

ENTANGLEMENT, QUANTUM ENTROPY AND MUTUAL INFORMATION

VIACHESLAV P BELAVKIN AND MASANORI OHYA

ABSTRACT. The operational structure of quantum couplings and entanglements is studied and classified for semifinite von Neumann algebras. We show that the classical-quantum correspondences such as quantum encodings can be treated as diagonal semi-classical (d-) couplings, and the entanglements characterized by truly quantum (q-) couplings, can be regarded as truly quantum encodings. The relative entropy of the d-compound and entangled states leads to two different types of entropy for a given quantum state: the von Neumann entropy, which is achieved as the maximum of mutual information over all d-entanglements, and the dimensional entropy, which is achieved at the standard entanglement – true quantum entanglement, coinciding with a d-entanglement only in the case of pure marginal states. The d- and q- information of a quantum noisy channel are respectively defined via the input d- and q- encodings, and the q-capacity of a quantum noiseless channel is found as the logarithm of the dimensionality of the input algebra. The quantum capacity may double the classical capacity, achieved as the supremum over all d-couplings, or encodings, bounded by the logarithm of the dimensionality of a maximal Abelian subalgebra.

1. INTRODUCTION

The entanglements, as specifically quantum correlations yet first considered by Schrödinger in [1], now are used to study quantum information processes, in particular, quantum computations, quantum teleportation and quantum cryptography [2, 3, 4]. There have been mathematical studies of the entanglements in [6, 5, 7, 8], in which the entangled state of two quantum systems is defined as a compound state which is not a convex combination $\sum_n \varrho_n \otimes \varsigma_n p(n)$ with some states ϱ_n and ς_n on the corresponding algebras \mathcal{A} and \mathcal{B} . However, it is obvious that there exist several types of correlated states, written as ‘separable’ forms above. Such correlated, or classically entangled states have also been discussed in several contexts in quantum probability, such as quantum measurement and filtering [9, 10], quantum compound states [11, 12] and lifting [13].

In this paper, we study the mathematical structure of classical-quantum and quantum-quantum couplings to provide a finer classification of quantum separable and entangled states. We also discuss the informational degree of entanglement and entangled quantum mutual entropy and quantum capacity. The latter are treated

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here solely as quantities arising in certain maximization problems for quantum mutual information which is generalized here for arbitrary semifinite algebras.

The term entanglement was introduced by Schrödinger in 1935 out of the need to describe correlations of quantum states not captured by mere classical statistical correlations which are always the convex combinations of noncorrelated states. In this spirit the by now standard definition [6] of the entanglement in physics is the state of a compound quantum system ‘which cannot be prepared by two separated devices with only correlated classical data as their inputs’. We show that the entangled states can be achieved by quantum (q-) encodings, the nonseparable couplings of states, in the same way as the separable states can be achieved by classical (c-) encodings.

The compound states, called o-coupled, are defined by orthogonal decompositions of their marginal states. This is a particular case of a so called diagonal (d-compound) state of a compound system which is achieved by the classical-quantum correspondences called encodings. The d-compound states as convex combination of the special product states are most informative among c-compound states, in the sense that maximum of the mutual entropy over all c-couplings of probe systems \mathcal{A} to the quantum system \mathcal{B} with a given normal state ς is achieved on the extreme d-coupled (even o-coupled) states. This maximum is the von Neumann entropy, which is bound by the rank-capacity $\ln \text{rank} \mathcal{B}$, the supremum of $S(\varsigma)$ over all ς . The rank $\text{rank} \mathcal{B}$ of the algebra \mathcal{B} is a topological characteristic of \mathcal{B} defined as the dimensionality of the maximal Abelian subalgebra $\mathcal{A} \subseteq \mathcal{B}$ (in the case of the simple \mathcal{B} it coincides with the dimensionality $\dim \mathcal{H}$ of the Hilbert space \mathcal{H} of representation for \mathcal{B}). The von Neumann capacity defined as the maximal von Neumann entropy, i.e. as the maximum $C_c = \ln \text{rank} \mathcal{B}$ of mutual entropy over all c-couplings of the classical probe systems \mathcal{A} to the quantum system \mathcal{B} , is finite only if $\text{rank} \mathcal{B} < \infty$. Due to $\dim \mathcal{B} \leq (\text{rank} \mathcal{B})^2$ (the equality is only for the simple algebras \mathcal{B}) it is achieved on the normal tracial density operator $\sigma = (\text{rank} \mathcal{B})^{-1} I$ only in the case of finite dimensional \mathcal{B} .

We prove that the truly entangled compound states are most informative, in the sense that, the maximum of the mutual entropy over all couplings including entanglements of the quantum probe systems \mathcal{A} to the quantum system \mathcal{B} is achieved on a non-separable q-compound state. It is given by the standard entanglement, an extreme entanglement of $\mathcal{A} = \tilde{\mathcal{B}}$ with the marginal state $\varrho = \tilde{\varsigma}$, where $(\tilde{\mathcal{B}}, \tilde{\varsigma})$ is the transposed (time inversed) system to (\mathcal{B}, ς) . The maximal information gained for such extreme q-compound states defines another type of entropy, the q-entropy $H(\varsigma)$, which is bigger than the von Neumann entropy $S(\varsigma)$ in the case of mixed ς . The maximum of the q-entropy $H(\varsigma)$ over all states ς defines the dimensional capacity $\ln \dim \mathcal{B}$. The dimensionality $\dim \mathcal{B}$ of the algebra \mathcal{B} is the major topological characteristic of \mathcal{B} , and it gives true quantum capacity of \mathcal{B} achieved at the standard entanglement with the maximal chaotic ς . Thus, the true quantum capacity is the maximum $C_q = \ln \dim \mathcal{B}$ of the mutual entropy over all, not only classical-quantum couplings of the probe systems \mathcal{A} to the quantum system \mathcal{B} , and it is finite only for the finite dimensional algebra \mathcal{B} . The q-entropy $H(\varsigma)$, called also the dimensional entropy, can be considered as the true quantum entropy, in contrast to the von Neumann entropy $S(\varsigma)$, called also rank-entropy, or c-entropy (semi-classical entropy) as the supremum of mutual entropy over couplings only with only classical probe systems \mathcal{A} . The capacity C_q coincides with C_c only in

the classical case of the Abelian \mathcal{B} , and it is strictly larger than the semi-classical capacity $C_c = \ln \text{rank} \mathcal{B}$ for any noiseless quantum channel. We shall show that the capacity $C_q = \ln \dim \mathcal{B}$ is achieved as the supremum of the quantum Shannon information for the noiseless channel over the entanglements as q-encodings similar to the capacity C_c which is achieved as the supremum over c-encodings described by the classical-quantum correspondences $\mathcal{A} \rightarrow \mathcal{B}$.

In this paper we consider the case of semifinite quantum systems which are described by the von Neumann algebras \mathcal{A} and \mathcal{B} with normal faithful semifinite trace. Such quantum systems include all simple quantum systems described by full operator algebras as well as all classical systems as the commutative case. The particular cases of simple and discrete decomposable algebras are considered in [14, 15].

2. PAIRINGS, COUPLINGS AND ENTANGLEMENTS

In this section we give mathematical characterization of entanglement in terms of quantum coupling which is described in terms of transpose-completely positive operations extending individual states to compound state of a composed quantum system. We show how any normal compound state can be achieved in this way, and introduce the standard entanglement as an operation giving rise to the standard entangled compound state.

Let \mathcal{H} denote the Hilbert space of a quantum system, and $\mathcal{B} = \mathcal{L}(\mathcal{H})$ be the algebra of all linear bounded operators on \mathcal{H} . Note that \mathcal{B} consists of all operators $A : \mathcal{H} \rightarrow \mathcal{H}$ having the adjoints A^\dagger on \mathcal{H} . A linear functional ς on \mathcal{B} with complex-values $\varsigma(B) \in \mathbb{C}$ is called a state on \mathcal{B} if it is positive (i.e., $\varsigma(B) \geq 0$ for any positive operator $B = A^\dagger A$ in \mathcal{B}) and normalized (i.e., $\varsigma(I) = 1$ for the identity operator I in \mathcal{A}). A *normal* state can be expressed as,

$$(2.1) \quad \varsigma(B) = \text{Tr} \varkappa^\dagger B \varkappa \equiv \langle B, \sigma \rangle, \quad B \in \mathcal{B},$$

where \varkappa is a linear Hilbert-Schmidt operator from \mathcal{H} to (another) Hilbert space \mathcal{G} , and \varkappa^\dagger is the adjoint operator from \mathcal{G} to \mathcal{H} . Here Tr stands for the usual trace in \mathcal{G} (in the case of ambiguity it will also be denoted as $\text{Tr}_{\mathcal{G}}$). This \varkappa is called the *amplitude operator* which can always be considered on $\mathcal{G} = \mathcal{H}$ as the square root of the operator $\varkappa \varkappa^\dagger$ (it is called simply the amplitude, if \mathcal{G} is the one dimensional space \mathbb{C} , $\varkappa = \eta \in \mathcal{H}$ with $\varkappa^\dagger \varkappa = \|\eta\|^2 = 1$, in which case \varkappa^\dagger is the functional η^\dagger from \mathcal{H} to \mathbb{C}).

We can always equip \mathcal{H} (and will equip all auxiliary Hilbert spaces, e.g. \mathcal{G}) with an isometric involution $J = J^\dagger$, $J^2 = I$ having the properties of complex conjugation

$$J \sum \lambda_j \eta_j = \sum \bar{\lambda}_j J \eta_j, \quad \forall \lambda_j \in \mathbb{C}, \eta_j \in \mathcal{H},$$

and denote by $\langle B, \sigma \rangle$ the tilde-pairing $\text{Tr} B \tilde{\sigma}$ of \mathcal{B} with the trace class operators $\sigma \in \mathcal{T}(\mathcal{H})$ such that $\tilde{\sigma} = J \sigma^\dagger J$. We shall call $\sigma = J \varkappa \varkappa^\dagger J = \tilde{\varkappa}^\dagger \tilde{\varkappa}$ the probability density of the state (2.1) with respect to this pairing and assume that the support E_σ of σ is the minimal projector $E = E^\dagger \in \mathcal{B}$ for which $\varsigma(E) = 1$, i.e. that $\tilde{E}_\sigma := J E_\sigma J = E_\sigma$. The latter can also be expressed as the symmetricity property $\tilde{E}_\sigma = E_\sigma$ with respect to the tilde operation (transposition) $\tilde{B} = J B^\dagger J$ on $\mathcal{L}(\mathcal{H})$. One can always assume that J is the standard complex conjugation in an eigen-representation of σ such that $\tilde{\sigma} = \varkappa \varkappa^\dagger = \tilde{\sigma}$ coincides with σ as the real element

of the invariant maximal Abelian subalgebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ of all diagonal (and thus symmetric) operators in this basis.

The auxiliary Hilbert space \mathcal{G} and the amplitude operator in (2.1) are not unique, however \varkappa is defined uniquely up to a unitary transform $\varkappa^\dagger \mapsto U\varkappa^\dagger$ in \mathcal{G} . \mathcal{G} can always be taken to be minimal by identifying it with the support $\mathcal{H}_\sigma = E_\sigma\mathcal{H}$ for σ defined as the closure of $\sigma\mathcal{H}$ (E_σ is the minimal orthoprojector in \mathcal{B} such that $\sigma E = \sigma$). In general, \mathcal{G} is not one dimensional, the dimensionality $\dim \mathcal{G}$ must not be less than $\text{rank} \varkappa^\dagger = \text{rank} \sigma$, the dimensionality of the range $\text{ran} \rho = \text{ran} \varkappa^\dagger$ of $\rho = \varkappa^\dagger \varkappa$ coinciding with the support \mathcal{G}_ρ for this $\rho \simeq \tilde{\sigma}$.

Given the amplitude operator $\varkappa : \mathcal{G} \rightarrow \mathcal{H}$, one can define not only the state ς but also the normal state,

$$(2.2) \quad \varrho(A) = \text{Tr} \tilde{\varkappa}^\dagger A \tilde{\varkappa} \equiv \langle A, \rho \rangle, \quad A \in \mathcal{A},$$

on $\mathcal{A} = \mathcal{L}(\mathcal{G})$, as the marginal of the *pure compound state*

$$\omega(A \otimes B) = \text{Tr} \tilde{A} \tilde{\varkappa}^\dagger B \varkappa = \text{Tr} \tilde{\varkappa}^\dagger A \tilde{\varkappa} \tilde{B},$$

where ω is defined on the algebra $\mathcal{A} \otimes \mathcal{B}$ of all bounded operators on the Hilbert tensor product space $\mathcal{G} \otimes \mathcal{H}$.

Indeed, the defined bilinear form, with $\tilde{A} = JA^\dagger J$, is uniquely extended to such a state given on $\mathcal{L}(\mathcal{G} \otimes \mathcal{H})$ by the amplitude $\psi = \varkappa'$, where \varkappa' is uniquely defined by $(\zeta \otimes \eta)^\dagger \varkappa' = \eta^\dagger \varkappa J \zeta$ for all $\zeta \in \mathcal{G}$, $\eta \in \mathcal{H}$.

This pure compound state ω is the so called *entangled state* [1] unless its marginal state ς (and ϱ) is pure corresponding to a rank one operator $\varkappa^\dagger = \zeta \eta^\dagger$, in which case $\omega = \varrho \otimes \varsigma$ is given by the amplitude $v = \zeta \otimes \eta$. The amplitude operator \varkappa corresponding to mixed states on \mathcal{A} and \mathcal{B} will be called the *entangling operator* of $\rho = \varkappa^\dagger \varkappa$ to $\sigma = \tilde{\varkappa}^\dagger \tilde{\varkappa}$.

