EVENTUM MECHANICS OF QUANTUM TRAJECTORIES: CONTINUAL MEASUREMENTS, QUANTUM PREDICTIONS AND FEEDBACK CONTROL

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ABSTRACT. Quantum mechanical systems exhibit an inherently probabilistic nature upon measurement which excludes in principle the singular direct observability continual case. Quantum theory of time continuous measurements and quantum prediction theory, developed by the author on the basis of an independent-increment model for quantum noise and nondemolition causality principle in the 80's, solves this problem allowing continual quantum predictions and reducing many quantum information problems like problems of quantum feedback control to the classical stochastic ones. Using explicit indirect observation models for diffusive and counting measurements we derive quantum filtering (prediction) equations to describe the stochastic evolution of the open quantum system under the continuous partial observation. Working in parallel with classical indeterministic control theory, we show the Markov Bellman equations for optimal feedback control of the *a posteriori* stochastic quantum states conditioned upon these two kinds of measurements. The resulting filtering and Bellman equation for the diffusive observation is then applied to the explicitly solvable quantum linear-quadratic-Gaussian (LQG) problem which emphasizes many similarities and differences with the corresponding classical nonlinear filtering and control problems and demonstrates microduality between quantum filtering and classical control.

CONTENTS

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1. Introduction

With technological advances now allowing the possibility of continuous monitoring and rapid manipulations of systems at the quantum level [1, 2], there is an increasing awareness of the importance of quantum feedback control in applications of quantum information such as the dynamical problems of quantum error corrections and quantum computations. The theory of quantum feedback control based upon a classical stochastic process formally developed by the author in the 80's [3, 4, 5] has been recently applied in many contexts including state preparation [6, 7, 8], purification [9, 10], risk-sensitive control [11, 12] and quantum error correction [13, 14]. It has also been studied from the practical point of view of stability theory [15] which contains a useful introduction to quantum probability and along with [16] gives a comprehensive discussion on the comparison of classical and quantum control techniques.

The main ingredients of quantum control are essentially the same as in the classical case. One controls the system by coupling to an external control field which modifies the system in a desirable manner. The desired objectives of the control can be encoded into a *cost function* along with any other stipulations or restrictions on the controls such that the minimization of this cost indicates optimality of the control process. We restrict ourselves to the more interesting case of quantum *closed-loop* or *feedback* control based on the indirect continuous in time observations, the quantum stochastic theory of which was initiated by Belavkin in a series of increasing generality papers [17, 18, 19, 20, 5] as a quantum analogy to the classical stochastic prediction theory which is based upon the nonlinear (Stratonovich) filtering equations. This work was developed then in the beginning of 90's [21, 22, 23, 24, 25, 26, 27], and this development laid down the foundations for Eventum Mechanics, a new quantum stochastic mechanics of continual observations giving a microscopic theory of continuous reductions [28, 29] and spontaneous localizations [30, 31] for quantum diffusions [32, 33], quantum jumps [34, 35], and other mixed stochastic quantum trajectories [36, 37, 38].

In order to demonstrate the power of this new event enhanced quantum mechanics, it was also applied right from the beginning [3, 39, 5] to solve the typical problems of quantum feedback control in parallel to the work on classical stochastic control with partial observations first introduced by Stratonovich [40, 41] and Mortensen [42]. Thus, the problem of optimal quantum feedback control was separated into quantum filtering which provides optimal estimates of the quantum state variables (density operators) and then a classical optimal control problem based on the output of the quantum filter. The classical noise which is filtered out by passing from the prior to the posterior quantum states comes from the irreducible disturbance to the quantum system during the observation (due to the interaction with measurement apparatus). Unlike in classical case, this is an unavoidable feature of quantum measurement since the state of an individual quantum system is not directly observable. However, the lack of urgency for such a theory and the novelty of the mathematical language at the time left this work relatively undiscovered only to be rediscovered recently in the physics and engineering community.

The purpose of this paper is to build on the original work of the author and present an accessible account of the theory of quantum continual measurements, quantum causality and predictions and optimal quantum feedback control. Firstly we introduce the necessary concepts and mathematical tools from modern quantum theory including quantum probability, continuous causal (non-demolition) measurements, quantum stochastic calculus and quantum filtering. Next the quantum Bellman equations for optimal feedback control with diffusive and counting measurement schemes are informally derived. The latter results were first stated in [5] without derivation and a consideration for the diffusive case was recently given in [43]. We conclude with an application of these results to the multi-dimensional quantum Linear-Quadratic-Gaussian (LQG) problem and a discussion on the comparison with the corresponding classical results. However we first start from a model example of LQG quantum filtering and feedback control problem which is important since it is one of the few exactly solvable control problems which emphasizes the similarities between the corresponding classical and quantum filtering and control theories. It allows us to set up notations and clearly demonstrates not only the similarity but also the difference of classical and quantum feedback control theories which can be observed in *microduality principle*, a more elaborated duality between quantum linear Gaussian filtering and classical linear optimal control.

1.1. **Model example: quantum free particle.** The quantum linear filtering and optimal quadratic control problem with quantum Gaussian noise was first studied and resolved by the author in a series of quantum measurement and filtering papers [17],[44],[28] and based on this quantum feedback control papers [18],[20],[5]. The simplest example of a single quantum Gaussian oscillator matched with a transmission line [44] as complex one-dimensional channel was taken as a quantum feedback model in the starting preprint [18], eventually published in [6]. However a more similar to the classical case quantum linear models require at least two real dimensions instead of single complex, and we may now use the multidimensional quantum LQG control solutions derived in the last Section of this paper for application on higher dimensional systems which do not have such complex representation. The optimal control of a continuously observed quantum free particle with quadratic cost is the simplest such example.

Let $\check{x}_\bullet = (\check{x}_1, \check{x}_2)$ be a pair of phase space operators $\check{x}_1 = \check{q}$, $\check{x}_2 = \check{p}$ for a quantum particle in one dimension, given by selfadjoint operators of position $\check{q}$ and momentum $\check{p}$ satisfying the canonical commutation relation (CCR)

$$[\check{q}, \check{p}] := \check{q}\check{p} - \check{p}\check{q} = \mathrm{i}\hbar\check{1}. \tag{1.1}$$

Here $\check{1}$ is the identity operator in a Hilbert space $\mathfrak{h}$ of the CCR representation (1.1) and $\hbar$ is called Planck constant, which could be for our purpose any positive constant $\hbar > 0$. Let us denote the row of initial expectations $\langle \check{x}_j \rangle$ of $\check{x}_j$ in a quantum Gaussian state by $x_\bullet = (q, p)$, and also denote the initial dispersions of $\check{q}$ and $\check{p}$ by σ_q

and σ_p respectively and the initial symmetric covariance $\mathrm{Re}\langle\check{q}\check{p}\rangle - qp$ by $\sigma_{qp} = \sigma_{pq}$. The Hamiltonian $\check{p}^2/2\mu$ of free particle is perturbed by a controlling force using the linear potential $\phi(t,\check{q}) = \beta u(t)\check{q}$ with $u(t) \in \mathbb{R}$ as $H(u) = \check{p}^2/2\mu + \beta u\check{q}$ where $\mu > 0$ is the mass of the particle. The particle is assumed to be coupled not only to control which can be realized by a quantum coherent (forward) channel, but also to a coherent observation (estimation) quantum channel such that its open Heisenberg dynamics is described by quantum Langevin equations as a case of (5.3):

$$\mathrm{d}Q(t) + \lambda Q(t)\,\mathrm{d}t = \frac{1}{\mu}P(t)\,\mathrm{d}t + \mathrm{d}W_q^t,\ Q(0) = \check{q} \tag{1.2}$$

$$\mathrm{d}P(t) + \lambda P(t)\,\mathrm{d}t = \mathrm{d}V_p^t - \beta u(t)\,\mathrm{d}t,\ \ P(0) = \check{p} \tag{1.3}$$

Here $\lambda = \frac{1}{2}(\alpha\varepsilon + \beta\gamma)$ and $V_q^t = \alpha V_e^t + \beta V_f^t$, $W_q^t = -\varepsilon W_e^t - \gamma W_f^t$ are given by two independent pairs $(V_\circ, W_\circ)$ (where $\circ = e, f$ stands for error and force) of Wiener noises $V_\circ = \hbar\Im(A_\circ^+)$, $W_\circ = 2\Re(A_\circ^+)$ due to the interaction with the coupled estimation and feedback channels. Note that these noises do not commute,

$$W_p^s V_q^r - V_q^r W_p^s = (r \wedge s)\,\mathrm{i}\hbar\lambda I, \tag{1.4}$$

if $\lambda \neq 0$, which is necessary and sufficient condition for preservation of the CCR (1.1) by the system (1.2), (1.3). It can be easily found by substituting the solution

$$P(t) = \mathrm{e}^{-\lambda t}\check{p} + \int_0^t \mathrm{e}^{(s-t)\lambda}\left(\mathrm{d}V_p^s - \beta u(s)\,\mathrm{d}s\right)$$

of the second equation (1.3) into the first (1.2) that

$$[Q(r), Q(s)] = \frac{\mathrm{i}\hbar}{\mu}|r-s|\,\mathrm{e}^{-\lambda|r-s|} \neq 0.$$

Therefore the family $\{Q(t)\}$ is incompatible, cannot be represented as a classical stochastic process and directly observed. However it can be indirectly observed by continuous measuring of the coupling operator $\alpha\check{q}$ with an error white noise in the estimation channel as it was suggested in [5],[25]. To this end we measure $W_e^t = 2\Re(A_e^+)$ as an input process evolved after an interaction with the particle into an output classical process given by a commutative family $[Y_e^t : t > 0]$ in the linear estimation channel

$$\mathrm{d}Y_e^t = \alpha Q(t)\,\mathrm{d}t + \mathrm{d}W_e^t, \tag{1.5}$$

in which the input process appears as measurement error noise with commutative independent increments $\mathrm{d}W_e^t$ representing the standard Wiener process such that $(\mathrm{d}W_e)^2 = \mathrm{d}t$, but noncommuting with the perturbative force V_p^t since, as it is explained in the last Section,

$$\mathrm{d}W_e^t\mathrm{d}V_p^t = \frac{\alpha\hbar}{2\mathrm{i}}\mathrm{d}t,\ \mathrm{d}W_e^t\mathrm{d}W_q^t = -\varepsilon\mathrm{d}t. \tag{1.6}$$

Thus the measurement error noise W_e^t satisfies the error-perturbation CCR

$$\left[V_p^r, W_e^s\right] = (r \wedge s)\,\mathrm{i}\hbar\alpha I, \tag{1.7}$$

which is necessary and sufficient condition of quantum causality (or quantum non-demolition condition) in the form

$$[Y_e^r, Q(s)] = 0 = [Y_e^r, P(s)]\ \ \forall r \leq s \tag{1.8}$$

requiring the statistical predictability of *quantum hidden in the future trajectories* $\{X_\bullet(s) : s \geq t\}$ with respect to the *classical observed in the past trajectories*

$\{Y_e^r : r \leq t\}$ for each t. From this we derive the Heisenberg *error-perturbation uncertainty principle* in the precise *Belavkin inequality* form [44],[28]

$$\left(\mathrm{d}V_p^t\right)^2 \geq \left(\frac{\alpha\hbar}{2}\right)^2 \mathrm{d}t, \quad \left(\mathrm{d}W_e^t\right)^2 = \mathrm{d}t \tag{1.9}$$

in terms of the perturbation V_p^t in (1.3) and standard error W_e^t in (1.5). Thus we have the case

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{\Lambda} = \begin{pmatrix} \frac{\alpha}{2} & \frac{\mathrm{i}\varepsilon}{\hbar} \\ \frac{\beta}{2} & \frac{\mathrm{i}\gamma}{\hbar} \end{pmatrix} \tag{1.10}$$

of the general quantum linear open system considered in the last Section, where $\mathbf{\Lambda}$ is the direct sum $\mathbf{\Lambda}_e \oplus \mathbf{\Lambda}_f$ of two rows $\mathbf{\Lambda}_e, \mathbf{\Lambda}_f$ corresponding to $\mathbf{B}_e = (\alpha, 0)$, $\mathbf{E} = (0, \varepsilon)$, $\mathbf{B} = (\beta, 0)$, $\mathbf{E}_f = (0, \gamma)$. From this we compute the matrices (5.10) and (5.24) satisfying the microduality principle, which are turned to be diagonal,

$$\mathbf{G} = \begin{pmatrix} \zeta_q & 0 \\ 0 & \zeta_p \end{pmatrix}, \ \ \mathbf{H} = \begin{pmatrix} \eta_q & 0 \\ 0 & \eta_p \end{pmatrix},$$

with eigenvalues $\zeta_q = \gamma^2$, $\zeta_p = (\hbar/2)^2 \left(\alpha^2 + \beta^2\right) = \eta_q$, $\eta_p = \varepsilon^2$.

1.2. **Quantum feedback control example.** We can now apply the results obtained in the last Section to demonstrate optimal quantum filtering and optimal feedback control and their microduality on this model example. The optimal estimates of the position and momentum based on a nondemolition observation of free quantum particle via the continuous measurement of Y_t, originally derived in [5],[25] in absence of control channel, are then given by the Belavkin Kalman filter (5.12) in the form of linear stochastic equation

$$\mathrm{d}\hat{q}_0^t + \hat{q}_0^t\lambda\mathrm{d}t = \frac{1}{\mu}\hat{p}_0^t\mathrm{d}t + \left(\alpha\sigma_q(t) - \varepsilon\right)\mathrm{d}\hat{W}_e^t \tag{1.11}$$

$$\mathrm{d}\hat{p}_0^t + \hat{p}_0^t\lambda\mathrm{d}t = \beta u(t)\,\mathrm{d}t + \alpha\sigma_{qp}(t)\,\mathrm{d}\hat{W}_e^t, \tag{1.12}$$

where the estimation innovation process $\hat{W}_e^t$ describes the gain of information due to measurement of Y_e^t given by

$$\mathrm{d}\hat{W}_e^t = \mathrm{d}Y_e^t - \alpha\hat{q}_0^t\mathrm{d}t, \tag{1.13}$$

and the error covariances satisfy the Riccati equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_q &= \zeta_q + 2\left(\frac{1}{\mu}\sigma_{qp} + \sigma_q\delta\right) - (\alpha\sigma_q)^2 \\ \frac{\mathrm{d}}{\mathrm{d}t}\sigma_{qp} &= \frac{1}{\mu}\sigma_p - (\lambda - \delta)\,\sigma_{qp} - \alpha^2\sigma_q\sigma_{qp} \\ \frac{\mathrm{d}}{\mathrm{d}t}\sigma_p &= \zeta_p - 2\lambda\sigma_p - (\alpha\sigma_p)^2, \end{aligned} \tag{1.14}$$

where we denote $\delta = \frac{1}{2}(\alpha\varepsilon - \gamma\beta)$, with initial conditions

$$\sigma_q(0) = \sigma_q, \quad \sigma_{qp}(0) = \sigma_{qp}, \quad \sigma_p(0) = \sigma_p.$$

The Riccati equations for the error covariance in the filtered free particle dynamics have an exact solution [25] with profound implications for the ultimate quantum limit satisfying the Heisenberg uncertainty relations for the accuracy of optimal quantum state estimation via the continuous indirect quantum particle coordinate measurement.

The dual optimal control problem can be found by identifying the corresponding dual matrices from the table (5.23) which give the quadratic control parameters

$$\begin{aligned} \check{c}(u) &= (u-\check{z})^2+\eta_q\check{q}^2+\eta_p\check{p}^2, \\ \check{s} &= \omega_q\check{q}^2+\omega_{qp}(\check{p}\check{q}+\check{q}\check{p})+\omega_p\check{p}^2 \end{aligned} \tag{1.15}$$

corresponding to the dual output process given by $\check{z}=\gamma\check{p}$. For the linear Gaussian system (5.19) gives the optimal control strategy

$$u\left(t\right)=\beta(\omega_{pq}\left(t\right)\hat{p}_0^t+\omega_p\left(t\right)\hat{q}_0^t) \tag{1.16}$$

where the coefficients are the solutions to the Riccati equations

$$\begin{aligned} -\tfrac{\mathrm{d}}{\mathrm{d}t}\omega_q\left(t\right) &= \eta_q-2\lambda\sigma_q-(\beta\sigma_q)^2 \\ -\tfrac{\mathrm{d}}{\mathrm{d}t}\omega_{qp}\left(t\right) &= \tfrac{1}{\mu}\omega_q-(\lambda+\delta)\,\omega_{qp}-\beta^2\omega_p\omega_{qp} \\ -\tfrac{\mathrm{d}}{\mathrm{d}t}\omega_p\left(t\right) &= \eta_p+2\left(\tfrac{1}{\mu}\omega_{qp}-\omega_p\delta\right)-(\beta\omega_p)^2 \end{aligned} \tag{1.17}$$

with terminal conditions

$$\omega_p\left(T\right)=\omega_p,\quad \omega_{qp}\left(T\right)=\omega_{qp},\quad \omega_q\left(T\right)=\omega_q.$$

Note that in this example, as well as identifying the dual matrices by transposition and time reversal according to the duality table (5.23), one must also symplecticly interchange the phase coordinates $(\check{q},\check{p})\leftrightarrow(\check{p},\check{q})$. This is because the matrix of coefficients $\mathbf{A}$ is non-symmetric and nilpotent, so it is dual to its transpose only when we interchange the coordinates in the dual picture. Thus the optimal coefficients $\{\omega_p,\omega_{qp},\omega_q\}(t)$ in the quadratic cost-to-go correspond to the minimal error covariances $\{\sigma_q,\sigma_{qp},\sigma_p\}(T-t)$ in the dual picture.

The minimal total cost for the experiment can be obtained from (5.22) by substitution of these solutions

$$\begin{aligned} &\mathsf{S}=\omega_q(q^2+\sigma_q)+2\omega_{qp}(qp+\sigma_{qp}) \\ &\quad+\omega_p\left(0\right)(p^2+\sigma_p)+\textstyle\int_0^T(\hbar^2\omega_p\left(t\right)+\omega_{pq}^2\left(t\right)\sigma_q\left(t\right))\mathrm{d}t \\ &\quad+\textstyle\int_0^T(\omega_p^2\left(t\right)\sigma_p\left(t\right)+2\omega_{qp}\left(t\right)\omega_p\left(t\right)\sigma_{pq}\left(t\right))\mathrm{d}t \end{aligned} \tag{1.18}$$

This demonstrates the linear microduality principle in the following specified form of the table (5.23)

Filtering $\check{q}$	$\lambda-\mu^{-1}$	α	ε	$\mathbf{J}^\intercal\mathbf{K}$	$\mathbf{GJ}$	$\mathbf{\Sigma J}$
Control $\check{p}$	$\lambda-\mu^{-1}$	β	γ	$\mathbf{L}^\intercal$	$\mathbf{JH}$	$\mathbf{J\Omega}$

showing the complete symmetry under the time reversal and exchange of (q,p), in which the coordinate observation is seen as completely dual to the feedback of momentum.

2. Quantum Dynamics with Trajectories

This section highlights the differences between quantum and classical systems and introduces the problem of quantum observation and its solution in the framework of open dynamics. In orthodox quantum mechanics which treats only closed quantum dynamics without observations, there is no such problem. However, it is meaningless to consider quantum feedback control without solution of this problem. After the appropriate setting of quantum mechanics with observation is given, the measurement problem is then restated as a statistical problem of quantum causality which can be resolved by optimal dynamical estimation on the output of an open quantum system called quantum filtering.

Quantum physics which deals with the unavoidable random nature of the microworld requires a new, more general, noncommutative theory of stochastic processes than the classical one based on Kolmogorov's axioms. The appropriate quantum probability theory was developed through the 70s and 80s by Accardi, Belavkin, Gardiner, Holevo, Hudson and Parthasarthy [45, 17, 19, 4, 46, 47, 48] amongst others.

The essential difference between classical and quantum systems is that classical states, including the mixed states, are defined by probability measures not on *properties* but *events*. This is because the properties of classical systems are described by measurable subsets $\Delta \subseteq \Omega$ forming a Boolean σ-algebra $\mathfrak{A}$ on the space of classical pure states, the points $\omega \in \Omega$ of a phase space. In principle they all can be tested simultaneously and identified with the events represented by the indicator functions $1_\Delta(\omega)$ of $\Delta \in \mathfrak{A}$ on the *universal observation* space Ω. They are building blocks for classical random variables described by essentially measurable functions with respect to a probability measure $\mathbb{P}$ on $\mathfrak{A}$. The algebra of all such complex functions $a : \Omega \to \mathbb{C}$ with pointwise operations is denoted by A, while $\mathrm{L}^p(\Omega, \mathbb{P})$ with $p = 1, 2, \infty$ stands for the subspaces of absolutely integrable, square-integrable and essentially bounded functions $f, g, b \in \mathrm{A}$ respectively. Note that the Banach space $\mathrm{M} = \mathrm{L}^\infty(\Omega, \mathbb{P})$ is a commutative C*-algebra (see Appendix 1.1) of the algebra A with involution $* : \mathrm{A} \ni a \mapsto a^* \in \mathrm{A}$ defined by the complex conjugation $a^* = \overline{a}$. Moreover, it is W*-algebra since M has the preadjoint space $\mathrm{M}_\star = \mathrm{L}^1(\Omega, \mathbb{P})$ such that $\mathrm{M}_\star^\star = \mathrm{M}$ with respect to the standard pairing

$$\langle a|f\rangle := \int_\Omega \overline{a(\omega)} f(\omega)\, \mathbb{P}(\mathrm{d}\omega) \equiv (a^*, f) \tag{2.1}$$

defining the expectation on $\mathrm{M}_\star$ as $\mathbb{E}[f] = (1, f)$.

In quantum world, unfortunately, there are *incompatible* properties corresponding to inconsistent but not orthogonal (i.e. not mutually excluding) *questions* such that, if the infimum $P \wedge Q$ is zero, it does not mean that $P \perp Q$. These questions cannot be *surely* answered simultaneously, i.e. tested with simultaneous events on any universal measurable space Ω, and they cannot be represented in any Boolean algebra. Since the incompatibility is measured by noncommutativity of orthoprojectors P and Q representing these questions as Hermitian idempotents on a Hilbert space $\mathcal{H}$ of quantum *vector-states*, the algebra $\mathcal{B}$ generated by all quantum properties must be noncommutative. The set $\mathfrak{P}(\mathcal{B})$ of all orthoprojectors $P \in \mathcal{B}$, called *property logic* of a noncommutative algebra $\mathcal{B}$, clearly extends any *eventum logic* of commuting orthoprojectors injectively representing the Boolean logic $\mathfrak{A}$ by a σ-homomorphism $E : \mathfrak{A} \to \mathfrak{P}(\mathcal{B})$ such that $\sum E(\Delta_j) = I$ for any measurable σ-partition $\Omega = \sum \Delta_j$. Two normal quantum variables are said to be *compatible* if their orthoprojectors commute, and therefore can be represented classically by measurable functions on their joint spectrum space Ω, however there is no such Ω if they do not commute. Since there are many incompatible quantum variables, e.g. the position and momentum in quantum mechanics, quantum properties cannot be identified with any commuting set $E(\mathfrak{A})$ representing a Boolean logic $\mathfrak{A}$.

