, i\xi) = i\phi_r(t)\xi$ .

An a posteriori linear quantum diffusion under the continuous measurement of the process (4.6) is described by the a posteriori characteristic operator-valued function  $\widehat{\chi}^t(\xi) = \widehat{\mu}^t(X(\xi))$ . Its Wick symbol is the operator-valued function (5.2) defining the characteristic function

$$\theta_g^t(\xi) = (\psi \otimes f^\otimes | \widehat{\chi}^t(\xi) (\psi \otimes f^\otimes)) \exp \{ - \|f\|^2 \}, \quad \psi \in \mathbf{H},$$

of  $g(t) = 2 \operatorname{Re} f(t)$  on the exponential normalised vectors  $f^\otimes \exp \{ - \frac{1}{2} \|f\|^2 \}$ . Consequently, the stochastic operator-function  $\widehat{\chi}^t(\xi)$ , is obtained (using a factorial substitution in (5.2), with the standard Wiener process  $\widehat{w}_t$  replacing  $\int_0^t g(r) dr$ ) along with the solution  $\widehat{\eta}_r = \widehat{\phi}_r(t, i\xi)$  of the backward linear stochastic equation

$$(5.4) \quad -d_- \widehat{\eta}_r + i\boldsymbol{\kappa}_r \mathbf{s} \widehat{\eta}_r dr = (\zeta_r + \zeta_r^\sharp) d\widehat{w}_r, \quad \widehat{\eta}_t = i\xi.$$

Substituting  $\eta_r = \phi_r^{(g)}(t, i\xi)$ , defines the solution of the filtering equation for the case of linear diffusion and  $X = X(\xi)$ .

In order to find the operator  $\widehat{\chi}^t(\xi)$  in the form of a stochastic function  $\chi^t(\xi, \omega) = \omega(\widehat{\chi}^t(\xi))$  (of the trajectories  $w_t$  of the observable process (4.6)), let us solve the equation (3.6) for the initial Gaussian state (4.2) with an arbitrary  $\vartheta(\eta) = \langle \eta, \vartheta \rangle$  instead of  $\vartheta_0(\eta)$ . In that way we shall find the Wick symbol

$$\widehat{\theta}^t(\vartheta, \xi) = (\psi_\eta | \widehat{\chi}^t(\xi) \psi_\eta), \quad \vartheta = \vartheta_0 + 2 \operatorname{Re}(\mathbf{g}\eta),$$

where  $\psi_\eta = \eta^\otimes \exp\{-\frac{1}{2} \|\eta\|^2\}$ . This symbol defines  $\widehat{\chi}^t(\xi)$  with the normal ordered substitution of  $R(\eta) = \vartheta_0(\eta) I + 2\Re A(\eta)$  into  $\widehat{\theta}^t(\vartheta, \xi)$  instead of  $\vartheta(\eta)$ . It can be obtained by solving the linear stochastic differential equation in partial (functional) derivatives of the first order

$$\begin{aligned} id\widehat{\theta}^t(\xi) &+ \left\{ s(v_t, \xi) + \frac{i}{2} \varepsilon_t(\mathbf{s}\xi, \mathbf{s}\xi) \right\} \widehat{\theta}^t(\xi) dt \\ (5.5) \quad &= \left( \langle \boldsymbol{\kappa}_t \mathbf{s}\xi, \vartheta \rangle dt + 2 \operatorname{Im} \langle \zeta_t d\widehat{w}_t, i\vartheta + \frac{1}{2} \mathbf{s}\xi \rangle \right) \widehat{\theta}^t(\xi), \end{aligned}$$

representing the filtering equation in terms of the nonnormalised a posteriori characteristic functional  $\widehat{\theta}^t(\xi) = \varphi(\widehat{\chi}^t(\xi))$ , for the initial Gaussian characteristic functional

$$\varphi(X(\xi)) = \exp \left\{ \vartheta(i\xi) - \frac{1}{2} \xi^2 \right\} = \theta(\xi).$$

**Theorem 3.** *The solution to the filtering equation (5.5) with the Gaussian  $\widehat{\theta}^0(\xi) = \theta(\xi)$  defines the stochastic characteristic functional  $\theta^t(\omega) = \omega(\widehat{\theta}^t)$  in the form*

$$(5.6) \quad \widehat{\theta}^t(\xi) = \widehat{\rho}^t \exp \left\{ \widehat{\vartheta}_t(i\xi) - \frac{1}{2} p_t(\xi, \xi) \right\}.$$

In this equation  $\widehat{\rho}^t = \widehat{\theta}^t(0)$  is the Gaussian probability density of the output process (4.6) with respect to  $\widehat{w}_t$ :

$$\widehat{\rho}^t = \exp \left\{ \int_0^t \left( \widehat{\vartheta}_r(2 \operatorname{Re} \zeta_r) d\widehat{w}_r - \frac{1}{2} \widehat{\vartheta}_r(2 \operatorname{Re} \zeta_r)^2 dr \right) \right\},$$