As follows from the next theorem, any pure entangled state

$$\omega(A \otimes B) = \psi^\dagger (A \otimes B) \psi, \quad A \otimes B \in \mathcal{L}(\mathcal{G} \otimes \mathcal{H})$$

given by an amplitude $\psi \in \mathcal{G} \otimes \mathcal{H}$, can be described by a unique entanglement \varkappa to the algebra $\mathcal{A} = \mathcal{L}(\mathcal{G})$ of the marginal state ς on $\mathcal{B} = \mathcal{L}(\mathcal{H})$.

Before formulating this theorem in the generality required for further considerations, let us introduce the following notation.

Let \mathcal{A} be a *-algebra on \mathcal{G} with a normal, faithful, semifinite trace μ , \mathcal{A}' denote the commutant $\{A' \in \mathcal{L}(\mathcal{G}) : [A', A] = 0, \forall A \in \mathcal{A}\}$ of \mathcal{A} , and $(\tilde{\mathcal{A}}, \tilde{\mu})$ denote the transposed algebra of the operators \tilde{A} with $\tilde{\mu}(A) = \mu(\tilde{A})$, which may not coincide with (\mathcal{A}, μ) (nor with \mathcal{A}'). We can always assume that $\tilde{A} = JA^\dagger J$ with respect to an involution J on \mathcal{G} representing $\tilde{\mathcal{A}}$ on the same Hilbert space \mathcal{G} and in most cases $\tilde{\mathcal{A}} = \mathcal{A}$ and $\tilde{\mu} = \mu$ but not in the *standard representation* unless \mathcal{A} is Abelian algebra. We denote by $\mathcal{A}_\mu \subseteq \mathcal{A}$ the space of all operators $A \in \mathcal{A}$ in the form $x^\dagger z$, where $x, z \in \mathfrak{a}_\mu$, with $\mathfrak{a}_\mu = \{x \in \mathcal{A} : \mu(x^\dagger x) < \infty\}$. $(\mathcal{G}_\mu, \iota, J_\mu)$ denotes the standard representation $\iota : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{G}_\mu)$ given by the left multiplication $\iota(A)x = Ax$ on \mathfrak{a}_μ , with the standard isometric involution $J_\mu : x \mapsto x^\dagger$ defining the representation $\tilde{\iota}(\tilde{A}) = J_\mu \iota(A^\dagger) J_\mu$ of $\tilde{\mathcal{A}}$ on the completion \mathcal{G}_μ of the module \mathfrak{a}_μ with respect to the inner product $(x|z)_\mu = \mu(x^\dagger z)$. We recall that the von Neumann algebra \mathcal{A} defined by $\mathcal{A}'' = \mathcal{A}$ is anti-isomorphic to $\iota(\mathcal{A})' = J_\mu \iota(\mathcal{A}) J_\mu$

and thus $\widetilde{\mathcal{A}} \simeq \iota(\mathcal{A})'$ and that $\widetilde{\mathcal{A}} = \mathcal{A}_\mu^*$ as the space of all continuous functionals $\widetilde{\mathcal{A}} : \phi \mapsto \langle \phi, \widetilde{\mathcal{A}} \rangle_\mu$ with respect to the $*$ -norm $\|\phi\|_* = \sup\{|\mu(A\phi)| : \|A\| \leq 1\}$ on \mathcal{A}_μ and the pairing

$$\langle x^\dagger z, \widetilde{\mathcal{A}} \rangle_\mu = \mu(zAx^\dagger) = \langle A, \widetilde{x^\dagger z} \rangle_\mu, \quad x^\dagger z \in \mathcal{A}_\mu, \widetilde{\mathcal{A}} \in \widetilde{\mathcal{A}}.$$

The completion of \mathcal{A}_μ with respect to the norm $\|\cdot\|_*$ is the predual Banach space, denoted as \mathcal{A}_* (if $\mu = \tau|_{\mathcal{A}}$ is the usual trace $\tau = \text{Tr}_{\mathcal{G}}$ on \mathcal{A} , then \mathcal{A}_μ coincides with \mathcal{A}_* as the class $\mathcal{A}_\tau = \mathcal{A} \cap \mathcal{T}(\mathcal{G})$ of trace operators $\mathcal{T}(\mathcal{G}) = \{x^\dagger z : x, z \in \mathcal{S}(\mathcal{G})\}$, where $\mathcal{S}(\mathcal{G}) = \{x \in \mathcal{L}(\mathcal{G}) : \text{Tr}_{\mathcal{G}} x^\dagger x < \infty\}$).

If \mathcal{A} is not the algebra of all operators $\mathcal{L}(\mathcal{G})$, the density operator ρ for a normal state (2.2) is not unique with respect to $\tau = \text{Tr}_{\mathcal{G}}$. However it is uniquely defined as the bounded *probability density* $\rho = Jx^\dagger xJ = \bar{x}^\dagger \bar{x}$ with respect to the restriction $\mu = \tau|_{\mathcal{A}}$ (i.e. as the density operator with respect to μ) describing this state as $\langle A, \rho \rangle_\mu = \mu(xAx^\dagger)$ by the additional condition $\varkappa = \bar{x} \in \widetilde{\mathcal{A}}_\mu$. Note that each probability density $\rho \in \widetilde{\mathcal{A}}_\mu$ describing the normal state $\varrho(A) = \langle A, \rho \rangle_\mu$ on $\mathcal{A} \ni A$ is positive and normalized as $\langle I, \rho \rangle_\mu = 1$. However the predual space $\widetilde{\mathcal{A}}_*$ as the $*$ -completion of $\widetilde{\mathcal{A}}_\mu$ may consist of not only the bounded densities with respect to μ (however each $\rho \in \widetilde{\mathcal{A}}_*$ can always be approximated by the bounded $\rho_n \in \widetilde{\mathcal{A}}_\mu$).

In the following formulation \mathcal{B} can also be the more general von Neumann algebra, rather than $\mathcal{L}(\mathcal{H})$, with a normal faithful semifinite trace $\nu : \mathcal{B}_\nu \mapsto \mathbb{C}$ defining the pairing $\langle B, u^\dagger u \rangle_\nu = \nu(\tilde{u}^\dagger B \tilde{u})$, where $u \in \mathfrak{b}_\nu$ ($\mathcal{B}_\nu = \mathfrak{b}_\nu^\dagger \mathfrak{b}_\nu$ coincides with \mathcal{B}_* in the case of the standard trace $\nu(B\tilde{\sigma}) = \text{Tr} B\tilde{\sigma} = \langle B, \sigma \rangle_\nu$ when \mathfrak{b}_ν is the space of Hilbert-Schmidt operators $y \in \mathcal{B}$ and $\tilde{\mathcal{B}} = \mathcal{B}$).

Theorem 1. *Let $\omega : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathbb{C}$ be a normal compound state*

$$(2.3) \quad \omega(A \otimes B) = \tau\left(v(\widetilde{A \otimes B})v^\dagger\right) := \langle A \otimes B, v^\dagger v \rangle,$$

described by an amplitude operator $v : \mathcal{G} \otimes \mathcal{H} \rightarrow \mathcal{E} \otimes \mathcal{F}$ on the tensor product of Hilbert spaces \mathcal{E} and \mathcal{F} , satisfying the condition

$$v^\dagger v \in \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{B}}, \quad \tau(vv^\dagger) = 1,$$

where $\tau \simeq \tilde{\mu} \otimes \tilde{\nu}$ is the trace $\tau(vv^\dagger) = \langle I \otimes I, v^\dagger v \rangle$ defined in (2.3) by the pairing for $\mathcal{A} \otimes \mathcal{B}$ with respect to $\mu \otimes \nu$. Then this state is achieved by an entangling operator $\varkappa : \mathcal{G} \otimes \mathcal{F} \rightarrow \mathcal{E} \otimes \mathcal{H}$ as

$$(2.4) \quad \langle A, v(\varkappa^\dagger(I \otimes B)\varkappa) \rangle_\mu = \omega(A \otimes B) = \langle B, \mu(\tilde{\varkappa}^\dagger(A \otimes I)\tilde{\varkappa}) \rangle_\nu$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that

$$\nu(\varkappa^\dagger(I \otimes B)\varkappa) \subseteq \widetilde{\mathcal{A}}, \quad \mu(\tilde{\varkappa}^\dagger(A \otimes I)\tilde{\varkappa}) \subseteq \widetilde{\mathcal{B}}.$$

The operator \varkappa together with $\tilde{\varkappa} = J\varkappa^\dagger J$ is uniquely defined by $v = U\varkappa'$, where

$$(2.5) \quad (\xi \otimes \eta')^\dagger \varkappa'(\zeta \otimes J\eta) = (\xi \otimes \eta)^\dagger \varkappa(\zeta \otimes J\eta'), \quad \xi \in \mathcal{E}, \eta' \in \mathcal{F}, \zeta \in \mathcal{G}, \eta \in \mathcal{H},$$

up to a unitary transformation U of the minimal subspace space $\text{ran} v \subseteq \mathcal{E} \otimes \mathcal{F}$.

Proof. Without loss of generality we can assume that $\mathcal{E} = \mathcal{G}_\rho$, $\mathcal{F} = \mathcal{H}_\sigma$ and $v^\dagger = v(E_\rho \otimes E_\sigma)$ as the support $(\mathcal{G} \otimes \mathcal{H})_{v^\dagger v} = \text{ran } v^\dagger$ for $v^\dagger v$ is contained in $\mathcal{G}_\rho \otimes \mathcal{H}_\sigma$. Due to $v^\dagger v \in (\widetilde{\mathcal{A}}' \otimes \widetilde{\mathcal{B}}')'$ the range of v is invariant under the action

$$(A \otimes B)v = v(AE_\rho \otimes BE_\sigma), \quad \forall A \in \widetilde{\mathcal{A}}', B \in \widetilde{\mathcal{B}}'$$

of the commutant $(\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{B}})' = \widetilde{\mathcal{A}}' \otimes \widetilde{\mathcal{B}}'$. Let us equip \mathcal{G} and \mathcal{H} with the involutions J leaving invariant $\mathcal{G}_\rho = E_\rho \mathcal{G}$ and $\mathcal{H}_\sigma = E_\sigma \mathcal{H}$ denoting $J_\rho = E_\rho J$, $J_\sigma = E_\sigma J$, and $\mathcal{E} \otimes \mathcal{F} = \mathcal{G}_\rho \otimes \mathcal{H}_\sigma$ with the induced involution $J(\zeta \otimes \eta) = J_\rho \zeta \otimes J_\sigma \eta$. It is easy to check for such v and $\varkappa = v'$ defined by $v = \varkappa'$ in (2.5) that for any $A \in \mathcal{A}'$ and $B \in \widetilde{\mathcal{B}}'$

$$\begin{aligned} (\widetilde{A}\xi \otimes \eta)^\dagger \varkappa(\zeta \otimes \overline{B}J\eta') &= (\widetilde{A}\xi \otimes B\eta')^\dagger v(\xi \otimes J\eta) = (\xi \otimes \eta')^\dagger v(\overline{A}\xi \otimes J\widetilde{B}\eta) \\ &= (\xi \otimes \widetilde{B}\eta)^\dagger \varkappa(\overline{A}\zeta \otimes J\eta') \end{aligned}$$

where $\overline{A} = JAJ \in \widetilde{\mathcal{A}}'$, $\overline{B} = JBJ \in \mathcal{B}'$. Hence for any $B \in \mathcal{B}$

$$(A \otimes B') \varkappa^\dagger (I \otimes B) \varkappa = \varkappa^\dagger (A \otimes B'B) \varkappa = \varkappa^\dagger (I \otimes B) \varkappa (A \otimes B'),$$

where $A \in \widetilde{\mathcal{A}}'_\rho := \widetilde{\mathcal{A}}' E_\rho$, $B' \in \mathcal{B}'_\sigma := \mathcal{B}' E_\sigma$, and for any $A \in \mathcal{A}$

$$(A' \otimes B) \tilde{\varkappa}^\dagger (A \otimes I) \tilde{\varkappa} = \varkappa(A'A \otimes B) \varkappa^\dagger = \tilde{\varkappa}^\dagger (A \otimes I) \tilde{\varkappa} (A' \otimes B),$$

where $A' \in \mathcal{A}'$ and $B \in \widetilde{\mathcal{B}}'$. Thus for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$

$$\varkappa^\dagger (I \otimes B) \varkappa \in (\widetilde{\mathcal{A}}'_\rho \otimes \mathcal{B}'_\sigma)', \quad \tilde{\varkappa}^\dagger (A \otimes I) \tilde{\varkappa} \in (\mathcal{A}'_\rho \otimes \widetilde{\mathcal{B}}'_\sigma)'$$

Moreover, due to $\mathcal{A}''_\rho = E_\rho \mathcal{A} E_\rho \equiv \mathcal{A}_\rho$ and $\mathcal{B}''_\sigma = E_\sigma \mathcal{B} E_\sigma \equiv \mathcal{B}_\sigma$

$$\varkappa^\dagger (I \otimes B) \varkappa \subseteq J_\rho \mathcal{A}_\mu J_\rho \otimes E_\sigma \mathcal{B}_\nu E_\sigma := (\widetilde{\mathcal{A}}_\rho \otimes \mathcal{B}_\sigma)_{\tilde{\mu} \otimes \nu},$$

$$\tilde{\varkappa}^\dagger (A \otimes I) \tilde{\varkappa} \subseteq E_\rho \mathcal{A}_\mu E_\rho \otimes J_\sigma \mathcal{B}_\nu J_\sigma := (\mathcal{A}_\rho \otimes \widetilde{\mathcal{B}}_\sigma)_{\mu \otimes \tilde{\nu}}$$

as bounded by $\|B\| \varkappa^\dagger \varkappa$ and by $\|A\| \tilde{\varkappa}^\dagger \tilde{\varkappa}$ respectively. The partial traces ν and μ on these reduced algebras are defined as

$$(2.6) \quad \nu(\varkappa^\dagger (I \otimes B) \varkappa) = \langle B, v^\dagger v \rangle_\nu, \quad \mu(\tilde{\varkappa}^\dagger (A \otimes I) \tilde{\varkappa}) = \langle A, v^\dagger v \rangle_\mu,$$

according to $\langle A, \langle B, v^\dagger v \rangle_\nu \rangle_\mu = \langle A \otimes B, v^\dagger v \rangle = \langle B, \langle A, v^\dagger v \rangle_\mu \rangle_\nu$, where

$$\langle B, v^\dagger v \rangle_\nu = \nu(\widetilde{(I \otimes B) v^\dagger v}), \quad \langle A, v^\dagger v \rangle_\mu = \mu(\widetilde{(A \otimes I) v^\dagger v})$$