2.1. **Quantum causality and predictions.** Almost simultaneously with Kolmogorov's functional formulation of classical probability theory von Neumann [49] gave another, more general operator formulation, aiming to lay down the foundation of quantum probability theory. It deals with not only commutative W*-algebras, called von Neumann algebras when they are represented as algebras of operators on a Hilbert space $\mathcal{H}$ with involution as Hermitian conjugation $*$ and unit as the identity operator I on $\mathcal{H}$. In order to understand the relation between these two formulations it is useful to reformulate Kolmogorov's axioms in terms of von Neumann (wise versa is impossible in the case of noncommutativity of the operator algebra). Any random variable $a \in \mathrm{M}$ can be represented by the diagonal operator $\hat{a}$ of pointwise multiplication $\hat{a}g = ag$ in the Hilbert space $\mathrm{H} = \mathrm{L}^2(\Omega, \mathbb{P})$ such that the abelian (commutative) operator algebra $\hat{\mathrm{M}} = \{\hat{a} : a \in \mathrm{M}\}$ is maximal in the algebra $\mathcal{B}(\mathrm{H})$ of all bounded operators on H in the sense that $\hat{\mathrm{M}} = \hat{\mathrm{M}}'$. Here $\hat{\mathrm{M}}' = \left\{B \in \mathcal{B}(\mathrm{H}) : \left[\hat{\mathrm{M}}, B\right] = 0\right\}$ with $[\hat{\mathrm{M}}, B] = \left\{AB - BA : A \in \hat{\mathrm{M}}\right\}$ stands for the *bounded commutant* of $\hat{\mathrm{M}}$, which obviously coincides on H with the commutant

$$\hat{1}'_{\mathfrak{A}} = \left\{B : \left[\hat{1}_\Delta, B\right] = 0, \Delta \in \mathfrak{A}\right\} \tag{2.2}$$

of the Boolean algebra $\hat{1}_{\mathfrak{A}} = \left\{\hat{1}_\Delta : \Delta \in \mathfrak{A}\right\}$ of all diagonal orthoprojectors (the multiplications by 1_Δ) generating $\hat{\mathrm{M}}$. Note that the commutant $\mathcal{B} = \mathcal{M}'$ of any *nonmaximal* abelian subalgebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a noncommutative W*-algebra with strict inclusion of $\mathcal{M}$ as the *center* $\mathcal{B} \cap \mathcal{B}'$ of $\mathcal{B}$. Thus the *simple algebra* $\mathcal{B} = \mathcal{B}(\mathcal{H})$ is the commutant of the abelian algebra of scalar multipliers $\mathcal{M} = \mathbb{C}I$ which is generated by the trivial Boolean algebra $\mathfrak{A} = \{\emptyset, \Omega\}$ represented by improper orthoprojectors $P_\emptyset = O$, $P_\Omega = I$. The noncommutative algebra $\mathcal{A}$ cannot be generated by any Boolean algebra of orthoprojectors as the commuting Hermitian idempotents $P^2 = P = P^*$ in $\mathcal{B}(\mathcal{H})$.

Quantum causality, assuming the existence of not only properties but also observable events, requires that all quantum properties related to *present and future* at each time-instant t must be compatible with all *passed* events. This makes an allowance for simultaneous predictability of incompatible properties at least in the statistical sense. However the usual quantum mechanics, dealing only with irreducible representations $\mathcal{B} = \mathcal{B}(\mathcal{H})$ of quantum properties but not with the events, is causal only for the trivial eventum algebra of improper orthoprojectors $\{O, I\}$ on $\mathcal{H}$. This is why any nontrivial causality requires an extension of the orthodox framework of quantum mechanics to *quantum stochastics* unifying it with the framework of classical stochastics in a minimalistic way allowing the distinction between the future quantum properties and past classical events. This program was completed in [4, 24] on the basis of quantum nondemolition (QND) principle [19, 20, 27] as an algebraic formulation quantum causality. The past events, corresponding to the measurable histories $\Delta \in \mathfrak{A}_{t]}$ up to each $t \in \mathbb{R}_+$, should be represented in the commutant of a noncommutative subalgebra $\mathcal{B}_{[t} \subseteq \mathcal{B}$ describing the present and future on a universal Hilbert space $\mathcal{H}$. Thus, instead of a single noncommutative algebra $\mathcal{B}$ extending the eventum W*-algebra $\mathcal{M}$ generated by $E(\mathfrak{A})$ one should consider a decreasing family $\left(\mathcal{B}_{[t}\right)$ of reduced subalgebras $\mathcal{B}_{[t} \subseteq \mathcal{B}_{[s}$ $\forall s < t$ in the commutants $E\left(\mathfrak{A}_{t]}\right)' = \left\{B \in \mathcal{B} : \left[E\left(\mathfrak{A}_{t]}\right), B\right] = 0\right\}$ of the past eventum logics $E\left(\mathfrak{A}_{t]}\right)$ representing the consistent histories of increasing probability spaces $\left(\Omega_{t]}, \mathfrak{A}_{t]}, \mathbb{P}_{t]}\right)$ in nonmaximal abelian W*-algebras $\mathcal{M}_{t]}$ generated by $E\left(\mathfrak{A}_{t]}\right)$.

The nondemolition principle makes quantum causality irreversible by allowing future observations represented by decreasing eventum algebras $E(\mathfrak{A}_t) \subset \mathcal{B}_{[t}$ to be incompatible with some present-plus-future questions $Q \in \mathcal{B}_{[t}$. Although any projectively increasing family of classical probability spaces can be obtained by

Kolmogorov construction from a single $(\Omega, \mathfrak{A}, \mathbb{P})$ with projections $\kappa_{s]} : \Omega \to \Omega_{s]}$ inverting the injections $\kappa_{s]}^{-1}(\mathfrak{A}_{s]}) \subseteq \kappa_{t]}^{-1}(\mathfrak{A}_{t]}) \subseteq \mathfrak{A}$ for all $s \leq t$ such that $\mathbb{P}_{s]} = \mathbb{P}_{t]} \circ \kappa_{s]}^{-1} = \mathbb{P} \circ \kappa_{s]}^{-1}$, however this projective limit may not be compatible with any noncommutative algebra $\mathcal{B}_{[t}$. Thus the maximal W*-algebra $\mathcal{B} = \mathcal{B}_{[0}$, satisfying the compatibility condition $\mathcal{B}' = \mathcal{M}_{0]}$ with the initial central algebra $\mathcal{M}_{0]} = \mathcal{B} \cap \mathcal{B}'$, coincides with the decomposable algebra $\mathcal{M}'_{0]}$ which is not compatible with the total eventum algebra $\mathcal{M} = \vee \mathcal{M}_{t]}$ except the case $\mathcal{M} = \mathcal{M}_{0]}$ of absence of innovation $\mathcal{M}_{r]} = \mathcal{M}_{t]}$ for all r and t. The latter with $\mathcal{M} = \mathbb{C}I$ is a standard assumption in the orthodox quantum mechanics dealing in the absence of observations with the constant $\mathcal{B}_{[t}$ equal to $\mathcal{B}(\mathcal{H})$. We may assume that $\mathcal{M}_{0]} = \mathbb{C}I$ corresponding to trivial initial history, $\mathfrak{A}_{0]} = \{\varnothing, \Omega\}$ with $\mathbb{P}_{0]} = 1$ on a single-point $\Omega_{0]} = \{0\}$, which allows $\mathcal{B}_{[0} = \mathcal{B}(\mathcal{H}_{[0})$.

Note that since all operators $B \in E(\mathfrak{A}_{t]})'$ commute with $\mathcal{M}_{t]}$, they are jointly decomposable, given in the diagonal representation of $\mathcal{M}_{t]}$ by $(\mathfrak{A}_{t]}, \mathbb{P}_{t]})$-essentially bounded functions $B_t : \omega \mapsto B(\omega)$ on $\Omega_{t]}$ with operator values $B(\omega) \in \mathcal{B}(\omega)$ on the Hilbert components $\mathcal{H}(\omega)$ of the orthogonal decomposition $\int_{\Omega_{t]}}^{\oplus} \mathcal{H}(\omega) \mathbb{P}_{t]}(\mathrm{d}\omega) \sim \mathcal{H}$ corresponding to the joint spectral representations

$$E(\Delta) \simeq \int_{\Omega_{t]}}^{\oplus} 1_{\Delta}(\omega) I(\omega) \mathbb{P}_{t]}(\mathrm{d}\omega) \equiv I_{t]}(\Delta) \tag{2.3}$$

of commuting orthoprojectors $E(\Delta)$, $\Delta \in \mathfrak{A}_{t]}$.

Quantum state (See Appendix 1.2) consistent with the trajectory probability space $(\Omega, \mathbb{P}, \mathfrak{A})$ is given as the linear positive functional $\langle \varpi | Q \rangle = (\varpi, Q)$ by a Hermitian-positive $\varpi = \varpi^*$ mass-one $\langle \varpi | I \rangle = 1$ operator $\varpi \vdash \mathcal{B}$ ($\in \mathcal{B}$ in usual or a generalized sense as affiliated to $\mathcal{B}$) defining the probability measure $\mathbb{P}$ as the projective limit of

$$\mathbb{P}_{t]}(\Delta) = \langle \varpi | E(\Delta) \rangle = \mathbb{P}(\Delta), \quad \Delta \in \mathfrak{A}_{t]} \tag{2.4}$$

(where $\langle \varpi, B \rangle = \mathrm{tr}[B\varpi]$ for the semifinite $\mathcal{B}$). It is called the (probability) density operator for $\mathcal{B}$ since it defines the probability $\Pr[Q] = \langle \varpi | Q \rangle \in [0,1]$ of any quantum property described by an orthoprojector $Q \in \mathcal{A}$. Since $Q \in \mathfrak{P}(\mathcal{B}_{[t})$ is compatible with each eventum projector $E(\Delta)$ for $\Delta \in \mathfrak{A}_{t]}$, the property Q is statistically predictable with respect to all past events due to the existence of *a posteriori* conditional probability

$$\Pr[Q|\Delta] = \frac{1}{\mathbb{P}(\Delta)} \langle \varpi | Q E(\Delta) \rangle \quad \forall \Delta : \mathbb{P}(\Delta) \neq 0 \tag{2.5}$$

such that $\Pr[Q] = \mathbb{P}(\Delta) \Pr[Q|\Delta] + \mathbb{P}(\Delta^{\perp}) \Pr[Q|\Delta^{\perp}]$. Note that $\langle \varpi | QE \rangle$ is not positive and even not real without the compatibility of Q and E. This leads to the existence of the *posterior quantum states* on $\mathcal{B}_{[t}$ given by the conditional expectations

$$\epsilon^t [B|\omega] = \langle \hat{\varpi}_{[t}(\omega) | B(\omega) \rangle \quad \forall B \in \mathcal{B} \cap \mathcal{M}'_{t]}. \tag{2.6}$$

The posterior states are defined as classical stochastic adapted processes $\omega \mapsto \hat{\varpi}_{[t}(\omega)$ with density operator values affiliated to the components $\mathcal{B}(\omega)$ of $\mathcal{B}_{[t}$, and the corresponding conditional expectations defined as positive normal projections $\mathcal{B} \cap \mathcal{M}'_{t]} \to \mathcal{M}_{t]}$ will be denoted as $\epsilon^t = \hat{\varpi}^{\star}_{[t}$.

Theorem 1. *Let ϖ be a normal state on $\mathcal{B}$. Then the induced state $\varpi_{[t}$ on the relative commutant $\mathcal{B}_{[t} \subseteq \mathcal{B} \cap \mathcal{M}_{t]}^{\prime}$ of the eventum algebra $\mathcal{M}_{t]}$ is given as classical expectation $\langle \varpi_{[t}, B\rangle = \mathbb{E}_{\Omega_{t]}}\left[\hat{\varpi}_{[t}^{\star}(B)\right]$ in terms of pairing*

$$\langle \varpi_{[t} | B\rangle := \int_{\Omega_{t]}} \left\langle \hat{\varpi}_{[t}(\omega) | B(\omega)\right\rangle \mathbb{P}(\mathrm{d}\omega) \tag{2.7}$$

on $\mathcal{B}_t = \mathcal{B} \cap \mathcal{M}_{t]}^{\prime}$ with the posterior density operators $\hat{\varpi}_{[t} \vdash \mathcal{B}_t$ uniquely defined by the conditional expectation $\epsilon^t : \mathcal{B}_t \to \mathcal{M}_{t]}$ as positive integrable function for almost all $\omega \in \Omega_{t]}$

Proof. Since ϖ_t is normal state on $\mathcal{B}_t$, equivalent to the space of essentially bounded functions on $\left(\Omega_{t]}, \mathfrak{A}_{t]}, \mathbb{P}_{t]}\right)$ with operator values in $\mathcal{B}(\omega) \subseteq \mathcal{B}(\mathcal{H}(\omega))$, it is uniquely defined by an essentially integrable function $\varpi_t(\omega)$ in (2.7) with operator values in $\mathcal{B}_{\star}(\omega)$. Obviously it is the density operator for the posterior states as the conditional expectations defined on $\mathcal{B}_t$ with respect to the central Abelian subalgebra $\mathcal{M}_{t]} \sim \mathrm{L}^{\infty}\left(\Omega_{t]}, \mathbb{P}_{t]}\right)$ by the Radon-Nikodym derivatives

$$\epsilon^t [B|\omega] := \lim_{\mathfrak{A}_{t]} \ni \Delta \searrow \{\omega\}} \frac{\langle \varpi_t | BE(\Delta)\rangle}{\mathbb{P}(\Delta)}, \quad B \in \mathcal{B}_t \tag{2.8}$$

where the limit is understood for and almost all $\omega \in \Omega_{t]}$ in the same way as in the classical case. □

Note that in the most important "white noise" cases considered in next sections, all $\mathcal{H}(\omega)$ with $\omega \in \Omega_{t]}$ are isomorphic to a single Hilbert space $\mathcal{H}_{[t}$ of a decreasing family $\left(\mathcal{H}_{[t}\right)$ such that $B_t(\omega) \in \mathcal{B}\left(\mathcal{H}_{[t}\right)$, uniquely defined for almost all $\omega \in \Omega_{t]}$, represents $B \in \mathcal{B}_t$ as a bounded operator on $\mathcal{H} = \mathrm{H}_{t]} \otimes \mathcal{H}_{[t}$ commuting with $I_{t]}(\Delta) = \hat{1}_{\Delta} \otimes I_{[t}$, where $\mathrm{H}_{t]} = \mathrm{L}^2\left(\Omega_{t]}, \mathbb{P}_{t]}\right)$. Representing $\mathcal{H}$ as $\mathrm{H}_{t]} \otimes \mathcal{H}_{[t}$ in the case $\mathcal{H}(\omega) = \mathcal{H}_{[t}\ \forall \omega \in \Omega_{t]}$, the posterior states, described by the positive mass one operators $\hat{\varpi}_{[t}(\omega) = \hat{\varpi}_{[t}^{\omega}$ in $\mathcal{H}_{[t}$, can be considered as the conditional states on the operator algebras $\mathcal{B}_{[t} = \mathcal{B}\left(\mathcal{H}_{[t}\right)$, controlled by the history trajectory ω. Thus the above quantum causality setting gives immediately the posterior states $\hat{\varpi}_{[t}^{\omega}$ as the states for quantum present and future conditioned by the classical past without any reference to the projection or other phenomenological reduction postulate of quantum measurement. This is main advantage of the extended, event enhanced quantum mechanics, or *eventum mechanics*, which allows treatment of the observable events on equal basis with other quantum properties of the system. It can be shown, see (7.5), that any reduction postulate of the operational quantum mechanics can all be derived from QND causality, and this principle is also applicable to the continuous measurements in both in time and spectrum where projection postulates fails.

Thus posterior states provide the optimal in mean quadratic sense Bayesian estimators for any number of unobservable quantum noncommuting variables $x \in \mathcal{B}\left(\mathcal{H}_{[t}\right)$ or properties from future, given the observable history $\omega \in \Omega_{t]}$. For this reason, one can consider quantum measurements in this "nondemolition" setup as a form of quantum *filtering*.

We now describe an appropriate Markovian model for the time-continuous interactions between the open quantum system and the field.

2.2. **Quantum open dynamics and input-output.** Quantum Markovian dynamics with observable trajectories, which entered into physics in the 90's in terms of stochastic transfer-operators or stochastic Master equations, define the phenomenological "instruments" of observation without giving any microscopic dynamical model in terms of the fundamental Hamiltonian interactions. In fact such approach is equivalent to the earlier operational approach based on the instrumental transfer-measures (See Appendix 1.3), and its starting point corresponds to already filtered Markov dynamics in the classical case. Here we describe the general scheme for underlying Hamiltonian interaction models with continuous observation for open quantum dynamical objects in terms of quantum stochastic evolutions in parallel to the classical stochastic models with partial observation, following the original Belavkin approach suggested in [4, 5, 24].

Let us fix a quantum probability space $(\mathcal{H}, \mathcal{B}, \varpi)$ and an increasing family $\mathcal{B}_{s]} \subseteq \mathcal{B}_{t]}$, $\forall s < t$ of W*-subalgebras $\mathcal{B}_{t]} \subseteq \mathcal{B}$ containing the *compatible histories* $E\left(\mathfrak{A}_{t]}\right) \subset \mathcal{B}'_{[t}$ and *nontrivial present* $\mathcal{B}_{[t]} = \mathcal{B}_{t]} \cap \mathcal{B}_{[t}$ for each t, assuming that each $\mathcal{B}_{t]}$ commutes with *future* $\mathcal{A}_t \subset \mathcal{B}_{[t}$, $\mathcal{B}_{t]} \subseteq \mathcal{A}'_t$, being generated by only *nonanticipating questions* $Q \in \mathcal{A}'_t \cap \mathcal{B}$. The future is *quantum noise* which is described by W*-subalgebras $\mathcal{A}_t \subseteq \mathcal{B}$, forming a decreasing family $\mathcal{A}_t \subseteq \mathcal{A}_{t+s}$ $\forall t, s > 0$ with trivial intersection such that we may assume that $\vee \mathcal{B}_{t]} = \mathcal{B}$. Moreover, we shall assume that the family $\left(\mathcal{B}_{t]}\right)$ as well as $\left(\mathcal{M}_{t]}\right)$ form the *W*-product systems* in the sense of W*-isomorphisms

$$\mathcal{B}_{t]} \bar{\otimes} \mathcal{A}_t^s \sim \mathcal{B}_{t+s]}, \quad \mathcal{M}_{t]} \bar{\otimes} \mathcal{M}_t^s \sim \mathcal{M}_{t+s]}, \tag{2.9}$$

where $\mathcal{A}_t^s = \mathcal{A}_t \cap \mathcal{B}_{t+s]}$, $\mathcal{M}_t^s = \mathcal{M}_t \cap \mathcal{M}_{t+s]}$ and $(\mathcal{M}_t)$ is decreasing family of W*-algebras $\mathcal{M}_t \subset \mathcal{B}_{[t}$ generated by future events $E\left(\mathfrak{A}_t\right)$. This implies that the family $(\mathcal{A}_t^s)$ satisfies the product condition such that $\mathcal{A}_0 \sim \mathcal{A}^t \bar{\otimes} \mathcal{A}_t^s \bar{\otimes} \mathcal{A}_{t+s}$ for any t and $s > 0$, and similar for $(\mathcal{M}_t^s)$, corresponding to the split property

$$\Omega = \Omega^t \times \Omega_t^r \times \Omega_{t+r}^s, \ \mathfrak{A} = \mathfrak{A}_t \otimes \mathfrak{A}_t^r \otimes \mathfrak{A}_{t+r} \tag{2.10}$$

of the measurable trajectory space.

Quantum open object under the observation is represented at each time t by a past-future boundary W*-subalgebra $\mathfrak{b}_t \subseteq \mathcal{B}_{[t]}$ such that it is adapted with respect to the family $\left(\mathcal{B}_{t]}\right)$ quantum stochastic process (in the genral sense [4]), nonanticipating futures $(\mathcal{A}_t)$ and satisfying causality condition with respect to the histories $\left(\mathcal{M}_{t]}\right)$. We may assume that each $\mathfrak{b}_t$ represents a fixed $\mathfrak{b}$ or a variable boundary W*-algebra $\mathfrak{b}\left(t\right)$ by a W*-homomorphism π_t of $\mathfrak{b}\left(t\right)$ onto $\mathfrak{b}_t$, with $\mathfrak{b}\left(t\right)$ taken in the initial algebra $\mathcal{B}_{[0]}$, say. Due to the causality condition the product

$$\Pi_t\left(\Delta, \check{q}\right) = E\left(\Delta\right)\pi_t\left(\check{q}\right) \ \ \forall \Delta \in \mathfrak{A}_{t]}$$

defines for each t an adapted transfer-measure $\Pi_t\left(\Delta\right) : \mathfrak{b}\left(t\right) \to \mathcal{B}_{t]}$ (see Appendix 1.4) with W*-homomorphic values normalized to the history eventum projectors $E\left(\Delta\right)$. Obviously W*-algebras $\mathfrak{b}_t$ and $\mathcal{A}_t^s$ are in $\mathcal{B}_{[t}^s = \mathcal{B}_{t+s]} \cap \mathcal{B}_{[t}$, as well as $\mathfrak{b}_{t+s}$ and W*-algebras $\mathcal{M}_t^s$.

Following [4] we shall say that quantum open object $\mathfrak{b}\left(t\right)$ with eventum history $E\left(\mathfrak{A}_{t]}\right)$ is *dynamical* with respect to $(\mathcal{A}_t)$ if

$$\mathcal{M}_t^s \vee \mathfrak{b}_{t+s} \subseteq \mathfrak{b}_t \vee \mathcal{A}_t^s \ \ \forall t, s > 0.$$

This is equivalent [4] to the existence of *quantum flow with observations* described as follows on the co-images $\mathfrak{b}\left(t\right) = \pi^t\left(\mathfrak{b}_t\right)$ of the boundary algebras $\mathfrak{b}_t$. Assuming that the W*-algebras $\mathcal{B}_{[t}^{s}$ are generated by $\mathfrak{b}_t$ and $\mathcal{A}_t^s$, we can always consider the dynamical quantum open object with $\mathfrak{b}_t = \mathcal{B}_{[t]}$.