$\widehat{\vartheta}_t(\eta) = \langle \eta, \widehat{\vartheta}_t \rangle$  is a linear stochastic functional of the a posteriori mathematical expectation of the operators  $R(t, \eta)$  satisfying the quantum linear filtering equation

$$(5.7) \quad d\widehat{\vartheta}_t(\eta) + \widehat{\vartheta}_t(i\boldsymbol{\kappa}_t \mathbf{s}\eta) dt = 2 \operatorname{Re} \langle \eta, \mathbf{k}_t \zeta_t^\# \rangle d\widehat{w}_t + v_t(\mathbf{s}\eta) dt$$

with the initial  $\widehat{\vartheta}_0 = \vartheta \in \operatorname{Re} \Xi$ ,  $d\widehat{w}_t = d\widehat{w}_t - \widehat{\vartheta}_t(2 \operatorname{Re} \zeta_t) dt$ ,  $\mathbf{k}_t = \mathbf{p}_t + (i/2)\mathbf{s}$ . Here  $p_t(\xi, \eta) = \langle \xi, \mathbf{p}_t \eta \rangle$  is a symmetric quadratic form of the a posteriori covariance  $R(t, \eta)$  and  $\mathbf{R}(t, \xi)$ , satisfying the Riccati equation with the initial  $p_0(\xi, \eta) = \langle \xi, \eta \rangle$ :

$$(5.8) \quad \frac{d}{dt} p_t(\eta, \eta) - 2p_t(\eta, i\boldsymbol{\kappa}_t \eta) = \varepsilon_t(\mathbf{s}\eta, \mathbf{s}\eta) - |2 \operatorname{Re} \langle \eta, \mathbf{k}_t \zeta_t^\# \rangle|^2.$$

PROOF Let us find from (5.5) the stochastic equation for  $\widehat{\sigma}^t(i\xi, \omega) = \ln \widehat{\theta}^t(\xi, \omega)$  using the logarithmic Ito formula

$$d\widehat{\sigma}^t(i\xi) = \frac{1}{\widehat{\theta}^t(\xi)} d\widehat{\theta}^t(\xi) - \frac{1}{2} (d\widehat{\sigma}^t(i\xi))^2,$$

where

$$\begin{aligned}
 (d\widehat{\sigma}^t(\eta))^2 &= \left( d\widehat{\theta}^t\left(\frac{1}{i}\eta\right) / \widehat{\theta}^t\left(\frac{1}{i}\eta\right) \right)^2 \\
 &= \left( \langle 2\operatorname{Re}\zeta_t, \partial\widehat{\sigma}^t(\eta) \rangle + \langle \operatorname{Im}\zeta_t^\sharp, \mathbf{s}\eta \rangle \right)^2 dt \\
 &= \left\{ \lambda_t(\partial\widehat{\sigma}^t(\eta), \partial\widehat{\sigma}^t(\eta)) - 2i\beta_t(\partial\widehat{\sigma}^t(\eta), \mathbf{s}\eta) + \nu(\mathbf{s}\eta, \mathbf{s}\eta) \right\} dt,
 \end{aligned}$$

because  $d\widehat{w}_t d\widehat{w}_t = dt$ . Here

$$\begin{aligned}
 \langle \xi, \partial\widehat{\sigma}^t(\eta) \rangle &= \frac{d}{d\varepsilon} \widehat{\sigma}^t(\eta + \varepsilon\xi)|_{\varepsilon=0}, \\
 \lambda_t(\eta, \eta) &= \langle 2\operatorname{Re}\zeta_t, \eta \rangle^2, \quad \nu_t(\mathbf{s}\eta, \mathbf{s}\eta) = \langle \operatorname{Im}\zeta_t, \mathbf{s}\eta \rangle^2, \\
 \beta_t(\eta, i\xi) &= \langle 2\operatorname{Re}\zeta_t, \eta \rangle \langle \operatorname{Im}\zeta_t, \xi \rangle \equiv \langle i\boldsymbol{\beta}_t \xi, \eta \rangle
 \end{aligned}$$

This gives a quasilinear stochastic equation of the first order for  $\widehat{\sigma}^t(\eta)$ :

$$\begin{aligned}
 d\widehat{\sigma}^t &+ \left\{ s(\eta, v_t) + \langle i\widetilde{\boldsymbol{\kappa}}_t \mathbf{s}\eta, \partial\widehat{\sigma}^t \rangle - \frac{1}{2} \widetilde{\varepsilon}_t(\mathbf{s}\eta, \mathbf{s}\eta) \right\} dt \\
 &= \left\{ \langle 2\operatorname{Re}\zeta_t, \partial\widehat{\sigma}^t \rangle + \langle \operatorname{Im}\zeta_t^\sharp, \mathbf{s}\eta \rangle \right\} d\widehat{w}_t - \frac{1}{2} \lambda_t(\partial\widehat{\sigma}^t, \partial\widehat{\sigma}^t) dt,
 \end{aligned}$$

where  $\widetilde{\boldsymbol{\kappa}}_t = \boldsymbol{\kappa}_t - \boldsymbol{\beta}_t$  and  $\widetilde{\varepsilon}_t = \varepsilon_t - \nu_t$ . This equation has a solution in quadratic form

$$\sigma^t(\eta) = \ln \widehat{\rho}_t + \widehat{\vartheta}_t(\eta) + \frac{1}{2} p_t(\eta, \eta), \quad \partial\sigma^t(\eta) = \widehat{\vartheta}_t + \mathbf{p}_t \eta, \quad \partial^2 \sigma^t = \mathbf{p}_t,$$