In particular

$$\nu(\varkappa^\dagger \varkappa) = \tilde{\nu}(v^\dagger v) = \rho, \quad \mu(\tilde{\varkappa}^\dagger \tilde{\varkappa}) = \tilde{\mu}(v^\dagger v) = \sigma.$$

Any other choice of v with the minimal $\mathcal{E} \otimes \mathcal{F} \simeq \mathcal{G}_\rho \otimes \mathcal{H}_\sigma$ is unitary equivalent to \varkappa' . \square

Note that the entangled state (2.3) is written in (2.4) as,

$$\langle B, \varpi(A) \rangle_\nu = \omega(A \otimes B) = \langle A, \varpi^\top(B) \rangle_\mu,$$

in terms of the mutually adjoint maps $\varpi : \mathcal{A} \rightarrow \widetilde{\mathcal{B}}_*$ and $\varpi^\top : \mathcal{B} \rightarrow \widetilde{\mathcal{A}}_*$. These maps are given in (2.6) as

$$(2.7) \quad \varpi(A) = \langle A, v^\dagger v \rangle_\mu = \widetilde{\pi^*(A)}, \quad \varpi^\top(B) = \langle B, v^\dagger v \rangle_\nu = \widetilde{\pi(B)},$$

where the linear map $\pi : \mathcal{B} \rightarrow \mathcal{A}_\mu$ and the adjoint $\pi^* : \mathcal{A} \rightarrow \mathcal{B}_\nu$ are defined as partial traces

$$\pi(B) = \nu \left((I \otimes B) \widetilde{v^\dagger v} \right), \quad \pi^*(A) = \mu \left((A \otimes I) \widetilde{v^\dagger v} \right).$$

The linear normal map ϖ in (2.6) is written in the Kraus-Stinespring form [16] and thus is completely positive (CP). It is not unital but normalized to the density operators $\sigma = \omega(I)$ with respect to the weight ν .

A linear map $\pi : \mathcal{B} \rightarrow \mathcal{A}_*$ is called *tilde-positive* if the map $\widetilde{\pi}$ defined as $\widetilde{\pi}(B) := J\pi(B)^\dagger J$ is positive for any positive (and thus Hermitian) operator $B \geq 0$ in the sense of non-negative definiteness of B . It is called *tilde-completely positive* (TCP) if the operator-matrix $\widetilde{\pi}(\mathbf{B}) = J\pi(\mathbf{B})^\dagger J$ is positive for every positive operator-matrix $\mathbf{B} = [B_{ik}] = \mathbf{B}^*$, where $\mathbf{A}^\dagger = [A_{ik}^\dagger]$, $\mathbf{B}^* = [B_{ki}^\dagger]$ (and thus $\mathbf{A}^\dagger = [A_{ki}]$ for $\mathbf{A} = [A_{ik}] \geq 0$, and $\mathbf{B}^* = \mathbf{B}$ for $\mathbf{B} \geq 0$). Obviously every tilde-positive and tilde-completely positive π is positive as positive is $\tilde{A} = JA^\dagger J$ for every positive A , but it is not necessarily completely positive unless $\tilde{A} = A$ for all $A \in \mathcal{A}$, in which case \mathcal{A} is Abelian (or the Abelian is \mathcal{B}).

The map π defined in (2.8) as a TCP \dagger -map, $\pi(B^\dagger) = \pi(B)^\dagger$, is obviously transpose-CP in the sense of positivity of $\pi(\mathbf{B})^\dagger = [\pi(B_{ki})] = \pi(\mathbf{B}^\dagger)$ for any $\mathbf{B} \geq 0$, but it is in general not CP. Because every transpose-CP map can be represented as tilde-CP: there might be a positive-definite matrix \mathbf{B} for which $\pi(\mathbf{B})$ is not positive. Note that the adjoint map $\pi^* = \widetilde{\pi}^\top$ is also TCP, as well as the maps $\widetilde{\pi} = \widetilde{\pi}$ and $\pi^\top = \widetilde{\pi}^*$, where $\widetilde{\pi}(B) = J\pi(\widetilde{B})J$, obtained from (2.6) as partial tracings

$$(2.8) \quad \widetilde{\pi}(B) = \nu \left(\varkappa^\dagger \left(I \otimes \widetilde{B} \right) \varkappa \right), \quad \pi^\top(A) = \mu \left(\tilde{\varkappa}^\dagger \left(\widetilde{A} \otimes I \right) \tilde{\varkappa} \right).$$

In these terms, the compound state (2.4) is written as,

$$\langle A | \pi(B) \rangle_\mu = \omega(A^\dagger \otimes B) = \langle \pi^*(A) | B \rangle_\nu,$$

where $\langle x | y \rangle = \langle y, \bar{x} \rangle$ defines an inner product which coincides in the case of traces with the GNS product $\langle x | y \rangle$.

In the following definition the predual space $\mathcal{B}_\top = \widetilde{\mathcal{B}}_*$ (as well as $\mathcal{A}_\top = \widetilde{\mathcal{A}}_*$) is identified by the pairing $\langle B, \sigma \rangle_\nu = \zeta(B)$ with the space of generalized density operators σ which are thus uniquely defined as selfadjoint, in general unbounded, operators in \mathcal{H} . Note that $\mathcal{B}_\top = \mathcal{B}_\nu$ if $\mathcal{B} = \widetilde{\mathcal{B}}$ and $\nu = \text{Tr}_{\mathcal{H}} = \tilde{\nu}$.

Definition 1. A TCP map $\pi : \mathcal{B} \rightarrow \mathcal{A}_*$ (or $\mathcal{B} \rightarrow \mathcal{A}_\mu \subseteq \mathcal{A}_*$) normalized as $\mu(\pi(I)) = 1$ and having an adjoint with $\pi^*(\mathcal{A}) \subseteq \mathcal{B}_*$ ($\pi^*(\mathcal{A}) \subseteq \mathcal{B}_\nu$) is called *normal coupling* (bounded coupling) of the state $\zeta = \mu \circ \pi$ on \mathcal{B} to the state $\varrho = \nu \circ \pi^*$ on \mathcal{A} . The CP map $\varpi : \mathcal{A} \rightarrow \mathcal{B}_\top$ (or $\mathcal{A} \rightarrow \widetilde{\mathcal{B}}_\nu \subseteq \mathcal{B}_\top$) normalized to the probability density $\sigma = \varpi(I)$ of ζ with $\varpi^\top(I) \in \mathcal{B}_*$ ($\varpi^\top(I) \in \widetilde{\mathcal{A}}_\mu$) will be called *normal entanglement* (bounded entanglement) of the system (\mathcal{A}, ϱ) with the probability density $\rho = \varpi^\top(I)$ to (\mathcal{B}, ζ) . The coupling π (entanglement ϖ) is called *truly quantum* if it is not CP (not TCP). The self-adjoint entanglement $\varpi_q = \varpi_q^*$ on $(\mathcal{A}, \varrho) = (\widetilde{\mathcal{B}}, \tilde{\zeta})$ (or

symmetric coupling $\pi_q = \pi_q^\top$ into $\mathcal{A}_* = \mathcal{B}_\top$) is called standard for the system (\mathcal{B}, ς) if it is given by

$$(2.9) \quad \varpi_q(A) = \sigma^{1/2} A \sigma^{1/2}, \quad \pi_q(B) = \sigma^{1/2} \tilde{B} \sigma^{1/2}.$$

Note that the standard entanglement is true as soon as the reduced algebra $\mathcal{B}_\sigma = E_\sigma \mathcal{B} E_\sigma$ on the support $\mathcal{H}_\sigma = E_\sigma \mathcal{H}$ of the state ς is not Abelian, i.e. is not one-dimensional in the case $\mathcal{B} = \mathcal{L}(\mathcal{H})$, corresponding to a pure normal ς on $\mathcal{B} = \mathcal{L}(\mathcal{H})$. Indeed, π^q restricted to \mathcal{B}_σ is the composition of the nondegenerated multiplication $\mathcal{B}_\sigma \ni B \mapsto \tilde{\sigma}^{1/2} B \tilde{\sigma}^{1/2}$ (which is CP) and the transposition $\tilde{B} = J B^\dagger J$ on \mathcal{B}_σ (which is TCP but not CP if $\dim \mathcal{H}_\sigma > 1$).

The standard entanglement in the purely quantum case $\mathcal{B} = \mathcal{B}(\mathcal{H}) = \tilde{\mathcal{B}}$, $\nu = \text{Tr} = \tilde{\nu}$ corresponds to the pure *standard compound state*

$$(2.10) \quad \text{Tr} A \sigma^{1/2} \tilde{B} \sigma^{1/2} = \omega_q(A \otimes B) = \text{Tr} B \tilde{\sigma}^{1/2} \tilde{A} \tilde{\sigma}^{1/2}$$

on the algebra $\mathcal{B} \otimes \mathcal{B}$. It is given by the amplitude $v' \simeq |\sigma^{1/2}\rangle \equiv \psi$, with $|\sigma^{1/2}\rangle^\dagger = \varkappa' \equiv \langle \sigma^{1/2}|$ defined in (2.5) as $\varkappa'(\zeta \otimes J\eta) = \eta^\dagger \varkappa \zeta$ for $\varkappa = \sigma^{1/2}$.

Any entanglement on $\mathcal{A} = \mathcal{L}(\mathcal{G})$, $\mu = \text{Tr}$ corresponding to a pure compound state is true if $\text{rank} \rho = \text{rank} \sigma$ is not one. If the space \mathcal{G} is also minimal, $\mathcal{G} = \mathcal{G}_\rho$, π^\top is unitary equivalent to the standard one π_q . Indeed, $\varpi(A) = \tilde{\varkappa}^\dagger A \tilde{\varkappa}$ can be decomposed as

$$\varpi(A) = \sigma^{1/2} U^\dagger A U \sigma^{1/2} = \varpi_q(U^\dagger A U),$$

where $U : \sigma^{1/2} \eta \mapsto \tilde{\varkappa} \eta$ is a unitary operator from \mathcal{H}_σ onto the support \mathcal{G}_ρ of $\rho = U \sigma U^\dagger$ with nonabelian $\mathcal{A}_\rho = \mathcal{L}(\mathcal{G}_\rho)$ and $\mathcal{B}_\sigma = U^\dagger \mathcal{A}_\rho U = \mathcal{L}(\mathcal{H}_\sigma)$.

Note that the compound state (2.4) with $\tilde{\varkappa} = \sigma^{1/2}$ corresponding to the standard $\varpi = \varpi_q$ can always be extended to a vector state on $\tilde{\mathcal{B}} \vee \mathcal{B}$ in the standard representation $(\mathcal{H}_\nu, \iota, J_\nu)$ of $\mathcal{B} \equiv \iota(\mathcal{B})$ when $\tilde{\mathcal{B}} = J_\nu \mathcal{B} J_\nu = \mathcal{B}'$. However it cannot be extended to a normal state on $\tilde{\mathcal{B}} \otimes \mathcal{B}$ in the case of nonatomic \mathcal{B} . If \mathcal{B} is a factor this state is pure, given in the standard representation $\tilde{\mathcal{B}} \vee \mathcal{B} = \mathcal{L}(\mathcal{H}_\nu)$ by the unit vector $y = \tilde{\sigma}^{1/2} \in \mathcal{H}_\nu$; however it is not normal on $\tilde{\mathcal{B}} \otimes \mathcal{B}$ unless \mathcal{B} is type I: $\mathcal{B} \simeq \mathcal{L}(\mathcal{H})$.

3. C-, D- AND O-COUPPLINGS AND ENCODINGS

In this section we discuss the operational meaning of couplings corresponding to different types of encodings which are treated here solely in terms of coupling maps on input of a quantum physical system. We hope that this mathematical treatment will provide a new physical insight for the corresponding asymptotic problems of quantum information.

The compound states play the role of joint input-output probability measures in classical information channels and can be pure in the quantum case, even if the marginal states are mixed. The pure compound states achieved by an entanglement of mixed input and output states exhibit new, non-classical type correlations, which are responsible for the EPR type paradoxes in the interpretation of quantum theory [6]. However, mixed, so called *separable* states on $\mathcal{A} \otimes \mathcal{B}$, defined as convex product combinations

$$\omega_c(A \otimes B) = \sum_n \varrho_n(A) \varsigma_n(B) p(n),$$

which we refer as the *c-compound states*, do not exhibit such paradoxical behavior. Here $p(n) > 0$, $\sum p_n = 1$, is a probability distribution, and $\varrho_n : \mathcal{A} \rightarrow \mathbb{C}$, $\varsigma_n : \mathcal{B} \rightarrow \mathbb{C}$

are usually normal states defined by the product densities $\rho_n \otimes \sigma_n \in \mathcal{A}_\top \otimes \mathcal{B}_\top$ of $\omega_n = \varrho_n \otimes \sigma_n$. Such compound states are achieved by *c-couplings* $\pi_c : \mathcal{B} \rightarrow \mathcal{A}_*$ given by $\pi_c = \varpi_c^\top$, where

$$\varpi_c(A) = \sum_n \varrho_n(A) \sigma_n p(n), \quad \varpi_c^\top(B) = \sum_n \varsigma_n(B) \rho_n p(n),$$

Here $\rho_n \in \mathcal{A}_*$ and $\sigma_n \in \mathcal{B}_*$ are the probability densities for ϱ_n and ς_n with respect to given traces μ and ν on \mathcal{A} and \mathcal{B} . Note that the *c-entanglement* ϖ_c , being the convex combinations of the primitive CP-TCP maps $\varpi_n(A) = \varrho_n(A) \sigma_n \in \mathcal{B}_\top$, is not truly quantum.

