QUANTUM ENTROPY AND INFORMATION IN DISCRETE ENTANGLED STATES

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Abstract. Quantum correspondences and entanglements, describing truly quantum couplings, are studied and classified for the discrete compound states. We show that classical-quantum correspondences such as quantum encodings can be treated as d-entanglements leading to a special class of separable compound states. The mutual information for the d-compound and for q-compound (entangled) states leads to two different types of entropies for a given quantum state. The first one is the von Neumann entropy, which is achieved as the supremum of the information over all d-entanglements. The second one is the dimensional entropy, which is achieved at the standard entanglement, the true quantum entanglement, coinciding with a d-entanglement only in the commutative case. The q-conditional entropy and q-capacity of a quantum noiseless channel, defined as the supremum over all entanglements, is given as the logarithm of the dimensionality of the input von Neumann algebra. It can double the classical capacity, achieved as the supremum over all semi-quantum couplings (d-entanglements, or encodings), which is bounded by the logarithm of the dimensionality of a maximal Abelian subalgebra. An entopic measure for the essential entanglement is introduced.

1. Introduction

Recently, specifically quantum correlations called in quantum physics entanglements, are used to study quantum information processes, in particular, quantum computation, quantum teleportation, quantum cryptography [1, 2, 3]. There have been mathematical studies of entanglements in [4, 5, 6], in which the entangled state is defined by a compound state which cannot be written as a convex combination $\sum_n \varrho_n \otimes \varsigma_n \gamma(n)$ with any states ϱ_n and ς_n. However, it is obvious that there exist several important applications with correlated states written as separable forms above. Such correlated, or entangled states have also been discussed in several contexts in quantum probability, such as quantum measurement and filtering [7, 8], quantum compound state[9, 10], and lifting [11]. In this paper, we study the mathematical structure of quantum entangled states so as to provide a finer classification of quantum sates, and we discuss the informational degree of entanglement and entangled quantum mutual entropy.

We show that pure entangled states can be treated as generalized compound states, the nonseparable states of quantum compound systems which are not representable by convex combinations of the product states.

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The mixed compound states, defined as convex combinations by orthogonal decompositions of their input marginal states ς_0, were introduced in [9] for studying information in a quantum channel with the general output C*-algebra $\mathcal{B}$. This o-entangled compound state is a particular case of the so-called separable state of a compound system, the convex combination of the arbitrary product states which we call c-entangled. We shall prove that the o-entangled compound states are most informative among the c-entangled states in the sense that the maximum of mutual information over all c-entanglements with the quantum system $(\mathcal{B}, \varsigma)$ is achieved on the extreme o-entangled states, defined by a Schatten decomposition of a given state ς on $\mathcal{B}$. This maximum coincides with the von Neumann entropy $\mathsf{S}(\varsigma)$ of the state ς, and it can also be achieved as the maximum of the mutual information over all couplings with classical probe systems described by a maximal Abelian subalgebra $\mathcal{B}^1 \subseteq \mathcal{B}$. Thus the couplings described by c-entanglements of (quantum) probe systems $\mathcal{A}$ with a given system $\mathcal{B}$ do not give an advantage in maximizing the mutual information in comparison with the quantum-classical couplings corresponding to the Abelian $\mathcal{A} = \mathcal{B}^1$. The achieved maximal information $\mathsf{S}(\varsigma)$ coincides with the classical entropy on the Abelian subalgebra $\mathcal{B}^1$ of the Schatten decomposition for ς, and is bounded by $\ln \operatorname{rank}\mathcal{B} = \ln \dim \mathcal{B}^1$, where $\operatorname{rank}\mathcal{B}$ is the rank of the von Neumann algebra $\mathcal{B}$ defined as the dimensionality of a maximal Abelian subalgebra. Owing to $\dim \mathcal{B} \leq (\operatorname{rank}\mathcal{B})^2$, it is achieved on the normal central $\sigma = (\operatorname{rank}\mathcal{B})^{-1} I$ only in the case of finite-dimensional $\mathcal{B}$.

More general than o-entangled states, the d-entangled states, are defined as c-entangled states by the orthogonal decomposition of only one marginal state on the probe algebra $\mathcal{A}$. They can give bigger mutual entropy for a quantum noisy channel than the o-entangled state which gains the same information as d-entangled extreme states in the case of a deterministic channel.