where

$$\begin{aligned}
 d \ln \widehat{\rho}^t &= d\widehat{\sigma}^t(0) = \langle 2\operatorname{Re}\zeta_t, \widehat{\vartheta}_t \rangle d\widehat{w}_t - \frac{1}{2} \lambda_t(\widehat{\vartheta}_t, \widehat{\vartheta}_t) dt, \\
 d\widehat{\vartheta}_t &= d\partial\widehat{\sigma}^t(0) = \langle 2\mathbf{p}_t \operatorname{Re}\zeta_t + \mathbf{s} \operatorname{Im}\zeta_t \rangle d\widehat{w}_t - \left\{ (\mathbf{p}_t \boldsymbol{\lambda}_t - i\mathbf{s}\boldsymbol{\kappa}_t^\top) \widehat{\vartheta}_t - \mathbf{s}v_t \right\} dt \\
 d\mathbf{p}_t &= d\partial^2 \widehat{\sigma}^t = -\left\{ \mathbf{p}_t \boldsymbol{\lambda}_t \mathbf{p}_t + \mathbf{s}\varepsilon_t \mathbf{s} + i(\mathbf{p}_t \widetilde{\boldsymbol{\kappa}}_t \mathbf{s} - \mathbf{s}\widetilde{\boldsymbol{\kappa}}_t^\top \mathbf{p}_t) \right\} dt,
 \end{aligned}$$

and  $\widetilde{\boldsymbol{\kappa}}^\top(\mathbf{p}\eta, \mathbf{s}\eta) = \widetilde{\boldsymbol{\kappa}}(\mathbf{s}\eta, \mathbf{p}\eta)$ . By explicit expression of the finite-dimensional operators  $\boldsymbol{\beta}_t$ ,  $\boldsymbol{\lambda}_t$  and  $\boldsymbol{\nu}_t$  in terms of  $\zeta_t, \zeta_t^\sharp$ , the stochastic integral  $\ln \widehat{\rho}^t = \int_0^t d\widehat{\sigma}^t(0)$  can be represented in the form given by theorem 3. The differentials  $d\partial\widehat{\sigma}^t(0)$  and  $d\partial^2 \widehat{\sigma}^t$  can be written in the form of the stochastic differential equation (5.7) for  $\widehat{\vartheta}_t(\eta) = \langle \eta, \widehat{\vartheta}_t \rangle$ , and in the form of the Riccati equation (5.8) for  $p_t(\eta, \eta) = \langle \eta, \mathbf{p}_t \eta \rangle$ , with  $\mathbf{k}_t = \mathbf{p}_t + (i/2)\mathbf{s}$  because  $\mathbf{s}^\top = -\mathbf{s}$ . Hence

$$\begin{aligned}
 \langle \eta, 2\mathbf{p}_t \operatorname{Re}\zeta_t + \mathbf{s} \operatorname{Im}\zeta_t \rangle &= 2\operatorname{Re} \left\langle \eta, \left( \mathbf{p}_t + \frac{i}{2} \mathbf{s} \right) \zeta_t^\sharp \right\rangle, \quad \eta \in \operatorname{Re}\Xi^0, \\
 \lambda_t(\mathbf{p}_t \eta, \mathbf{p}_t \eta) - 2i\beta_t(\mathbf{p}_t \eta, \mathbf{s}\eta) + \nu_t(\mathbf{s}\eta, \mathbf{s}\eta) &= \left( 2\operatorname{Re} \langle \eta, \mathbf{k}_t \zeta_t^\sharp \rangle \right)^2.
 \end{aligned}$$

Theorem 3 is proved.

**Remark 3** The quantum linear filtering equation (5.7), (5.8) can be written in the form of the classical Kalman-Bucy filter

$$\begin{aligned}
 d\vartheta_t(\omega) + (\boldsymbol{\alpha}_t^\top \vartheta_t(\omega) + \mathbf{s}v_t) dt &= 2\operatorname{Re}(\mathbf{k}_t \zeta_t^\sharp) dw_t, \\
 \frac{d}{dt} \mathbf{p}_t + \boldsymbol{\alpha}_t^\top \mathbf{p}_t + \mathbf{p}_t \boldsymbol{\alpha}_t + \mathbf{s}\widetilde{\varepsilon}_t \mathbf{s} &= \mathbf{p}_t \boldsymbol{\lambda}_t \mathbf{p}_t,
 \end{aligned} \tag{5.9}$$

where  $\alpha_t = \lambda_t \mathbf{p}_t + i \tilde{\mathbf{k}}_t \mathbf{s}$ ,  $\alpha_t^\dagger = \mathbf{p}_t \lambda_t - i \mathbf{s} \tilde{\mathbf{k}}_t^\dagger$ ,  $\lambda_t = 4 \operatorname{Re} \zeta_t \operatorname{Re} \zeta_t^\dagger$ . The solution of this system of equations with the initial  $\vartheta_0(\omega) = \vartheta$ ,  $\mathbf{p}_0 = \mathbf{1}$  gives  $\hat{\sigma}^t(\eta)$  in the form of the integral