Theorem 2. *Let $\pi^t : \mathfrak{b}_t \to \mathfrak{b}(t)$ be normal injections inverted by the dynamical representations π_t, and let $\mathcal{B}_{t]}, \mathcal{M}_{t]}$ form the product systems (2.9). Then there exists a transitional spectral measure*

$$\Upsilon_r^t(\Delta, \check{q}) = E^t(\Delta)\,\gamma_r^t(\check{q}), \quad \check{q} \in \mathfrak{b}(t+r) \tag{2.11}$$

on $\mathfrak{A}_t^r$ with values in $\mathcal{B}_{[t}^r = \mathfrak{b}(t)\,\bar{\otimes}\mathcal{A}_t^r$ given by adapted σ-homomorphisms $E^t : \mathfrak{A}_t \to \mathfrak{b}(t)\,\bar{\otimes}\mathcal{A}_t$ and a Heisenberg flow (γ_r^t) of causal tensor-adapted W-homomorphisms $\gamma_r^t : \mathfrak{b}(t+r) \to E^t(\mathfrak{A}_t^r)' \cap \mathcal{B}_{[t}^r$ such that*

$$\gamma_r^t \circ \gamma_s^{t+r} = \gamma_{r+s}^t \quad \forall r, s > 0, \tag{2.12}$$

$$\gamma_r^{t-r}\left(E^t(\Delta)\right) = E^{t-r}(\Delta) \quad \forall t > r \tag{2.13}$$

under the trivial extensions onto $\mathcal{B}_{[t} = \mathfrak{b}(t)\,\bar{\otimes}\mathcal{A}_t$.

Proof. The representations π_t as well as π^{t-r} can be trivially extended to the adapted W*-homomorphisms with respect to the identity maps id respectively on $\mathcal{A}_t$ and $\mathcal{A}_{t-r}$ by virtue of commutativity $\mathcal{B}_{t]} \subseteq \mathcal{A}_t'$ as $\pi_t(\check{q} \otimes A_t) = \pi_t(\check{q})\,A_t$ and $\pi^{t-r}(\check{q}_{t-r}A_{t-r}) = \pi^{t-r}(\check{q}_{t-r}) \otimes A_{t-r}$ respectively for all $\check{q} \in \mathfrak{b}(t)$ and $\check{q}_t \in \mathfrak{b}_t$, $A_t \in \mathcal{A}_t$. This defines the compositions $\gamma_r^{t-r} = \pi^{t-r} \circ \pi_t$ of thus extended W*-representations as tensor-adapted W*-homomorphisms $\gamma_r^{t-r} : \mathcal{B}_{[t} \to \mathcal{B}_{[t-r}$ on $\mathcal{B}_{[t} = \mathfrak{b}(t) \otimes \mathcal{A}_t$ trivially extending the $\mathfrak{b}(t) \to \mathfrak{b}(t) \otimes \mathcal{A}_{t-r}^r$ and satisfying the hemigroup condition (2.12) such that $\gamma_0^t = \mathrm{id}\left(\mathcal{B}_{[t}\right)$ for each t. Obviously these extensions satisfy causality condition

$$\pi_t\left(\mathcal{B}_{[t]}\right) \subseteq E^t\left(\mathfrak{A}_{t]}\right)',\ \gamma_r^t\left(\mathcal{B}_{[t+r]}\right) \subseteq E^t\left(\mathfrak{A}_t^r\right)',$$

where the eventum projectors $E^t(\Delta) \in \mathcal{B}_{[t}$ are defined for any $\Delta \in \mathfrak{A}_t$ as $\pi^t(E(\Delta))$ by the extended injections $\pi^t : \mathfrak{b}_t \vee \mathcal{A}_t \to \mathfrak{b}(t) \otimes \mathcal{A}_t$ as right inverse of the extended π_t on $E = \pi_t(E^t) \in E(\mathfrak{A}_t)$. The second condition (2.13) simply follows from $\pi_t \circ \gamma_r^t = \pi_{t+r}$ due to $\pi_{t-r}(E^t) = E$ for any $r \in [0, t]$ and $E \in E(\mathfrak{A}_t)$. Thus the QS flow with nondemolition observations can be described in terms of the homomorphic transitional measures (2.11) with (2.12) and (2.13) satisfying the hemigroup composition low

$$\Upsilon_r^{t-r}\left(\Delta_{t-r}^r, \Upsilon_s^t(\Delta_t^s, \check{q})\right) = \Upsilon_{r+s}^{t-r}\left(\Delta_{t-r}^{r+s}, \check{q}\right) \tag{2.14}$$

where $\Delta_{t-r}^{r+s} = \Delta_{t-r}^r \times \Delta_t^s \in \mathfrak{A}_{t-r}^{r+s}$ and $\check{q} \in \mathfrak{b}(t+s)$. □

Corollary 1. *The dynamical QS object is Markovian in the usual sense* [4] *if the initial state $\varpi = \varpi_{[0}$ on $\mathcal{B} = \mathcal{B}_{[0}$ is product state $\varpi \sim \varpi_{t]} \otimes \varrho_t$ for any t such that*

$$\left\langle \varpi | B_{t]} A_t^s \right\rangle = \left\langle \varpi | B_{t]} \right\rangle \left\langle \varrho_t | A_t^s \right\rangle \ .$$

It is operationally described is such state by the hemigroup of reduced transitional measures

$$\mathcal{T}_r^t(\Delta, \check{q}) = \varrho_t^\star\left[\Upsilon_r^t(\Delta, \check{q})\right], \tag{2.15}$$

where $\varrho_t^\star : \mathcal{B}_{[0} \to \mathcal{B}_{t]}$ is conditional expectation defined as

$$\left\langle \varpi_{[0} | B_{t]} \right\rangle = \left\langle \varpi_{t]} | \varrho_t^\star\left[B_{t]}\right] \right\rangle \quad \forall \varpi_{t]}, B_{t]} \in \mathcal{B}_{t]}. \tag{2.16}$$

They satisfy the operational Chapman-Kolmogorov equation

$$\mathcal{T}_r^{t-r}\left(\Delta_{t-r}^r, \mathcal{T}_s^t\left(\Delta_t^s, \check{q}\right)\right) = \mathcal{T}_{r+s}^{t-r}\left(\Delta_{t-r}^{r+s}, \check{q}\right) \tag{2.17}$$

as a normal completely positive map $\mathfrak{b}(t+s) \to \mathfrak{b}(t-r)$ *for each product* Δ_{t-r}^{r+s} *of* $\Delta_{t-r}^r \in \mathfrak{A}_{t-r}^r$ *and* $\Delta_t^s \in \mathfrak{A}_t^s$*.*

Remark 1. *The event representations* $E^t = \pi^t(E)$ *are usually given by* input *σ-homomorphisms* $I : \mathfrak{A}_t^s \to \mathcal{A}_t^s$, $I(\Delta) = \iota(1_\Delta)$ *as* $E^t(\Delta) = \upsilon^t(I(\Delta))$ *in terms of a two side adapted W*-representation* $\iota : \mathrm{M}_t^s \to \mathcal{A}_t^s$ *and a hemigroup* (υ_r^t) *of* interaction isomorphisms $\upsilon_r^t : \mathfrak{b}(t+r)\bar{\otimes}\mathcal{A}_t^r \to \mathfrak{b}(t)\bar{\otimes}\mathcal{A}_t^r$ *such that* $\upsilon_r^t|\mathfrak{b}(t+r) = \gamma_r^t$*. Here* υ^t *defines the* output representation *induced on an* input eventum algebra $\iota(\mathrm{M}_t) \subset \mathcal{A}_t$ *for* $\mathrm{M}_t = \mathrm{L}^\infty(\Omega_t, \mathbb{P}_t)$ *by the limit* $\upsilon^t = \lim_r \upsilon_r^t|\mathcal{A}_t$ *of* $\upsilon_r^t|\mathcal{A}_t^r$ *which is well-defined on each* $\mathcal{A}_t = \vee_r\mathcal{A}_t^r$ *due to the* localization property $\upsilon_{s+r}^t|\mathcal{A}_t^r = \upsilon_r^t|\mathcal{A}_t^r$ *for all* $r, s > 0$*. Note that the localization property simply follows from the hemigroup condition and the normalization* $\upsilon^t(I_0) = I_0$ *for these* υ_r^t*, extended adaptively also on* $A_0^t \in \mathcal{A}_0^t$ *and* $A_{t+r} \in \mathcal{A}_{t+r}$ *such that*

$$\upsilon_r^t\left(A_0^t \otimes B \otimes A_{t+r}\right) = A_0^t \otimes \upsilon_r^t(B) \otimes A_{t+r}.$$

The quantum *free evolution* is usually described by a semigroup (θ_s) of endomorphisms $\theta_s : \mathcal{B}_{[0} \to \mathcal{B}_{[0}$ shifting isomorphically any $\mathcal{A}_t$ onto $\mathcal{A}_{t+s}$ with trivial action on $\mathfrak{b}$. QS Heisenberg flow (γ_s^t) with observation over a constant algebra $\mathfrak{b}(t) = \mathfrak{b}$ is called *covariant* with respect to a shift semigroup (θ_s) acting also on $\mathfrak{A}_0$ by the shift of any $\mathfrak{A}_t$ onto $\mathfrak{A}_{t+s} = \theta_s(\mathfrak{A}_t)$ if

$$\gamma_s^t \circ \theta_{t+s} = \theta_t \circ \vartheta_s, \quad E^t \circ \theta_t = \theta_t \circ E^0, \tag{2.18}$$

where $\vartheta_s = \gamma_s^0 \circ \theta_s$ and θ_t is extended on the W*-algebra $\mathcal{B}_{[0} = \mathfrak{b} \otimes \mathcal{A}_0$ by $\theta_t(\check{q} \otimes A_0) = \check{q} \otimes \theta_t(A_0)$. This defines a Heisenberg θ-cocycle $\gamma_s = \gamma_s^0$ corresponding to the semigroup (ϑ_s) of W*-endomorphisms $\vartheta_s = \gamma_s \circ \theta_s$ of the algebra $\mathcal{B}$, satisfying causality condition $\vartheta_s(\mathcal{B}) \subseteq E\left(\mathfrak{A}_{s]}\right)'$. Note that the shift semigroup can be extended to a group $\{\theta_t : t \in \mathbb{R}\}$ on $\mathcal{B} = \widetilde{\mathcal{A}}_0\bar{\otimes}\mathfrak{b}\bar{\otimes}\mathcal{A}_0$, where $\widetilde{\mathcal{A}}_0$ is an independent copy of the algebra $\mathcal{A}_0$, with θ_t transforming each segment $\widetilde{\mathcal{A}}_r^\tau$ onto $\mathcal{A}_s^\tau$ for any positive τ, r, s and $t = \tau + r + s$, shifting $\widetilde{\mathcal{A}}_t$ onto $\widetilde{\mathcal{A}}_0$ similar to the inverted shift $\theta_{-t} : \mathcal{A}_t \to \mathcal{A}_0$, with backward transformation of each $\mathcal{A}_s^\tau$ onto $\widetilde{\mathcal{A}}_r^\tau$ for $\tau+r+s = -t$. This free group dynamics θ_t defines the reversible quantum dynamics interaction on such $\mathcal{B}$ by one-parametric group of $\vartheta_t = \upsilon_t \circ \theta_{t+s}$ extending ϑ_t onto $\mathcal{B}$ by interaction W*-automorphisms $\upsilon_t = \upsilon_t^0$ on $\mathcal{B}_{[0} = \mathfrak{b} \otimes \mathcal{A}_0$, acting identically on $\widetilde{\mathcal{A}}_0$ for any $t > 0$, with the identical action on $\mathcal{A}_0$ and the reflected cocycle action $\upsilon_t = \theta^{-t} \circ \widetilde{\upsilon}_{-t} \circ \theta_{t+s}$ on $\widetilde{\mathcal{B}}_{0]} = \widetilde{\mathcal{A}}_0 \otimes \mathfrak{b}$ for $t < 0$, where $\widetilde{\upsilon}_s$ is defined on $\widetilde{\mathcal{B}}_{0]}$ exactly as υ_s^{-1} on $\mathcal{B}_{[0}$ for any $s > 0$. However the reversible quantum dynamics on such noncommutative $\mathcal{B}$ cannot satisfy the causality in both directions of time with respect to a nontrivial eventum algebra $E(\mathfrak{A})$, except the case of absence innovation as in the conservative quantum mechanics without observation. To keep the causality in the positive direction of time one must replace the nonabelian $\widetilde{\mathcal{A}}_0$ by the smaller, abelian subalgebra $\check{\mathrm{M}}_0$, a copy of the eventum algebra $\hat{\mathrm{M}}_0 = I(\mathrm{M}_0)$, which makes θ_s and ϑ_s irreversible on $\mathcal{B} = \check{\mathrm{M}}_0\bar{\otimes}\mathfrak{b}\bar{\otimes}\mathcal{A}_0$ with noncommutative $\mathcal{A}_0$.

3. Quantum Stochastics of Eventum Mechanics

In this section we consider quantum noise models defining quantum Markovian dynamics with continuous nondemolition observation and show the quantum filtering equations derived from this models. Such observation can be based only on indirect measurement of a quantum open object via a coupled channel representing a classical measured process m_0^t in a bath $\mathcal{A}_0$ which is usually assumed to be initially independent of the quantum object $\mathfrak{b}$. We shall consider the measurement processes having initially independent increments $m_t^s = m_0^{t+s} - m_0^t \equiv m(\mathrm{I}_t^s)$ as random measure on the intervals $\mathrm{I}_t^s = [t, t+s)$. They generate independent W*-algebras $\mathrm{M}_t^s = \mathrm{L}_{\mathfrak{A}}^\infty(\Omega_t^s, \mathbb{P}_t^s) \equiv \mathrm{M}(\mathrm{I}_t^s)$, and an input quantum process with the increments $M_t^s = \iota(m_t^s) \equiv M(\mathrm{I}_t^s)$ generates independent eventum algebras $\iota(\mathrm{M}_t^s) \subset \mathcal{B}(\mathcal{F}_t^s)$ on a Hilbert space $\mathcal{F}_0$ with respect to an initial unit state vector χ_0 satisfying the divisibility condition

$$\mathcal{F}_0 \sim \mathcal{F}_0^t \bar{\otimes} \mathcal{F}_t^s \bar{\otimes} \mathcal{F}_{t+s}, \ \chi_0 \sim \chi_0^t \otimes \chi_t^s \otimes \chi_{t+s}, \tag{3.1}$$

such that it induces the *initial product probability measure* $\mathbb{P}_0 = \mathbb{P}_0^t \otimes \mathbb{P}_t^s \otimes \mathbb{P}_{t+s}$ on the measurable space of observable trajectories under the split condition (2.10).

An appropriate candidate for such Hilbert space suitable to accommodate any kind of classical independent increment process is Guichardet-Fock space $\mathcal{F}_0 = \Gamma(\mathcal{E}_0)$ over the Hilbert space $\mathcal{E}_0$ of L^2-functions $\xi : t \mapsto \mathfrak{k}$ on $\mathbb{R}_0^+ = \{t \geq 0\}$ with values in a Hilbert space $\mathfrak{k}$ such that $\mathcal{F}_t^s = \Gamma(\mathcal{E}_t^s)$ with $\mathcal{E}_t^s = \mathrm{L}^2(\mathrm{I}_t^s \to \mathfrak{k})$ (see the definitions in [24],[50] summarized in the Appendix2). There are sufficiently many product vectors in $\mathcal{F}_0$, called *exponential vectors* $\{\xi^\otimes : \xi \in \mathcal{E}_0\}$, generating Fock space $\mathcal{F}_0$ such that any *coherent state vector*

$$\chi_\xi = \mathrm{e}^{-\frac{1}{2}\|\xi\|_{\mathcal{E}}^2} \xi^\otimes, \ \|\xi\|_{\mathcal{E}}^2 = \int_0^\infty \|\xi(t)\|_{\mathfrak{k}}^2 \,\mathrm{d}t \tag{3.2}$$

defines a product state $\langle \varphi_\xi | A \rangle = \langle \chi_\xi | A \chi_\xi \rangle$ on any subalgebra $\mathcal{A}_0 \subseteq \mathcal{B}(\mathcal{F}_0)$ satisfying the divisibility condition $\mathcal{A}_0 \sim \mathcal{A}_0^t \otimes \mathcal{A}_t^s \otimes \mathcal{A}_{t+s}$ for any $t, s > 0$ such that each $\mathcal{A}_t^s = \mathcal{A}(\mathrm{I}_t^s)$ is represented in $\mathcal{B}(\mathcal{F}_t^s)$. However there is only one shift-invariant such state which is given by the *vacuum vector* $\chi_0 = \delta_\emptyset$ corresponding to $\xi = 0$. In fact, not only shift invariant but any infinitely divisible normal state on $\mathcal{A}_0$ can be induced by the vacuum state φ_0 on $\mathcal{B}(\mathcal{F}_0)$ by choosing in general time dependent Hilbert space $\mathfrak{k}(t)$ in a canonical way [51],[52]. In particular, since the state on the abelian algebra M_0 defined by the probability $\mathbb{P}_0$ of any classical process with independent increments is infinitely divisible, any such state can be induced from the quantum vacuum state by restricting it to the abelian part $\iota(\mathrm{M}_0) \subset \mathcal{A}_0$ given by an adapted input W*-representation $\iota : \mathrm{M}_0 \to \mathcal{A}_0$ such that $\iota(\mathrm{M}_t^s) \subset \mathcal{A}_t^s$ for any $t, s > 0$. It defines the infinitely divisible probability measure $\mathbb{P}_0$ as

$$\mathbb{P}_t^s(\Delta) := \langle \delta_\emptyset | I(\Delta) \delta_\emptyset \rangle \equiv \langle \varphi_0 | I(\Delta) \rangle \ \forall \Delta \in \mathfrak{A}_t^s, \tag{3.3}$$

where $I(\Delta) = \iota(1_\Delta)$, by the vacuum vectors $\chi_t^s = \delta_\emptyset$ of $\mathcal{F}_t^s$.

There are two basic processes with additive independent increments which can be realized in Fock space with finite-dimensional $\mathfrak{k} = \mathbb{C}^d$: Wiener vector-valued process $\boldsymbol{w}^t = (w_i^t)_{i \in I_W}$ which we index by a subset $I_W \subseteq \{1, \ldots, d\}$, and Poisson compound process $\boldsymbol{n}(\mathrm{I}_0^t) = (n_i^t)_{i \in I_N}$ which we index by another subset $I_N \subseteq \{1, \ldots, d\}$ (It should be thought as diagonal matrix-valued rather than vector). Their differential increments $\mathrm{d}m = m(\mathrm{d}t)$ satisfy quite different Itô multiplication tables

$$\mathrm{d}w_i \mathrm{d}w_k = \delta_{ik} \mathrm{d}t, \ \mathrm{d}n_i \mathrm{d}n_k = \delta_i^j \delta_k^j \mathrm{d}n_j, \tag{3.4}$$

where the summation rule is applied over j. Their canonical input representations $M_i^t = \iota(m_i^t)$ in Fock space over $\mathcal{E}_0 = \mathbb{C}^d \otimes \mathrm{L}^2(\mathbb{R}_0^+)$ are defined as

$$W_i^t = A_i^+(t) + A_-^i(t) \equiv \Re\left[A_i^+(t)\right], \tag{3.5}$$

$$N_i^t = A_i^i(t) + A_i^+(t) + A_-^i(t) + A_-^+(t) \tag{3.6}$$

in terms of four basic operator-processes in $\mathcal{F}_0$ defined in Appendix B. These are creation $A_\circ^+(t) := A_\circ^+(\mathrm{I}_0^t)$ (row-valued, $A_\circ^+ = (A_i^+)$), annihilation $A_-^\circ(t) := A_-^\circ(\mathrm{I}_0^t)$ (column-valued, $A_-^\circ := (A_-^k)$), exchange $A_\circ^\circ(t) := A_\circ^\circ(\mathrm{I}_0^t)$ (matrix-valued, $A_\circ^\circ = (A_i^k)$) and preservation $A_+^-(t) := A_+^-(\mathrm{I}_0^t)$ (scalar-valued, $A_-^+(t) = tI$). These canonical processes forming a pseudo-Hermitian matrix $\boldsymbol{A} = (A_\iota^\kappa)_{\iota=-,\circ}^{\kappa=\circ,+} = \boldsymbol{A}^\star$ under the involution $(A_{-\iota}^\kappa)^\star = (A_{-\kappa}^{\iota *})$ with respect to the index reflection $-(-,\circ,+) = (+,\circ,-)$, satisfy pseudo-Poisson multiplication table

$$\mathrm{d}A_\mu^\iota \mathrm{d}A_\kappa^\nu = \delta_\kappa^\iota \mathrm{d}A_\mu^\nu \quad \forall \iota,\nu \in \{i,+\}, \mu,\kappa \in \{-,k\} \tag{3.7}$$

of *quantum stochastic calculus* discovered as a noncommutative generalization of the classical Itô-Poisson table by Belavkin in [24]. Note that from (3.7) it follows that

$$\mathrm{d}W_i \mathrm{d}N_k = \delta_k^i \mathrm{d}A_-^i + \delta_{ik}\mathrm{d}t,\ \mathrm{d}N_i \mathrm{d}W_k = \delta_k^i \mathrm{d}A_k^+ + \delta_{ik}\mathrm{d}t$$

which cannot be realized in the classical category of commutative processes in which it always $\mathrm{d}w_i \mathrm{d}n_k = 0 = \mathrm{d}n_k \mathrm{d}w_i$. Thus the *joint* operator representation of two types of basic classical processes with independent increments is possible only in splitted Fock spaces such that $I_W \cap I_N = \emptyset$ corresponding to orthogonal subspaces $\mathfrak{k}_W \perp \mathfrak{k}_N$, which reflects the classical split

$$\Omega = \Omega_W \times \Omega_N,\ \mathfrak{A} = \mathfrak{A}_W \otimes \mathfrak{A}_N,\ \mathbb{P} = \mathbb{P}_W \otimes \mathbb{P}_N.$$

The four basic processes A_μ^ν form a linear basis of quantum Itô $\star$-algebra as noncommutative integrators for the increments

$$M_t^s = \sum_{\mu,\nu} \int_t^{t+s} K_\nu^\mu(r)\,\mathrm{d}A_\mu^\nu \equiv \mathfrak{i}_t^s(\boldsymbol{K}) \tag{3.8}$$

defined by four integrants $\boldsymbol{K} = (K_\nu^\mu)_{\nu=\circ,+}^{\mu=-,\circ}$ as operator-valued functions integrable in a quantum-stochastic [24]. These quantum stochastic integrals satisfy $\star$-property such that $\mathfrak{i}_t^s(\boldsymbol{K})^* = \mathfrak{i}_t^s(\boldsymbol{K}^\star)$ under the involution $(K_{-\nu}^\mu)^\star = (K_{-\mu}^{\nu *})$, and the Itô product rule

$$\mathrm{d}(M^* M) = \mathrm{d}(M^*)\,M + M^*(\mathrm{d}M) + (\mathrm{d}M^*)\,\mathrm{d}(M), \tag{3.9}$$

where $\mathrm{d}M = K_\nu^\mu \mathrm{d}A_\mu^\nu$ (the usual summation convention is assumed), with the Itô correction calculated as

$$\mathrm{d}\mathfrak{i}_t^s(\boldsymbol{K})^* \,\mathrm{d}\mathfrak{i}_t^s(\boldsymbol{K}) = \mathrm{d}\mathfrak{i}_t^s(\boldsymbol{K}^\star \boldsymbol{K}) \tag{3.10}$$

for adapted quantum stochastic integrators $K_\nu^\mu(t)$.