The separable states of the particular form

$$(3.1) \quad \omega_d(A \otimes B) = \sum_n \langle n|A|n \rangle \varsigma(n, B),$$

where $\varrho_n(A) = \langle n|A|n \rangle$ are pure states on $\mathcal{A} = \mathcal{L}(\mathcal{G}) = \tilde{\mathcal{A}}$ given by an ortho-normal system $\{|n\rangle\} \subset \mathcal{G}$, and $\varsigma(n, B) = \langle B, \sigma(n) \rangle_\nu$ with $\sigma(n) = \sigma_n p(n)$, are usually considered as the proper candidates for the input-output states in the communication channels involving the classical-quantum (c-q) encodings. Such a separable state was introduced by Ohya [11, 21] using a Schatten decomposition $\rho = \sum |n\rangle\langle n| p(n)$ of the input density operator $\rho \in \mathcal{T}(\mathcal{G})$ into the orthogonal one-dimensional projectors $\rho_n = |n\rangle\langle n|$. Here we note that such a state is the mixture of the classical-quantum correspondences $n \mapsto |n\rangle\langle n| \otimes \sigma_n$ which can be described as the composition of quantum channeling $|n\rangle\langle n| \mapsto \sigma_n$ and the errorless *encodings* $n \mapsto |n\rangle\langle n|$ in the sense that they can be inverted by the measurements $|n\rangle\langle n| \mapsto n$ as input *decodings*. We shall call such separable states *d-compound* as they are achieved by the diagonal couplings $\pi_d = \varpi_d^\top$ (*d-couplings*) to the subalgebra $\mathcal{A}_d \subseteq \mathcal{A}$ of the diagonal operators $A = \sum a(n) |n\rangle\langle n|$, where

$$(3.2) \quad \varpi_d(A) = \sum_n \langle n|A|n \rangle \sigma(n), \quad \varpi_d^\top(B) = \sum_n \varsigma(n, B) |n\rangle\langle n|$$

with respect to the standard transposition $\langle n|\tilde{A}|m \rangle = \langle m|A|n \rangle$ in the eigenbasis of ρ .

Actually Ohya obtained the compound states ω_d as the result of the composition

$$\omega_d(A \otimes B) = \omega_o(A \otimes \Lambda(B)),$$

of quantum channels as normal unital CP maps $\Lambda : \mathcal{B} \rightarrow \mathcal{A}$ and the special, *o-compound* states

$$(3.3) \quad \omega_o(A \otimes B) = \sum_n \langle n|A|n \rangle p(n) \langle n|B|n \rangle$$

corresponding to the orthogonal decompositions

$$(3.4) \quad \varpi_o(A) = \sum_n \langle n|A|n \rangle p(n) |n\rangle\langle n| = \varpi_o^\top(A)$$

such that $\varsigma_n(B) = \langle n|\Lambda(B)|n \rangle$, $\sigma_n = \Lambda^\top(|n\rangle\langle n|)$, where $\langle B, \Lambda^\top(\rho) \rangle_\nu = \text{Tr}_{\mathcal{G}} \Lambda(B) \tilde{\rho}$.

Assuming that $\langle A, \rho \rangle = \text{Tr}_{\mathcal{G}} A \tilde{\rho}$, we can extend this construction to any discretely-decomposable algebra $\mathcal{A} = \tilde{\mathcal{A}}$ on the Hilbert sum $\mathcal{G} = \oplus \mathcal{G}_i$ with invariant components \mathcal{G}_i under the standard complex conjugation J in the eigen-basis of the density operator $\tilde{\rho} = J\rho J = \rho$. In particular, the von Neumann algebra \mathcal{A} might be Abelian, as it is in the case $\tilde{A} = A$ for all $A \in \mathcal{A}$, e.g. when $\mathcal{A} = \tilde{\mathcal{A}}$ is the

diagonal algebra of pointwise multiplications $Ag = ag = \tilde{A}g$ by the bounded functions $n \mapsto a(n) \in \mathbb{C}$ on the functional Hilbert space $\mathcal{G} = \ell^2 \ni g$ with the standard complex conjugation $Jg = \bar{g}$. In this case the densities $\rho \in \mathcal{A}_*$ are given by the summable functions $p \in \ell^1$ with respect to the standard trace $\mu(\rho) = \sum p(n)$, and any compound state has the separable form with $\varrho_n(A) = a(n)$ corresponding to the Kronecker δ -densities $\rho_n \simeq \delta_n$. The normal states on the $\mathcal{A} \simeq \ell^\infty$ are described by the probability densities $p(n) \geq 0$, $\sum p(n) = 1$ with respect to the standard pairing

$$\langle A, \rho \rangle_\mu = \sum a(n) p(n), \quad p \in \ell^1, a \in \ell^\infty$$

of $\mathcal{A}_\mu = \mathcal{A}_*$ with the commutative algebra \mathcal{A} . Every normal compound state ω on $\mathcal{A} \otimes \mathcal{B}$ is defined by

$$\omega_c(A \otimes B) = \sum_n a(n) \langle B, \sigma(n) \rangle_\nu,$$

where $\sigma(n) = \sigma_n p(n)$ is the function with positive values $\sigma(n) \in \mathcal{B}_+$ normalized to the probability density $p(n) = \langle I, \sigma(n) \rangle_\nu$. Thus all normal compound states on $\ell^\infty \otimes \mathcal{B}$ are achieved by *c-couplings* $\pi_c = \varpi_c^\dagger : \mathcal{B} \rightarrow \ell^1$ with $\pi_c^\dagger = \varpi_c$ given by convex combinations of the primitive CP (and TCP) maps $\varpi_n(a) = a(n) \sigma_n \in \mathcal{B}_*$,

$$\varpi_c(A) = \sum_n a(n) \sigma(n), \quad \varpi_c^\dagger(B) = \sum_n \varsigma(n, B) \delta_n,$$

where $\varsigma(n, B) = \langle B, \sigma(n) \rangle_\nu$.

Note that any d-coupling can be regarded as quantum-classical c-coupling, achieved by the identification $a(n) = \langle n|A|n\rangle$ of $\ell^\infty \ni a$ and the reduced diagonal algebra $\mathcal{A}^0 = \{\sum |n\rangle a(n) \langle n| : A \in \mathcal{A}\}$. This simply follows from the commutativity of the density operators $\rho = \sum |n\rangle \langle n| p(n)$ for the induced states $\varrho(A) = \omega_d(A \otimes I)$ identified with $p \in \ell^1$.

In the case $\mathcal{A} = \mathcal{L}(\mathcal{G})$ and pure elementary states ω_n described by probability amplitudes $v_n = \chi_n \otimes \psi_n$, where $\tilde{\chi}_n \equiv |\chi_n\rangle \in \mathcal{G}$, $\tilde{\psi}_n \equiv |\psi_n\rangle \in \mathcal{H}$, we have density operators $\rho_n = \chi_n^\dagger \chi_n$ and $\sigma_n = \psi_n^\dagger \psi_n$ of rank one. The total compound amplitude is obviously $v = \sum |n\rangle v(n)$, where $v(n) = \chi_n \otimes \psi_n p(n)^{1/2}$ are the amplitude operators $\mathcal{G} \otimes \mathcal{H} \rightarrow \ell^2$ satisfying the orthogonality relations

$$v(n)^\dagger v(m) = \rho_n \otimes \sigma_n p(n) \delta_n^m$$

corresponding to the decomposition $v^\dagger v = \sum \rho_n \otimes \sigma_n p(n)$. The “entangling” operator for the separable state \varkappa can be chosen as either as $\varkappa = \sum |n\rangle \varkappa(n)$ or as $\varkappa = \sum \varkappa(n) \langle n|$ or even as $\varkappa = \sum |n\rangle \varkappa(n) \langle n|$ with $\varkappa(n) = \chi_n \otimes \tilde{\psi}(n)$, where $\tilde{\psi}(n) = \tilde{\psi}_n p(n)^{1/2}$. In particular a d-entangling operator \varkappa corresponding to d-encodings (3.2) is diagonal, $\varkappa = \sum |n\rangle \tilde{\psi}(n) \langle n|$ on $\mathcal{G} = \ell^2$, corresponding to the orthogonal $\tilde{\chi}_n = |n\rangle$. Thus, we have proved the Theorem 2 below in the case of pure states ς_n and ϱ_n . But, before formulating this theorem in a natural generality let us introduce the following notations.

The general c-compound states on $\mathcal{A} \otimes \mathcal{B}$ are defined as integral convex combinations

$$\omega(A \otimes B) = \int \varrho_x(A) \varsigma_x(B) p(dx)$$

given by a probability distribution p on the product-states $\varrho_x \otimes \varsigma_x$. Such compound states are achieved by convex combinations of the primitive CP (and TCP) maps

$\pi_x^\sim = \varpi_x^\top$ with $\varpi_x(A) = \varrho_x(A) \sigma_x$:

$$(3.5) \quad \varpi_c(A) = \int \varrho_x(A) \sigma_x p(\mathrm{d}x), \quad \varpi_c^\top(B) = \int \varsigma_x(B) \rho_x p(\mathrm{d}x).$$

This is always the case when the von Neumann algebra \mathcal{A} is Abelian, and thus can be identified with the diagonal algebra of multiplications $(Ag)(x) = a(x)g(x)$ by the functions $a \in L_\mu^\infty$ on the functional Hilbert space $\mathcal{G} = L_\mu^2$ with respect to a (not necessarily finite) measure μ on X . It defines trace μ on $\mathcal{A}_\mu \simeq L_\mu^1 \cap L_\mu^\infty$ as the integral $\mu(\rho) = \int p(x) \mu(\mathrm{d}x)$ for the bounded multiplication densities $(\rho g)(x) = p(x)g(x)$. The normal states on \mathcal{A} are given by the probability densities $p \in L_\mu^1$ with respect to the standard pairing

$$\langle A, \rho \rangle_\mu = \int a(x) p(x) \mu(\mathrm{d}x), \quad p \in L_\mu^1, a \in L_\mu^\infty$$

of $\mathcal{A}_* = \mathcal{A}_\top \simeq L_\mu^1$ and $\mathcal{A} = \tilde{\mathcal{A}} \simeq L_\mu^\infty$ corresponding to the trivial transposition $\tilde{a} = a$. Any normal compound state ω on $\mathcal{A} \otimes \mathcal{B} \simeq L_\mu^\infty(X \rightarrow \mathcal{B})$ is the c-compound state, defined on the diagonal algebra \mathcal{A} by

$$(3.6) \quad \omega_d(A \otimes B) = \int a(x) \varsigma(x, B) \mu(\mathrm{d}x),$$

where $\varsigma(x, B) = \langle B, \sigma(x) \rangle_\nu$ is an absolutely integrable function with density operator values $\sigma(x) = \sigma_x p(x)$ normalized to the probability density $p(x) = \langle I, \sigma(x) \rangle_\nu = \varsigma(x, I)$. It corresponds to d-couplings $\pi_d = \varpi_d^\top = \pi_d$ with $\pi_d^\top = \varpi_d$ decomposing into $\varpi(x, A) = a(x) \sigma(x)$:

$$(3.7) \quad \varpi_d(A) = \int a(x) \sigma(x) \mu(\mathrm{d}x), \quad \varpi_d^\top(B) = \int \varsigma(x, B) \delta_x \mu(\mathrm{d}x),$$

where δ_x is the (generalized) density operator of the Dirac state $\varrho_x(A) = \langle A, \delta_x \rangle_\mu = a(x)$ on the diagonal algebra \mathcal{A} .