We prove that the truly (strongest) entangled states are most informative in the sense that the maximum of mutual entropy over all entanglements with the quantum system $(\mathcal{B}, \varsigma)$ is achieved on the quasi-compound state given by an extreme entanglement of the probe system $\mathcal{A} = \mathcal{B}$ with coinciding marginals, called standard for a given ς. The standard entangled state is o-entangled only in the case of an Abelian $\mathcal{B}$ or pure marginal state ς. The gained information for such an extreme q-compound state defines another type of entropy, the quasi-entropy $\mathsf{S}_q(\varsigma)$ which is bigger than the von Neumann entropy $\mathsf{S}(\varsigma)$ in the case of non-Abelian $\mathcal{B}$ (and mixed ς.) The maximum of mutual entropy over all quantum couplings, described by true quantum entanglements of probe systems $\mathcal{A}$ with the system $\mathcal{B}$ is bounded by $\ln \dim\mathcal{B}$, the logarithm of the dimensionality of the von Neumann algebra $\mathcal{B}$, which is achieved on a normal tracial σ in the case of finite-dimensional $\mathcal{B}$. Thus the q-entropy $\mathsf{S}_q(\varsigma)$, which can be called the dimensional entropy, is the true quantum entropy, in contrast to the von Neumann rank entropy $\mathsf{S}(\varsigma)$, which is a semi-classical entropy as it can be achieved as the supremum over all couplings with the classical probe systems $\mathcal{A}$. These entropies coincide in the classical case of an Abelian $\mathcal{B}$ in which $\operatorname{rank}\mathcal{B} = \dim \mathcal{B}$. In the case of non-Abelian finite-dimensional $\mathcal{B}$ the q-capacity $\mathsf{C}_q = \ln \dim\mathcal{B}$ is achieved as the supremum of mutual entropy over all q-encodings (correspondences), described by entanglements. It is strictly larger than the semi-classical capacity $\mathsf{C} = \ln \operatorname{rank}\mathcal{B}$ of the identity channel, which is achieved as the supremum over usual encodings described by the classical-quantum correspondences $\mathcal{B}^1 \to \mathcal{B}$.

In this paper we consider the case of a discrete decomposable W*-algebra $\mathcal{B}$ for which the results are achieved by relatively simple proofs. The purely quantum case of a simple algebra $\mathcal{B} = \mathcal{L}(\mathcal{H})$, for which some proofs are rather obvious was considered in a short paper [12]. The general case of arbitrary W*-algebra $\mathcal{B}$ will be published elsewhere.

2. Compound States and Entanglements

Notations. Let $\mathcal{H}$ denote the (separable) Hilbert space of a quantum system, and $\mathcal{L}(\mathcal{H})$ be the algebra of all linear bounded operators on $\mathcal{H}$. In order to include the classical discrete systems as a particular quantum case, we shall fix a decomposable subalgebra $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ of bounded observables $B \in \mathcal{B}$ of the block-diagonal form $B = \left[B(j)\,\delta^i_j\right]$, where $B(j) \in \mathcal{L}(\mathcal{H}_j)$ are arbitrary bounded operators in Hilbert subspaces $\mathcal{H}_i$ corresponding to an orthogonal decomposition $\mathcal{H} = \oplus_j \mathcal{H}_j$.

A bounded linear functional $\varsigma : \mathcal{B} \to \mathbb{C}$ is called the *state* on $\mathcal{B}$ if it is positive (i.e., $\varsigma(B) \geq 0$ for any positive operator B in $\mathcal{B}$) and normalized $\varsigma(I) = 1$ for the identity operator I in $\mathcal{B}$. A *normal* state can be expressed as

$$\varsigma(B) = \mathrm{Tr}_{\mathcal{G}} \chi^\dagger B \chi = \mathrm{Tr}\, B\sigma, \qquad B \in \mathcal{B} \tag{2.1}$$

where $\mathcal{G}$ is another separable Hilbert space, χ is the Hilbert–Schmidt operator from $\mathcal{G}$ to $\mathcal{H}$, $\chi^\dagger$ is the adjoint operator of χ from $\mathcal{H}$ to $\mathcal{G}$ and $\mathrm{Tr}_{\mathcal{G}}$ (or simply Tr if there is no ambiguity) denotes the standard trace in $\mathcal{G}$ (or in $\mathcal{H}$) normalized to one-dimensional projectors $\sigma = \psi\psi^\dagger$. This χ is called the amplitude operator, or simply amplitude given by a $\chi = \psi \in \mathcal{H}$ with $\chi^\dagger\chi = \|\psi\|^2 = 1$ in the case of one dimensional $\mathcal{G} = \mathbb{C}$, corresponding to the pure state $\varsigma(B) = \psi^\dagger B \psi$, in which case $\chi^\dagger$ is the functional $\psi^\dagger$ from $\mathcal{H}$ to $\mathbb{C}$.