Note that in the case of a single degree of freedom $d = 1$ this quantum Itô table reads in the Hudson-Parthasarathy (HP) form [48] as

$$\mathrm{d}A_-^\circ \mathrm{d}A_\circ^+ = \mathrm{d}tI,\quad \mathrm{d}A_-^\circ \mathrm{d}A_\circ^\circ = \mathrm{d}A_-^\circ,$$

$$\mathrm{d}A_\circ^\circ \mathrm{d}A_\circ^\circ = \mathrm{d}A_\circ^\circ,\quad \mathrm{d}A_\circ^\circ \mathrm{d}A_\circ^+ = \mathrm{d}A_\circ^+,$$

with all other increment multiplications vanishing, in terms of four scalar-operator processes A^μ_ν with pseudo-Hermitian property written as

$$A^+_-(t) = A^+_-(t)^*,\ A^+_\circ(t) = A^\circ_-(t)^*,\ A^\circ_\circ(t) = A^\circ_\circ(t)^*.$$

3.1. **Quantum mechanics with observations.** Interaction automorphic evolutions $(\upsilon_r)_{r>0}$ on the the tensor product $\mathcal{B}_{[0} = \mathfrak{b}\otimes\mathcal{A}_0$ of simple algebras $\mathfrak{b} = \mathcal{B}(\mathfrak{h})$ and $\mathcal{A}_0 = \mathcal{B}(\mathcal{F}_0)$ are usually described by right unitary cocycles $\{U_r : r > 0\}$ as $\upsilon_t(B) = U_t B U_t^*$. The cocycles satisfy the operator hemigroup condition $U^t_r U^{t+r}_s = U^t_{r+s}$ similar to (2.12) in terms of the shifted operators $U^s_r = \theta_s(U_r)$, where (θ_s) is the semigroup of right shift W*-endomorphisms $\theta_s : \mathcal{A}_0 \to \mathcal{A}_s \subseteq \mathcal{A}_0$ describing free evolution of the bath by trivial placing of each subalgebra $\mathcal{A}\left(\mathrm{I}^s_{t-s}\right)$ onto $\mathcal{A}(\mathrm{I}^s_t)$ as $\mathcal{A}^s_t = \mathcal{B}(\mathcal{F}^s_t)$. Hudson and Parthasarathy [48],[50] derived a QS *forward equation*

$$\mathrm{d}U_t = U_t\left(R\mathrm{d}A^\circ_\circ + R_+\mathrm{d}A^+_\circ + R^-\mathrm{d}A^\circ_- + R^-_+\mathrm{d}t\right) \tag{3.11}$$

for the unitary cocycles in Fock space $\mathcal{F}_0 = \Gamma\left(\mathrm{L}^2\left(\mathbb{R}^+_0\right)\right)$ defined as the solution of QS integral equation $U_t = U_0 + \mathfrak{i}^t_0(U\boldsymbol{R})$ with four adapted operator-valued coefficients R^μ_ν and $U_0 = I$. They gave the necessary conditions for unitarity of this solution written in terms $S = I + R$ as

$$S^* = S^{-1},\ R^- = -R^*_+S,\ 2\Re\left(R^-_+\right) = -R^*_+R_+, \tag{3.12}$$

where $\Re(A)$ denotes the Hermitian part $(A + A^*)/2$ of an operator A (the sufficiency was shown only for the constant bounded initial-valued coefficients R^μ_ν).

It is important for control to have also the sufficient unitarity conditions in the case of time-dependent and multi-dimensional noise A. As it was proved in [53] under the natural QS-integrability conditions, the relations (3.12) are also sufficient for the uniqueness of unitary solution even in the case of time-dependent infinite dimensional coefficients with values $R^\mu_\nu(t) = \left(\check{s}^\mu_\nu(t) - \delta^\mu_\nu\check{1}\right)\otimes I$ in the initial operator algebra $\mathfrak{b}$. In terms of these $\check{s}^\mu_\nu(t)$ the multidimensional version of HP equation can be simply written in Belavkin's $\star$-algebraic notations [54, 55] as

$$\mathrm{d}U_t = U_t\left(\check{s}^\mu_\nu(t) - \delta^\mu_\nu\check{1}\right)A^\nu_\mu(\mathrm{d}t) \tag{3.13}$$

where usual summation convention over all $\mu,\nu \in \{-,\circ,+\}$ can be restricted to the domain $\mu \leq \nu$ under the order $- < \circ < +$ of the the triangular matrix $\check{\boldsymbol{s}} = (\check{s}^\mu_\nu)^{\mu=-,\circ,+}_{\nu=-,\circ,+}$ with zero operator entries for $\mu > \nu$ and $\check{s}^-_- = \check{1} = \check{s}^+_+$ (the usual summation convention then can be applied). In this notations the algebraic relations between nonzero matrix elements $\check{s}^\mu_\nu$ generalizing the unitarity conditions (3.12) are simply expressed as the pseudo-unitarity $\check{\boldsymbol{s}}^\star = \check{\boldsymbol{s}}^{-1}$ of the operator matrix $\check{\boldsymbol{s}}$ in terms of the pseudo-Hermitian adjoint matrix $\check{\boldsymbol{s}}^\star = \left(\check{s}^{-\nu*}_{-\mu}\right)$.

From the quantum Itô rule (3.9) applied to Heisenberg QS flow $X(t) = \gamma_t(\check{x})$ given by interaction dynamics as $\gamma_t(\check{x}) = U_t(\check{x}\otimes I)U^*_t \equiv \upsilon_t(\check{x}\otimes I)$, and the quantum Itô multiplication table (3.7), we obtain the general *QS Langevin equation*

$$\mathrm{d}X = \left(\Sigma^\mu_\nu(X) - X\delta^\mu_\nu\right)\mathrm{d}A^\nu_\mu \equiv \left(\boldsymbol{\Sigma}(X) - X\mathbf{1}\right)\cdot\mathrm{d}\mathbf{A}, \tag{3.14}$$

Here $\boldsymbol{\Sigma}(t,X) = \boldsymbol{S}(t)(X\otimes\mathbf{1})\boldsymbol{S}(t)^\star$, called *QS germ*, is given on $X = X(t)$ by matrix-function $\boldsymbol{\Sigma}(t)$ which is defined as a triangular matrix $(\Sigma^\mu_\nu)^{\mu=-,\circ,+}_{\nu=-,\circ,+}$ of six time evolved maps

$$\Sigma^\mu_\nu(t, U_t(\check{x}\otimes I)U^*_t) = U_t(\check{\sigma}^\mu_\nu(t,\check{x})\otimes I)U^*_t$$

given on $\mathfrak{b}$ by $\check{\boldsymbol{\sigma}}(\check{x}) = \check{\boldsymbol{s}}(\check{x} \otimes \mathbf{1})\check{\boldsymbol{s}}^{\star}$, where $\mathbf{1} = (\delta^{\mu}_{\nu})^{\mu=-,\circ,+}_{\nu=-,\circ,+}$, with trivial $\check{\sigma}^{-}_{-} = \mathrm{id}(\mathfrak{b}) = \check{\sigma}^{+}_{+}$ and $\check{\sigma}^{\mu}_{\nu}(\check{x}) = 0$ for $\mu > \nu$ such that the summation convention can be applied only for $\mu \leq \nu$. It was proved in [53] under the natural QS-integrability conditions that the unitality and $\star$-multiplicativity of the germ

$$\check{\boldsymbol{\sigma}}(t, \check{x}^{*}\check{x}) = \check{\boldsymbol{\sigma}}(t, \check{x})^{\star}\check{\boldsymbol{\sigma}}(t, \check{x}), \quad \check{\boldsymbol{\sigma}}\left(t, \check{1}\right) = \check{\mathbf{1}}, \tag{3.15}$$

are the necessary and sufficient conditions for the existence and uniqueness of the unital $*$-homomorphic solutions $X(r+s) = \gamma^{r}_{s}(\check{x})$ to the Langevin equation (3.14) with $X(t) = \check{x}$.

The composition $\check{\tau}^{r}_{t-r}(\check{x}) = \varrho^{\star}_{r}\left[\gamma^{r}_{t-r}(\check{x})\right]$ with noise conditional expectation (2.16), defined by the vacuum state $\varrho_{r} = \varphi_{0}$, describes a *dynamical hemigroup* $(\check{\tau}^{r}_{s})$ (or semigroup $(\check{\tau}_{s})$ in the stationary case) of unital completely positive maps $\check{\tau}^{r}_{s} : \mathfrak{b} \to \mathfrak{b}$ on operator algebra $\mathfrak{b} \subseteq \mathcal{B}(\mathfrak{h})$. This bath expectation, given by solutions $X(r+s) = \gamma^{r}_{s}(\check{x})$ of QS flow equation (3.14) with $X(r) = \check{x} \otimes I$ for the evolved on time interval I^{s}_{r} operators $\check{x} \in \mathfrak{b}$, satisfies the *master equation* [56, 57] $\frac{\mathrm{d}}{\mathrm{d}t}\check{\tau}_{t}(\check{x}) = \check{\tau}_{t}\left(\check{\lambda}(\check{x})\right)$ with a *Lindblad generator* $\check{\lambda}$ which can be written as a linear conditionally positive map in $\check{x} \in \mathfrak{b}$ in the form

$$\check{\lambda}(\check{x}) = \sum_{i,k} K^{i*}\check{\sigma}^{i}_{k}(\check{x})K^{k} + L\check{x} + \check{x}L^{*}. \tag{3.16}$$

Here $K^{i} = -R^{i}_{+}$, $L = R^{-}_{+}$ satisfies the condition $L + L^{*} = -\sum_{i} K^{i*}K^{i}$ and $\check{\sigma}^{i}_{k}(\check{x}) = \sum_{j} S^{i}_{j}\check{x}S^{k*}_{j}$ ($R^{i}_{+} = \check{s}^{i}_{+}$, $R^{-}_{+} = \check{s}^{-}_{+}$, $S^{i}_{j} = \check{s}^{i}_{j}$ to denote that these operators belong or affiliated to the algebra $\mathfrak{b}$). In fact, for quantum coherent control we need a time dependent version of this equation in the following decomposed form.

Theorem 3. *Let $\varphi^{\star}_{r,\xi}$ be coherent conditional expectation $\mathcal{B}_{[0} \to \mathcal{B}_{r]}$ defined on $\mathcal{B}_{[0} = \mathfrak{b} \otimes \mathcal{A}_{0}$ as normal positive projection such that*

$$\left\langle \varpi_{r]} | \varphi^{\star}_{r,\xi}[B] \right\rangle = \left\langle \psi_{r]} \otimes \chi_{\xi_{r}} | B | \psi_{r]} \otimes \chi_{\xi_{r}} \right\rangle$$

for any vector product-state $\varpi_{r]} = \varsigma_{\eta} \otimes \varphi_{\xi^{r}_{0}}$ with $\eta \in \mathfrak{h}$ and coherent vector (3.2) in $\mathcal{F}^{r}_{0}$. Then the dynamical map $\check{\tau}(t, \check{x}) = \varphi^{\star}_{r,\xi}\left[\gamma^{r}_{t-r}(\check{x})\right] \equiv \check{\tau}^{r}_{t-r}(\xi, \check{x})$ satisfies forward evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\check{\tau}(t, \check{x}) = \check{\tau}\left(t, \check{\lambda}(t, \check{x})\right), \quad \check{\tau}(r) = \mathrm{Id}(\mathfrak{b}) \tag{3.17}$$

with Lindblad type generator decomposed as

$$\check{\lambda}(\check{x}) = \frac{\mathrm{i}}{\hbar}[H, \check{x}] + \check{\lambda}_{R}(\check{x}) + K^{\circ*}\left(\check{\sigma}^{\circ}_{\circ}(\check{x}) - \check{x}\delta^{\circ}_{\circ}\right)K^{\circ} \tag{3.18}$$

where $H(t) = \hbar\Im\left[\left(2\xi(t)^{*}R^{\circ}_{+} + R^{-}_{+}\right)\right]$, $K^{\circ}(t) = \xi(t) - R^{\circ}_{+}$, $\check{\sigma}^{\circ}_{\circ}(\check{x}) = S^{\circ}_{\circ}\check{x}S^{\circ*}_{\circ}$ and

$$\check{\lambda}_{R}(x) = \frac{1}{2}\sum_{i}\left(R^{i*}_{+}\left[\check{x}, R^{i}_{+}\right] + [R^{i*}_{+}, \check{x}]R^{i}_{+}\right).$$

Proof. Indeed, it can be shown [58] that conditional coherent expectations $\check{\tau}^{r}_{t-r}(\xi, \check{x})$, evaluated from the QS flow equation (3.14), satisfy Lindblad type equation with generator

$$\check{\lambda}(t) = \check{\sigma}^{-}_{+} + \check{\sigma}^{-}_{\circ}\xi(t) + \xi(t)^{*}\check{\sigma}^{\circ}_{+} + \xi(t)^{*}\check{\rho}^{\circ}_{\circ}\xi(t)$$

where $\check{\rho}^{i}_{k}(\check{x}) = \check{\sigma}^{i}_{k}(\check{x}) - \check{x}\delta^{i}_{k}$. Using HP conditions (3.12) in multidimensional form

$$S^{\circ*}_{\circ}S^{\circ}_{\circ} = \delta^{\circ}_{\circ}1, \; R^{\circ*}_{+}S^{\circ}_{\circ} = -R^{-}_{\circ}, R^{\circ*}_{+}R^{\circ}_{+} = -R^{-*}_{+} - R^{-}_{+}$$

as pseudo-unitarity conditions $T^{\mu}_{-\nu} = S^{\nu *}_{-\mu}$ in terms of operator matrix elements of inverse $\boldsymbol{T} = \boldsymbol{S}^{-1}$ to the triangular operator matrix $\boldsymbol{S}$ for $\check{\sigma}^{\mu}_{-\nu}(\check{x}) = \sum_{\iota} S^{\mu}_{\iota} \check{x} S^{\nu *}_{-\iota}$, this generator can be written as

$$\check{\lambda}[\check{x}] = \sum_{\iota=-,\circ,+} \left(S^{-}_{-\iota} + \xi^{*} S^{\circ}_{-\iota}\right) \check{x} \left(S^{-}_{\iota} + \xi^{*} S^{\circ}_{\iota}\right)^{*} - \xi^{*}\xi\check{x}$$

where $\xi(t)^{*}\xi(t) = \|\xi(t)\|^{2}_{\mathfrak{k}}$. This gives (3.18) after taking into account again the unitarity conditions. □

Given a normal quantum state $\varsigma(r)$ on $\mathfrak{b}$ at time r, the hemigroup $(\check{\tau}^{r}_{s})$ defines an averaged coherent controlled non-Hamiltonian state evolution $[\varsigma(t) : t \geq r]$ of the quantum dynamical object by composing it with the adjoint CP maps $\tau^{t}_{r} = \check{\tau}^{r\star}_{t-r}$ as $\varsigma(t) = \tau^{t}_{r}(\varsigma(t-r))$. It satisfies the *hemigroup master equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varsigma + K\varsigma + \varsigma K^{*} = \sum_{i} L^{i}\varsigma L^{i*}, \tag{3.19}$$

where the operators $K = -L_{\star}$ and $L^{i} = L_{i\star}$ are defined by left adjoints $L_{\star} = L^{\sharp}, L_{i\star} = L^{\sharp}_{i}$ to

$$L = R^{-}_{+} + \xi^{*} R^{\circ}_{+} - \frac{1}{2}\xi^{*}\xi,\ L_{i} = R^{-}_{i} + \xi^{*} S^{\circ}_{i}$$

with respect to the $\langle \mathfrak{b}^{\star} | \mathfrak{b} \rangle$-pairing: $\left\langle L^{\sharp}\varsigma | \check{x} \right\rangle = \langle \varsigma | L\check{x} \rangle$ (which are usual Hilbert space adjoints, $L_{\star} = L^{*}$, $L_{i\star} = L^{*}_{i}$ in the case of the trace pairing $\langle \varsigma | \check{x} \rangle = \operatorname{tr}[\varsigma^{*}\check{x}]$). This master equation is usually written in the Lindbladian form as $\frac{\mathrm{d}}{\mathrm{d}t}\varsigma = \lambda(t, \varsigma)$, and it is a particular case of the general QS Master equation (7.13) derived in [59],[60].

3.2. **Quantum nonlinear filtering equations.** A time continuous measurement of each Wiener process w^{t}_{i} represented by the field quadratures (3.5) after interaction with quantum object as $Y^{t}_{i} = U_{t'} W^{t}_{i} U^{*}_{t'}$ due to locality for any $t' \geq t$, realizes an indirect measurement of the evolved generalized coordinate $Q_{i}(t) = 2\Re[L_{i}(t)]$, where $L_{i}(t) = U_{t}\left(\check{s}^{-}_{i} \otimes I\right) U^{*}_{t}$. This can be seen from the quantum Itô formula (3.9), (3.10) applied to the output operators $\upsilon_{t}(W^{t}_{i}) = U_{t} W^{t}_{i} U^{*}_{t}$:

$$\mathrm{d}\upsilon_{t}\left(W^{t}_{i}\right) = 2\Re\left(L^{*}_{i}(t)\,\mathrm{d}t + \mathrm{d}A^{+}_{i}\right) = Q_{i}(t)\,\mathrm{d}t + \mathrm{d}W^{t}_{i} \tag{3.20}$$

Similarly, the output process corresponding to the field counting process (3.6) as $Y^{t}_{i} = U_{t'} N^{t}_{i} U^{*}_{t'}$ for any $t' \geq t$ is given by $\upsilon_{t}(N^{t}_{i}) = U_{t} N^{t}_{i} U^{*}_{t}$ as the QS integral of

$$\mathrm{d}\upsilon_{t}\left(N^{t}_{i}\right) = L_{i}(t)\, L^{*}_{i}(t)\,\mathrm{d}t + 2\Re\left(L^{*}_{i}(t)\,\mathrm{d}A^{+}_{i}\right) + \mathrm{d}A^{i}_{i}. \tag{3.21}$$

Classically, filtering equations are used when we need to estimate the value of dynamical variables about which we have incomplete knowledge due to an indirect observation. For example, the Kalman-Bucy filter [61],[62] gives a continuous least-squares estimator for a Gaussian classical random variable with linear dynamics when we only have access to a correlated, noisy output signal. Since closed quantum systems are fundamentally unobservable (hidden) unless they are open, e.g. disturbed by quantum noise processes (c.f. (3.14) such that equations (3.20)) and (3.21) have nontrivial input from the quantum object in terms of the non-Hamiltonian part $L^{i} = \check{s}^{*}_{i}$ of the Lindblad generator, filtering of quantum noise plays an important role in quantum measurement. As it follows immediately from the localization property of quantum interaction evolution due to the hemigroup property the output operators Y_i^t are self non-demolition (i.e. mutually compatible at all times) and satisfy the quantum non-demolition (QND) condition

$$[X(s), Y_i^t] = 0 \quad \forall s \le t, i \in \{1, \dots, d\} \tag{3.22}$$

with respect to any evolved quantum object process $X(s) = \upsilon_s(\check{x} \otimes I) \equiv \gamma_s(\check{x})$. Belavkin was the first to realize that an optimal estimation without further disturbance is possible in the general quantum open dynamical models when based on any output QND measurements [18],[44],[20],[5]. He constructed the quantum filtering equation which describes the evolution of the optimal estimate given by the density matrix conditioned on a classical output of the noisy quantum channel. This is used to estimate arbitrary object variable $X(t) \vdash \mathfrak{b}_t$ which are driven by environmental quantum noises. The QND condition insists that the expectation of $X(t)$ is not disturbed when we measure Y_i^s for $s \le t$. As it was already pointed out by the Section 1 Theorem, this is necessary and sufficient for the existence of a well defined conditional expectation of $X(t)$ with respect to past measurement results of $Y^{t]}$.

Let $\mathcal{M}_{t]}$ be the the history abelian W*-algebra $\mathcal{W}_{t]} \subset \mathcal{B}(\mathcal{F}_{t]})$ generated by the output operators $\{\upsilon_r(W_i^r) : r \in \mathrm{I}_0^t, i \in I_W\}$ for an index subset $I_W \subseteq \{1, \dots, d\}$, or another abelian W*-algebra $\mathcal{N}_{t]} \subset \mathcal{B}(\mathcal{F}_{t]})$ generated by $\{\upsilon_r(N_i^r) : r \in \mathrm{I}_0^t, i \in I_N\}$ for the same or another index subset I_N, or the product algebra $\mathcal{M}_{t]} \sim \mathcal{W}_{t]} \bar{\otimes} \mathcal{N}_{t]}$ generated by $Y_e^{t]} = \{Y_i^r : t \in \mathrm{I}_0^t, i \in I_e\}$ corresponding to the union $I_e \subseteq \{1, \dots, d\}$ of disjoint index subsets I_W and I_N. Also let $\mathcal{B}_{[t}$ denote the future nonabelian W*-algebra generated by the system operators $X(s) \vdash \gamma_s(\mathfrak{b})$ for $s \ge t$. From the compatibility of the output operators, we have quantum causality condition $\mathcal{M}_{t]} \subset \mathcal{B}_{[t}'$, so again we have a unique well defined conditional expectation $\epsilon^t = \hat{\varpi}_{[t}^\star$ given by the posterior states $\hat{\varpi}_{[t}$ on $\mathcal{B}_{[t}$ onto $\mathcal{M}_{t]}$.

The conditional expectation $\hat{x}_r^t = \epsilon^t[X(t+r)]$ gives for any $r \ge 0$ the best prediction of $X(t+r) = \upsilon_r^t(X(t))$ as least squares estimator of any operator $X(t) = \upsilon_t(\check{x} \otimes I)$ evolved to the time $t' = t+r$ conditional on the output operators $Y_e^{t]}$ and so is equivalent to a classical random variable on the space of measurement trajectories $\Omega_{t]} = \{\omega_i^r : r \in \mathrm{I}_0^t, i \in I_e\}$ such that ω_i^r is an eigenvalue of Y_i^r. This conditional expectation for is most conveniently written in the Schrödinger picture as $\hat{x}_r^t = \langle \hat{\varsigma}^t | \check{\tau}_r^t(\check{x}) \rangle$ in terms of the expected CP hemigroup $\check{\tau}_r^t$ and posterior states $\hat{\varsigma}^t$ is defined by the relation

$$\langle \hat{\varsigma}^t | \check{x} \rangle := \epsilon^t[\upsilon_t(\check{x} \otimes I)] = \langle \hat{\varpi}_{[t} | \upsilon_t(\check{x} \otimes I) \rangle .$$

In the case of product state $\varpi_{[r} = \varsigma \otimes \varrho_r$ with infinitely divisible ϱ_r realized on $\mathcal{A}_r = \mathcal{B}(\mathcal{F}_r)$ by the vacuum state, the posterior state is given for any $t > r$ as $\hat{\varsigma}(t) = \phi_r^{t-r}(\varsigma)$ by a hemigroup $\{\phi_r^s\}$ of nonlinear transformations of a starting state $\varsigma = \hat{\varsigma}(r)$ on the object algebra $\mathfrak{b}$, resolving the quantum nonlinear filtering (*Belavkin*) equation

$$\mathrm{d}\hat{\varsigma}(t) = \lambda[\hat{\varsigma}](t)\,\mathrm{d}t + \sum_{i \in I_e} \delta^i(\hat{\varsigma})(t)\,\hat{M}_i(\mathrm{d}t) . \tag{3.23}$$

Here λ is Lindblad generator of the adjoint equation (3.19), and quantum filtering coefficients δ^i against the innovation martingales

$$\hat{M}_i(\mathrm{I}_r^s) = Y_i(\mathrm{I}_r^s) - \int_r^{r+s} \epsilon_r^t[Y_i(\mathrm{d}t)]$$

on the time intervals I_t^s where first specified in [63] as the functionals of the posterior quantum states $\hat{\varsigma}_r^s = \hat{\varsigma}_r(r+s) = \phi_r^s(\varsigma)$ resolving Itô equation (3.23) for $\hat{\varsigma}(r) = \varsigma$.