Theorem 2. *Let $\omega_c : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathbb{C}$ be a normal c-compound state given as*

$$(3.8) \quad \omega_c(A \otimes B) = \int \mu_x(\tilde{\chi}^\dagger A \tilde{\chi}) \nu_x(\tilde{\psi}^\dagger B \tilde{\psi}) p(\mathrm{d}x),$$

where $\chi_x : \mathcal{G} \rightarrow \mathcal{E}_x$, $\psi_x : \mathcal{H} \rightarrow \mathcal{F}_x$ are linear operators having bounded transpose $\tilde{\chi} = J \chi^\dagger J$, $\tilde{\psi} = J \psi^\dagger J$ on Hilbert spaces $\mathcal{E} = \int^\oplus \mathcal{E}_x p(\mathrm{d}x)$, $\mathcal{F} = \int^\oplus \mathcal{F}_x p(\mathrm{d}x)$ with respect to pointwise involution $J = J^\dagger$. We also assume that

$$\chi_x^\dagger \chi_x \in \tilde{\mathcal{A}}, \quad \psi_x^\dagger \psi_x \in \tilde{\mathcal{B}}, \quad \mu_x(\tilde{\chi}^\dagger \tilde{\chi}) = 1 = \nu_x(\tilde{\psi}^\dagger \tilde{\psi})$$

with respect to the traces

$$(3.9) \quad \mu_x(\tilde{\chi}^\dagger \tilde{\chi}) = \langle I, \chi_x^\dagger \chi_x \rangle_\mu, \quad \nu_x(\tilde{\psi}^\dagger \tilde{\psi}) = \langle I, \psi_x^\dagger \psi_x \rangle_\nu.$$

Then this state is achieved by a decomposable entangling operator $\varkappa = \int^\oplus \chi_x \otimes \tilde{\psi}_x p(\mathrm{d}x)$ defining c-entanglement (3.5) with

$$(3.10) \quad \varrho_x(A) = \mu_x(\tilde{\chi}^\dagger A \tilde{\chi}), \quad \varsigma_x(B) = \nu_x(\tilde{\psi}^\dagger A \tilde{\psi}),$$

corresponding to the probability densities $\rho_x = \chi_x^\dagger \chi_x$, $\sigma_x = \psi_x^\dagger \psi_x$. In particular, every d-compound state (3.6) corresponding to $p(\mathrm{d}x) = p(x) \mu(\mathrm{d}x)$ with the Abelian

algebra \mathcal{A} can be achieved by the orthogonal sum of entangling operators $\varkappa_x = \delta_x \otimes \tilde{\psi}_x$ defining d -entanglement (3.7) with

$$\sigma(x) = \psi_x^\dagger \psi_x p(x), \quad \varsigma(x, B) = \nu_x \left(\tilde{\psi}^\dagger A \tilde{\psi} \right) p(x).$$

Proof. The amplitude operator $v = \int^\otimes v_x p(dx)$ corresponding to c -compound state (3.8) is defined as the orthogonal sum of $v_x = \chi_x \otimes \psi_x$ on $\mathcal{G} \otimes \mathcal{H}$ into $\int^\oplus \mathcal{E}_x \otimes \mathcal{F}_x p(dx)$. Without loss of generality we can assume that $\mathcal{E}_x = \mathcal{G}_\rho$, $\mathcal{F}_x = \mathcal{H}_\sigma$ and $v_x^\dagger = v_x (E_\rho \otimes E_\sigma)$ because the support $(\mathcal{G} \otimes \mathcal{H})_{v_x^\dagger v_x} = \text{ran } v_x^\dagger$ for

$$v_x^\dagger v_x = \chi_x^\dagger \chi_x \otimes \psi_x^\dagger \psi_x = \rho_x \otimes \sigma_x$$

is in $\mathcal{G}_\rho \otimes \mathcal{H}_\sigma$. Due to $\chi_x^\dagger \chi_x \in \tilde{\mathcal{A}}'$, $\psi_x^\dagger \psi_x \in \tilde{\mathcal{B}}'$ for almost all x , the operators χ_x and ψ_x commute with $A \in \tilde{\mathcal{A}}'$ and $B \in \tilde{\mathcal{B}}'$ respectively, and $\tilde{\psi}_x$ commutes with $B \in \mathcal{B}'$ for almost all x . Thus,

$$\tilde{\chi}_x^\dagger A \tilde{\chi}_x \in \mathcal{A}, \quad \tilde{\psi}_x B \tilde{\psi}_x \in \mathcal{B}$$

which defines the traces (3.9) on $L_p^\infty \otimes \mathcal{A}$ and $L_p^\infty \otimes \mathcal{B}$ for almost all x . The rest of the proof is a repetition of the proof of Theorem 1 for each x , with the addition that \varkappa_x is the product $v_x' = \chi_x \otimes \tilde{\psi}_x$ for each x . The total entangling operator $\varkappa : \mathcal{G} \otimes \mathcal{F} \rightarrow \mathcal{E} \otimes \mathcal{H}$ acts componentwise as $\varkappa_x(\zeta \otimes \eta) = \chi_x \zeta \otimes \tilde{\psi}_x \eta_x$.

In the case of d -compound state (3.6) one should take $\mathcal{G} = L_\mu^2$, $\mathcal{E}_x = \mathbb{C}$, and $\chi_x g = g(x)$. Thus the entangling operator in this case is given as

$$\varkappa(g \otimes \eta) = \int^\otimes g(x) \tilde{\psi}_x \eta_x \mu(dx), \quad \forall g \in L_\mu^2, \eta = \int^\oplus \eta_x \mu(dx) \in \mathcal{F}.$$

□

Note that c -entanglements ϖ_c in (3.5) are both CP and TCP and thus are not true quantum entanglements. The map $\varpi_c : \mathcal{A} \rightarrow \mathcal{B}_\top$ with Abelian algebra \mathcal{A} in (3.7) is described by a \mathcal{B}_\top -valued measure $\sigma(dx) = \sigma(x) \mu(dx)$ normalized to the input probability measure as $p(dx) = \langle I, \sigma(dx) \rangle_\nu$. This gives the concise form for the description of random classical-quantum state correspondences $x \mapsto \sigma_x$ with the given probability measure p , called *encodings* of $\sigma = \int \sigma(dx)$.

Definition 2. Let both algebras \mathcal{A} and \mathcal{B} be non-Abelian. The map $\varpi : \mathcal{A} \rightarrow \mathcal{B}_\top$ is called a c -encoding of (\mathcal{B}, ς) if it is a convex combination of the primitive maps $\sigma_n \varrho_n$ given by the probability densities $\sigma_n \in \mathcal{B}_\top$ and normal states $\varrho_n : \mathcal{A} \rightarrow \mathbb{C}$. It is called d -encoding if it has the diagonalizing form (3.2) on \mathcal{A} , and it is called o -encoding if all density operators σ_n are mutually orthogonal: $\sigma_m \sigma_n = 0$ for all $m \neq n$ as in (3.4). The entanglement which is described by non-separable CP map $\varpi : \mathcal{A} \rightarrow \mathcal{B}_\top$ will be called q -encoding.

Note that due to the commutativity of the operators $A \otimes I$ with $I \otimes B$ on $\mathcal{G} \otimes \mathcal{H}$, one can treat the encodings as nondemolition measurements [10] in \mathcal{A} with respect to \mathcal{B} . The corresponding compound state is the state prepared for such measurements on the input \mathcal{G} . It coincides with the mixture of the states corresponding to those after the measurement, without reading the message sent. The set of all d -encodings for a Schatten decomposition of the input state ρ on \mathcal{A} is obviously convex with the extreme points given by the pure output states ς_n on \mathcal{B} , corresponding to the

not necessarily orthogonal (not Schatten) decompositions $\sigma = \sum \sigma(n)$ into the one-dimensional density operators $\sigma(n) = p(n) \sigma_n$.

The Schatten decompositions $\sigma = \sum_n q(n) \sigma_n$ correspond to o-encodings, the extreme d-encodings $\sigma_n = \eta_n \eta_n^\dagger$, $p(n) = q(n)$ characterized by the orthogonality $\sigma_m \sigma_n = 0$, $m \neq n$. For each Schatten decomposition of σ they form a convex subset of d-encodings with mixed commuting σ_n .

4. QUANTUM VERSUS VON NEUMANN ENTROPY.

As we have seen in the previous section, the encodings $\varpi : \mathcal{A} \rightarrow \mathcal{B}_\tau$, which are described in (3.7) usually with a discrete Abelian \mathcal{A} , correspond to the case (3.2) when the general entanglement (2.7) is *d-encoding*, with the diagonal coupling $\pi = \varpi^\top$ in the eigen-representation of a discrete probability density ρ on non-Abelian \mathcal{A} . The true quantum entanglements with non-Abelian \mathcal{A} cannot be achieved by d-, or more generally, c-encodings even in the case of discrete \mathcal{A} . The nonseparable, true entangled states ω called in [21] *q-compound* states, can be achieved by *q-encodings*, the quantum-quantum nonseparable correspondences (2.6) which are not diagonal in the eigen-representation of ρ .

As we shall prove in this section, the self-dual standard true entanglement $\varpi_q = \varpi_q^\top$ to the probe system $(\mathcal{A}^0, \varrho_0) = (\tilde{\mathcal{B}}, \tilde{\zeta})$, which is defined in (2.9), is the most informative for a quantum system (\mathcal{B}, ς) , in the sense that it achieves the maximal mutual information in the coupled system $(\mathcal{A} \otimes \mathcal{B}, \omega)$ when $\omega = \omega_q$ is given in (2.10).

Let us consider entangled mutual information and quantum entropies of states by means of the above three types of compound states. To define the quantum mutual entropy we need to apply a quantum version of the relative entropy to compound states on the algebra $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$, called also the information divergency of the state ω with respect to a reference state φ on \mathcal{M} . The relative entropy was defined in [17, 18, 19], even for the most general von Neumann algebra \mathcal{M} , but for our purposes we need the following explicit formulation.

Let \mathcal{M} be a semi-finite algebra with normal states ω and φ having the density operator $v^\dagger v$ and $\phi \in \tilde{\mathcal{M}}$ with respect to the pairing

$$\langle M, v^\dagger v \rangle = \tau \left(v \tilde{M} v^\dagger \right), \quad M \in \mathcal{M}, v v^\dagger \in \tilde{\mathcal{M}}$$

given by a normal faithful trace τ on the transposed algebra $\tilde{\mathcal{M}} = J\mathcal{M}J$ (not necessary decomposable as $\tau = \tilde{\mu} \otimes \tilde{\nu}$ in (2.3) in the case of $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$). Then the *relative entropy* $R(\omega; \varphi)$ of the state ω with respect to φ is given by the formula

$$(4.1) \quad R(\omega; \varphi) = \tau \left(v \left(\ln v^\dagger v - \ln \phi \right) v^\dagger \right) = \tau \left(\omega \left(\ln \omega - \ln \phi \right) \right).$$

(For notational simplicity here and below we identify the state ω with its density operator $v^\dagger v$). It has a positive value $R(\omega; \varphi) \in [0, \infty]$ if the states are equally normalized, say (as usual) $\tau(\omega) = 1 = \tau(\phi)$, and it can be finite only if the state ω is absolutely continuous with respect to the reference state φ , i.e. iff $\omega(E) = 0$ for the maximal null-orthoprojector $E \in \mathcal{M}$, $E\phi = 0$. This definition does not depend on the choice of the semi-finite trace τ , and it can be extended also to the arbitrary normal ω and φ with unbounded self-adjoint density operators $v^\dagger v$ and ϕ .

The most important property of the information divergency R is its monotonicity property [17, 20], i.e. nonincrease in the divergency $R(\omega_0; \varphi_0)$ after the application

of the pre-dual of a normal completely positive unital map $K : \mathcal{M} \rightarrow \mathcal{M}^0$ to the states ω_0 and φ_0 on a von Neumann algebra \mathcal{M}^0 :

$$(4.2) \quad (\omega = \omega_0 K, \varphi = \varphi_0 K) \quad \Rightarrow \quad \mathbf{R}(\omega : \varphi) \leq \mathbf{R}(\omega_0 : \varphi_0).$$

The *mutual information* $I(\pi) = I(\pi^*)$ in a compound state ω , achieved by a coupling $\pi : \mathcal{B} \rightarrow \mathcal{A}_*$, or by $\pi^* : \mathcal{A} \rightarrow \mathcal{B}_*$ with the marginals

$$\varrho(A) = \omega(A \otimes I) = \langle A, \rho \rangle_\mu, \quad \varsigma(B) = \omega(I \otimes B) = \langle B, \sigma \rangle_\nu,$$

is defined by the relative entropy

$$(4.3) \quad I(\pi) = \tau(\omega(\ln \omega - \ln(\rho \otimes I) - \ln(I \otimes \sigma))) = \mathbf{R}(\omega : \tilde{\varrho} \otimes \varsigma)$$

of the state ω on $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$ with respect to the product state $\varphi = \varrho \otimes \varsigma$ for $\tau = \tilde{\mu} \otimes \tilde{\nu}$. This quantity, generalizing the classical mutual information corresponding to the case of Abelian \mathcal{A}, \mathcal{B} , describes an information gain in a quantum system (\mathcal{A}, ϱ) via the entanglement $\varpi^\top = \pi^\top$, or in (\mathcal{B}, ς) via an entanglement $\varpi : \mathcal{A} \rightarrow \mathcal{B}_*$. It is naturally treated as a measure of the strength of the generalized entanglement having zero value only for completely disentangled states $\omega = \varrho \otimes \varsigma$.