$$(5.10) \quad \begin{aligned} \hat{\sigma}^t(\eta) &= \vartheta(\hat{\eta}_0) + \frac{1}{2} \hat{\eta}_0^2 + \int_0^t \left\{ \langle \mathbf{s} \hat{\eta}_r, d\Upsilon_r \rangle \right. \\ &\quad \left. + \frac{1}{2} [\tilde{\varepsilon}_r(\mathbf{s} \hat{\eta}_r, \mathbf{s} \hat{\eta}_r) - \lambda_r(\hat{p}_r(\hat{\eta}_r), \hat{p}_r(\hat{\eta}_r))] dr \right\}, \end{aligned}$$

over the stochastic trajectories  $\hat{\eta}_r = \hat{\phi}_r(t, \eta)$  of the adjoint equation (5.4) with  $\hat{p}_r = \hat{\vartheta}_r + \mathbf{p}_r \hat{\eta}_r$ ,  $\hat{\eta}_0 = \hat{\phi}_0(t, \eta)$ , and

$$d\Upsilon_t = v_t dt + \operatorname{Im} \zeta_t^\sharp d\hat{w}_t.$$

Indeed, if  $d_- \hat{\eta}_r = \hat{\eta}_r - \hat{\eta}_{r-dr}$  is the backward stochastic differential, then  $d \langle \hat{\vartheta}_r, \hat{\eta}_r \rangle = \langle \hat{\eta}_r, d\hat{\vartheta}_r \rangle + \langle \hat{\vartheta}_r, d_- \hat{\eta}_r \rangle$ , and

$$d [p_r(\hat{\eta}_r, \hat{\eta}_r)] = 2p_r(\hat{\eta}_r, d_- \hat{\eta}_r) + \dot{p}_r(\hat{\eta}_r, \hat{\eta}_r) dr.$$

Using the equations (5.9) and writing the equation (5.4) in the form  $d_- \hat{\eta}_r = \alpha_r \hat{\eta}_r dr - 2 \operatorname{Re} \zeta_r d\hat{w}_r$  with respect to

$$d\hat{w}_r = d\hat{w}_r + \langle \mathbf{p}_r \hat{\eta}_r, 2 \operatorname{Re} \zeta_r \rangle dr,$$

one can obtain, by integrating by parts, the difference  $\hat{\sigma}^t(\eta) - \ln \hat{\rho}^t - \vartheta(\eta_0) - \frac{1}{2} \hat{\eta}_0^2$ :

$$\begin{aligned} \hat{\vartheta}_t(\eta) &- \vartheta(\hat{\eta}_0) + \frac{1}{2} p_t(\eta, \eta) - \frac{1}{2} \hat{\eta}_0^2 \\ &= \int_0^t \left\{ \langle \hat{\eta}, d\hat{\vartheta} \rangle + \frac{1}{2} \dot{\mathbf{p}} \hat{\eta} dr - \langle d_- \hat{\eta}, \hat{p}(\hat{\eta}) \rangle \right\} \\ &= \int_0^t \left\{ \langle \mathbf{s} \hat{\eta}, d\Upsilon \rangle - \langle 2 \operatorname{Re} \zeta d\hat{w}, \hat{\vartheta} \rangle + \langle \alpha \hat{\eta}, \mathbf{p} \hat{\eta} \rangle + \frac{1}{2} (\dot{\hat{\eta}}, \hat{\eta}) - \lambda(\mathbf{p} \hat{\eta}, \mathbf{p} \hat{\eta}) dr \right\} \\ &= \int_0^t \left\{ \langle \mathbf{s} \hat{\eta}, d\Upsilon \rangle - \hat{\vartheta} (2 \operatorname{Re} \zeta) d\hat{w} + \frac{1}{2} [\tilde{\varepsilon}(\mathbf{s} \hat{\eta}, \mathbf{s} \hat{\eta}) + \lambda(\hat{\vartheta}, \hat{\vartheta}) - \lambda(\hat{p}, \hat{p})] \right\} \\ &= \int_0^t \left\{ \langle \mathbf{s} \hat{\eta}, d\Upsilon \rangle + \frac{1}{2} [\varepsilon(\mathbf{s} \hat{\eta}, \mathbf{s} \hat{\eta}) - \lambda(\hat{p}, \hat{p})] dr \right\} - \ln \hat{\rho}^t, \end{aligned}$$

which gives (5.10) with  $\ln \hat{\rho}^t = \int_0^t (\hat{\vartheta} (2 \operatorname{Re} \zeta) d\hat{w} - \frac{1}{2} \lambda(\hat{\vartheta}, \hat{\vartheta}) dr)$ .