We present here two separate cases of Belavkin quantum filtering equation corresponding to the diffusive and counting measurements. For rigorous derivation see [36], and for the most general mixed case we refer to [24].

3.2.1. *Diffusive Belavkin equation.* The diffusive version [63],[32],[36] of Belavkin quantum filtering equation corresponding to continuous observation of the diffusive row-vector $\boldsymbol{Y}_W^t = \upsilon_t\left(W_{I_e}^t\right) = \left(Y_i^t\right)_{i\in I_W}$ indexed by a fixed set $I_e = I_W$ specifying the *estimation channels* is a classical non-linear stochastic differential equation given by

$$\mathrm{d}\hat{\varsigma} = \lambda[\hat{\varsigma}]\,\mathrm{d}t + \sum_{i\in I_W} \delta^i(\hat{\varsigma})(Y_i(\mathrm{d}t) - \langle\hat{\varsigma}|L_i + L_i^*\rangle\,\mathrm{d}t). \tag{3.24}$$

Here $\lambda(\varsigma) = \sum_i L^i\varsigma L^{i*} - K\varsigma - \varsigma K^*$ is defined by the left adjoints $L^i = L_i^\sharp$, $K = -L^\sharp$ of the Lindblad operators in (3.16), and

$$\delta^i(\varsigma) = \varsigma L^{i*} + L^i\varsigma - \langle\varsigma|L_i + L_i^*\rangle\,\varsigma$$

is the nonlinear *fluctuation coefficient* such that $\left\langle\delta^i(\varsigma)\,|\check{1}\right\rangle = 0$ for any $\varsigma \in \mathfrak{b}_\star$ with respect to the pairing of $\mathfrak{b}$ and $\mathfrak{b}_\star$.

3.2.2. *Counting Belavkin equation.* The counting, or quantum jump version [63, 34], [36] of Belavkin quantum filtering equation corresponding to counting observation of the number processes $\boldsymbol{Y}_N^t = \upsilon_t\left(N_{I_e}^t\right) = \left(Y_i^t\right)_{i\in I_N}$ indexed by $I_e = I_N$ is given as a stochastic differential equation in the classical Itô form by

$$\mathrm{d}\hat{\varsigma} = \lambda[\hat{\varsigma}]\,\mathrm{d}t + \sum_{i\in I_N} \delta^i(\hat{\varsigma})\,(Y_i(\mathrm{d}t) - \langle\hat{\varsigma}|L_iL_i^*\rangle\mathrm{d}t) \tag{3.25}$$

for a counting measurement in the field. Here $\lambda(\varsigma) = \sum_i L^i\varsigma L^{i*} - K\varsigma - \varsigma K^*$ such that $\left\langle\lambda(\varsigma)\,|\check{1}\right\rangle = 0$ for any $\varsigma \in \mathfrak{b}_\star$ and

$$\delta^i(\varsigma) = \alpha^i(\varsigma) - \varsigma, \quad \alpha^i(\varsigma) = \frac{L^i\varsigma L^{i*}}{\langle\varsigma|L_iL_i^*\rangle} \tag{3.26}$$

is the non-linear *normalized difference coefficient* defined by $L^i = L_i^\sharp$ ($=L_i^*$ for the trace pairing of $\mathfrak{b}$ with $\mathfrak{b}_\star$) such that $\left\langle\delta^i(\varsigma)\,|\check{1}\right\rangle = 0$ for any $\varsigma \in \mathfrak{b}_\star$ with respect to the standard pairing of $\mathfrak{b}_\star$ and $\mathfrak{b}$.

4. Optimal Quantum Feedback Control

We now couple the system to a control force (row-vector) $\boldsymbol{u} = (u_i)_{i\in I_f}$ via *forward* (feedback) channels indexed by a finite set I_f, $|I_f| = d_f$. The force perturbs open quantum dynamics described by unitary cocycle $\{U_r\}$ by making it causally dependent on each time interval $[r, r+s)$ on the control segment $\boldsymbol{u}_r^s = u(t) : t \in [r, r+s))$ such that $U_{\boldsymbol{u}}^r(r+s) = U_s^r(\boldsymbol{u}_r^s)$. The family $\{U_s^r(\boldsymbol{u}_r^s)\}$ is assumed to satisfy QS equation (3.13) with controlled pseudo-unitary germ $\check{\boldsymbol{s}}(t,\boldsymbol{u}) = \check{\boldsymbol{s}}(\boldsymbol{u}(t))$ such that its solution forms a hemigroup

$$U_r^t(\boldsymbol{u}_t^r)\,U_s^{t+r}\left(\boldsymbol{u}_{t+r}^s\right) = U_{r+s}^t\left(\boldsymbol{u}_t^{r+s}\right).$$

for any composed segment $\boldsymbol{u}_t^{r+s} = \left(\boldsymbol{u}_t^r, \boldsymbol{u}_t^{r+s}\right)$. This defines a hemigroup (γ_s^r) of time-dependent interaction dynamics

$$\gamma_{\boldsymbol{u}}^r(t, \check{x}) = U_{\boldsymbol{u}}^r(t)(\check{x} \otimes I)U_{\boldsymbol{u}}^{r*}(t) = \gamma_{t-r}^r\left(\boldsymbol{u}_r^{t-r}, \check{x}\right)$$

satisfying the controlled Langevin equation (3.14) with $\boldsymbol{\Sigma}_{\boldsymbol{u}}(t)$ given by time evolved germ $\check{\boldsymbol{\sigma}}_{\boldsymbol{u}}(t) = \check{\boldsymbol{\sigma}}(\boldsymbol{u}(t))$.

Following the original Belavkin's formulation [18],[3],[6] of quantum optimal control theory, we assume that the quality of a control process on a quantum object over a finite period $[r, T)$ with starting product state $\varpi_{[r} = \varsigma \otimes \varrho_r$ on the object plus noise algebra $\mathfrak{b} \otimes \mathcal{A}_r$ is judged by the integral expectation

$$\mathsf{J}_r(\varsigma, \boldsymbol{u}_r) = \int_r^T \left\langle \varpi_{[r} | C(\boldsymbol{u}_r, t) \right\rangle \mathrm{d}t + \left\langle \varpi_{[r} | S(\boldsymbol{u}_r, T) \right\rangle \tag{4.1}$$

of the *operator-valued cost functionals* of $\boldsymbol{u}_r$ given by the evolved object operator-valued measurable positive cost function $\check{c}(t) : \mathbb{U} \to \mathfrak{b}$ and a terminal positive cost operator $\check{s} \in \mathfrak{b}$ in

$$C(\boldsymbol{u}_r, t) = \gamma_{r,\boldsymbol{u}_r}^{t-r}(\check{c}(\boldsymbol{u}))\,,\ S(\boldsymbol{u}_r, T) = \gamma_{r,\boldsymbol{u}_r}^{T-r}(\check{s}) \tag{4.2}$$

for self-adjoint positive system operators $\check{c}(\boldsymbol{u}(t)), \check{s} \vdash \mathcal{A}$. An alternative problem of *risk-sensitive* control has also been studied by James [11, 12] where the cost is exponentiated to enforce higher penalties for undesirable behavior.

Coherent control of open quantum dynamics uses field channels indexed by a subset $I_f \subseteq \{1, \ldots, d\}$. It is realized by controlling quantum noise in these channels by $\boldsymbol{u} \in \mathbb{R}^{d_f}$ via their coherent states. One can start with uncontrolled dynamics described by Hamiltonian and Lindbladian operators $H_0 = \hbar\Im\left(R_+^-\right)$ and $K_0^i = -R_+^i$ for $R_+^\iota = \check{s}_+^\iota$, $\iota = -, i$, defining QS unitary evolution by (3.13) with arbitrary unitary scattering $S_\circ^\circ = \check{s}_\circ^\circ$, and apply coherent conditional expectation $\varphi_{r,\xi}^\star$ to corresponding QS flow $\gamma_s^r(\check{x})$ with $\xi(t) = \frac{\mathrm{i}}{\hbar}\boldsymbol{u}^\intercal(t)$ defining controlling field expectations

$$\left\langle \varphi_{t,\xi} | \hbar\Im\left[A_+^i\left(\mathrm{I}_t^s\right)\right] \right\rangle = \int_0^s u_i(t+s)\,\mathrm{d}s \ \ \forall i \in I_f.$$

This effectively results in change of the Hamiltonian H_0 and all operators K_0^i with $i \in I_f$ to

$$H_{\boldsymbol{u}} = H_0 + u_i\Re\left(K_0^i\right),\ K_{\boldsymbol{u}}^i = K_0^i + \frac{\mathrm{i}}{\hbar}u_i, \tag{4.3}$$

and no change for other K_0^i with $i \notin I_f$ (The summation is taken only over $i \in I_f$.) The resulting conditioned dynamics $\check{\tau}_{\boldsymbol{u}}^r(t, \check{x})$ satisfies time dependent Lindblad equation which can be written in the form (3.18) as

$$\check{\lambda}_{\boldsymbol{u}}(\check{x}) = \frac{\mathrm{i}}{\hbar}\left[H_{2\boldsymbol{u}}, \check{x}\right] + \check{\lambda}_R(\check{x}) + \sum_{i,k \in I_f} K_{\boldsymbol{u}}^{i*}\check{\rho}_k^i(\check{x})\,K_{\boldsymbol{u}}^k \tag{4.4}$$

with $\check{\rho}_k^i(\check{x}) = \check{\sigma}_k^i(\check{x}) - \delta_k^i\check{x}$, without change of the part $\check{\lambda}_R$ (but with doubled $\boldsymbol{u}$ in the Hamiltonian $H_{\boldsymbol{u}}$).

We are going to consider quantum feedback control problem in which it is natural to assume that the forward feedback control channels are disjoint to the set $I_e = I_W \cup I_N$ of *estimation.* This is achieved by considering coherent controls in the channels I_f such that $I_f \cap I_e = \emptyset$. In this case the controlling amplitude $\xi(t) \in \mathfrak{k}_f$ is orthogonal to the subspace $\mathfrak{k}_e = \mathfrak{k}_W \oplus \mathfrak{k}_N$ of observation channels, so the output equations (3.20), (3.21) are not affected by the coherent control which will simplify optimal feedback control problem which we solve by applying dynamical programming to coherent controlled quantum states. The controlled posterior density operator $\hat{\varsigma}(t) = \hat{\varsigma}^{t-r}_{r,\boldsymbol{u}}$ can then be obtained from the relevant uncontrolled filtering equation by replacing Lindblad generator $\lambda = \check{\lambda}_\star$ in (3.23) by time dependent $\lambda_{\boldsymbol{u}}(t) = \lambda(\boldsymbol{u}(t))$, and so we now have a controlled time dependent nonlinear filtering dynamics $\hat{\varsigma}^s_{r,\boldsymbol{u}} = \hat{\varsigma}^s_r(\boldsymbol{u}^s_r)$ satisfying Belavkin equation of the form, say (3.24) or (3.25), in which the fluctuating part under the above coherent control assumption is independent of $\boldsymbol{u}(t)$.

4.1. **Dynamical programming of quantum states.** In the search for optimal control inputs, it is desirable to allow the control to be determined in terms of measurement results on the system, particularly in the quantum setting where quantum noises introduce an inevitable stochastic nature. A feedback strategy $\varkappa_{[0}$ consists of measurable maps $\varkappa(t)$ which give for each $0 \le t < T$ an operator-valued control law $\boldsymbol{u}(t) = \varkappa(t, \boldsymbol{Y}^{t]}_e)$ as a function of the current and previous commuting output operators $\boldsymbol{Y}^{t]}_e = \{\boldsymbol{Y}^r_e : r \in (0,t]\}$. Thus the control law $\varkappa(t)$ is realized by an adapted random vector variable $\varkappa(t, \omega_{t]})$ on the probability space $(\Omega_{t]}, \mathfrak{A}_{t]}, \mathbb{P}_{t]})$ of output measurement results in the value space $\mathbb{U}$ of admissible control inputs in the spectral representation

$$\boldsymbol{u}(t) = \int_{\Omega_{t]}} \varkappa(t, \omega_{t]}) E_0\left(\mathrm{d}\omega_{t]}\right) \equiv \varkappa(t, \boldsymbol{Y}^{t]}_e).$$

We denote the space of admissible operator-valued feedback controls on the interval $[r, r+s)$ by $\mathbb{U}^s_r(\boldsymbol{Y}_e)$. Note that no measurement results are available initially, so the initial control $\boldsymbol{u}(0)$ is deterministic and also no controls are applied at the termination time T. It is too restrictive to consider only continuous sample paths $\{\varkappa(t, \omega^t_e) : t > 0\}$, since for example the Poisson process $\{\boldsymbol{Y}^t_N\}$ certainly does not have continuous sample paths. Instead we give the following definition of an admissible strategy.

Definition 1. *An admissible feedback control strategy* $\varkappa_r = \{\varkappa(t) : r \le t < T\}$ *determines randomized control laws* $\boldsymbol{u}(t) = \varkappa(t, \boldsymbol{Y}^{t]}_e)$ *at each time* $t > 0$ *which realize values in* $\mathbb{U}$ *and form càdlàg sample paths* $\{\varkappa(t,\omega) : t \ge r\}$ [1]*. Moreover, an admissible strategy* $\varkappa^o_t$ *shall be called* optimal *if it realizes the infimum*

$$\mathsf{S}(r, \varsigma) = \inf_{\varkappa_r \in \mathcal{K}_r} \mathsf{J}_r(\varsigma, \varkappa_r) = \mathsf{J}_r(\varsigma, \varkappa^o_r) \tag{4.5}$$

over the space $\mathcal{K}_r$ *of admissible feedback control strategies, where* $\mathsf{J}_r(\varsigma, \varkappa_r)$ *is the expected cost for the control process determined by the feedback strategy* $\varkappa_r$*.*

It is a simple exercise to show that under a feedback strategy, the output operators once again form a QND measurement with respect to the controlled system operators which justifies the existence of the conditional expectation

$$\epsilon^t\left[\gamma^{t-r}_{r,\boldsymbol{u}}(\check{x})\right] = \left\langle \hat{\varsigma}^{t-r}_{r,\boldsymbol{u}} | \check{x} \right\rangle \tag{4.6}$$

with respect to the output operators $\boldsymbol{Y}^{t]}_e$ for $t > r$. It is given by the posterior state $\hat{\varsigma}^{t-r}_{r,\boldsymbol{u}} = \hat{\varsigma}_{r,\boldsymbol{u}}(t)$ as solution of Belavkin equation (3.23) which now has dependence on

[1] Right-continuous paths $(\lim_{h\to 0} \mu_{t+h}(\omega^{t+h}) = \mu_t(\omega^t))$ having well defined left limits $\mu_t(\omega^t)_- = \lim_{h\to 0} \mu_{t-h}(\omega^{t-h})$

the chosen control $\boldsymbol{u}_s^r$ inputs through the dynamics $\gamma_r^s(\boldsymbol{u}_s^r, \check{x})$, and, given an initial condition $\hat{\varsigma}_{r,\boldsymbol{u}}(r) = \varsigma$, it does not really depend on $\boldsymbol{Y}_e^{r]}$ due to Markovianity of the process $\hat{\varsigma}_{\boldsymbol{u}}^t$ proved in [3]. The existence of this conditional expectation permits the following theorem which lies at the heart of quantum feedback control.

Theorem 4. *The expectation (4.1) of the operator valued cost (4.2) when a feedback control strategy $\varkappa_r$ is in operation can be written as a classical expectation*

$$\mathsf{J}_r(\varsigma, \varkappa) = \int_\Omega \mathsf{J}_r^\omega(\varsigma, \varkappa)\mathbb{P}(\mathrm{d}\omega) \equiv \mathbb{E}_\Omega[\mathsf{J}_r^\bullet(\varsigma, \varkappa)] \tag{4.7}$$

of the random cost-to-go function

$$\mathsf{J}_r^\omega(\varsigma, \varkappa) = \int_r^T \langle \hat{\varsigma}_{r,\varkappa}^{t-r}(\omega) | \check{c}(\varkappa(t, \omega_{t]})) \rangle \mathrm{d}t + \langle \hat{\varsigma}_{r,\varkappa}^{T-r}(\omega) | \check{s} \rangle \tag{4.8}$$

where $\hat{\varsigma}_{r,\varkappa}^{t-r}(\omega)$ is the solution $\hat{\varsigma}(t, \omega)$ to the controlled filtering equation corresponding to the chosen measurement process $\boldsymbol{Y}_e^t$ classically represented as $\boldsymbol{\omega}_e^t$ for the feedback strategy $\varkappa$ with the initial condition $\hat{\varsigma}(r) = \varsigma$.

Proof. Using the existence and state invariance of the conditional expectation and the classical isomorphism proved in the first Section, it is straight forward application of the formula (4.6) to $\check{x}(t) = \check{c}(\boldsymbol{u}(t))$ and $\check{x}(T) = \check{s}$ in 4.1. We can write the expected cost as

$$\begin{aligned} \mathsf{J}_r(\varsigma, \varkappa_r) &= \int_r^T \left\langle \varsigma \otimes \varrho_r | \gamma_{r,\boldsymbol{u}}^{t-r}(\check{c}(\boldsymbol{u}(t))) \right\rangle \mathrm{d}t + \left\langle \varsigma \otimes \varrho_r | \gamma_{r,\boldsymbol{u}}^{T-r}(\check{s}) \right\rangle \\ &= \int_r^T \left\langle \varpi_{[r}, \epsilon_r^t \left[\gamma_{r,\boldsymbol{u}}^{t-r}(\check{c}(\boldsymbol{u}(t))) \right] \right\rangle \mathrm{d}t + \left\langle \varpi_{[r} | \epsilon_r^T \left[\gamma_{r,\boldsymbol{u}}^{T-r}(\check{s}) \right] \right\rangle \\ &= \int_r^T \mathbb{E}_\Omega \left[\left\langle \hat{\varsigma}_{r,\boldsymbol{u}}^{t-r} | \check{c}(\boldsymbol{u}(t)) \right\rangle \right] \mathrm{d}t + \mathbb{E}_\Omega \left[\left\langle \hat{\varsigma}_{r,\boldsymbol{u}}^{T-r} | \check{s} \right\rangle \right]. \end{aligned}$$

□

This allows us to treat the quantum control problem as a classical control problem on the space of quantum states. We define the expected cost-to-go by the classical expression (4.7) for a truncated admissible strategy $\varkappa_r \in \mathcal{K}_r$ when starting in an arbitrary state ς at time r where $\hat{\varsigma}_{r,\varkappa}^{t-r}(\omega)$ is evaluated at $\omega \in \Omega_{t]}$ solution $\hat{\varsigma}(t) = \phi_{r,\varkappa}^{t-r}(\varsigma)$ to the controlled filtering equation for these initial conditions at $t = r$ and the initial strategy $\varkappa_r$ for $r < t \leq T$.

Theorem 5. *Suppose that $\mathsf{S}(t, \varsigma)$ is a functional which is continuously differentiable in t, has continuous Frèchet derivatives of all order with respect to ς and satisfies*

$$\inf_{\boldsymbol{u} \in \mathbb{U}(Y^t)} \left\{ \langle \varsigma, \check{c}(\boldsymbol{u}) \rangle + \mathbb{E}_\Omega^t \left[\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{S}(t, \hat{\varsigma}) \right] \right\} = 0. \tag{4.9}$$

for all $0 < t < T$ and $\mathsf{S}(T, \varsigma) = \langle \varsigma, \check{s} \rangle$ for all $\varsigma \in \mathcal{S}$. Suppose also that $\varkappa^o$ is the strategy built from the control laws attaining these minima within a convex space $\mathbb{U}$ of admissible control values, then $\mathsf{S}(t, \varsigma)$ is the functional which minimizes (4.7) *and $\varkappa^o$ is the optimal strategy for the control problem*[2]*.*

[2] Additional technical assumptions and mathematical rigour are required to formalise the proof of this theorem when dealing with unbounded operators which is beyond the scope of this paper. See recommended texts e.g. [64],[65] for a formal classical treatment.

Sketch proof. Let $\{\boldsymbol{u}(t)\}$, $\{\hat{\varsigma}_0^t\}$ be any control and state trajectories resulting from an admissible strategy $\varkappa$ on the initial state ς, then from (4.9), we have the inequality

$$\langle\hat{\varsigma}_0^t, \check{c}(\boldsymbol{u}(t))\rangle + \mathbb{E}_\Omega\big[\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{S}(t,\hat{\varsigma}_0^t)\big] \geq 0$$

which we integrate over $[0,T]$ and take the expectation $\mathbb{E}_\Omega$ to obtain due to convexity

$$\mathbb{E}_\Omega\big[\int_0^T\langle\hat{\varsigma}_0^t, \check{c}(\boldsymbol{u}(t))\rangle\mathrm{d}t + \mathsf{S}(T,\hat{\varsigma}_0^T) - \mathsf{S}(0,\hat{\varsigma}_0^0)\big] \geq 0$$

Since we have $\hat{\varsigma}_0^0 = \varsigma$ initially, we can rearrange and use the terminal condition to obtain

$$\mathsf{S}(0,\varsigma) \leq \mathbb{E}_\Omega\big[\int_0^T\langle\hat{\varsigma}_0^t, \check{c}(\boldsymbol{u}(t))\rangle\mathrm{d}t + \langle\hat{\varsigma}_0^T, \check{s}\rangle\big] = \mathsf{J}_0(\varsigma,\varkappa).$$

with equality when $\varkappa = \varkappa^o$ and so the lower bound $\mathsf{S}(0,\varsigma)$ is attained, proving optimality of $\varkappa^o$. □

A choice of controlled filtering equation is required to determine the stochastic trajectories $\mathrm{d}\hat{\varsigma}_0^t$ along which to differentiate candidate solutions. The next two sections are concerned with the examples of feedback control with respect to QND measurements of the diffusive process $\{\boldsymbol{Y}_W^t\}$ and the counting process $\{\boldsymbol{Y}_N^t\}$ respectively.