The amplitude operator is not unique, however it is defined uniquely up to a unitary transform in $\mathcal{G}$ as a *probability amplitude* by the additional condition $\chi\chi^\dagger = \mathrm{P}_{\mathcal{B}} \in \mathcal{B}$. Such a χ always exists, and the corresponding density operator $\sigma = \mathrm{P}_{\mathcal{B}}$ is a decomposable trace one operator $\mathrm{P}_{\mathcal{B}} = \oplus\sigma(i)$ called the *probability operator* with the components $\sigma(j) = \mathrm{P}_{\mathcal{B}}(j) \in \mathcal{L}(\mathcal{H}_j)$ normalized as

$$\mathrm{Tr}_{\mathcal{H}_j} \mathrm{P}_{\mathcal{B}}(j) = \varkappa(j) \geq 0, \quad \sum_i \varkappa(i) = 1.$$

Thus the predual space $\mathcal{B}_*$ can be identified with the direct sum $\oplus\mathcal{T}(\mathcal{H}_i) \subset \mathcal{B}$ of the Banach spaces $\mathcal{T}(\mathcal{H}_i)$ of trace class operators in $\mathcal{H}_i$. The probability operators $\mathrm{P}_{\mathcal{A}} \in \mathcal{A}_*$, $\mathrm{P}_{\mathcal{B}} \in \mathcal{B}_*$ of the states ϱ, ς on different algebras $\mathcal{A}$, $\mathcal{B}$ will be usually denoted by different letters ρ, σ corresponding to their Greek variations ϱ, ς.

In general, $\mathcal{G}$ is not one-dimensional, the dimensionality $\dim \mathcal{G}$ must be not less than $\operatorname{rank}\sigma$, the dimensionality of the range $\operatorname{ran}\sigma \subseteq \mathcal{H}$ of the density operator σ. We shall equip it with an isometric involution $J = J^\dagger$, $J^2 = I$, having the properties of complex conjugation on $\mathcal{G}$,

$$J \sum_k \lambda_k \zeta_k = \sum_k \bar{\lambda}_k J\zeta_k, \quad \forall \lambda_k \in \mathbb{C}, \zeta_k \in \mathcal{G}$$

with respect to which $J\rho = \rho J$ for the positive (and thus self-adjoint) operator $\rho = \chi^\dagger\chi = \rho^\dagger$ on $\mathcal{G}$. The latter can also be expressed as the symmetry property $\tilde{\varrho} = \varrho$ of the state $\varrho(A) = \mathrm{Tr}\, A\rho$ given by the real (and thus symmetric) density operator $\bar{\rho} = \rho = \tilde{\rho}$ on $\mathcal{G}$ with respect to the complex conjugation $\bar{A} = JAJ$ and the tilde operation ($\mathcal{G}$-transposition) $\tilde{A} = JA^{\dagger}J$ on the algebra $\mathcal{L}(\mathcal{G})$, and thus on any tilde invariant decomposable subalgebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{G})$ containing $\chi^{\dagger}\mathcal{B}\chi \ni \rho$.

The space $\mathcal{G}$ can always be realized as a subspace of $\ell^2(\mathbf{N})$ of complex sequences $\mathbf{N} \ni m \mapsto \zeta(m) \in \mathbb{C}$, with $\sum |\zeta(n)|^2 < +\infty$ in the diagonal representation $\rho = [\mu(n)\,\delta_n^m]$. The involution J can be identified then with the complex conjugation $C\zeta(n) = \bar{\zeta}(n)$, i.e.,

$$C : \zeta = \sum_n |n\rangle\zeta(n) \mapsto C\zeta = \sum_n |n\rangle\bar{\zeta}(n)$$

in the canonical basis $|n\rangle$ of $\mathcal{G} \subseteq \ell^2(\mathbf{N})$. In this case $\chi = \sum \chi(n)\langle n|$ is given by orthogonal eigen-amplitudes $\chi(n) \in \mathcal{H}$, $\chi(n)^{\dagger}\chi(m) = 0$, $m \neq n$, normalized to the eigen-values $\nu(n)$ $(= \mu(n) = \chi(n)^{\dagger}\chi(n))$ of the density operator σ such that $\sigma = \sum \chi(n)\chi(n)^{\dagger}$ is a Schatten decomposition, i.e. the spectral decomposition of σ into one-dimensional orthogonal projectors. In any other ortho-normal basis $\{\zeta_n\} \subset \mathcal{G}$ the operator J is defined then by $J = U^{\dagger}CU$, where $U = \sum |n\rangle\zeta_n^{\dagger}$ is the corresponding unitary transformation $\mathcal{G} \to \ell^2(\mathbb{N})$, and $\chi = \sum \chi(n)\zeta_n^{\dagger}$. One can also identify $\mathcal{G}$ with the whole space $\mathcal{H}$ by choosing $\zeta_n = \nu(n)^{-1/2}\chi(n)$ for those n where $\nu(n) = \|\chi(n)\|^2 \neq 0$ such that $J\chi_n = \chi_n$. The operator σ becomes then real and symmetric, $\sigma = J\sigma J = \sigma$ in $\mathcal{H} = \mathcal{G}$ with respect to any such involution J in $\mathcal{H}$ given by an isometric operator $U : \mathcal{H} \to \ell^2(\mathbb{N})$ diagonalizing the operator σ: $U\sigma U^{\dagger} = \sum |n\rangle\nu(n)\langle n|$. The amplitude operator

$$\chi = \sum \nu(n)^{1/2}\zeta_n\zeta_n^{\dagger} = \sigma^{1/2}$$

corresponding to this case $\mathcal{A} = \mathcal{B}$, $\rho = \sigma$ is called *standard*.