**Example.** Let us consider the quantum linear filtering equations (5.9) for the case of indirect nondemolition observation of the coordinate of free quantum Brownian motion, described in the above example. Since  $\zeta_t = \zeta_t^\sharp$  we obtain  $\beta_t = 0$ ,  $\gamma_t = 0$ ,  $\nu_t = 0$ , and  $\lambda_t = \lambda \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 4\varepsilon_t$ . This allows us to define the unique positive

$$\mathbf{p}_t = \frac{1}{\operatorname{Re} \omega_t} \begin{pmatrix} 2 |\omega_t|^2 & \operatorname{Im} \omega_t \\ \operatorname{Im} \omega_t & 1/2 \end{pmatrix} = \frac{1}{2} (\mathbf{k}_t + \mathbf{k}_t^\sharp),$$

solution of the Riccati equation (5.9) with the initial  $\mathbf{p}_0 = \mathbf{1}$ . Here

$$\mathbf{k}_t = \frac{1}{\operatorname{Re} \omega_t} \begin{pmatrix} 2 |\omega_t|^2 & i\omega_t^* \\ -i\omega_t & 1/2 \end{pmatrix}, \quad \mathbf{k}_t^\sharp = \frac{1}{\operatorname{Re} \omega_t} \begin{pmatrix} 2 |\omega_t|^2 & -i\omega_t \\ i\omega_t^* & 1/2 \end{pmatrix},$$

correspond to the degenerate form  $k_t(\eta^\sharp, \eta) = |2\omega_t \eta_p + i\eta_q|^2 / (2 \operatorname{Re} \omega_t)$  of the solution

$$\frac{d}{dt} \mathbf{k}_t + \mathbf{k}_t \lambda_t \mathbf{k}_t = i \left\{ \mathbf{s} \left( \frac{1}{2} \lambda_t + i\omega_t \right) \mathbf{k}_t + \mathbf{k}_t \left( \frac{1}{2} \lambda_t - i\omega_t \right) \mathbf{s} \right\}$$

with the initial  $\mathbf{k}_0 = \mathbf{1} + (i/2)\mathbf{s}$  and fixed  $\mathbf{k}_t - \mathbf{k}_t^\# = i\mathbf{s}$ . In addition, the parameter  $\omega_t \in \mathbf{C}$ , that defines the a posteriori wave function

$$\widehat{\psi}^t(x) = (\sqrt{\operatorname{Re} \omega_t / \pi \widehat{\rho}^t})^{1/2} \exp \left\{ -\frac{1}{2} \omega_t (x - \widehat{q}_t)^2 + \frac{i}{2} \widehat{p}_t x \right\}$$

for  $\widehat{\varphi}^t = (\widehat{\psi}^t | X(\xi) \widehat{\psi}^t)$  satisfies the one-dimensional complex Riccati equation

$$\frac{d}{dt} \omega_t + \frac{i}{2m} \omega_t^2 = \frac{1}{2} \lambda, \quad \omega_0 = \frac{1}{2},$$

which has (only) one positive solution

$$\omega_t = \frac{\alpha}{2} \frac{1 + \alpha \operatorname{th}(\lambda t / \alpha)}{\operatorname{th}(\lambda t / \alpha) + \alpha}, \quad \alpha = \sqrt{\lambda m / 2} (1 - i).$$

The a posteriori mathematical expectations

$$\begin{aligned} \widehat{p}_t &= \frac{1}{\widehat{\rho}^t} \int \widehat{\psi}^t(x)^* \frac{2}{i} \frac{d}{dx} \widehat{\psi}^t(x) dx = \frac{1}{\widehat{\rho}^t} (\widehat{\psi}^t | P \widehat{\psi}^t), \\ \widehat{q}_t &= \frac{1}{\widehat{\rho}^t} \int \widehat{\psi}^t(x)^* x \widehat{\psi}^t(x) dx = \frac{1}{\widehat{\rho}^t} (\widehat{\psi}^t | Q \widehat{\psi}^t), \end{aligned}$$

of the momentum and the coordinate of the quantum Brownian particle satisfy the linear system of stochastic equations

$$\begin{aligned} d\widehat{p}_t + \sqrt{\lambda} \frac{\operatorname{Im} \omega_t}{\operatorname{Re} \omega_t} d\widehat{w}_t &= 0, & \widehat{p}_0 &= p, \\ d\widehat{q}_t - \frac{1}{m} \widehat{p}_t dt &= \frac{i}{2} \sqrt{\lambda} \frac{1}{\operatorname{Re} \omega_t} d\widehat{w}_t, & \widehat{q}_0 &= q. \end{aligned}$$

The solution of this system gives the probability density

$$\widehat{\rho}^t = \exp \left\{ \int_0^t \left( \sqrt{\lambda} \widehat{q}_r d\widehat{w}_r - \frac{1}{2} \widehat{q}_r^2 dr \right) \right\}$$

for the indirect nondemolition observation  $Y(t)$  of the coordinate of the free quantum Brownian particle with the mass  $m$ . Note, that the solution  $\omega_t$  tends exponentially to the limit  $\omega_\infty = \alpha$  which defines the finite a posteriori dispersions

$$\begin{aligned} \sigma_t^2 &= \frac{\|(P - \widehat{p}I) \widehat{\psi}_0^t\|^2}{\|\widehat{\psi}_0^t\|^2} = 2 \frac{|\omega_t|^2}{\operatorname{Re} \omega_t} \\ \tau_t^2 &= \frac{\|(Q - \widehat{q}I) \widehat{\psi}_0^t\|^2}{\|\widehat{\psi}_0^t\|^2} = \frac{1}{2 \operatorname{Re} \omega_t} \end{aligned}$$

and the correlation  $\rho_t = \|\widehat{\psi}_0^t\| - 2 \operatorname{Re}((P - \widehat{p}I) \widehat{\psi}_0^t | (Q - \widehat{q}I) \widehat{\psi}_0^t)$  in the limit  $t \rightarrow \infty$ :

$$\sigma_\infty^2 = \sqrt{2\lambda m}, \quad \tau_\infty^2 = \sqrt{2/(\lambda m)}, \quad \rho_\infty = -1.$$

It is well known that the unobserved free quantum particle becomes ‘fuzzy’ and has an infinite a posteriori limit dispersions. The contradiction of the localised motion of the quantum particle, which is seen during the continuous observation, and of the impossibility of localising this particle on the basis of the von Neumann or Lindblad equation, has provided a source for quantum paradoxes such as the Zeno paradox [4]. The derived quantum filtering (3.6) and a posteriori von Neumann (3.10) equations resolves these quantum paradoxes not only on the qualitative level,

but also on the quantitative level of the microscopic quantum stochastic model for the continuous observation.