4.2. **Quantum state Bellman equations.** First let us introduce notations of differential calculus on the quantum state space $\mathfrak{S} \subset \mathfrak{b}_\star$. Let $\mathsf{F} = \mathsf{F}[\cdot]$ be a (nonlinear) functional $\varsigma \mapsto \mathsf{F}[\varsigma]$ on $\mathfrak{S}$, then we say it admits a (Frèchet) derivative if there exists an $\mathfrak{b}$-valued function $\nabla_\varsigma\mathsf{F}[\cdot]$ on $\mathfrak{b}_\star$ such that

$$\lim_{h\to 0}\frac{1}{h}\{\mathsf{F}[\cdot + h\tau] - \mathsf{F}[\cdot]\} = \langle\tau|\nabla_\varsigma\mathsf{F}[\cdot]\rangle = (\tau, \nabla_\varsigma\mathsf{F}[\cdot]) \tag{4.10}$$

for each $\tau = \tau^* \in \mathfrak{b}_\star$. In the same spirit, a Hessian $\nabla_\varsigma^{\otimes 2} \equiv \nabla_\varsigma \otimes \nabla_\varsigma$ can be defined as a mapping from the functionals on to the $\mathfrak{b}_{sym}^{\otimes 2}$-valued functionals, via

$$\begin{aligned}\lim_{h,h'\to 0}\frac{1}{hh'}\{\mathsf{F}[\cdot + h\tau + h'\tau'] - \mathsf{F}[\cdot + h\tau] - \mathsf{F}[\cdot + h'\tau'] + \mathsf{F}[\cdot]\} \\ = \langle\tau\otimes\tau'|\nabla_\varsigma\otimes\nabla_\varsigma\mathsf{F}[\cdot]\rangle.\end{aligned} \tag{4.11}$$

and we say that the functional is twice continuously differentiable whenever $\nabla_\varsigma^{\otimes 2}\mathsf{F}[\cdot]$ exists and is continuous in the trace norm topology.

With the customary abuses of differential notation, we have for instance

$$\nabla_\varsigma f(\langle\varsigma|X\rangle) = f'(\langle\varsigma|X\rangle)X$$

for any differentiable function f of the scalar $x = \langle\varsigma|X\rangle$.

4.2.1. *Diffusive Bellman equation.* We have the Itô rule $\mathrm{d}Y_i^t\mathrm{d}Y_k^t = \delta_{ik}I\mathrm{d}t$ with $i,k \in I_e = I_W$ for the increments of the diffusive processes $Y_i^t = \upsilon_t(W_i^t)$ which have the expectations $\mathbb{E}_\Omega^t[\mathrm{d}Y_i^t] = \langle\varsigma|2\Re(L_i)\rangle\mathrm{d}t$, so using the classical Itô formula for the diffusive process Y_i^t we can show

$$\begin{aligned}\mathbb{E}_\Omega^t[\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{S}(t,\varsigma)] =& \frac{\partial}{\partial t}\mathsf{S}(t,\varsigma) + \langle\varsigma|\check{\lambda}(\boldsymbol{u})[\nabla_\varsigma\mathsf{S}(t,\varsigma)]\rangle \\ &+ \frac{1}{2}\sum_{i\in I_W}\langle\delta^i(\varsigma)\otimes\delta^i(\varsigma)|\nabla_\varsigma^{\otimes 2}\mathsf{S}(t,\varsigma)\rangle\end{aligned} \tag{4.12}$$

where $\nabla_\varsigma \mathsf{S} \in \mathfrak{b}$ denotes Frèchet derivative with respect to $\varsigma \in \mathfrak{b}_\star$, and $\nabla_\varsigma^{\otimes 2}\mathsf{S} \in \mathfrak{b}^{\bar{\otimes}2} := \mathfrak{b}\bar{\otimes}\mathfrak{b}$ denotes Hessian applied to S. Observing $\frac{\partial}{\partial t}\mathsf{S}(t,\varsigma)$ is independent of $\boldsymbol{u}$ leads to the following corollary.

Corollary 2. *Suppose there exists a functional $\mathsf{S}(t,\varsigma)$ which is continuously differentiable in t, has continuous first and second order Frèchet derivatives with respect to ς and satisfies the following Bellman equation*

$$-\frac{\partial}{\partial t}\mathsf{S}(t,\varsigma) = \inf_{\boldsymbol{u}\in\mathbb{U}}\left\{\langle\varsigma|\check{c}(\boldsymbol{u}) + \check{\lambda}(\boldsymbol{u})[\nabla_\varsigma \mathsf{S}(t,\varsigma)]\rangle\right\} + \frac{1}{2}\sum_i \langle\delta^i(\varsigma)\otimes\delta^i(\varsigma)|\nabla_\varsigma^{\otimes 2}\mathsf{S}(t,\varsigma)\rangle \tag{4.13}$$

for all $t>0$, $\varsigma\in\mathcal{S}$ with the terminal condition $\mathsf{S}(T,\varsigma) = \langle\varsigma,\check{s}\rangle$. Then the strategy $\varkappa^o\left(t,\mathbf{Y}_W^{t]}\right) = \boldsymbol{u}^o(t,\hat{\varsigma})$ built from the control laws

$$\boldsymbol{u}^o(t,\varsigma) = \arg\inf_{\boldsymbol{u}\in\mathbb{U}}\left\{\langle\varsigma|\check{c}(\boldsymbol{u}) + \check{\lambda}(\boldsymbol{u})[\nabla_\varsigma \mathsf{S}(t,\varsigma)]\rangle\right\} \tag{4.14}$$

for $0\le t<T$ is optimal for the feedback control problem based on diffusive output measurements.

Note that last line in 4.13 is precisely half of the Laplace operator

$$\triangle\mathsf{S}(\varsigma) = \sum_{i\in I_W}\left\langle\delta^i(\varsigma)\otimes\delta^i(\varsigma)|\nabla_\varsigma^{\otimes 2}\mathsf{S}(t,\varsigma)\right\rangle$$

in the quantum state 'coordinates' $\varsigma\in\mathfrak{b}_\star$ as the sufficient coordinates from the preadjoint space of the algebra $\mathfrak{b}$.

4.2.2. *Counting Bellman equation.* We have the Itô rules $\mathrm{d}Y_i^t\mathrm{d}Y_k^t = \delta_i^j\delta_k^j\mathrm{d}Y_j^t$ for the increments of the counting processes $Y_j^t = \upsilon\left(N_j^t\right)$, $j\in I_e : I_N$ which have the expectations $\mathbb{E}_\Omega^t[\mathrm{d}Y_i^t] = \langle\varsigma|L_i(t)L_i^*(t)\rangle\mathrm{d}t$, so using Itô formula for the counting processes Y_i^t we we can show

$$\begin{aligned}\mathbb{E}_\Omega^t\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{S}(t,\varsigma)\right] =& \frac{\partial}{\partial t}\mathsf{S}(t,\varsigma) + \left\langle\varsigma|\check{\lambda}(\boldsymbol{u})[\nabla_\varsigma\mathsf{S}(t,\varsigma)]\right\rangle \\ &-\sum_{i\in I_N}\langle\varsigma|L_iL_i^*\rangle\left\langle\alpha^i(\varsigma)-\varsigma|\nabla_\varsigma\mathsf{S}(t,\varsigma)\right\rangle \\ &+\sum_{i\in I_N}\langle\varsigma|L_iL_i^*\rangle\left(\mathsf{S}\left(t,\alpha^i(\varsigma)\right)-\mathsf{S}(t,\varsigma)\right)\end{aligned} \tag{4.15}$$

The last two lines can be written in the Feller form as $\frac{1}{2}\mu_i(\varsigma)\triangle^i\mathsf{S}(t,\varsigma)$ as in the diffusive case in terms of doubled difference combination $\mu_i(\varsigma)\triangle^i\mathsf{S}(t,\varsigma)$ of the linear combination of differences

$$\mathsf{S}\left(t,\alpha^i(\varsigma)\right)-\mathsf{S}(t,\varsigma)\,,\quad \mu_i(\varsigma) = \langle\varsigma|L_iL_i^*\rangle$$

and $\mu_i(\varsigma)\left\langle\delta^i(\varsigma)|\nabla_\varsigma\mathsf{S}(t,\varsigma)\right\rangle$ in terms of $\delta^i(\varsigma) = \alpha^i(\varsigma)-\varsigma$. The formal Taylor expansion

$$\triangle^i\mathsf{S}(t,\varsigma) = 2\sum_{n=2}^{\infty}\frac{1}{n!}\left\langle\delta^i(\varsigma)^{\otimes n}|\nabla_\varsigma^{\otimes n}\mathsf{S}(t,\varsigma)\right\rangle$$

of each $\triangle^i\mathsf{S}(t,\varsigma) = 2\left(\mathsf{S}(\alpha(\varsigma))-\mathsf{S}(\varsigma)-\left\langle\delta^i(\varsigma)|\nabla_\varsigma\right\rangle\right)$ in terms of the higher order Frèchet derivatives $\nabla_\varsigma^{\otimes n}$ starts from the Hessian $\nabla_\varsigma^{\otimes 2} = \nabla_\varsigma\otimes\nabla_\varsigma$, determining the

Laplace operators $\left\langle \delta^{i}\left(\varsigma\right)^{\otimes 2}|\nabla_{\varsigma}^{\otimes 2}\right\rangle$ for the diffusive approximation of this counting measurement case.

Let us introduce the Pontryagin's 'Hamiltonian' in the 'coordinates' $\check{q}\left(\varsigma\right)=\varrho-\varsigma$ and 'momenta' $\check{p}\in\mathfrak{b}$ as the Legendre-Fenchel transform

$$\mathsf{H}\left(\check{q},\check{p}\right)=\sup_{\boldsymbol{u}\in\mathbb{U}}\left\{\left\langle\lambda\left(\boldsymbol{u}\right)\left[\check{q}\right]\mid\check{p}\right\rangle-\mathsf{L}\left(\check{q},\boldsymbol{u}\right)\right\}$$

of the 'Lagrangian' $\mathsf{L}\left(\check{q},\boldsymbol{u}\right)=\left\langle\varrho-\check{q}|\check{c}\left(\boldsymbol{u}\right)\right\rangle$, where $\varrho=\check{q}-\varsigma$ is any stationary element $\varrho=\varrho^{*}$ in $\mathfrak{b}_{\star}$ such that $\lambda\left(\boldsymbol{u}\right)\left[\varrho\right]=0$ for any $\boldsymbol{u}\in\mathbb{U}$ (e.g. $\varrho=0$). Then one can write Bellman equation defining minimal expected cost-to-go (4.5) as the action functional in the compact Jacobi-Feller form as follows, similar as it was done for the diffusive case.

Corollary 3. *Suppose there exists a functional $\mathsf{S}(t,\varsigma)$ which is continuously differentiable in t, has continuous first order Frèchet derivatives with respect to ς and satisfies the following Bellman-Jacobi-Feller equation*

$$-\frac{\partial}{\partial t}\mathsf{S}(t,\varsigma)+\mathsf{H}\left(\check{q}\left(\varsigma\right),\nabla_{\varsigma}\mathsf{S}(t,\varsigma)\right) \tag{4.16}$$

$$=\frac{1}{2}\sum_{i\in I_{N}}\left\langle\varsigma\mid L_{i}L_{i}^{*}\right\rangle\triangle^{i}\,\mathsf{S}\left(t,\varsigma\right) \tag{4.17}$$

for all $t>0$, $\varsigma\in\mathfrak{b}_{\star}$ with the terminal condition $\mathsf{S}(T,\varsigma)=\langle\varsigma,\check{s}\rangle$. Then the strategy $\varkappa^{o}\left(t,\mathbf{Y}_{N}^{t]}\right)=\boldsymbol{u}^{o}\left(t,\hat{\varsigma}\right)$ built from the control laws

$$\boldsymbol{u}^{o}\left(t,\varsigma\right)=\arg\inf_{\boldsymbol{u}\in\mathbb{U}}\left\{\left\langle\varsigma|\check{c}(\boldsymbol{u})+\check{\lambda}(\boldsymbol{u})[\nabla_{\varsigma}\mathsf{S}(t,\varsigma)]\right\rangle\right\} \tag{4.18}$$

is optimal for the feedback control problem based on counting output measurements.

Thus we have shown that without loss in optimality, one can reformulate the unobservable quantum feedback control problem into a feedback problem based on indirect QND measurements with feedback of the controlled conditional density matrix. However, the corresponding Hamilton-Jacobi-Bellman equation resulting from the minimization rarely has a regular solution $\mathsf{S}(t,\varsigma)$ from which to construct the optimal feedback laws. We now study a specific quantum filtering and feedback case where such a control solution can be explicitly found, which is familiar as the only such example in the classical case.

5. Application to a Linear Quantum Dynamical System

Let $\check{x}_{\bullet}=(\check{x}_{1},\ldots,\check{x}_{m})$ be the row-vector of self-adjoint operators $\check{x}_{j}=\check{x}_{j}^{*}$, $j=1,...,m$ and $\mathbf{J}=(\mathrm{J}_{ik})$ be an anti-symmetric real valued matrix defining the canonical commutation relations (CCRs)

$$[\check{x}_{i},\check{x}_{k}]:=\check{x}_{i}\check{x}_{k}-\check{x}_{i}\check{x}_{i}=\mathrm{i}\hbar\mathrm{J}_{ik}\check{1}, \tag{5.1}$$

written in the matrix form as $[\check{x}_{\bullet}^{\intercal},\check{x}_{\bullet}]=\mathrm{i}\hbar\mathbf{J}\check{1}$ where $\check{1}$ is the identity operator on $\mathfrak{h}$. Usually this is the standard symplectic matrix $\mathbf{J}^{\intercal}=-\mathbf{J}=\mathbf{J}^{-1}$, but we may not assume that $\mathbf{J}$ is standard or non-degenerate in order to include also the commuting random variables $\check{x}_{\bullet}=(\check{x}_{1},\ldots,\check{x}_{m})$ as a special (classical) case. It is worth remarking at this point that the noncommuting operators $\check{x}_{j}$ are secondary commuting (in the sense of commutativity with all the commutants (5.1)), and therefore they must be unbounded in the Hilbert space $\mathfrak{h}$, affiliated to the the generated algebra $\mathfrak{b}$.

We couple the open quantum system to $d_e = |I_e|$ estimation (side) channels with linear combinations $L_i = \sum_j \Lambda_{ij}\check{x}_j$ indexed by a subset $I_e \subseteq \{1,\dots,d\}$, given by a complex-valued $d_e \times m$ matrix $\mathbf{\Lambda}_e = (\Lambda_{ij})$ with $\Lambda_{ij} = 0$ for $i \notin I_e$. In the i-th estimation channel, we perform a measurement of the output $Y_i^t = \upsilon_\infty \left(2\Re\left[A_i^-(t)\right]\right)$ defined by $\upsilon_t|\mathcal{A}_0^t = \upsilon_\infty|\mathcal{A}_0^t$ where $2\Re\left[A_i^-\right] = A_i^+ + A_-^i$, $i \in I_e$ and the output transformations is given by a quantum stochastic evolution υ_t on the algebra $\mathfrak{b}\otimes\mathcal{A}_0$ generated by the canonical independent variables $\check{x}_\bullet$ and $A_\circ^+$. The system is also coupled to $d_f = |I_f|$ feedback (input) channels by the operators $K^i = L_i^*$ as linear combinations of $(\check{x}_j)$, given by the row $\boldsymbol{L}_f = \check{x}_\bullet\mathbf{\Lambda}_f^\intercal$ of operators L_i, $i \in I_f$ for a complex-valued $d_f \times m$ matrix $\mathbf{\Lambda}_f = (\Lambda_{ij})$ with $\Lambda_{ij} = 0$ for $i \notin I_f$, and we apply input controls with real-valued components $u_i(t) \in \mathbb{R}$ in the row $\boldsymbol{u} = (u_i)_{i\in I_f}$ via the i-th feedback channel at time t. Both matrices $\mathbf{\Lambda}_e$, $\mathbf{\Lambda}_f$ may depend on t, but they are always orthogonal such that $\mathbf{\Lambda}^\dagger\mathbf{\Lambda} = \mathbf{\Lambda}_e^\dagger\mathbf{\Lambda}_e + \mathbf{\Lambda}_f^\dagger\mathbf{\Lambda}_f$, where $\mathbf{\Lambda} = \mathbf{\Lambda}_e + \mathbf{\Lambda}_f$ and $\mathbf{\Lambda}^\dagger$ is transposed to complex conjugated matrix $\mathbf{\Lambda}^* = \left(\overline{\Lambda_{ij}}\right)$.

Let the free dynamics of the quantum system be described by a quadratic Hamiltonian $H_0 = \frac{1}{2}\check{x}_\bullet\mathbf{M}^{-1}\check{x}_\bullet^\intercal$ for a symmetric real $m\times m$ matrix $\mathbf{M}^{-1}$. We now introduce a coherent control source separating the controls $\boldsymbol{u}(t)\mathbf{B}_f\mathrm{d}t$ in coherent superposition with the noise $\hbar\Im\left(\mathrm{d}A_\circ^+\right)\mathbf{B}_f$ in control channel, where $2\mathrm{i}\Im\left(A_i^+\right) = A_i^+ - A_-^i$, coming from the feedback channel with $\mathbf{B}_f = 2\operatorname{Re}\mathbf{\Lambda}_f$, so that the Hamiltonian in (4.4) is modelled by

$$H_{2\boldsymbol{u}}(t) = \frac{1}{2}\check{x}_\bullet\mathbf{M}^{-1}\check{x}_\bullet^\intercal + \boldsymbol{u}(t)\mathbf{B}_f\check{x}_\bullet^\intercal. \tag{5.2}$$

Using CCR's (5.1) and assuming that the QS dynamics is purely diffusive (no scattering, $S_k^i = \check{1}\delta_k^i$ in (3.13)), we can easily evaluate Lindblad generator $\check{\lambda}_{\boldsymbol{u}}(\check{x}_i)$ for controlled CP hemigroup dynamics $\check{\tau}_{\boldsymbol{u}}^r(t,\check{x}) = \check{\tau}_{t-r}^r(\boldsymbol{u},\check{x})$ arising from time dependent quadratic Hamiltonian (5.2). Substituting $\check{x}_j$ into the decomposed generator (4.4) with linear L^i in $\check{x}_j$ and zero jump part, $\check{\rho}_\nu^\mu(\check{x}) = 0$, we obtain $\check{\lambda}_{\boldsymbol{u}}(\check{x}_i)$ as linear transformation of $\check{x}_\bullet = (\check{x}_i)$ written in vector form as

$$\check{\lambda}_{\boldsymbol{u}}(t,\check{x}_\bullet^\intercal) = \mathbf{J}(\mathbf{M}^{-1} + \hbar\operatorname{Im}\left(\mathbf{\Lambda}^\dagger\mathbf{\Lambda}\right))\check{x}_\bullet^\intercal - \mathbf{C}_f\boldsymbol{u}^\intercal(t)$$

where $\mathbf{C}_f^\intercal = \mathbf{B}_f\mathbf{J}$. From this, we obtain the quantum Langevin equation for $X_\bullet(t) = \upsilon_t(\check{x}_\bullet\otimes I)$ in the linear form

$$\mathrm{d}X_\bullet(t) + \left(X_\bullet(t)\mathbf{A}^\intercal + \boldsymbol{u}(t)\mathbf{C}_f^\intercal\right)\mathrm{d}t = \mathrm{d}U_\bullet^t \tag{5.3}$$

derived in [28], where $\mathbf{A}^\intercal := (\hbar\operatorname{Im}(\mathbf{\Lambda}^\intercal\mathbf{\Lambda}^*) + \mathbf{M}^{-1})\mathbf{J}$ with quantum noise

$$U_\bullet^t := 2\hbar\Im\left[A_\circ^+(t)\mathbf{\Lambda}\right]\mathbf{J} = V_\bullet^t + W_\bullet^t, \tag{5.4}$$

for $\mathbf{\Lambda} = \mathbf{\Lambda}_e + \mathbf{\Lambda}_f$. Here $V_\bullet^t = V_\circ^t\mathbf{C}^\intercal$ is total Langevin force and $W_\bullet^t = W_\circ^t\mathbf{F}_e^\intercal$ is total Wiener noise given by left action of matrices

$$\mathbf{C}^\intercal = 2\operatorname{Re}\mathbf{\Lambda}\mathbf{J}, \quad \mathbf{F}^\intercal = \hbar\operatorname{Im}\mathbf{\Lambda}\mathbf{J}, \tag{5.5}$$

on rows $V_\circ^t = 2\Im\left(A_1^+,\dots A_d^+\right)(t)$, $W_\circ^t = 2\Re\left(A_1^+,\dots A_d^+\right)(t)$ of all quantum Langevin forces and all conjugate Wiener noises respectively, coming both from the estimation and feedback channels.

5.1. **Quantum filtering of linear, Gaussian dynamics.** The linear output equation for the row $\boldsymbol{Y}_e^t = (Y_i^t)_{i\in I_e}$ of observable processes in estimation channel satisfies

$$\mathrm{d}\boldsymbol{Y}_e^t = X_\bullet(t)\,\mathbf{B}_e^\intercal \mathrm{d}t + \mathrm{d}\boldsymbol{W}_e^t \tag{5.6}$$

where $\mathbf{B}_e = 2\,\mathrm{Re}\,\boldsymbol{\Lambda}_e$. The quantum measurement noise is given by the row $\boldsymbol{W}_e = (W_i)_{i\in I_e}$ having Gaussian independent increments on each measurement channel with zero mean and standard variance given by $d_e \times d_e$ identity matrix $\mathbf{I}_e$. Considered alone, this noise represents the standard d_e-dimensional classical Wiener process which we measure in the field after interaction with the quantum object by the output isomorphic transformation $\boldsymbol{Y}_e^t = 2\Re\upsilon_\infty\left[\boldsymbol{A}_e^+(t)\right]$ for $\boldsymbol{A}_e^+ = \left(A_i^+\right)_{i\in I_d}$. However, it does not commute with the quantum Langevin force $V_\bullet^t$ as

$$[V_\bullet^{r\intercal}, \boldsymbol{W}_e^s] = (r\wedge s)\,\mathrm{i}\hbar\mathbf{C}_e I, \tag{5.7}$$

due to the noncommutativity with $\boldsymbol{V}_e^t = (V_i^t)_{i\in I_e}$ and commutativity with $\boldsymbol{V}_f^t = \hbar\Im\left[\boldsymbol{A}_f^+(t)\right]$, resulting form independence of $\boldsymbol{A}_f^+ = \left(A_i^+\right)_{i\in I_f}$. The fundamental CCR (5.7), defined by nonzero (if $\mathbf{J}\neq 0$) matrix $\mathbf{C}_e^\intercal = \mathbf{B}_e\mathbf{J}$, was first derived in a complex form in [18],[44], and in even more general infinite dimensional setting in [28]. It expresses the Heisenberg error-perturbation uncertainty principle in a precise form

$$\mathrm{d}V_\bullet^\intercal \mathrm{d}V_\bullet \geq \frac{\hbar^2}{4}\mathbf{C}_e\mathbf{C}_e^\intercal \mathrm{d}t,\ \mathrm{d}\boldsymbol{W}_e\mathrm{d}\boldsymbol{W}_e^\intercal = \mathbf{I}_e\mathrm{d}t \tag{5.8}$$

derived by Belavkin in [44],[28] as necessary and sufficient condition for nondemolition causality of the observable past $\boldsymbol{Y}_e^{t]} = \{\boldsymbol{Y}_e^r : r\leq t\}$ and quantum future described by $\{X_\bullet(s) : s\geq t\}$.