Given the amplitude operator χ, one can define not only the states ς and ϱ on the algebras $\mathcal{B} = \mathcal{L}(\mathcal{H})$ and $\mathcal{A} = \mathcal{L}(\mathcal{G})$ but also a pure *entanglement* state ϖ on the algebra $\mathcal{A} \otimes \mathcal{B}$ of all bounded operators on the tensor product Hilbert space $\mathcal{G} \otimes \mathcal{H}$ by

$$\mathrm{Tr}_{\mathcal{G}}\tilde{A}\chi^{\dagger}B\chi = \varpi(A \otimes B) = \mathrm{Tr}_{\mathcal{H}}\chi\tilde{A}\chi^{\dagger}B.$$

Indeed, thus defined ϖ is uniquely extended by linearity to a normal state on the algebra $\mathcal{A} \otimes \mathcal{B}$ generated by all the linear combinations $C = \sum \lambda_k A_k \otimes B_k$ due to $\varpi(I \otimes I) = \mathrm{Tr}\,\chi^{\dagger}\chi = 1$ and

$$\begin{aligned}\varpi\left(C^{\dagger}C\right) &= \sum_{i,k}\bar{\lambda}_i\lambda_k\mathrm{Tr}_{\mathcal{G}}\tilde{A}_k\tilde{A}_i^{\dagger}\chi^{\dagger}B_i^{\dagger}B_k\chi \\ &= \sum_{i,k}\bar{\lambda}_i\lambda_k\mathrm{Tr}_{\mathcal{G}}\tilde{A}_i^{\dagger}\chi^{\dagger}B_i^{\dagger}B_k\chi\tilde{A}_k = \mathrm{Tr}_{\mathcal{G}}X^{\dagger}X \geq 0,\end{aligned}$$

where $X = \sum \lambda_k B_k\chi\tilde{A}_k$.

This state is pure on $\mathcal{L}(\mathcal{G} \otimes \mathcal{H})$ as it is given by an amplitude $\psi \in \mathcal{G} \otimes \mathcal{H}$ defined as

$$(\zeta \otimes \eta)^{\dagger}\psi = \eta^{\dagger}\chi J\zeta, \quad \forall\zeta \in \mathcal{G}, \eta \in \mathcal{H},$$

and it has the states ς and ϱ as the marginals of ϖ:

$$\varpi(A \otimes I) = \mathrm{Tr}_{\mathcal{G}}A\rho, \quad \varpi(I \otimes B) = \mathrm{Tr}_{\mathcal{H}}B\sigma. \tag{2.2}$$

As follows from the next theorem for the case $\mathcal{F} = \mathbb{C}$, any pure state

$$\varpi(A \otimes B) = \psi^{\dagger}(A \otimes B)\psi, \quad A \in \mathcal{A}, B \in \mathcal{B}$$

given on $\mathcal{L}(\mathcal{G}\otimes\mathcal{H})$ by an amplitude $\psi\in\mathcal{G}\otimes\mathcal{H}$ with $\psi^\dagger\psi=1$, can be achieved by a unique entanglement of its marginal states ϱ and ς.

Theorem 1. *Let $\varpi:\mathcal{A}\otimes\mathcal{B}\to\mathbb{C}$ be a compound state*

$$\varpi(A\otimes B)=\mathrm{Tr}_{\mathcal{F}}\upsilon^\dagger(A\otimes B)\upsilon,$$

defined by an amplitude operator $\upsilon:\mathcal{F}\to\mathcal{G}\otimes\mathcal{H}$ on a separable Hilbert space $\mathcal{F}$ into the tensor product Hilbert space $\mathcal{G}\otimes\mathcal{H}$ with

$$\upsilon\upsilon^\dagger\in\mathcal{A}\otimes\mathcal{B},\quad \mathrm{Tr}_{\mathcal{F}}\upsilon^\dagger\upsilon=1.$$

Then this state can be achieved as an entanglement

$$\mathrm{Tr}_{\mathcal{G}}\tilde{A}\chi^\dagger(I\otimes B)\chi=\varpi(A\otimes B)=\mathrm{Tr}_{\mathcal{F}\otimes\mathcal{H}}\chi\tilde{A}\chi^\dagger(I\otimes B)\tag{2.3}$$

of the states (2.2) with $\rho=\chi^\dagger\chi$ and $\sigma=\mathrm{Tr}_{\mathcal{F}}\chi\chi^\dagger$, where χ is an operator $\mathcal{G}\to\mathcal{F}\otimes\mathcal{H}$ with

$$\mathrm{Tr}_{\mathcal{F}}\chi\mathcal{A}\chi^\dagger\subset\mathcal{B},\ \chi^\dagger(I\otimes\mathcal{B})\chi\subset\mathcal{A}.\tag{2.4}$$