## 6. APPENDIX

**1.** Let  $\{\mathbf{H}_\xi | \xi > 1\}$  be a continuous family of Hilbert subspaces  $\mathbf{H}_\xi \subseteq \mathbf{H}$  with nondecreasing norms:  $\eta \leq \xi \Leftrightarrow \|\psi\|_\eta \leq \|\psi\|_\xi, \forall \psi \in \mathbf{H}$ . It will be called a *scale of the Hilbert space*  $\mathbf{H}$  with a scalar product  $(\psi|\psi) = \lim_{\xi \downarrow 1} \|\psi\|_\xi^2$ . An *inductive limit*  $\lim_{\xi \downarrow 1} \mathbf{H}_\xi$  of the Hilbert scale  $\{\mathbf{H}_\xi\}$  is defined as a pre-Hilbert space  $\mathbf{D} = \cup_{\xi > 1} \mathbf{H}_\xi$ , provided with the inductive convergence:

$$\psi_n \rightarrow 0 \Leftrightarrow \exists \xi > 1 : \|\psi_n\|_\xi \rightarrow 0.$$

The operator  $X : \mathbf{D} \rightarrow \mathbf{D}$  is called (*inductively*) *continuous* if  $X\psi_n \rightarrow 0$  for any convergent sequence  $\{\psi_n\}$  that tends to zero. This means, that the restriction of  $X$  to any subspace  $\mathbf{H}_\zeta \in \{\mathbf{H}_\xi\}$  is a continuous map into some  $\mathbf{H}_\xi \subseteq \mathbf{D}$ , i.e. for any  $\zeta > 1$  there exists  $\xi > 1$ , such that

$$(A.1) \quad \|X\|_\zeta^\xi := \sup_{\psi \in \mathcal{H}_\zeta} \left\{ \frac{\|X\psi\|_\xi}{\|\psi\|_\zeta} \right\} < \infty.$$

The set  $\mathcal{B}(\mathbf{D})$  of all continuous operators  $X$ , having the Hermitian-conjugate operators  $X^*$  on the pre-Hilbert space  $\mathbf{D}$  form an associative algebra with the identity  $I \in \mathcal{B}(\mathbf{D})$  and the involution  $X^{**} = X$ . This algebra is a  $C^*$ -algebra of the bounded operators only if  $\mathbf{D} = \mathbf{H}$ .

**2.** Let  $\Gamma_+$  be the set of all the chains  $\tau_n = (t_1, \dots, t_n), t_i \in \mathbf{R}_+, t_1 < \dots < t_n$  of length  $n < \infty$ , identified with finite subsets  $\tau \subset \mathbf{R}_+, \tau = \{t_1, \dots, t_n\}$  of the cardinality  $|\tau| = n \in \{0, 1, \dots\}$ . We denote by  $d\tau = \prod_{t \in \tau} dt$  the positive  $\sigma$ -finite measure on  $\Gamma_+ = \sum_{n=0}^{\infty} \Gamma_n$ , which is defined as a sum  $\sum_{n=0}^{\infty} d\tau_n$  of the measures  $d\tau_n = dt_1 \cdots dt_n$  on  $\Gamma_n = \{|\tau| = n\}$  with the only atom  $d\tau_0 = 1$  on the empty chain  $\tau_0 = \emptyset$  corresponding to  $|\tau| = 0$ . The Hilbert space  $\mathbf{H} = L^2(\Gamma_+)$  of the square-integrable functions  $f : \Gamma_+ \rightarrow \mathbf{C}, (f|f) < \infty$ , where  $(f|f) = \int |f(\tau)|^2 d\tau$ ,

$$(A.2) \quad \int g(\tau) d\tau := \sum_{n=0}^{\infty} \int_{0 < t_1 < \dots < t_n < \infty} \cdots \int g(t_1, \dots, t_n) dt_1 \cdots dt_n,$$

is naturally identified with the *Fock space*  $\mathcal{F} = \Gamma(\mathcal{K})$ , of the sequences  $f = \{\varphi_n | n = 0, 1, \dots\}$  of symmetric continuations  $\varphi_n : \mathbf{R}_+^n \rightarrow \mathbf{C}$  of the functions  $f(\tau_n)$ , with the scalar product

$$(f|f) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^\infty \cdots \int_0^\infty |\varphi(t_1, \dots, t_n)|^2 dt_1 \cdots dt_n.$$

The Hilbert scale  $\{\mathcal{F}_\xi | \xi > 1\}$  of the dense subspaces

$$\mathcal{F}_\xi = \left\{ f \in \mathcal{F} \mid \|f\|_\xi^2 = \int \xi^{|\tau|} |f(\tau)|^2 d\tau < \infty \right\}$$

is called Fock [9]–[11] over the Hilbert space  $\mathcal{K} = L^2(\mathbf{R}_+)$ . The function  $f(\tau) = \delta_\emptyset(\tau)$  (where  $\delta_\emptyset(\tau) = 1$  for  $\tau = \emptyset, \delta_\emptyset(\tau) = 0$  for  $\tau \neq \emptyset$ ), normalised ( $\|f\|_\xi^2 = \int \xi^{|\tau|} \delta_\emptyset(\tau) d\tau = 1$ ) with respect to any  $\xi$ , is called the *vacuum* function.