Usually in classical filtering theory the process $U_\bullet^t$ and measurement noises W_e^t are considered to be independent, although in the quantum setting the Heisenberg principle, implying the dependence of $U_\bullet^t = V_\bullet^t + W_\bullet^t$ and $\boldsymbol{W}_e^t$, may result in a nonzero covariance matrix $\mathbf{F}_e = \hbar\mathbf{J}\,\mathrm{Im}\left(\boldsymbol{\Lambda}_e^\dagger\right)$ describing the real part of quantum Itô table

$$\mathrm{d}U_\bullet^\intercal \mathrm{d}\boldsymbol{W}_e = \left(\mathbf{F}_e + \frac{\mathrm{i}\hbar}{2}\mathbf{C}_e\right)\mathrm{d}t \tag{5.9}$$

as the sum of $\mathrm{d}V_\bullet^\intercal \mathrm{d}\boldsymbol{W}_e = \frac{\mathrm{i}\hbar}{2}\mathbf{C}_e\mathrm{d}t$ and $\mathrm{d}W_\bullet^\intercal \mathrm{d}\boldsymbol{W}_e = \mathbf{F}_e\mathrm{d}t$. Note that although each component $U_i = U_i^*$ of the row $U_\bullet = (U_1,\dots,U_m)$ for vector quantum noise (5.4) having the independent increments can also be realized as a classical Wiener process, these components mutually do not commute, having complex multiplication table $\mathrm{d}U_\bullet^\intercal \mathrm{d}U_\bullet = \hbar^2\boldsymbol{\Lambda}^\dagger\boldsymbol{\Lambda}\mathrm{d}t$ with imaginary part defining the commutation relations

$$[U_\bullet^{r\intercal}, U_\bullet^s] = (r\wedge s)\,2\mathrm{i}\hbar^2\mathbf{J}^\intercal\,\mathrm{Im}\left(\boldsymbol{\Lambda}^\dagger\boldsymbol{\Lambda}\right)\mathbf{J}.$$

The symmetrized multiplication $\Re\left[\mathrm{d}U_\bullet^\intercal \mathrm{d}U_\bullet\right] = \hbar^2\,\mathrm{Re}\left[\boldsymbol{\Lambda}^\dagger\boldsymbol{\Lambda}\right]\mathrm{d}t$ results in the symmetric covariance

$$\mathrm{Re}\,\langle V_\bullet^{r\intercal}V_\bullet^s\rangle = (r\wedge s)\,\hbar^2\mathbf{J}^\intercal\,\mathrm{Re}\left(\boldsymbol{\Lambda}^\dagger\boldsymbol{\Lambda}\right)\mathbf{J}^\intercal$$

defined by $\boldsymbol{\Lambda}^\dagger\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_e^\dagger\boldsymbol{\Lambda}_e + \boldsymbol{\Lambda}_f^\dagger\boldsymbol{\Lambda}_f$. It can be parametrized as $\mathbf{F}_e^\intercal\mathbf{F}_e + \mathbf{G}$ with positive matrix

$$\mathbf{G} = \frac{\hbar^2}{4}\mathbf{C}_e\mathbf{C}_e^\intercal + \hbar^2\mathbf{J}^\intercal\,\mathrm{Re}\left(\boldsymbol{\Lambda}_f^\dagger\boldsymbol{\Lambda}_f\right)\mathbf{J} \tag{5.10}$$

implying the error-perturbation uncertainty relation

$$\Re\left[\mathrm{d}U_\bullet^\intercal \mathrm{d}U_\bullet\right] \geq \left(\mathbf{F}_e^\intercal \mathbf{F}_e + (\hbar/2)^2 \mathbf{C}_e \mathbf{C}_e^\intercal\right) \mathrm{d}t$$

in terms of the total perturbative noise (5.4) in the Langevin equation (5.3) with respect to the standard normalized error noise in estimation channel (5.6).

Let us denote the initial mean $\langle \check{x}_\bullet \rangle$ of the phase space operator vector by the component wise expectation $x_\bullet = \langle \varsigma, \check{x}_\bullet \rangle$ and covariance matrix

$$\mathbf{\Sigma} := \left(\mathrm{Re}\langle \varsigma, \check{x}_i \check{x}_k \rangle - x_i^\intercal x_k\right)$$

where $\mathrm{Re}\langle \varsigma, \check{x}_i \check{x}_k \rangle = \frac{1}{2}\langle \varsigma, \check{x}_i \check{x}_k + \check{x}_k \check{x}_i \rangle$, which is symmetric positive definite matrix $\mathbf{\Sigma} = (\Sigma_{ik})$ satisfying the Heisenberg uncertainty inequality

$$\mathbf{\Sigma} \geq \pm \frac{\mathrm{i}\hbar}{2} \mathbf{J} \tag{5.11}$$

As it was shown by Belavkin in [5],[66], and even in infinite dimensions in [28], the filtering equation (3.24) preserves the Gaussian nature of the posterior state [28], so the posterior mean $\hat{x}_\bullet^t = \langle \hat{\varsigma}_0^t, \check{x}_\bullet \rangle$ and the matrix $\mathbf{\Sigma}(t)$ of symmetric error covariances as real part of

$$\langle \hat{\varsigma}_0^t, \left(\check{x}_i - \hat{x}_i^t\right)\left(\check{x}_k - \hat{x}_k^t\right) \rangle = \langle \hat{\varsigma}_0^t, \check{x}_i \check{x}_k \rangle - \hat{x}_i^t \hat{x}_k^t$$

form a set of sufficient coordinates for the quantum LQG system which agree with the initial mean and covariance for $\hat{\varsigma}(0) = \varsigma$. Applying (3.24) to the first and second symmetrized moments of $\check{x}_\bullet$, i.e., rigorously speaking, to the spectral projectors of these unbounded operators affiliated to $\mathfrak{b} = \mathcal{B}(\mathfrak{h})$, provides posterior expectations of these sufficient coordinates for diffusive non-demolition measurement of the output operators $\boldsymbol{Y}_e^t$. These can be found as solutions to Belavkin's Kalman filter equation [44],[28] written in vector form as

$$\mathrm{d}\hat{x}_\bullet^t + (\hat{x}_\bullet^t \mathbf{A}^\intercal + \boldsymbol{u}(t)\mathbf{C}_f^\intercal)\mathrm{d}t = \mathrm{d}\hat{\boldsymbol{W}}_e^t \mathbf{K}^\intercal(t) \tag{5.12}$$

with initial condition $\hat{x}_\bullet^0 = x_\bullet$ for the posterior mean,

$$\mathbf{K}(t) = \mathbf{\Sigma}(t)\mathbf{B}_e^\intercal + \mathbf{F}_e,\ \mathrm{d}\hat{\boldsymbol{W}}_e^t = \mathrm{d}\boldsymbol{Y}_e^t - \hat{x}_\bullet^t \mathbf{B}_e^\intercal \mathrm{d}t \tag{5.13}$$

and for the symmetric error covariance we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Sigma} = \mathbf{G} - \mathbf{\Sigma}\mathbf{A}_e^\intercal - \mathbf{A}_e \mathbf{\Sigma} - \mathbf{\Sigma}\mathbf{B}_e^\intercal \mathbf{B}_e \mathbf{\Sigma}, \tag{5.14}$$

$$\mathbf{A}_e = \mathbf{A} + \mathbf{F}_e \mathbf{B}_e, \quad \mathbf{\Sigma}(\mathbf{0}) = \mathbf{\Sigma}. \tag{5.15}$$

5.2. **Quantum LQG control and microduality.** We aim to control an *output* quantum stochastic linear evolution $\boldsymbol{Z}(t) = \upsilon_0^t(\check{\boldsymbol{z}} \otimes I)$ of a d_f-dimensional linear combination $\check{\boldsymbol{z}} = \check{x}_\bullet \mathbf{E}_f^\intercal$, where $\mathbf{E}_f$ is a real,in general time dependent $d_f \times m$ matrix, represented in the Heisenberg picture as a row $(Z_i)_{i \in I_f} = X_\bullet \mathbf{E}_f^\intercal$ of $Z_i(t) = \upsilon_t(\check{z}_i)$ by forcing $\boldsymbol{Z}(t)$ to follow the classical input trajectory of the feedback controlling force $\boldsymbol{u}(t)$ whilst constraining for energy considerations a positive quadratic functional of phase space operators $X_\bullet(t) = \upsilon_t(\check{x}_\bullet \otimes I)$. Thus, our control objectives and restraints can be described by the general quadratic operator valued risk (4.1) in the canonical form

$$\check{c}(\boldsymbol{u}) = (\boldsymbol{u} - \check{\boldsymbol{z}})(\boldsymbol{u} - \check{\boldsymbol{z}})^\intercal + \check{x}_\bullet \mathbf{H} \check{x}_\bullet^\intercal \tag{5.16}$$

and $\check{s} = \check{x}_\bullet \mathbf{\Omega} \check{x}_\bullet^\intercal$ for positive real symmetric $m \times m$ matrices $\mathbf{\Omega}, \mathbf{H}$.

Since $\hat{x}_\bullet$ and $\mathbf{\Sigma}$ are generators for the full probability distribution given by the Gaussian posterior state $\hat{\varsigma}$, they form a set of sufficient coordinates, so we may consider derivations of $\mathsf{S}(t,\varsigma)$ as partial derivative of $\mathsf{S}(t,x_\bullet,\mathbf{\Sigma})$

$$\begin{array}{l} \langle\delta\varsigma,\nabla_\varsigma \mathsf{S}(t,\varsigma)\rangle \\ +\dfrac{1}{2}\langle\delta\varsigma\otimes\delta\varsigma,\nabla_\varsigma^{\otimes 2}\mathsf{S}(t,\varsigma)\rangle \end{array} = \begin{array}{l} \mathrm{d}x_\bullet\partial_\bullet^\intercal \mathsf{S}(t,x_\bullet,\mathbf{\Sigma}) + \big(\mathrm{d}\mathbf{\Sigma},\partial^{\bullet\bullet}\mathsf{S}(t,x_\bullet,\mathbf{\Sigma})\big) \\ +\dfrac{1}{2}\big(\mathrm{d}x_\bullet^\intercal\mathrm{d}x_\bullet,\partial_\bullet^\intercal\partial_\bullet\mathsf{S}(t,x_\bullet,\mathbf{\Sigma})\big). \end{array}$$

We use the notation $\big(\cdot,\cdot\big)$ to denote the matrix trace inner product $\big(\mathbf{D},\mathbf{F}\big) = \mathrm{Tr}[\mathbf{D}^\intercal\mathbf{F}]$ on the vector space of matrix configurations for the multi-dimensional system. This gives the directional derivatives along $\mathrm{d}x_j$ and $\mathrm{d}\Sigma_{ik}$ as functionals of the column $\partial_\bullet^\intercal\mathsf{S}$ of partial derivatives $\partial^i\mathsf{S} = \partial\mathsf{S}/\partial x_i$ and the matrices

$$\partial^{\bullet\bullet}\mathsf{S} = \left(\frac{\partial}{\partial\mathbf{\Sigma}_{ik}}\mathsf{S}\right), \qquad \partial_\bullet^\intercal\partial_\bullet\mathsf{S} = \left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}\mathsf{S}\right)$$

which are evaluated at $(\hat{x}_\bullet,\mathbf{\Sigma}(t))$.

Inserting this parametrization into the Bellman equation (4.13) and minimizing gives the optimal control strategy

$$\boldsymbol{u}(t) = \frac{1}{2}\partial^\bullet\mathsf{S}\left(t,\hat{x}_\bullet^t,\mathbf{\Sigma}\right)\mathbf{C} + \hat{x}_\bullet^t\mathbf{E}_f^\intercal \tag{5.17}$$

where $\mathsf{S}(t,x_\bullet,\mathbf{\Sigma})$ at $x_\bullet = \hat{x}_\bullet^t, \mathbf{\Sigma} = \mathbf{\Sigma}(t)$ now satisfies the non-linear partial differential equation

$$\begin{aligned} -\tfrac{\partial}{\partial t}\mathsf{S}(t,x_\bullet,\mathbf{\Sigma}) = \;& \tfrac{1}{2}(x_\bullet\mathbf{A}^\intercal\partial_\bullet^\intercal\mathsf{S} + \partial_\bullet\mathsf{S}\mathbf{A}x_\bullet^\intercal) + x_\bullet\mathbf{H}x_\bullet^\intercal \\ & + (\mathbf{G} - \mathbf{A\Sigma} + \mathbf{\Sigma A}^\intercal, \partial^{\bullet\bullet}\mathsf{S}) + (\mathbf{\Sigma},\mathbf{H}) \\ & - (\tfrac{1}{2}\partial_\bullet\mathsf{S}\mathbf{C} + x_\bullet\mathbf{E}_f^\intercal)(\tfrac{1}{2}\partial_\bullet\mathsf{S}\mathbf{C} + x_\bullet\mathbf{E}_f^\intercal)^\intercal \\ & + \big((\mathbf{\Sigma B}^\intercal + \mathbf{F}_e)(\mathbf{\Sigma B}^\intercal + \mathbf{F}_e)^\intercal, \tfrac{1}{2}\partial_\bullet^\intercal\partial_\bullet\mathsf{S} - \partial^{\bullet\bullet}\mathsf{S}\big) \end{aligned} \tag{5.18}$$

which is the Hamilton-Jacobi-Bellman equation for this example.

It is well known from classical control theory that LQG control has a minimum cost-to-go which is quadratic in the state, so we try the candidate solution

$$\mathsf{S}(t,x_\bullet,\mathbf{\Sigma}) = x_\bullet\mathbf{\Omega}(t)x_\bullet^\intercal + \big(\mathbf{\Omega}(t),\mathbf{\Sigma}\big) + \alpha(t)$$

in the HJB equation (5.18). This separates the HJB equation into a set of coupled ordinary differential equations and gives the optimal feedback control strategy

$$\boldsymbol{u}(t) = \hat{x}_\bullet(t)\,\mathbf{L}^\intercal(t),\; \mathbf{L}^\intercal(t) = \mathbf{\Omega}(t)\mathbf{C}_f + \mathbf{E}_f^\intercal \tag{5.19}$$

which is linear in the solution to the filtering equation $\hat{x}_\bullet^t$ at time t where $\mathbf{\Omega}(t)$ satisfies the matrix Riccati equation

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Omega} &= \mathbf{H} - \mathbf{\Omega A}_f - \mathbf{A}_f^\intercal\mathbf{\Omega} - \mathbf{\Omega C}_f\mathbf{C}_f^\intercal\mathbf{\Omega} \\ \mathbf{A}_f &= \mathbf{A} + \mathbf{C}_f\mathbf{E}_f,\;\; \mathbf{\Omega}(T) = \mathbf{\Omega} \end{aligned} \tag{5.20}$$

and $\alpha(t)$ satisfies

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) &= \Big((\mathbf{\Omega}(t)\mathbf{C}_f + \mathbf{E}_f^\intercal)(\mathbf{\Omega}(t)\mathbf{C}_f + \mathbf{E}_f^\intercal)^\intercal, \mathbf{\Sigma}(t)\Big) + (\mathbf{\Omega}(t),\mathbf{G}) \\ \alpha(T) &= 0. \end{aligned} \tag{5.21}$$

From this we obtain the total minimal cost for the control experiment

$$\mathsf{S}(0, x_\bullet, \boldsymbol{\Sigma}) = x_\bullet \boldsymbol{\Omega}_0 x_\bullet^\intercal + \mathrm{Tr}[\boldsymbol{\Omega}_0 \boldsymbol{\Sigma}] + \int_0^T \mathrm{Tr}[\boldsymbol{\Omega}(t)\mathbf{G}]\mathrm{d}t + \int_0^T \mathrm{Tr}[(\boldsymbol{\Omega}(t)\mathbf{C}_f + \mathbf{E}_f^\intercal)^\intercal \boldsymbol{\Sigma}(t)(\boldsymbol{\Omega}(t)\mathbf{C}_f + \mathbf{E}_f^\intercal)]\mathrm{d}t \tag{5.22}$$

where $\boldsymbol{\Omega}_0$ is the solution to (5.20) at time $t = 0$.

The equations (5.13)-(5.14) and (5.19)-(5.20) demonstrate the intrinsic duality between optimal quantum linear filtering and optimal classical linear control, which we call *microduality*. To make this duality more transparent let us introduce real matrices $\mathbf{E}$ and $\mathbf{F}$ defining the matrices $\mathbf{F}_e$ and $\mathbf{E}_f$ in (5.5) and (5.16) by $\mathbf{EJ} = \mathbf{F}_e^\intercal$ and $\mathbf{E}_f\mathbf{J} = \mathbf{F}^\intercal$ in the similar to $\mathbf{BJ} = \mathbf{C}_e^\intercal$ and $\mathbf{B}_f\mathbf{J} = \mathbf{C}^\intercal$ in terms of the estimation and feedback channel matrices $\mathbf{B} = \mathbf{B}_e$ and $\mathbf{C} = \mathbf{C}_f$. Then the microduality is summarized in the table

(5.23)

Filt	$\mathbf{AJ}$	$\mathbf{BJ}$	$\mathbf{EJ}$	$\mathbf{K}(T-t)$	$\mathbf{G}(T-t)$	$\boldsymbol{\Sigma}(T-t)$
Con	$\mathbf{JA}^\intercal$	$\mathbf{C}^\intercal$	$\mathbf{F}^\intercal$	$\mathbf{JL}^\intercal(t)$	$\mathbf{JH}(t)\mathbf{J}^\intercal$	$\mathbf{J}\boldsymbol{\Omega}(t)\mathbf{J}^\intercal$

in which the duality notations are made in filtering-control alphabetical order $(\mathbf{B}, \mathbf{C})$, $(\mathbf{E}, \mathbf{F})$, $(\mathbf{G}, \mathbf{H})$ and $(\mathbf{K}, \mathbf{L})$ and the matrices $\mathbf{A}$ $\mathbf{B}, \mathbf{E}$ should be also taken at $t^\intercal = T - t$ if they depend on t for the duality with $\mathbf{A}^\intercal, \mathbf{C}^\intercal, \mathbf{F}^\intercal$ evaluated at t. This duality allows us to formulate and solve the dual classical control problem given the solution to quantum filtering problem with dual parameters. The duality can be understood when we examine the nature of each of the methods used. Both methods involve the minimization of a quadratic function for linear, Gaussian systems, (i.e. the least squares error for filtering and the quadratic cost for control). The time reversal in the dual picture is explained by the interchange of the input (feedback) and the output (estimation) channels together with the linear canonical transformation given by the symplectic matrix $\mathbf{J}$. Note that $\mathbf{E} = \hbar \,\mathrm{Im}\, \boldsymbol{\Lambda}_e$, and $\mathbf{G}$ must be positive definite satisfying the relation (5.10) due to Heisenberg uncertainty relation corresponding to the error-perturbation CCR's $[\mathrm{d}\boldsymbol{W}_e^\intercal, \mathrm{d}\boldsymbol{U}_e] = \mathrm{i}\hbar \mathbf{F}_e \mathbf{C}_e^\intercal \mathrm{d}t$ in the Itô multiplication table (**??**). In order to complete this filtering-control microduality we may set $\mathbf{F}^\intercal = \hbar \,\mathrm{Im}\, \boldsymbol{\Lambda}_f \mathbf{J}$ to have the relation between $\mathbf{E}$ and $\mathbf{F}$ similar to the duality of $\mathbf{B} = 2\,\mathrm{Re}\, \boldsymbol{\Lambda}_e$ and $\mathbf{C}^\intercal = 2\,\mathrm{Re}\, \boldsymbol{\Lambda}_f^* \mathbf{J}$, and also assume that matrix $\mathbf{H}$ is also positive, satisfying

$$\mathbf{H} = (\hbar/2)^2 \,\mathbf{B}_f^\intercal \mathbf{B}_f + \hbar^2 \,\mathrm{Re}\left(\boldsymbol{\Lambda}_e^\dagger \boldsymbol{\Lambda}_e\right), \tag{5.24}$$

where $\mathbf{B}_f = 2\,\mathrm{Im}\, \boldsymbol{\Lambda}_f$. Note that although the condition (5.24) is not a requirement in this classical-quantum setting, in which only positivity of the combination $\mathbf{H} + \mathbf{E}_f^\intercal \mathbf{E}_f$ suffices, this relation may be required for the fully quantum setting when the both the input $\boldsymbol{u}$ and the output $\boldsymbol{Y}_e$ are allowed to be noncommutative, which will be studied and published elswhere.

6. Discussion

The Bellman equations for quantum systems having separate diffusive and counting measurement schemes have been derived in continuous time under a general setup. This presents original derivations of the results stated in [5] (a derivation of the diffusive case has also appeared in [43]) and allows us to reformulate the optimal control problem for a fundamentally unobservable quantum system into a classical control problem on the Banach space of observable filtered states.

The multi-dimensional quantum LQG problem which finishes the paper demonstrates the first application of the general quantum Bellman equation from which one can obtain the special cases of the Gaussian quantum oscillator [6] and quantum free particle [67, 25]. Note that the fundamental difference between this example and the corresponding well studied classical case is in the observability of the system. The quantum noises introduced act only to account for the quantum backaction due to the incompatibility of quantum events. No further restrictions on the observability or additional classical noise are introduced. As such, the corresponding classical problem (when $\hbar \to 0$) admits direct observations of $\hat{x}^t_\bullet = \check{x}^t_\bullet$ for a deterministic classical system and has an optimal direct feedback strategy $\boldsymbol{u}(t) = \hat{x}^t_\bullet \mathbf{L}(t)$ and minimal cost $\mathsf{S}(0, x_\bullet) = x_\bullet \boldsymbol{\Omega}_0 x^\intercal_\bullet$. Also this example clearly demonstrates the micro duality between quantum linear filtering and classical feedback control as a more elaborated duality involving also the linear symplectic transformation $\mathbf{J}$.

During the publication of this paper, there have appeared a number of recent works on quantum filtering and feedback control by Bouten et al. [68, 69, 70] to which we refer the interested reader for a more detailed introduction to these subjects.