Proposition 1. *Let (\mathcal{A}^0, μ_0) be a quantum system with a normal faithful semifinite trace, and $\pi_0 : \mathcal{A}^0 \rightarrow \mathcal{B}_*$ be a normal coupling of the state $\varrho_0 = \nu \circ \pi_0$ on \mathcal{A}^0 to $\varsigma = \mu \circ \pi$, defining an entanglement $\varpi = \pi^{*\top}$ of (\mathcal{A}, ϱ) to (\mathcal{B}, ς) by the composition $\pi^* = \pi_0 K$ with a normal completely positive unital map $K : \mathcal{A} \rightarrow \mathcal{A}^0$. Then $I(\pi) \leq I(\pi^0)$, where $\pi^0 = \pi_0^*$. In particular, for each normal c-coupling given by (3.5) such as $\pi^\top = \varpi_c^\top$ there exists a not less informative d-coupling $\pi^0 = \varpi_0^\top$ with Abelian \mathcal{A}^0 corresponding to the encoding $\varpi_0 = \pi_0^\top$ of (\mathcal{B}, ς) , and the standard q-coupling $\pi^0 = \pi_q$, $\pi_q(B) = \sigma^{1/2} \tilde{B} \sigma^{1/2}$ to $\varrho_0 = \tilde{\varsigma}$ on $\mathcal{A}^0 = \tilde{\mathcal{B}}$ is the maximal coupling in this sense.*

Proof. The first follows from the monotonicity property (4.2) applied to the extension $K(A \otimes B) = K(A) \otimes B$ of the CP map K from $\mathcal{A} \rightarrow \mathcal{A}^0$ to $\mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}^0 \otimes \mathcal{B}$. The compound state $\omega_0(K \otimes I)$ (I denotes the identity map $\mathcal{B} \rightarrow \mathcal{B}$) is achieved by the entanglement $\varpi = \varpi_0 K$ and $\varphi_0(K \otimes I) = \varrho \otimes \varsigma$, $\varrho = \varrho_0 K$ corresponding to $\varphi_0 = \varrho_0 \otimes \varsigma$. It corresponds to the coupling $\pi = K^* \pi_0$ which is defined by $K^* : \mathcal{A}_*^0 \rightarrow \mathcal{A}_*$ as $K^* \tilde{\rho}_0 = J(K^\top \rho_0)^\dagger J$, where

$$\langle A, K^\top \rho_0 \rangle_\mu = \langle KA, \rho_0 \rangle_{\mu_0}, \quad \forall A \in \mathcal{A}, \rho_0 \in \mathcal{A}_*^0.$$

This monotonicity property proves, in particular, that for any separable compound state (3.8) on $\mathcal{A} \otimes \mathcal{B}$, which is prepared by the c-entanglement $\pi_c^\top = \varpi_c^\top$, there exists a d-entanglement $\varpi_0^\top = \pi_0$ with $(\mathcal{A}^0, \varrho_0)$ having the same, or even larger information gain (4.3). One can even take a classical system $(\mathcal{A}^0, \varrho_0)$, say the diagonal subalgebra $\mathcal{A}^0 \simeq L_p^\infty$ on $\mathcal{G}_0 = L_p^2$ with the state ϱ_0 , induced by the measure $\mu = p$, and consider the classical-quantum correspondence (encoding)

$$\varpi_0(A^0) = \int a(x) \sigma_x p(dx), \quad A^0 = \int^{\oplus} a(x) p(dx), \quad a \in L_p^\infty$$

assigning the states $\varsigma_x(B) = \langle B, \sigma_x \rangle_\nu$ to the letters x with the probabilities $p(dx)$. In this case the state ϱ is described by the density $\rho = I$, the multiplication by identity function in L_p^2 , ω is multiplication by $\sigma = (\sigma_x)$ in $L_p^2 \otimes \mathcal{H}$ and the mutual

information (4.3) is given as

$$(4.4) \quad I(\pi^0) = \int \tilde{\nu}_x(\sigma_x(\ln \sigma_x - \ln \sigma)) p(dx) = S(\sigma) - \int S(\sigma_x) p(dx),$$

where $S(\sigma) = -\tilde{\nu}(\sigma \ln \sigma)$. The achieved information gain $I(\pi^0)$ is larger than $I(\pi)$ corresponding to $\omega = \int \rho_x \otimes \sigma_x p(dx)$ because the c-entanglement ϖ_c in (3.5) is represented as the composition $\varpi_0 K$ of the encoding $\varpi_0 : \mathcal{A}^0 \rightarrow \mathcal{B}_\top$ with the CP map

$$K(A) = \int^{\oplus} \varrho_x(A) p(dx), \quad A \in \mathcal{A}$$

given by $a(x) = \varrho_x(A)$ for each $A \in \mathcal{A}$. Hence

$$\pi^*(A) = \widetilde{\varpi(A)} = \widetilde{\varpi_0 K A} = \pi_0(KA), \quad \forall A \in \mathcal{A}$$

where $\pi_0 = \widetilde{\varpi_0}$, and thus $I(\pi^0) \geq I(K^* \pi^0) = I(\pi)$, where $\pi^0 = \pi_0^* = \widetilde{\varpi_0^\top}$.

The inequality (4.2) can also be applied to the standard entanglement corresponding to the compound state (2.10) on $\widetilde{\mathcal{B}} \otimes \mathcal{B}$. Indeed, any normal entanglement $\varpi(A) = \mu(\tilde{\kappa}^\dagger(A \otimes I)\tilde{\kappa})$, on \mathcal{A} into \mathcal{B}_\top described by a CP map $\mathcal{A} \rightarrow \widetilde{\mathcal{B}}_\nu$, can be decomposed as

$$\mu(\tilde{\kappa}^\dagger(A \otimes I)\tilde{\kappa}) = \sigma^{1/2} \mu(X^\dagger(A \otimes I)X) \sigma^{1/2} = \varpi_0(KA),$$

where $KA = \mu(X^\dagger(A \otimes I)X)$ is a normal unital CP map $\mathcal{A} \rightarrow \widetilde{\mathcal{B}}$. It is uniquely given by an operator $X : \mathcal{E} \otimes \mathcal{H} \rightarrow \mathcal{G} \otimes \mathcal{F}$ with $\mathcal{E} = \mathcal{G}_\rho$, $\mathcal{H} = \mathcal{F}_\sigma$ satisfying the condition $X(I \otimes \sigma)^{1/2} = \tilde{\kappa}$, and thus $X \in \mathcal{A} \otimes \mathcal{B}'$ due to the commutativity of $\tilde{\kappa}$ with $\mathcal{A}' \otimes \mathcal{B}$ and σ with \mathcal{B} . Moreover, the partial trace μ of $X^\dagger X$ is well-defined by $\mu(\tilde{\kappa}^\dagger \tilde{\kappa}) = \sigma$ as $\mu(X^\dagger X) = I$. Thus $\varpi = \varpi_q K$ and $\pi = K^* \pi_q$, where K is a normal unital CP map $\mathcal{A} \rightarrow \widetilde{\mathcal{B}}$, and $K^* : \mathcal{B}_\top = \widetilde{\mathcal{B}}_* \rightarrow \mathcal{A}_*$. Hence the standard entanglement (coupling) (2.9) corresponds to the maximal mutual information, $I(\pi_q) \geq I(K^* \pi_q) = I(\pi)$. \square

Note that the mutual information (4.3) is written as

$$I(\pi) = S(\rho) + S(\sigma) - S(\omega/\varphi),$$

where $\varphi = \mu \otimes \nu$, $S(\rho) = S(\varrho/\mu)$, $S(\sigma) = S(\varsigma/\nu)$ and

$$(4.5) \quad S(\omega/\varphi) = -\tilde{\varphi}(v(\ln v^\dagger v) v^\dagger) \equiv -\tilde{\varphi}(v^\dagger v \ln v^\dagger v)$$

denotes the *entropy of the density operator* $v^\dagger v \in \widetilde{\mathcal{M}}$ of the state ω with respect to the trace φ on \mathcal{M} . Note that the entropy $S(\omega/\varphi)$, coinciding with $-R(\omega : \varphi)$ (cf. with (4.1) in the case $\tau = \tilde{\varphi}$), is not in general positive, and may not even be bounded from below as a function of ω . However, in the case of irreducible \mathcal{M} it can always be made positive by the choice of the standard trace $\tau = \text{Tr}$ on \mathcal{M} , in which case it is called the von Neumann entropy of the state $\omega (= v^\dagger v)$, denoted simply as $S(\omega)$:

$$(4.6) \quad S(\omega/\tau) = -\text{Tr} \omega \ln \omega \equiv S(\omega).$$

In the following we shall assume that \mathcal{B} is a discrete decomposition of the irreducible $\mathcal{B}_i = \mathcal{L}(\mathcal{H}_i) = \widetilde{\mathcal{B}}_i$ with the trace $\nu = \text{Tr}_{\mathcal{H}} = \tilde{\nu}$ induced on $\mathcal{B}_* = \mathcal{B}_\top$. The entropy $S(\sigma) = S(\varsigma/\nu)$ of the density operator σ for the normal state ς on \mathcal{B} can be found in this case as the maximal information $S(\varsigma) = \sup I(\pi_c)$ achieved via all c-encodings $\varpi : \mathcal{A} \mapsto \mathcal{B}_\top$ of the system (\mathcal{B}, ς) such that, $\varpi(I) = \sigma$, $\varpi^\top = \pi_c$. Indeed,

as follows from the proposition above, it is sufficient to find the maximum of $I(\pi)$ over all d-couplings $\pi = \varpi^\top$ mapping \mathcal{B} into Abelian \mathcal{A} with fixed $\varpi(I) = \sigma$, i.e. to find maximum of (4.4) under the condition $\int \sigma_x p(dx) = \sigma$. Due to positivity of the *d*-conditional entropy

$$(4.7) \quad S(\pi_d) = - \int \text{Tr}(\sigma_x \ln \sigma_x) p(dx) = \int S(\sigma_x) p(dx)$$

the information $I(\pi^0) = I(\pi_d)$ has the maximum $S(\sigma)$ which is achieved on an extreme d-coupling π_d^0 when almost all $S(\sigma_x)$ are zero, i.e. when almost all σ_x are one-dimensional projectors $\sigma_x^0 = P_x$ corresponding to pure states ζ_x . One can take for example, the maximal Abelian subalgebra $\mathcal{A}^0 \subseteq \mathcal{B}$ generated by $P_n = |n\rangle\langle n| \in \mathcal{B}$ for a Schatten decomposition $\sigma = \sum_n |n\rangle\langle n| p(n)$ of $\sigma \in \mathcal{B}_\tau$. The maximal value $\ln \text{rank } \mathcal{B}$ of the von Neumann entropy is defined by the dimensionality $\text{rank } \mathcal{B} = \dim \mathcal{A}^0$ of the maximal Abelian subalgebra of the decomposable algebra \mathcal{B} , i.e. by $\dim \mathcal{H}$.

However, if π is not c-coupling, the difference $S(\pi) = S(\sigma) - I(\pi)$ can achieve the negative value, and may not serve as a measure of conditional entropy in such a case.

Definition 3. *The supremum of the mutual information*

$$(4.8) \quad H(\zeta) = \sup \{I(\pi) : \mu \circ \pi = \zeta\} = I(\pi_q),$$

which is achieved on $\mathcal{A} = \tilde{\mathcal{B}}$ for a fixed state $\zeta(B) = \text{Tr}_{\mathcal{H}} B \sigma$ by the standard *q*-coupling $\pi_q(B) = \sigma^{1/2} \tilde{B} \sigma^{1/2}$, is called *q*-entropy of the state ζ . The maximum

$$S(\zeta) = \sup \{I(\pi_c) : \mu \circ \pi_c = \zeta\} = I(\pi_d^0)$$

over all c-couplings π_c corresponding to c-encodings (3.5), which is achieved on an extreme d-coupling π_d^0 , is called the *c*-entropy of the state ζ . The differences,

$$H(\pi) = H(\zeta) - I(\pi), \quad S(\pi) = S(\zeta) - I(\pi),$$

are called respectively, the *q*-conditional entropy on \mathcal{B} with respect to \mathcal{A} and the (degree of) disentanglement for the coupling $\pi : \mathcal{B} \rightarrow \mathcal{A}$. A compound state is said to be essentially entangled if $S(\pi) < 0$, and $S(\pi) \geq 0$ for a c-coupling $\pi = \pi_c$ (this is called the *c*-conditional entropy on \mathcal{B} with respect to \mathcal{A}).

Obviously, $H(\zeta)$ and $S(\zeta)$ are both positive, do not depend, unlike $S(\sigma) = S(\zeta/\nu)$, on the choice of the faithful trace ν on \mathcal{B} and obey the inequality $H(\zeta) \geq S(\zeta)$. The same is true for the conditional entropies $H(\pi)$ and $S(\pi)$, where $S(\pi)$ always has a positive value

$$S(\pi) \geq S(\pi^0) \geq 0$$

in the case of a c-coupling $\pi = \pi_c$ due to $\pi_c^* = \pi_d^* K$ for a normal unital CP map $K : \mathcal{A} \rightarrow \mathcal{A}^0$, where $\pi^0 = \pi_d$ is a d-coupling with Abelian \mathcal{A}^0 . But the disentanglement $S(\pi)$ can also achieve the negative value

$$(4.9) \quad \inf \{S(\pi) : \mu \circ \pi = \zeta\} = S(\zeta) - H(\zeta) = - \sum_i \varkappa(i) S(\sigma_i)$$

as the following theorem states in the case of discrete \mathcal{B} . Here the $\sigma_i \in \mathcal{L}(\mathcal{H}_i)$ are the density operators of the normalized factor-states $\zeta_i = \varkappa(i)^{-1} \zeta|_{\mathcal{L}(\mathcal{H}_i)}$ with $\varkappa(i) = \zeta(I^i)$, where I^i are the orthoprojectors onto \mathcal{H}_i . Note that $H(\zeta) = S(\zeta)$ if the algebra \mathcal{B} is completely decomposable, i.e. Abelian. In this case the maximal

value $\ln \text{rank } \mathcal{B}$ of $S(\varsigma)$ can be written as $\ln \dim \mathcal{B}$. The disentanglement $S(\pi)$ is always positive in this case, and $S(\pi) = H(\pi)$ as in the case of Abelian \mathcal{A} .