**3.** The most important examples of the continuous operators, and their conjugates  $\mathcal{F}_+ \rightarrow \mathcal{F}_+$  on the inductive limit  $\mathcal{F}_+ = \cup_{\xi > 1} \mathcal{F}_\xi$ , are the ‘quantum annihilation’ operators  $\widehat{a}_t$  on the interval  $[0, t)$ , which act as the integrals

$$[\widehat{a}_t f](\tau) = \int_0^t f(\tau \sqcup r) dr, \quad \forall f \in \mathcal{F}.$$

Here  $\tau \sqcup r$  is a chain, defined almost everywhere ( $r \notin \tau$ ) as a union  $\{\tau, r\}$  of the single-point chain  $r \in \mathbf{R}_+$  with some chain  $\tau = \{t_1, \dots, t_n\}$ . Although the operators  $\widehat{a}_t$  possess a complete system of exponential eigenfunctions

$$f(\tau) = k^\otimes(\tau) := \prod_{t \in \tau} k(t), \quad k \in l^2(\mathbf{R}_+),$$

they are not normal, i.e. they do not commute with their adjoint operators  $\widehat{a}_t^*$ , defined by the finite sums

$$[\widehat{a}_t^* f](\tau) = \sum_{r \in \tau}^{r < t} f(\tau \setminus r), \quad \forall f \in \mathcal{F}_+,$$

where  $\tau \setminus r = \{t \in \tau \mid t \neq r\}$ . Polynomials of the operators  $\{\widehat{a}_t^*, \widehat{a}_t \mid t \in \mathbf{R}_+\}$  form the *Weyl algebra* over the simple functions  $g \in L^2(\mathbf{R}_+)$ , generating the whole algebra  $\mathcal{B}(\mathcal{F}_+)$  through the trivality of the commutant:

$$\{Y \in \mathcal{B}(\mathcal{F}_+) \mid X \in \{\widehat{a}_t^*, \widehat{a}_t\} \Leftrightarrow [X, Y] = 0\} = \mathbf{C}\widehat{1}.$$

Linear elements of the algebra  $\mathcal{B}(\mathcal{F}_+)$ , defined as the quantum Wiener integrals

$$\widehat{r}(f, g) = \int_0^\infty (f(t) d\widehat{a}_t + g(t) d\widehat{a}_t^*) = \widehat{a}(f) + \widehat{a}^*(g)$$

of the square-integrable functions  $f, g : \mathbf{R}_+ \rightarrow \mathbf{C}$ , together with the identity operator  $\widehat{1}$ , form the \*-representation  $\widehat{r}(f, g)^* = \widehat{r}(g^*, f^*)$  of the canonical commutation relations

$$(A.3) \quad [\widehat{r}(f, g), \widehat{r}(f, g)^*] = (\|f\|^2 - \|g\|^2) \widehat{1}$$

with respect to the involution  $(f, g)^\sharp = (g^*, f^*)$ . The vacuum function  $\delta_\emptyset \in \mathcal{F}$  induces a Gaussian state on  $\mathcal{B}(\mathcal{F}_+)$ , defined by the characteristic functional

$$(\delta_\emptyset \mid e^{i\widehat{r}(f, g)} \delta_\emptyset) = \exp \left\{ \frac{1}{2} (\|f\|^2 - \|g\|^2) \right\}$$

on the Hilbert space of pairs  $\eta = (f, g)$  with the norm<sup>2</sup>  $|\eta|^2 = \frac{1}{2} (\|f\|^2 + \|g\|^2)$  and symplectic bilinear form  $s(\eta', \eta)$  which equals  $s(\eta^\sharp, \eta) = \frac{1}{i} (\|f\|^2 - \|g\|^2)$  for  $\eta' = (g^*, f^*)$ .