The entangling operator χ is uniquely defined by $\tilde{\chi}U=\upsilon$, where

$$(\zeta\otimes\eta)^\dagger\tilde{\chi}\xi=(J\xi\otimes\eta)^\dagger\chi J\zeta,\quad\forall\xi\in\mathcal{F},\zeta\in\mathcal{G},\eta\in\mathcal{H},\tag{2.5}$$

up to a unitary transformation U of the minimal space $\mathcal{F}=\mathrm{ran}\upsilon^\dagger$ equipped with an isometric involution J.

Proof. Without loss of generality we can assume that the space $\mathcal{F}$ is a subspace of $\ell^2(\mathbf{N})$ for the diagonal representation of $\upsilon^\dagger\upsilon$ equipped with the complex conjugation C just as the space $\mathcal{G}$ is in the diagonal representation of $\chi^\dagger\chi$. In these canonical basises of $\mathcal{F}$ and $\mathcal{G}$ the amplitude operator $\chi=\sum\chi(n)\langle n|$ can be defined as the block-matrix $\sum|k\rangle\otimes\chi_k(n)\langle n|$ transposed to $\sum|n\rangle\otimes\chi_k(n)\langle k|$, where the amplitudes $\psi_k(n)\in\mathcal{H}$ are given by the matrix elements $\eta^\dagger\chi_k(n)=\left(\langle n|\otimes\eta^\dagger\right)\upsilon|k\rangle$:

$$\begin{aligned}\mathrm{Tr}_{\mathcal{G}}\tilde{A}\chi^\dagger(I\otimes B)\chi &= \sum_{k,m,n}\langle n|\tilde{A}|m\rangle\chi_k^\dagger(m)B\chi_k(n)\\ &= \sum_{k,m,n}\chi_k^\dagger(m)\langle m|A|n\rangle B\chi_k(n)=\mathrm{Tr}_{\mathcal{F}}\upsilon^\dagger(A\otimes B)\upsilon\ .\end{aligned}$$

In any other ortho-normal basis $\{\xi_k\}\subset\mathcal{F}$ the involution $J:\mathcal{F}\to\mathcal{F}$ satisfying $J\xi_k=\xi_k$ is defined as $U^\dagger CU$, and $\upsilon=\sum|n\rangle\otimes\psi_k(n)\xi_k^\dagger=\tilde{\chi}U$, where $U=\sum|k\rangle\xi_k^\dagger$. The isometric transformation U of $\{\xi_k\}$ into the canonical basis $\{|k\rangle\}\subset\ell^2(\mathbf{N})$ is real in the sense $\bar{U}:=CUJ=U$, and thus $\tilde{U}:=CU^\dagger J=U^\dagger$. Hence amplitude operator $\chi:\mathcal{G}\to\mathcal{F}\otimes\mathcal{H}$ which was defined above by the transposition of $\upsilon U^\dagger=\upsilon\tilde{U}\equiv\tilde{\chi}$, is equivalent to $\tilde{\upsilon}$: $\chi=(U\otimes I)\tilde{\upsilon}$. Thus

$$\chi^\dagger\chi=\mathrm{Tr}_{\mathcal{H}}\upsilon\upsilon^\dagger=\rho,\quad\mathrm{Tr}_{\mathcal{F}}\chi\chi^\dagger=\mathrm{Tr}_{\mathcal{G}}\upsilon\upsilon^\dagger=\sigma.$$

Moreover, it satisfies the conditions (2.4) since $\omega=\upsilon\upsilon^\dagger\in\mathcal{A}\otimes\mathcal{B}$:

$$J\chi^\dagger(I\otimes B)^\dagger\chi J=\mathrm{Tr}_{\mathcal{H}}(I\otimes B)\omega\in\mathcal{A},\ \mathrm{Tr}_{\mathcal{F}}\chi\tilde{A}\chi^\dagger=\mathrm{Tr}_{\mathcal{G}}(A\otimes I)\omega\in\mathcal{B}.$$