Theorem 3. *Let \mathcal{B} be a discrete decomposable algebra on $\mathcal{H} = \oplus_i \mathcal{H}_i$, with a normal state given by the density operator $\sigma = \oplus \sigma(i)$ with respect to the trace $\mu = \text{Tr}_{\mathcal{H}}$ on \mathcal{B} , and $\mathcal{C} \subseteq \mathcal{B}$ be its center with the state $\varkappa = \varsigma|_{\mathcal{C}}$ induced by the probability distribution $\varkappa(i) = \text{Tr } \sigma(i)$. Then the c-entropy $S(\varsigma)$ is given as the von Neumann entropy (4.6) of the density operator σ and the q-entropy (4.8) is given by the formula*

$$(4.10) \quad H(\varsigma) = \sum_i (\varkappa(i) \ln \varkappa(i) - 2 \text{Tr}_{\mathcal{H}_i} \sigma(i) \ln \sigma(i)).$$

This can be written as $H(\varsigma) = H_{\mathcal{B}|\mathcal{C}}(\varsigma) + H_{\mathcal{C}}(\varsigma)$, where $H_{\mathcal{C}}(\varsigma) = -\sum_i \varkappa(i) \ln \varkappa(i)$, and

$$H_{\mathcal{B}|\mathcal{C}}(\varsigma) = -2 \sum_i \varkappa(i) \text{Tr}_{\mathcal{H}_i} \sigma_i \ln \sigma_i = 2S_{\mathcal{B}|\mathcal{C}}(\varsigma),$$

with $\sigma_i = \sigma(i) / \varkappa(i)$. $H(\varsigma)$ is finite iff $S(\varsigma) < \infty$, and if \mathcal{B} is finite-dimensional, it is bounded, with maximal value $H(\varsigma^\circ) = \ln \dim \mathcal{B}$, achieved for $\sigma^\circ = \oplus \sigma_i^\circ \varkappa^\circ(i)$

$$\sigma_i^\circ = (\dim \mathcal{H}_i)^{-1} I^i, \quad \varkappa^\circ(i) = \dim \mathcal{B}(i) / \dim \mathcal{B},$$

where $\dim \mathcal{B}(i) = (\dim \mathcal{H}_i)^2$, $\dim \mathcal{B} = \sum_i \dim \mathcal{B}(i)$.

Proof. We have already proven that $S(\varsigma) = S(\sigma)$, where

$$S(\sigma) = -\sum_i \text{Tr}_{\mathcal{H}_i} \sigma(i) \ln \sigma(i) = S_{\mathcal{C}}(\varsigma) + S_{\mathcal{B}|\mathcal{C}}(\varsigma),$$

with $S_{\mathcal{C}}(\varsigma) = H_{\mathcal{C}}(\varsigma)$, $S_{\mathcal{B}|\mathcal{C}}(\varsigma) = \sum \varkappa(i) S(\sigma_i) = \frac{1}{2} H_{\mathcal{B}|\mathcal{C}}(\varsigma)$.

The q-entropy $H(\varsigma)$ is the supremum (4.8) of the mutual information (4.3) which is achieved on the standard entanglement, corresponding to the density operator $\omega = \oplus \omega(i, k)$ with $\omega(i, k) = \varkappa(i) |\sigma_i^{1/2}\rangle \langle \sigma_i^{1/2}| \delta_k^i$ of the standard compound state (2.10) with $\tilde{\mathcal{B}} = \mathcal{B}$, $\rho = \sigma$. Thus $H(\varsigma) = I(\pi_q)$, where

$$\begin{aligned} I(\pi_q) &= \text{Tr} \omega (\ln \omega - \ln(\sigma \otimes I) - \ln(I \otimes \sigma)) = S(\omega) - 2S(\sigma) \\ &= \sum_i \varkappa(i) \ln \varkappa(i) - 2 \text{Tr} \sigma \ln \sigma = -\sum_i \varkappa(i) (\ln \varkappa(i) + 2 \text{Tr}_{\mathcal{H}_i} \sigma_i \ln \sigma_i). \end{aligned}$$

Here we used that $\text{Tr } \omega \ln \omega = \sum_i \varkappa(i) \ln \varkappa(i)$ due to

$$\omega \ln \omega = \oplus_{i,k} \omega(i, k) \ln \omega(i, k) = \oplus_i \varkappa(i) |\sigma_i^{1/2}\rangle \langle \sigma_i^{1/2}| \ln \varkappa(i),$$

and that $\text{Tr } \sigma \ln \sigma = \sum_i \varkappa(i) (\ln \varkappa(i) - S_{\mathcal{B}_i}(\varsigma_i))$ due to

$$\sigma \ln \sigma = \oplus_i \sigma(i) \ln \sigma(i) = \oplus_i \varkappa(i) \sigma_i (\ln \varkappa(i) + \ln \sigma_i)$$

for the orthogonal decomposition $\sigma = \oplus_i \varkappa(i) \sigma_i$, where $\varkappa(i) = \text{Tr } \sigma(i)$.

Thus $H(\varsigma) = H_{\mathcal{B}|\mathcal{C}}(\varsigma) + H_{\mathcal{C}}(\varsigma) = 2S_{\mathcal{B}|\mathcal{C}}(\varsigma) + S_{\mathcal{C}}(\varsigma) \leq 2S(\varsigma)$, and it is bounded by

$$\begin{aligned} C_{\mathcal{B}} &= \sup_{\varkappa} \sum_i \varkappa(i) \left(2 \sup_{\varsigma_i} S_{\mathcal{B}(i)}(\varsigma_i) - \ln \varkappa(i) \right) \\ &= -\inf_{\varkappa} \sum_i \varkappa(i) (\ln \varkappa(i) - 2 \ln \dim \mathcal{H}_i) = \ln \dim \mathcal{B}. \end{aligned}$$

Here we used the fact that the supremum of von Neumann entropies $S(\sigma_i)$ for the simple algebras $\mathcal{B}(i) = \mathcal{L}(\mathcal{H}_i)$ with $\dim \mathcal{B}(i) = (\dim \mathcal{H}_i)^2 < \infty$ is achieved on the tracial density operators $\sigma_i = (\dim \mathcal{H}_i)^{-1} I^i \equiv \sigma_i^\circ$, and the infimum of the relative entropy

$$R(\varkappa : \varkappa^\circ) = \sum_i \varkappa(i) (\ln \varkappa(i) - \ln \varkappa^\circ(i)),$$

where $\varkappa^\circ(i) = \dim \mathcal{B}(i) / \dim \mathcal{B}$, is zero, achieved at $\varkappa = \varkappa^\circ$. \square

Note that as shown in [22] for the case of the simple algebra \mathcal{B} , the quantum entropy $H(\varsigma)$ can be also achieved as the supremum of the von Neumann entropy $S(\varrho)$ over all pure couplings given by the isometries $X : \mathcal{H} \rightarrow \mathcal{G} \otimes \mathcal{H}$, $X^\dagger X = I$ preserving the state ς . The latter means that the density operator ω of the corresponding compound states with the marginals $\rho = \text{Tr}_{\mathcal{H}} \omega$ and $\sigma = \text{Tr}_{\mathcal{G}} \omega$ is given as $\omega = X \sigma X^\dagger$.

5. QUANTUM CHANNEL AND ENTROPIC CAPACITIES

In this section we describe quantum noisy channel in terms of normal unital CP maps and their duals, and introduce an analog of Shannon information for general semifinite algebras. We consider the maximization problems for this quantity with various operational constrains on encodings, and define the entropic capacities which serve as upper bounds for the operational capacities corresponding to these constrains. The question of asymptotic equivalence of the entropic and operational capacities is not touched here.

Let \mathcal{H}_1 be a Hilbert space describing a quantum input system and \mathcal{H} describe its output Hilbert space. A quantum channel is an affine operation sending each input state defined on \mathcal{H}_1 to an output state defined on \mathcal{H} such that the mixtures of states are preserved. A deterministic quantum channel is given by a linear isometry $U : \mathcal{H}_1 \rightarrow \mathcal{H}$ with $U^\dagger U = I^1$ (I^1 is the identify operator in \mathcal{H}_1) such that each input state vector $\eta_1 \in \mathcal{H}_1$, $\|\eta_1\| = 1$, is transmitted into an output state vector $\eta = U\eta_1 \in \mathcal{H}$, $\|\eta\| = 1$. The orthogonal sums $\varsigma_1 = \oplus \varsigma_1(n)$ of pure input states $\varsigma_1(B, n) = \eta_1(n)^\dagger B \eta_1(n)$ are sent into the orthogonal sums $\varsigma = \oplus \varsigma(n)$ of pure states on $\mathcal{B} = \mathcal{L}(\mathcal{H})$ corresponding to the orthogonal state vectors $\eta(n) = U\eta_1(n)$.

A noisy quantum channel sends pure input states ς_1 on an algebra $\mathcal{B}^1 \subseteq \mathcal{L}(\mathcal{H}_1)$ into mixed ones $\varsigma = \varsigma_1 \Lambda$ given by the composition with a normal completely positive unital map $\Lambda : \mathcal{B} \rightarrow \mathcal{B}^1$. We shall assume that \mathcal{B}^1 (as well as \mathcal{B}) is equipped with a normal faithful semifinite trace ν_1 defining the pairing $\langle B, u^\dagger u \rangle_1 = \nu_1(\tilde{u}^\dagger B \tilde{u})$ of \mathcal{B}^1 and $\mathcal{B}_\top^1 = \widetilde{\mathcal{B}_*^1}$. Then the input-output state transformations are described by the transposed map $\Lambda^\top : \mathcal{B}_\top^1 \rightarrow \mathcal{B}_\top$

$$\langle B, \Lambda^\top(\sigma_1) \rangle = \langle \Lambda(B), \sigma_1 \rangle_1, \quad B \in \mathcal{B}, \sigma_1 \in \mathcal{B}_\top^1$$

defining the output density operators $\sigma = \Lambda^\top(\sigma_1)$ for any input normal state $\varsigma_1(B) = \langle B, \sigma_1 \rangle_1$. Without loss of generality the input algebra \mathcal{B}^1 can be assumed to be the smallest decomposable algebra generated by the range $\Lambda(\mathcal{B})$ of the channel map Λ (\mathcal{B}^1 is Abelian if $\Lambda(\mathcal{B})$ consists of only commuting operators on \mathcal{H}_1).

The input generalized entanglements $\varpi^1 : \mathcal{A} \rightarrow \mathcal{B}_\top^1$, including encodings of the state ς_1 with the density $\sigma_1 = \varpi^1(I)$, will be defined by the couplings $\kappa^* : \mathcal{B}^1 \rightarrow \mathcal{A}_*$ as $\varpi^1 = \kappa^*$. Here $\kappa : \mathcal{A} \rightarrow \mathcal{B}_*^1$ is a normal TCP map defining the state $\varrho = \nu_1 \circ \kappa$

of a probe system (\mathcal{A}, μ) which is entangled to $(\mathcal{B}^1, \varsigma_1)$ by $\kappa^-(A) = J\kappa(A^\dagger)J$, and the adjoint map κ^* is defined as usual by

$$\langle A|\kappa^*(B)\rangle_\mu = \omega_1(A^\dagger \otimes B) = \langle \kappa(A)|B\rangle_1, \quad \forall A \in \mathcal{A}, B \in \mathcal{B}_1,$$

where ω_1 is the corresponding compound state on $\mathcal{A} \otimes \mathcal{B}^1$.

These (generalized) entanglements describe the quantum-quantum correspondences (q-, c-, or o-encodings) of the probe systems (\mathcal{A}, ϱ) with the density operators $\rho = \kappa^\top(I^1)$, to the input $(\mathcal{B}^1, \varsigma_1)$ of the channel Λ . In particular, the most informative standard input entanglement $\varpi_q^1 : \widetilde{\mathcal{B}}^1 \rightarrow \mathcal{B}_1^1$ is the entanglement of the transposed input system $(\mathcal{A}^0, \varrho_0) = (\widetilde{\mathcal{B}}^1, \widetilde{\varsigma}_1)$ corresponding to the TCP map $\kappa_q(A) = J\sigma_1^{1/2}A^\dagger\sigma_1^{1/2}J$. In the case of discrete decomposable $\mathcal{A}^0 = \widetilde{\mathcal{B}}^1 = \mathcal{B}^1$ with the density operator $\sigma_1 = \oplus_i \sigma_1(i)$ this extreme input q-encoding defines the following density operator

$$(5.1) \quad \omega_q = (\mathbf{I} \otimes \Lambda^\top)(\omega_{q1}), \quad \omega_{q1} = \oplus_i |\sigma_1(i)^{1/2}\rangle(\sigma_1(i)^{1/2}|$$

of the input-output compound state $\omega_{q1}\Lambda$ on $\mathcal{A}^0 \otimes \mathcal{B} = \mathcal{B}^1 \otimes \mathcal{B}$.

The other extreme case of the generalized input entanglements, the pure c-encodings corresponding to (3.2), are less informative than the pure d-encodings $\varpi_d^1 = \kappa_d^1$ given by the decompositions $\kappa_d^* = \sum |n\rangle\langle n|_{\varsigma_1}(n)$ with pure states $\varsigma_1(B, n) = \eta(n)^\dagger B\eta(n)$ on \mathcal{B}_1 . They define the density operators

$$(5.2) \quad \omega_d = (\mathbf{I} \otimes \Lambda^\top)(\omega_{d1}), \quad \omega_{d1} = \sum_n |n\rangle\langle n| \otimes \eta_1(n)\eta_1(n)^\dagger,$$

of the $\mathcal{B}^1 \otimes \mathcal{B}$ -compound state $\omega_{d1}\Lambda = \omega_{d1} \circ (\mathbf{I} \otimes \Lambda)$. These are the Ohya compound states $\omega_o = \omega_{o1}\Lambda$ [11] in the case

$$\sigma_1(n) = \eta_1^o(n)\eta_1^o(n)^\dagger, \quad \eta_1^o(n)^\dagger \eta_1^o(m) = p_1(n)\delta_n^m,$$

of orthogonality of the density operators $\sigma_1(n)$ normalized to the eigen-values $p_1(n)$ of σ_1 . The o-compound states are achieved by pure o-encodings $\varpi_o^1 = \kappa_o^1$ described by the couplings $\kappa_o = \sum |n\rangle\langle n|_{\varsigma_1^o}(n)$ with ς_1^o corresponding to η_1^o . The input-output density operator

$$(5.3) \quad \omega_o = (\mathbf{I} \otimes \Lambda^\top)\omega_{o1}, \quad \omega_{o1} = \sum_n |n\rangle\langle n| \otimes \eta_1^o(n)\eta_1^o(n)^\dagger$$

of the Ohya compound state ω_o is achieved by the coupling $\lambda = \kappa^*\Lambda$ of the output (\mathcal{B}, ς) to the extreme probe system $(\mathcal{A}^0, \varrho_0) = (\mathcal{B}^1, \varsigma_1)$ as the composition of κ^* and the channel Λ .