**4.** The measurable function  $F : t \mapsto F(t)$  whose values are the operators  $\mathcal{D} \rightarrow \mathcal{D}$  of the inductive limit  $\mathcal{D} = \cup_{\xi > 1} \mathcal{H}_\xi$  is called locally *p-integrable in the inductive scale*  $\{\mathcal{H}_\xi = \mathbf{H}_\xi \otimes \mathcal{F}_\xi\}$ , if, for any  $\zeta > 1$ , there exists a  $\xi > 1$ , such that

$$\|F\|_{\zeta, t}^{\xi, p} := \left( \int_0^t (\|F(r)\|_\zeta^\xi)^p dr \right)^{1/p} < \infty, \quad \forall t \in \mathbf{R}_+.$$

In particular, this condition means that the operators  $F(t) : \mathcal{D} \rightarrow \mathcal{D}$  are continuous for almost all  $t \in \mathbf{R}_+$ . For such square-integrable functions the quantum stochastic

integrals  $\widehat{a}_t^*(F)$ ,  $\widehat{a}_t(F)$  on  $\{\widehat{a}_r^*|r < t\}$  and  $\{\widehat{a}_r|r < t\}$  are defined to be the operators  $\mathcal{D} \rightarrow \mathcal{D}$ :

$$(A.4) \quad \begin{aligned} [\widehat{a}_t^*(F)h](\tau) &= \sum_{r \in \tau}^{r < t} [F(r)h](\tau), \quad \forall h \in \mathcal{D}, \\ [\widehat{a}_t(F)h](\tau) &= \int_0^t [F(r)\dot{h}(r)](\tau). \end{aligned}$$

In this equation  $h \mapsto \dot{h}(r)$  is a Mal'inen derivative, defined in the Fock representation  $h : \Gamma_+ \rightarrow \mathcal{D}$  of the elements  $h \in \mathcal{D}$  almost everywhere ( $r \notin \tau$ ) by the vector-function  $\dot{h}(r) \in \mathcal{D}$  as  $\dot{h}(r, \tau) = h(r \sqcup \tau)$ . The continuity of the operators  $\widehat{a}_t(F)$ ,  $\widehat{a}_t^*(F)$  in  $\mathcal{D}$  follows directly from the estimates

$$\begin{aligned} \|\widehat{a}_t(F)\|_{\zeta+\varepsilon}^\xi &\leq \sqrt{1/\varepsilon} \|F\|_{\zeta,t}^{\xi,2}, \quad \forall \varepsilon > 0, \\ \|\widehat{a}_t^*(F)\|_{\zeta}^{\xi-\varepsilon} &\leq \sqrt{\xi/\varepsilon} \|F\|_{\zeta,t}^{\xi,2}, \quad \forall \varepsilon < \xi. \end{aligned}$$

obtained in [5], [9]–[11]. From these estimates, if there exists  $\xi > 1$  for any  $\zeta > 1$  such that  $\|F\|_{\zeta,t}^{\xi,2}$ ,  $\|D\|_{\zeta,t}^{\xi,2} < \infty$  then, for any  $\zeta^+ > 1$ , there exists a  $\xi_- > 1$ , for which the operator

$$i_0^t(F, D) = \widehat{a}_t(F) + \widehat{a}_t^*(D)$$

is bounded from  $\mathcal{H}_{\zeta^+}$  to  $\mathcal{H}_{\xi_-}$ . To be precise, by choosing for every  $\zeta^+ > 1$  a  $\zeta > 1$ , such that  $\zeta < \zeta^+$ , and an  $\varepsilon > \xi - 1$ , such that  $\varepsilon < \zeta^+ - \zeta$ , we obtain

$$\|i_0^t(F, D)\|_{\zeta^+}^{\xi_-} \leq \sqrt{\xi/\varepsilon} (\|F\|_{\zeta,t}^{\xi,2} + \|D\|_{\zeta,t}^{\xi,2})$$

for any  $\xi_-$  on the nonempty interval  $(1, \xi - \varepsilon]$ .

**5.** If  $D(t) \in \mathcal{B}(\mathcal{D})$  for almost all  $t \in \mathbf{R}_+$  and the adjoint function  $D^*(t) = D(t)^*$  is also locally integrable on  $\mathcal{D}$ , then  $\widehat{a}_t^*(D) \in \mathcal{B}(\mathcal{D})$  and  $\widehat{a}_t^*(D)^* = \widehat{a}_t(D^*)$ . This means that the integral  $i_0^t(D^*, D)$  is (formally) self-adjoint. Moreover, the set of integrals

$$X(t) = i_0^t(D^*, D) + \int_0^t G(r) dr,$$

where  $G : \mathbf{R}_+ \rightarrow \mathcal{B}(\mathcal{D})$  is locally integrable ( $p = 1$ ), together with the  $G^*$ , function, forms a  $*$ -algebra with respect to the pointwise operator product  $(X^*X)(t) = X(t)^*X(t)$ . This product is defined by the quantum nonadapted Ito formula [10]

$$(A.5) \quad \begin{aligned} X(t)^*X(t) &= i_0^t(D^*X + X^*D, F^*D + X^*D) \\ &+ \int_0^t [G^*X + X_r^*D + D^*D + D^*X_r + X^*G](r) dr, \end{aligned}$$

which corresponds to the case of the locally square-integrable operator function  $t \mapsto X_t(t)$ ,  $X_t^*(t) = X(t)_t^*$ . Here  $X \mapsto X_t$  means a derivation, which is defined by

$$[X_t h](\tau) = [Xh](\tau \sqcup t) - [X\dot{h}(t)](\tau)$$

for almost all  $\tau \in \Gamma$ ,  $t \notin \tau$ . In this instance, the case  $X_t(t) = 0 = X_t^*(t)$  for all  $t \in \mathbf{R}_+$  corresponds to the adaptive property of the operator functions  $F(t)$ ,  $D(t)$  and  $G(t)$ .

Comprehensive information on explicit stochastic integration in the Fock scale is given in [20].

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