The uniqueness up to the U follows from the obvious isometricity of the families

$$\left\{\sum_k|k\rangle\eta^\dagger\psi_k(n):n\in\mathbb{N},\eta\in\mathcal{H}\right\},\quad\left\{\sum_k\eta^\dagger\psi_k(n)\xi_k^\dagger:n\in\mathbb{N},\eta\in\mathcal{H}\right\}$$

of vectors $\left(I\otimes\eta^{\dagger}\right)\chi|n\rangle$ in $\mathcal{F}\subseteq\ell^{2}\left(\mathbf{N}\right)$ and of $\left(\langle n|\otimes\eta^{\dagger}\right)\upsilon$ in $\mathcal{F}^{\dagger}$ which follows from

$$\mathrm{Tr}_{\mathcal{G}}|n\rangle\langle m|\chi^{\dagger}\left(I\otimes\eta\eta^{\dagger}\right)\chi=\mathrm{Tr}_{\mathcal{F}}\upsilon^{\dagger}\left(|m\rangle\langle n|\otimes\eta\eta^{\dagger}\right)\upsilon.$$

Thus they are unitary equivalent in the minimal space $\mathcal{F}$. So the entangling operator χ is defined in the minimal $\mathcal{F}$ up to unitary equivalence corresponding to the unitary operator U in $\mathcal{F}$ intertwining the involutions C and J. □

Note that the entangled state (2.3) is written as

$$\mathrm{Tr}_{\mathcal{G}}A\pi\left(B\right)=\varpi\left(A\otimes B\right)=\mathrm{Tr}_{\mathcal{H}}\pi^{*}\left(A\right)B,$$

where the operator $\pi^{\top}\left(A\right)=\mathrm{Tr}_{\mathcal{F}}\chi\tilde{A}\chi^{\dagger}\in\mathcal{B}$ for any $A\in\mathcal{L}\left(\mathcal{G}\right)$ is in the predual space $\mathcal{B}_{*}$ as bounded by $\|A\|\,\sigma\in\mathcal{B}_{*}$, and

$$\pi\left(B\right)=J\chi^{\dagger}\left(I\otimes B^{\dagger}\right)\chi J\equiv\chi^{\dagger}\,\widetilde{\left(I\otimes B\right)}\,\chi$$

is in $\mathcal{A}_{*}$ as a trace-class operator in $\mathcal{G}$, bounded by $\|B\|\,\rho\in\mathcal{A}_{*}$. The linear map π is written in the Steinspring form [18] of the normal completely positive (CP) map $B\mapsto\widetilde{\pi\left(B\right)}$, while $\pi^{*}:\mathcal{A}\rightarrow\mathcal{B}_{*}$ is written in the Kraus form [19] of the normal CP map $A\mapsto\pi^{*}\left(\tilde{A}\right)$ in the canonical orthonormal basis $|k\rangle$ of $\mathcal{F}\subseteq\ell^{2}\left(\mathbb{N}\right)$:

$$\pi^{*}\left(A\right)=\sum_{k}\left(\langle k|\otimes I\right)\chi\tilde{A}\chi^{\dagger}\left(|k\rangle\otimes I\right).$$

A linear map $\pi:\mathcal{B}\rightarrow\mathcal{A}_{*}$ is called transpose n-positive (transpose-completely positive, or TCP) if the operator matrix

$$\pi\left(\mathbf{B}\right)^{\prime}=\left[\pi\left(B_{ki}\right)\right]=\left[\pi\left(\mathbf{B}\right)\right]^{\dagger}$$

is positive for every $n\times n$ positive-definite operator-matrix $\mathbf{B}=\left[B_{ik}\right]=\mathbf{B}^{*}$ (respectively for any $n\in\mathbb{N}$) , where $\mathbf{B}^{\dagger}=\left[B_{ik}^{\dagger}\right]=\widetilde{\mathbf{B}}$:

$$\sum_{ik=1}^{n}\zeta_{i}^{\dagger}\pi\left(Y_{k}Y_{i}^{\dagger}\right)\zeta_{k}\geq0,\quad\forall\zeta_{j}\in\mathcal{G},Y_{j}\in\mathcal{B}.$$

Every transpose n-positive map is positive but not necessarily completely positive (CP) even if it is TCP, unless $\widetilde{A}=A$ (i.e. $\mathcal{A}$ is Abelian), in which case a positive map is both CP and TCP.

The entanglements π and π^{*} as linear maps: $\mathcal{B}\rightarrow\mathcal{A}_{*}$, $\mathcal{A}\rightarrow\mathcal{B}_{*}$ are dual to each other, $\pi^{**}=\pi$, both are positive, but they are not completely positive but transpose-completely positive as the compositions of the transposition $A\mapsto\tilde{A}$ and the completely positive maps. In terms of the compound density operator $\omega=\upsilon\upsilon^{\dagger}$ for the entangled state $\varpi\left(A\otimes B\right)=\mathrm{Tr}\left(A\otimes B\right)\omega$ they can be written simply as

$$\pi^{*}\left(A\right)=\mathrm{Tr}_{\mathcal{G}}\left(A\otimes I\right)\omega,\quad\pi\left(B\right)=\mathrm{Tr}_{\mathcal{H}}\left(I\otimes B\right)\omega,$$

and in any orthonormal basis of $\mathcal{G}$, e.g. in the diagonal representation of ρ, they have the form