If $K : \mathcal{A} \rightarrow \mathcal{A}^0$ is a normal completely positive unital map

$$K(A) = \text{Tr}_{\mathcal{F}_-} \widetilde{X}A\widetilde{X}^\dagger, \quad A \in \mathcal{A},$$

where X is a bounded operator $\mathcal{F}_- \otimes \mathcal{G}_0 \rightarrow \mathcal{G}$ with $\text{Tr}_{\mathcal{F}_-} X^\dagger X = I^0$, the compositions $\kappa = \kappa_0 K$, $\pi = \Lambda^* \kappa$ describe the entanglements of the probe system (\mathcal{A}, ϱ) to the channel input $(\mathcal{B}^1, \varsigma_1)$ and the output (\mathcal{B}, ς) via this channel respectively. The state $\varrho = \varrho_0 K$ is given by

$$K^\top(\rho_0) = X(I^- \otimes \rho_0)X^\dagger \in \mathcal{A}_*$$

for each density operator $\rho_0 \in \mathcal{A}_*^0$, where I^- is the identity operator in \mathcal{F}_- . The resulting entanglement $\pi = \lambda^* K$ defines the compound state $\omega = \omega_{01} \circ (K \otimes \Lambda)$ on

$\mathcal{A} \otimes \mathcal{B}$ with

$$\omega_{01} (A^0 \otimes B^1) = \text{Tr} \tilde{A}^0 \kappa_0^* (B^1) = \text{Tr} \tilde{v}_{01}^\dagger (A^0 \otimes B^1) \tilde{v}_{01}$$

on $\mathcal{A}^0 \otimes \mathcal{B}^1$. Here $v_{01} : \mathcal{G}_0 \otimes \mathcal{H}_1 \rightarrow \mathcal{F}_{01}$ is the amplitude operator uniquely defined by the input compound density operator $\omega_{01} \in \mathcal{A}_\dagger^0 \otimes \mathcal{B}_\dagger^1$ up to a unitary operator U^0 on \mathcal{F}_{01} . The effect of the input entanglement κ and the output channel Λ can be written in terms of the amplitude operator of the state ω as

$$v = (X \otimes Y) (I^- \otimes v_{01} \otimes I^+) U$$

up to a unitary operator U in $\mathcal{F} = \mathcal{F}_- \otimes \mathcal{F}_{01} \otimes \mathcal{F}_+$. Thus the density operator of the input-output compound state ω is given by $\omega_{01} (\mathbf{K} \otimes \Lambda)$ with the density

$$(5.4) \quad (\mathbf{K} \otimes \Lambda)^* (\omega_{01}) = (X \otimes Y) \omega_{01} (X \otimes Y)^\dagger,$$

where $\omega_{01} = v_{01} v_{01}^\dagger$.

Let \mathcal{K}_q^1 be the set of all normal TCP maps $\kappa : \mathcal{A} \rightarrow \mathcal{B}_*^1$ with any probe algebra \mathcal{A} normalized as $\text{Tr} \kappa(I) = 1$ and $\mathcal{K}_q(\varsigma_1)$ be the subset of all $\kappa \in \mathcal{K}_q^1$ with $\kappa(I) = \varsigma_1$. Each $\kappa \in \mathcal{K}_q^1$ can be decomposed as $\kappa_q \mathbf{K}$, where $\kappa_q : \mathcal{A}^0 \rightarrow \mathcal{B}^1$ defines the standard input entanglement $\varpi_q^1 = \kappa_q^\sim$, and \mathbf{K} is a normal unital CP map $\mathcal{A} \rightarrow \widetilde{\mathcal{B}}^1$.

Further let \mathcal{K}_c^1 be the set of all CP-TCP maps κ described as the combinations

$$(5.5) \quad \kappa(A) = \sum_n \varrho_n(A) \sigma_1(n)$$

of the primitive maps $A \mapsto \varrho_n(A) \sigma_1(n)$, and \mathcal{K}_d^1 be the subset of the diagonalizing entanglements κ , i.e. the decompositions

$$(5.6) \quad \kappa(A) = \sum_n \langle n|A|n \rangle \sigma_1(n).$$

As in the first case $\mathcal{K}_c(\varsigma_1)$ and $\mathcal{K}_d(\varsigma_1)$ denote the subsets corresponding to a fixed $\kappa(I) = \varsigma_1$. Each $\mathcal{K}_c(\varsigma_1)$ can be represented as the composition $\kappa = \kappa_d \mathbf{K}$, where κ_d normalized to ς_1 describes a pure d-encoding $\varpi_d^1 = \kappa_d^\sim$ of $(\mathcal{B}^1, \varsigma_1)$ for a proper choice of the CP map $\mathbf{K} : \mathcal{A} \rightarrow \mathcal{B}^1$.

Furthermore let \mathcal{K}_o^1 (and $\mathcal{K}_o(\varsigma_1)$) be the subset of all decompositions (5.6) with orthogonal $\sigma_1(n)$ (and fixed $\sum_n \sigma_1(n) = \sigma_1$):

$$\sigma_1(m) \sigma_1(n) = 0, \quad m \neq n.$$

Each $\kappa \in \mathcal{K}_o(\varsigma_1)$ can also be represented as $\kappa = \kappa_o \mathbf{K}$, with κ_o describing the pure o-encoding $\varpi_o^1 = \kappa_o^\sim$ of $(\mathcal{B}^1, \varsigma_1) = (\mathcal{A}^0, \varrho_0)$.

Now, let us maximize the entangled mutual entropy for a given quantum channel Λ (and a fixed input state ς_1 on the decomposable $\mathcal{B}^1 = \widetilde{\mathcal{B}}^1$) by means of the above four types of entanglement κ . The mutual information (4.3) was defined in the previous section by the density operators of the corresponding compound state ω on $\mathcal{A} \otimes \mathcal{B}$ and the product-state $\varphi = \varrho \otimes \varsigma$ of the marginals ϱ, ς for ω . In each case

$$\omega = \omega_{01} (\mathbf{K} \otimes \Lambda), \quad \varphi = \varphi_{01} (\mathbf{K} \otimes \Lambda),$$

where \mathbf{K} is a CP map $\mathcal{A} \rightarrow \mathcal{A}^0 = \mathcal{B}^1$, ω_{01} is one of the corresponding extreme compound states $\omega_{q1}, \omega_{c1} = \omega_{d1}, \omega_{o1}$ on $\mathcal{B}^1 \otimes \mathcal{B}^1$, and $\varphi_{01} = \rho_0 \otimes \varsigma_1$. The density operator $\omega = (\mathbf{K} \otimes \Lambda)^\dagger (\omega_{01})$ is written in (5.4), and $\phi = \rho \otimes \sigma$ can be written as

$$\phi = \kappa^\dagger(I) \otimes \lambda^\dagger(I),$$

where $\lambda^\dagger = \Lambda^\dagger \pi_1^0$. This proves the following proposition.

Proposition 2. *The entangled mutual informations achieve the following maximal values*

$$(5.7) \quad \sup_{\kappa \in \mathcal{K}_q(\varsigma_1)} I(\kappa^* \Lambda) = I_q(\varsigma_1, \Lambda) := I(\kappa_q^* \Lambda),$$

$$I_c(\varsigma_1, \Lambda) := \sup_{\kappa \in \mathcal{K}_c(\varsigma_1)} I(\kappa^* \Lambda) = \sup_{\kappa_d} I(\kappa_d^* \Lambda) \equiv I_d(\varsigma_1, \Lambda),$$

$$(5.8) \quad \sup_{\kappa \in \mathcal{K}_o(\varsigma_1)} I(\kappa^* \Lambda) = I_o(\varsigma_1, \Lambda) := \sup_{\kappa_o} I(\kappa_o^* \Lambda),$$

where κ . are the corresponding extremal input couplings $\mathcal{A}^0 \rightarrow \mathcal{B}_*^1$ with $\mu \circ \kappa^* = \varsigma_1$. They are ordered as

$$(5.9) \quad I_q(\varsigma_1, \Lambda) \geq I_c(\varsigma_1, \Lambda) = I_d(\varsigma_1, \Lambda) \geq I_o(\varsigma_1, \Lambda).$$

In the following definition the maximal information $I_c(\varsigma_1, \Lambda) = I_d(\varsigma_1, \Lambda)$ is simply denoted as $I_1(\varsigma_1, \Lambda)$.

Definition 4. *The suprema*

$$(5.10) \quad C_q(\Lambda) = \sup_{\kappa \in \mathcal{K}_q^1} I(\kappa^* \Lambda) = \sup_{\varsigma_1} I_q(\varsigma_1, \Lambda),$$

$$\sup_{\kappa \in \mathcal{K}_d^1} I(\kappa^* \Lambda) = C_1(\Lambda) := \sup_{\varsigma_1} I_1(\varsigma_1, \Lambda),$$

$$C_o(\Lambda) = \sup_{\kappa \in \mathcal{K}_o^1} I(\kappa^* \Lambda) = \sup_{\varsigma_1} I_o(\varsigma_1, \Lambda),$$

are called the q -, c - or d -, and o -capacities respectively for the quantum channel defined by a normal unital CP map $\Lambda : \mathcal{B} \rightarrow \mathcal{B}^1$.

Obviously, the capacities (5.10) satisfy the inequalities

$$C_o(\Lambda) \leq C_1(\Lambda) \leq C_q(\Lambda).$$

Theorem 4. *Let $\Lambda(B) = U^\dagger B U$ be a unital CP map $\mathcal{B} \rightarrow \mathcal{B}^1$ describing a quantum deterministic channel. Then*

$$I_1(\varsigma_1, \Lambda) = I_o(\varsigma_1, \Lambda) = S(\varsigma_1), \quad I_q(\varsigma_1, \Lambda) = S_q(\varsigma_1),$$

where $S_q(\varsigma_1) = H(\varsigma_1)$, and thus in this case

$$C_1(\Lambda) = C_o(\Lambda) = \ln \text{rank } \mathcal{B}^1, \quad C_q(\Lambda) = \ln \dim \mathcal{B}^1.$$

Proof. It was proved in the previous section for the case of the identity channel $\Lambda = I$ and is thus also valid for any isomorphism $\Lambda : B \mapsto U^\dagger B U$ describing the state transformations $\Lambda^\top : \sigma \mapsto Y \sigma Y^\dagger$ by a unitary operator $U = \bar{Y}$. In the case of non-unitary Y we can use the identity

$$\text{Tr } Y (\sigma_1 \otimes I^+) Y^\dagger \ln Y (\sigma_1 \otimes I^+) Y^\dagger = \text{Tr } S (\sigma_1 \otimes I^+) \ln S (\sigma_1 \otimes I^+),$$

where $S = Y^\dagger Y$. Due to this $S(\varsigma_1 \Lambda) = -\text{Tr } S (\sigma_1 \otimes I^+) \ln S (\sigma_1 \otimes I^+)$, and $S(\omega_{01}(\mathbf{K} \otimes \Lambda)) =$

$$-\text{Tr } (R \otimes S) (I^- \otimes \omega_{01} \otimes I^+) \ln (R \otimes S) (I^- \otimes \omega_{01} \otimes I^+),$$

where $R = X^\dagger X$. Thus $S(\varsigma_1 \Lambda) = S(\varsigma_1)$, $S(\omega_{01}(\mathbf{K} \otimes \Lambda)) = S(\omega_{01}(\mathbf{K} \otimes I))$ if $Y^\dagger Y = I$, and

$$I((\pi_1 \Lambda)) = S(\varrho_0 \mathbf{K}) + S(\varsigma_1) - S(\omega_{01}(\mathbf{K} \otimes I))$$

$$\leq S(\varrho_0) + S(\varsigma_1) - S(\omega_{01}) = I(\omega_{01})$$

for $\kappa = \kappa_0 K$ with any normal unital CP map $K : \mathcal{A} \rightarrow \mathcal{A}^0$ and a compound state ω_{01} on $\mathcal{A}^0 \otimes \mathcal{B}^1$. The supremum (5.7), which is achieved at the standard entanglement, corresponding to $\omega_{01} = \omega_{q1}$, coincides with q-entropy $H(\zeta_1)$ and the supremum (5.8), coinciding with $S(\zeta_1)$, is achieved for a pure o-entanglement, corresponding to $\omega_{01} = \omega_{o1}$ given by any Schatten decomposition for σ_1 . Moreover, the entropy $H(\zeta_1)$ is also achieved by any pure d-entanglement, corresponding to $\omega_{01} = \omega_{d1}$ given by any extreme decomposition for σ_1 and thus is the maximal mutual information $I_1(\zeta_1, \Lambda)$ in the case of deterministic Λ . Thus the capacity $C_1(\Lambda)$ of the deterministic channel is given by the maximum $C_o = \ln \dim \mathcal{H}_1$ of the von Neumann entropy S , and the q-capacity $C_q(\Lambda)$ is equal $C_{\mathcal{B}^1} = \ln \dim \mathcal{B}^1$. \square

In the general case, d-entanglements can be more informative than o-entanglements as can be shown by an example of a quantum noisy channel for which

$$I_1(\zeta_1, \Lambda) > I_o(\zeta_1, \Lambda), \quad C_1(\Lambda) > C_o(\Lambda).$$

The last equalities of the above theorem are related to the work on entropy by Voiculescu [23].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTTINGHAM, NG7 2RD NOTTINGHAM, UK

DEPARTMENT OF INFORMATION SCIENCES, SCIENCE UNIVERSITY OF TOKYO, 278 NODA CITY, CHIBA, JAPAN