7. Appendix

7.1. A. Some definitions and facts on W*-algebras.

(1) A complex Banach algebra A with involution $a \mapsto a^*$ such that $\|a^* a\| = \|a\|^2$ is called C*-algebra, and W*-algebra if it is dual to a linear subspace $\mathrm{L} \subseteq \mathrm{A}^\star$ (called preadjoint of $\mathrm{A} = \mathrm{L}^\star$ if it is closed, denoted as $\mathrm{L} = \mathrm{A}_\star$). They all can be realized as operator algebras on a complex Hilbert space $\mathcal{H}$, and an operator W*-algebra is called von Neumann algebra if its unit is the identity operator I in $\mathcal{H}$. The simplest example of W*-algebra is the von Neumann algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators acting in a complex Hilbert space $\mathcal{H}$. A von Neumann algebra $\mathcal{A}$ is called semisimple if $\mathcal{H}$ has an orthogonal decomposition into invariant subspaces $\mathcal{H}_i$ in which $\mathcal{A}$ is $\mathcal{B}(\mathcal{H}_i)$. Let $\{Q_i\}$ (or $\{\mathcal{A}_i\}$) be a family of self-adjoint operators (operator algebras $\mathcal{A}_i$) acting in $\mathcal{H}$, e.g. orthoprojectors $Q_i^2 = Q_i = Q_i^*$. The W*-algebra generated by this family is defined as the smallest weakly closed self-adjoint sub-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ containing these operators, or the spectral projectors of these operators if Q_i are unbounded in $\mathcal{H}$ (or the algebras $\mathcal{A}_i$, $\mathcal{A} = \vee \mathcal{A}_i$). It is not necessarily semisimple but in the case $I \in \mathcal{A}$ it consists of all bounded operators that commute with the bounded commutant $\mathcal{B} = \{B \in \mathcal{B}(\mathcal{H}) : BQ_i = Q_i B \quad \forall i\} \equiv \{Q_i\}'$ (or with $\mathcal{B} = \cap \mathcal{A}_i'$), i.e., it is the second commutant $\mathcal{A} = \{Q_i\}''$ of the family $\{Q_i\}$. The latter can be taken as the definition of the von Neumann algebra generated by the family $\{Q_i\}$. Note that the commutant $\mathcal{B}$ is a von Neumann algebra such that $\mathcal{B} = \mathcal{A}'$, called the commutant algebra of $\mathcal{A}$. [71].

(2) A (normal) state on a von Neumann algebra $\mathcal{A}$ is defined as a linear ultra-weakly continuous functional $\mathcal{A} \to \mathbb{C}$, satisfying the positivity and normalization conditions

$$(\varrho, Q) \geq 0, \qquad \forall Q \geqq 0, \qquad (\varrho, I) = 1 \tag{7.1}$$

[$Q \geqq 0$ signifies the nonnegative definiteness $\langle\psi|Q\psi\rangle \geqq 0\ \forall\psi \in \mathcal{H}$ called Hermitian positivity of Q]. The linear span of all *normal* states is isometric with the preadjoint space $\mathcal{A}_\star$. The latter is usually described as the space of density operators ς uniquely defined as (generalized, or affiliated) elements of the algebra $\mathcal{A}$ with respect to a *standard pairing* $(\varrho^*, Q) := \langle\varrho|Q\rangle \equiv \varrho^\star(Q)$ of $\mathcal{A}_\star$ and $\mathcal{A}$ given by the mass $\mu(\varrho) = \langle\varrho|I\rangle$ on the positive ϱ such that $\varrho^* = \varrho \geq 0$ is state density iff $\langle\varrho|I\rangle = 1$. A state ϱ is called vector state if $(\varrho, Q) = \langle\psi|Q\psi\rangle \equiv \langle\varrho_\psi|Q\rangle$ (denoted $\varrho = \varrho_\psi$) for some $\psi \in \mathcal{H}$, and pure state if it is an extreme point of the convex set $\mathcal{S}(\mathcal{A})$ of all normal states on $\mathcal{A}$. Every normal state is in the closed convex hull of vector states ϱ_ψ with $\|\psi\| = 1$ but there might be no pure state in $\mathcal{S}(\mathcal{A})$. If algebra $\mathcal{A}$ is semifinite (there exists a faithful normal semi-finite trace $Q \mapsto \mathrm{tr}Q$, then the states on $\mathcal{A}$ can be described by unit trace operators $\varrho \in \mathcal{A}$ (or $\varrho \vdash \mathcal{A}$ if the are only affiliated to $\mathcal{A}$), by means of the tracial pairing $\langle\varrho|Q\rangle = \mathrm{tr}[\varrho^* Q]$. In the simple case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ the density operator ϱ is any nuclear positive operator normalized with respect to the usual trace [71].

(3) Let $\mathcal{A}$, $\mathcal{B}$ be von Neumann algebras in respective Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$, and let $\Phi : \mathcal{B} \to \mathcal{A}$ be a linear map that transforms the operators $B \in \mathcal{B}$ into operators $A \in \mathcal{A}$ (called sometimes superoperator). The map Φ is called a *transfer map* if it is ultraweakly continuous, completely positive (CP) in the sense

$$\sum_{i,k=1}^{\infty} \langle\psi_i|\Phi(B_i^* B_k)\psi_k\rangle \geqq 0, \qquad \forall B_j, \psi_j \tag{7.2}$$

$(i = 1, \ldots, d_e < \infty)$, and unity-preserving: $\Phi(I_1) = I_0$ (or $\Phi(I_1) \leq I_0$). The CP condition is obviously satisfied if Φ is normal homomorphism (or W*-representation) $\pi : \mathcal{B} \to \mathcal{A}$, which is defined by the additional multiplicativity property $\pi(B^*B) = \pi(B)^* \pi(B)$. A composition $(\varrho_0, \Phi(B)) = \langle\varrho_0|\Phi(B)\rangle$ with any state $\varrho_0 = \varrho_0^*$ is a state ϱ_1 on $\mathcal{B}$ described by the adjoint action of the superoperator Φ on ϱ_0:

$$\langle\varrho_0|\Phi(B)\rangle = \langle\Phi^\star(\varrho_0)|B\rangle, \forall B \in \mathcal{B}, \varrho_0 \in \mathcal{A}_\star.$$

A transfer map Φ is called spatial if

$$\Phi(B) = FBF^* \quad \text{or} \quad \Phi^\star(\varrho_0) = F_\star \varrho_0 F_\star^*, \tag{7.3}$$

where F is a linear coisometric operator $\mathcal{H}_1 \to \mathcal{H}_0$, $FF^* = I_0$ (or $FI_1F^* \leq I_0$) called the *propagator* and $F_\star = F^\sharp$ is defined as left adjoint $\langle F^\sharp\varrho|Q\rangle = \langle\varrho|FQ\rangle$ with respect to the standard pairings (which is usual adjoint, $F^\sharp = F^*$, in the case of tracial pairing $\langle\varrho|Q\rangle = \mathrm{tr}[\varrho^* Q]$). Every transfer map is in the closed convex hull of spatial transfer maps, but there might be no extreme point in this hull.

(4) Let $\mathbb{V}$ be a measurable space, and $\mathfrak{B}$ its Borel σ-algebra. A mapping $\Pi : \mathrm{d}v \in \mathfrak{B} \mapsto \Pi(\mathrm{d}v)$ with values $\Pi(\mathrm{d}v)$ in ultraweakly continuous, completely positive superoperators $\mathcal{B} \to \mathcal{A}$ is called a *transfer measure* if for any $\varphi \in \mathcal{A}_\star$, $B \in \mathcal{B}$ the $\mathbb{C}$-valued function

$$\langle\Pi(\mathrm{d}v)^\star \varrho|B\rangle = \langle\varrho|\Pi(\mathrm{d}v)B\rangle$$

of the set $\mathrm{d}v \subseteq \mathbb{V}$ is a countably additive measure normalized to unity for $B = I$. In other words, $\Pi(\mathrm{d}v)$ is a CP map valued measure that is

σ-additive in the weak (strong) operator sense and for $\mathrm{d}v = \mathbb{V}$ is equal to some transfer-map Φ. In particular, $\Pi(\mathrm{d}v, B) = M(\mathrm{d}v)\Phi(B)$ with $M(\mathrm{d}v) = \Pi(\mathrm{d}v, I)$ is transfer map iff $[M(\mathrm{d}v), \Phi(B)] = 0$ for all $\mathrm{d}v \in \mathfrak{B}$ and $B \in \mathcal{B}$ as it is the case of the *nondemolition measurements* given by representations $M = E$ of $\mathfrak{B}$ in $\mathcal{A}$ and $\Phi = \pi$ in $E(\mathfrak{B})' \cap \mathcal{A}$. The quantum state transformations $\varrho \mapsto \Pi^\star(\Delta, \varrho)$ corresponding to the results $v \in \Delta$ of an *ideal measurement* are described by transfer-operator measures of the form

$$\Pi(\Delta, B) = \int_\Delta F(v) BF(v)^* \mathrm{d}v, \tag{7.4}$$

where $F(v)$ denote linear operators $\mathcal{H}_1 \to \mathcal{H}_0$, the integral with respect to a positive Borel measure λ on $\mathbb{V}$ is interpreted in strong operator topology, and $\int F(v) F(v)^* \mathrm{d}v = I_0$. Every transfer-operator $\Phi : \mathcal{B} \to \mathcal{A}$ can be represented by the integral (7.4) on $\Delta = \mathbb{V}$ of an ideal measurement $\Pi(\mathrm{d}v)$ as the compression

$$\Pi(\Delta, B) = FE(\Delta)\pi(B) F^* \tag{7.5}$$

of the nondemolition measurement $E(\Delta)\pi(B) = \hat{1}_\Delta \otimes B$ on the extended Hilbert space $\mathcal{H} = \int_{\mathbb{V}}^{\oplus} F(v)^* \mathcal{H}_0 \mathrm{d}v$ with the isometric embedding $F^*\psi_0 = \int_{\mathbb{V}}^{\oplus} F(v)^* \psi_0 \lambda(\mathrm{d}v)$ of $\psi_0 \in \mathcal{H}_0$ into $\mathcal{H}$ adjoint to the coisometry $F\psi = \int_{\mathbb{V}} F(\nu)\psi(\nu)\lambda(\mathrm{d}v)$ of $\psi \in \mathcal{H}$ into $\mathcal{H}_0$.

7.2. B. Notations of quantum stochastic calculus.

(1) Guichardet Fock space $\mathcal{F}_t^s = \Gamma(\mathcal{E}_t^s)$ over the space $\mathcal{E}_t^s = L^2(\mathrm{I}_t^s)$ of square integrable complex functions $\xi(r)$ on the interval $\mathrm{I}_t^s = [t, t+s)$ is built as the Hilbert sum $\oplus_{n=0}^\infty \mathcal{E}_n$ of the spaces $\mathcal{E}_n = \mathrm{L}^2(\Gamma_n)$ of square integrable functions $\chi_n : \tau \in \Gamma_n \mapsto \mathbb{C}$ of *finite chains* $\tau = \{t_1 < \ldots < t_n\} \subset \mathrm{I}_t^s$ for all $n = 0, 1, \ldots$ where $\chi_0(\emptyset) = c$ is a constant corresponding to only one – empty chain $\tau_0 = \emptyset$ of Γ_0, with $c = 1$ for the vacuum vector state $\chi_n = \delta_0^n$. It is $\mathrm{L}^2(\Gamma(\mathrm{I}_t^s))$, where $\Gamma(\mathrm{I}_t^s)$ is disjoint union $\sum_{n=0}^\infty \Gamma_n(\mathrm{I}_t^s)$ of the n-simplices $\Gamma_n(\mathrm{I}_t^s)$. The integration on Γ is assumed over the Lebesgues sum $\mathrm{d}\tau = \sum \mathrm{d}\tau_n$ of measures $\mathrm{d}\tau_n = \mathrm{d}t_1 \ldots \mathrm{d}t_n$ on the simplices Γ_n of chains $\tau_n : |\tau_n| = n$ with the only atom $\mathrm{d}\tau_0 = 1$ at $\Gamma_0 = \{\emptyset\}$ such that

$$\sum_{n=0}^\infty \int_{\Gamma_n} \left\| \xi^{\otimes n}(\tau_n) \right\|^2 \mathrm{d}\tau_n = \exp\left[\int_0^\infty \|\xi(t)\|^2 \mathrm{d}t\right]$$

for the exponential vector-functions $\xi^{\otimes n}(\tau_n) = \xi(t_1)\ldots\xi(t_n)$ given by a single-point function $\xi \in \mathcal{E}$. It is isomorphic to both usual Fock spaces $\oplus_{n=0}^\infty \mathcal{E}_n^\pm$ of symmetric and antisymmetric functions $\chi_n(r_1, \ldots, r_n)$ extending $\chi(\tau_n)$ on $(\mathrm{I}_t^s)^n$ with respect to the measure $(n!)^{-1} \mathrm{d}r_1 \cdots \mathrm{d}r_n$. Fock space is infinitely divisible in the multiplicative sense $\mathcal{F}_{t-r}^{r+s} \sim \mathcal{F}_{t-r}^r \otimes \mathcal{F}_t^s$ for any $r, s > 0$, which is a reflection of the additive divisibility $\mathcal{E}_{t-r}^{r+s} \sim \mathcal{E}_{t-r}^r \oplus \mathcal{E}_t^s$ of $L^2\left(\mathrm{I}_{t-r}^{r+s}\right)$. The generalization to the multiple Fock-Guichardet case over the space $\mathcal{E}_t^s = \mathrm{L}^2(\mathrm{I}_t^s \to \mathfrak{k})$ of vector-valued functions $\xi(t)$ in a Hilbert space $\mathfrak{k}$ $(= \mathbb{C}^d$, say) is straight forward and can be found in [54],[24]. All properties remain the same, and the only difference is that $\mathcal{F}_0^t$ is not L^2-space of scalar-valued functions χ on Γ but is Hilbert integral $\int_{\tau \in \Gamma(\mathrm{I}_t^s)}^{\oplus} \mathcal{K}(\tau) \mathrm{d}\tau$

of $\mathcal{K}(\tau) \sim \mathfrak{k}^{\otimes|\tau|} \otimes \mathrm{L}^2(\mathrm{I}_t^s)$, the spaces of square integrable tensor-valued functions $\chi : \Gamma_n \to \mathfrak{k}^{\otimes n}$.

(2) Four basic *integrators* A_-^+, A_-°, $A_\circ^+$ and $A_\circ^\circ$ of the universal quantum stochastic (QS) calculus [54],[24] are operator-valued measures $A_\mu^\nu(\mathrm{I})$ of preservation, annihilation, creation and exchange respectively, defining the basic QS integrals of the total QS integral as sum-integral

$$\mathfrak{i}_0^t(\boldsymbol{K}) = \sum_{\mu=-,\circ;\nu=\circ,+} \int_0^t K_\nu^\mu(r)\, A_\mu^\nu(\mathrm{d}r) \tag{7.6}$$

of four basic integrants K_+^-, $K_\circ^-$, K_+° and $K_\circ^\circ$ as measurable operator-valued functions $K_\nu^\mu(r)$ in $\mathcal{F}$ by the following explicit formulas:

$$\left[\mathfrak{i}_0^t(K)_-^+\right]\chi(\tau) = \int_0^t [K(r)\chi](\tau)\,\mathrm{d}r, \tag{7.7}$$

$$\left[\mathfrak{i}_0^t(K)_-^\circ\right]\chi(\tau) = \int_0^t [K(r)\mathring{\chi}(r)](\tau)\,\mathrm{d}r, \tag{7.8}$$

$$\left[\mathfrak{i}_0^t(K)_\circ^+\chi\right](\tau) = \sum_{r\in\tau_0^t} [K(r)\chi](\tau\backslash r), \tag{7.9}$$

$$\left[\mathfrak{i}_0^t(K)_\circ^\circ\right]\chi(\tau) = \sum_{r\in\tau_0^t} [K(r)\mathring{\chi}(r)](\tau\backslash r). \tag{7.10}$$

Here $[\mathring{\chi}(r)](\tau) = \chi(r \sqcup \tau)$, where $r \sqcup \tau$ is union of disjoint $\tau \in \Gamma$ and $r \notin \tau$, $\tau\backslash r$ is difference of τ and a singleton $r \equiv \{r\} \subseteq \tau$ and $\tau_0^t = \tau \cap \mathrm{I}_0^t$. The functions $K_\nu^\mu(r)$ should be L^p-integrable in a uniform operator topology [54],[24], with $p = 2/(\nu - \mu)$ where $- = -1$, $\circ = 0$, $+ = 1$. Note that these definitions do not assume adaptedness of integrants as they generalize Hitsuda-Skorochod extended stochastic integral. The multiple version of this explicit QS-integration is straight forward and can be found also in [54],[24], and the adapted version based on coherent vectors is in [50].

(3) Itô rule (3.9) for QS integrals $M(t) = M(0) + \mathfrak{i}_0^t(\boldsymbol{K})$ with adapted four-integrant $\boldsymbol{K}(t) = (K_\nu^\mu)(t)$ is based on the noncommutative Itô table (3.10) which uses $\star$-matrix algebra of the canonical triangular representation

$$\boldsymbol{K} = \begin{bmatrix} 0 & K_\circ^- & K_+^- \\ 0 & K_\circ^\circ & K_+^\circ \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{K}^\star = \begin{bmatrix} 0 & R_\circ^- & R_+^- \\ 0 & R_\circ^\circ & R_+^\circ \\ 0 & 0 & 0 \end{bmatrix}, \tag{7.11}$$

where $R_{-\nu}^\mu = K_{-\mu}^{\nu *}$, for any *noncommutative Itô algebra* [54],[24]. It was derived in [48] for simple bounded integrants, and therefore can not be rigorously applied for multiple integration of quantum stochastic equations except the special unitary case. In the general form presented here QS Itô formula was proved for unbounded integrants in [53] where it was also extended to nonadapted integrants, and the *functional noncommutative Itô formula* was also obtained in the pseudo-Poisson form as

$$\mathrm{d}f(M(t)) = (f(\boldsymbol{M}(t+)) - f(M(t)) \otimes \boldsymbol{I})_\nu^\mu\, A_\mu^\nu(\mathrm{d}t), \tag{7.12}$$

where $\boldsymbol{M}(t+) = M(t) \otimes \boldsymbol{I} + \boldsymbol{K}(t)$ is QS *germ* [72] of the QS integral $M(t)$ which is defined by its four QS-derivatives $K_\nu^\mu(t)$ and unite matrix $\boldsymbol{I} = (\delta_\nu^\mu)$, and the summation convention over $\mu, \nu = -, \circ, +$ is applied. Using this formula the HP differential conditions (3.12) of QS unitarity of QS interaction evolution U_t were obtained as pseudo-unitarity condition in terms of germ $\boldsymbol{U}_{t+} = U_t \otimes \boldsymbol{S}$, and also QS differential conditions of complete positivity, contractivity and projectivity were found in [72],[58] respectively as its pseudo complete positivity, pseudo contractivity and pseudoprojectivity of the corresponding QS germs $\boldsymbol{M}(t+)$.

(4) Using quantum Itô formula the general QS *evolution equation* for a quantum stochastic density operator $\hat{\varrho}(t)$ was obtained in the form of quantum stochastic Master equation [59],[60]

$$\mathrm{d}\hat{\varrho}(t) = \left(G^{\iota}_{\nu}\hat{\varrho}(t)\, G^{\star\nu}_{\kappa} - \hat{\varrho}(t)\,\delta^{\iota}_{\kappa}\right)\mathrm{d}A^{\kappa}_{\iota},\ \ \hat{\varrho}(0) = \varrho \tag{7.13}$$

Here $\boldsymbol{G} = \boldsymbol{I} + \boldsymbol{L}$ is the germ of QS evolution equation $\mathrm{d}V_t = V_t L^{\mu}_{\nu}\mathrm{d}A^{\nu}_{\mu}$ which is not assumed to be pseudo-unitary. In the case of normalization condition $S^{-}_{\iota} S^{\star\iota}_{+} = O$ in terms of left adjoint operators $S^{\mu}_{-\nu} = G^{\nu\sharp}_{-\mu}$ with respect to a standard pairing $\langle \mathfrak{b}_\star | \mathfrak{b} \rangle$, this equation is called QS *decoherence, or entangling equation* for quantum states satisfying normalization $\langle(\hat{\varrho}(t), I)\rangle_{\emptyset} = \left(\varrho, \check{1}\right)$ with respect to the pairing $(\hat{\varrho}^{*}, A)_{\emptyset} = \langle \hat{\varrho}\delta_{\emptyset} | A\delta_{\emptyset}\rangle \equiv \langle \hat{\varrho} | A\rangle_{\emptyset}$ on the noise algebra $\mathcal{A}$ given by the vacuum vector $\delta_{\emptyset} = 0^{\otimes}$. This is the most general QS equation preserving complete positivity and normalization in this mean form. Denoting $K^{-}_{\iota} = -G^{-}_{\iota} \equiv K_{\iota}$ such that $G^{\star\iota}_{+} = -K^{*}_{\iota}$, this can be written [59],[58] as

$$\mathrm{d}\hat{\varrho}(t) + 2\Re\left[K_{\iota}\hat{\varrho}(t)\,\mathrm{d}A^{\iota}_{-}\right] = \left(\sum_{j} G^{j}_{\kappa}\hat{\varrho}(t)\, G^{j*}_{\iota} - \hat{\varrho}(t)\,\delta^{\iota}_{\kappa}\right)\mathrm{d}A^{\kappa}_{\iota}.$$

More explicitly this Belavkin equation can be written in terms of $K = K_{+}$, $L^{i}_{+} = G^{i}_{+} \equiv L^{i}$ such that $G^{j*}_{+} = L^{j*}$ as

$$\begin{aligned} &\mathrm{d}\hat{\varrho}(t) + \left(2\Re\left[K\hat{\varrho}(t)\right] - \sum_{j} L^{j}\hat{\varrho}(t)\, L^{j*}\right)\mathrm{d}t \\ = \ &\sum_{k} 2\Re\left[\left(\sum_{j} G^{j}_{k}\hat{\varrho}(t)\, L^{j*} - K_{k}\hat{\varrho}(t)\right)\mathrm{d}A^{k}_{-}\right] \\ &+ \sum_{ik}\left(\sum_{j} G^{j}_{k}\hat{\varrho}(t)\, G^{j*}_{i} - \hat{\varrho}(t)\,\delta^{i}_{k}\right)\mathrm{d}A^{k}_{i}. \end{aligned}$$

The weak normalization condition can be written as $L + L^{*} + \sum_{i} L_{i}L^{*}_{i} = 0$ in terms of left adjoint L, L_{i} to $-K, L^{i}$ such that $K = -L^{\sharp}, L^{i} = L^{\sharp}_{i}$ ($K = -L^{*}, L^{i} = L^{*}_{i}$ in the case of trace pairing) for any number of i's, and arbitrary K_{i}, G^{i}_{k}, $i, k = 1, \ldots, d$. This is QS generalization of Lindblad equation [57] given by the generator (3.16) corresponding to the case $d = 0$.

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