(2.6)

$$\pi^{*}\left(A\right)=\sum_{m,n}\langle n|A|m\rangle\mathrm{Tr}_{\mathcal{F}}\chi\chi\left(m\right)\chi\left(n\right)^{\dagger},\ \pi\left(B\right)=\sum_{m,n}|m\rangle\,\langle n|\,\chi\left(n\right)^{\dagger}\left(I\otimes B\right)\chi\left(m\right)$$

corresponding to

$$\omega=\sum_{m,n}|m\rangle\langle n|\otimes\mathrm{Tr}_{\mathcal{F}}\chi\left(m\right)\chi\left(n\right)^{\dagger}.\tag{2.7}$$

Note that the eigen-basis of ρ is characterized by weak orthogonality property

$$\mathrm{Tr}_{\mathcal{F}}\psi(m)^{\dagger}\psi(n)=\mu(n)\delta_{n}^{m}. \tag{2.8}$$

in terms of the amplitude operators $\psi(n)=(I\otimes\langle n|)\tilde{\chi}$.

Definition 1. *The TCP map $\pi:\mathcal{B}\to\mathcal{A}_{*}$ (or its dual map $\pi^{*}:\mathcal{A}\to\mathcal{B}_{*}$), normalized as $\mathrm{Tr}_{\mathcal{G}}\pi(I)=1$ (or $\mathrm{Tr}_{\mathcal{H}}\pi^{*}(I)=1$) is called the (generalized) entanglement of the state ς on $\mathcal{B}$ to $\rho=\pi^{*}(I)$ (or of ϱ on $\mathcal{A}$ to $\sigma=\pi(I)$). The entanglement π is called true quantum if it is not completely positive, i.e. if there exists a positive operator-matrix $\mathbf{B}=[B_{ik}]$ with $B_{ik}\in\mathcal{B}$ for which $\pi(\mathbf{B})=[\pi(B_{ik})]$ is not positive. All other entanglements, described by the maps π which are both CP and TCP, are not true quantum. The entanglement $\pi=\pi^{q}=\pi^{*}$ by*

$$\pi^{q}(B)=\sigma^{1/2}\tilde{B}\sigma^{1/2},\quad B\in\mathcal{B} \tag{2.9}$$

of the state $\varrho=\varsigma$ on the algebra $\mathcal{A}=\mathcal{B}$ is called standard for the system $(\mathcal{B},\varsigma)$.

The standard entanglement defines the *standard compound state*

$$\mathrm{Tr}_{\mathcal{H}}A\sigma^{1/2}\tilde{B}\sigma^{1/2}=\varpi_{q}(A\otimes B)=\mathrm{Tr}_{\mathcal{H}}\sigma^{1/2}\tilde{A}\sigma^{1/2}B$$

on the algebra $\mathcal{B}\otimes\mathcal{B}$. In the case of the simple algebra $\mathcal{B}=\mathcal{L}(\mathcal{H})$ this state is pure, given by the amplitude $\psi=|\sigma^{1/2}\rangle$, where $|\sigma^{1/2}\rangle=\tilde{\chi}$ with $\chi=\sigma^{1/2}$ and $\tilde{\chi}$ defined in (2.5) as $(\zeta\otimes\eta)^{\dagger}\tilde{\chi}=\eta^{\dagger}\chi J\zeta$.

In the general case of a decomposable $\mathcal{B}=\oplus\mathcal{L}(\mathcal{H}_{i})$ with the density operator $\sigma=\oplus\sigma(i)$ having more than one components $\sigma(j)=\sigma_{j}\varkappa(j)$ with $\varkappa(j)=\mathrm{Tr}\sigma(j)\neq0$ and positive $\sigma_{j}\in\mathcal{L}(\mathcal{H}_{j})$, the standard compound state is mixed, described by the decomposable density operator

$$\omega_{q}=\oplus_{i,j}\delta_{j}^{i}|\sigma_{j}^{1/2}\rangle\langle\sigma_{j}^{1/2}|\varkappa(j),\quad A,B\in\mathcal{B} \tag{2.10}$$

with zero components $\omega(i,j)=\delta_{j}^{i}\varkappa(j)\omega_{qj}$ at $i\neq j$, corresponding to the pure compound states $\omega_{qj}=\psi_{j}\psi_{j}^{\dagger}$. The standard amplitudes $\psi_{j}=|\sigma_{j}^{1/2}\rangle\in\mathcal{H}_{j}\otimes\mathcal{H}_{j}$ defining the orthogonal decomposition

$$\upsilon=\oplus_{i,j}\varkappa(j)^{1/2}|\sigma_{j}^{1/2}\rangle\delta_{j}^{i}\langle i|=\oplus_{i,j}\psi(j)\delta_{j}^{i}\langle i|$$

of the standard amplitude operator $\upsilon:\mathcal{F}\to\oplus\mathcal{H}_{i}\otimes\mathcal{H}_{k}$ on $\mathcal{F}=\ell^{2}(\mathbb{N})$ with zero components $\upsilon(i,j)=\psi(j)\delta_{j}^{i}\langle i|$ at $i\neq j$, corresponds to the block-diagonal entangling operator $\chi=\left[\chi(j)\delta_{j}^{i}\right]$ with $\chi(j)=|j\rangle\otimes\sigma(j)^{1/2}$.

[Example] In quantum physics entangled states are usually obtained by a unitary transformation U of an initial disentangled state described by the density operator $\rho_{0}\otimes\sigma_{0}\otimes\tau_{0}$ on the tensor product Hilbert space $\mathcal{G}\otimes\mathcal{H}\otimes\mathcal{K}$, that is,

$$\varpi(A\otimes B)=\mathrm{Tr}\,U_{1}^{\dagger}(A\otimes B\otimes I)U_{1}(\rho_{0}\otimes\sigma_{0}\otimes\tau_{0}).$$

In the simple case $\mathcal{K}=\mathbb{C}$, $\tau_{0}=1$, the joint amplitude operator υ is defined on the tensor product $\mathcal{F}=\mathcal{G}\otimes\mathcal{H}_{0}$ with $\mathcal{H}_{0}=\operatorname{ran}\sigma_{0}$ as $\upsilon=U_{1}(\rho_{0}\otimes\sigma_{0})^{1/2}$. The entangling operator χ describing the entangled state ϖ is constructed as in the proof of Theorem 1 by transposition of the operator $\upsilon U^{\dagger}$, where U is an arbitrary isometric operator $\mathcal{F}\to\mathcal{G}\otimes\mathcal{H}_{0}$. The dynamical procedure of such entanglement described in terms of completely positive maps $\pi:\mathcal{B}\to\mathcal{A}_{*}$ is the subject of Belavkin quantum filtering theory [17]. The quantum filtering dilation theorem [17] proves that any entanglement π can be obtained as the unitary entanglement in the result of quantum filtering by tracing out nonobservable degrees of freedom of a quantum environment described by the density operator τ_0 on the Hilbert space $\mathcal{K}$, even in the continuous time case.

3. C- and D-Entanglements and Encodings

The compound states play the role of joint input-output probability measures in classical information channels, and can be pure in quantum case even if the marginal states are mixed. The pure compound states achieved by an entanglement of mixed input and output states exhibit new, non-classical type of correlations which are responsible for the EPR type paradoxes in the interpretation of quantum theory. The mixed compound states on $\mathcal{A}\otimes\mathcal{B}$ which are given as the convex combinations

$$\varpi=\sum_n \varrho_n\otimes\varsigma_n\gamma(n), \quad \gamma(n)\geq 0,\ \sum_n\gamma(n)=1$$

of tensor products of pure or mixed normalized states $\varsigma_n\in\mathcal{B}_*$, $\varrho_n\in\mathcal{A}_*$ as in classical case, do not exhibit such paradoxical behavior, and are usually considered as the proper candidates for the input-output states in the communication channels. Such convex combinations of primitive disentangled states are called *c-compound* states. We can say that they are achieved by c-entanglements $\pi=\pi_c^*$, where

$$\pi_c(A)=\sum_n\sigma_n\mathrm{Tr}_{\mathcal{G}}A\rho_n\gamma(n), \quad \pi_c^*(B)=\sum_n\rho_n\mathrm{Tr}_{\mathcal{H}}B\sigma_n\gamma(n) \tag{3.1}$$

are the convex combinations of the *primitive entanglements* $A\mapsto\sigma_n\mathrm{Tr}A\tilde{\rho}_n$ and $B\mapsto\tilde{\rho}_n\mathrm{Tr}B\sigma_n$ given by the density operators $\omega_n=\rho_n\otimes\sigma_n$ of the product states $\varpi_n=\varrho_n\otimes\varsigma_n$. Note that such maps π (and $\pi^*=\pi_c$) are not only co-positive but also completely positive as it follows from the positive-definiteness of c-matrices

$$\left[\mathrm{Tr}_{\mathcal{G}}\widetilde{A_k^*A_i}\rho_n\right]=\left[\mathrm{Tr}_{\mathcal{G}}\tilde{A}_i\tilde{A}_k^*\rho_n\right]=\left[\mathrm{Tr}_{\mathcal{G}}A_k^*A_i\tilde{\rho}_n\right].$$

A compound state of this sort was introduced by Ohya [9, 13] in order to define the quantum mutual entropy expressing the amount of information transmitted from an input quantum system to an output quantum system through a quantum channel. He used a Schatten decomposition $\rho=\sum_n\rho_n\mu(n)$ into one-dimensional density operators ρ_n of the input states ϱ_n. In the canonical basis $|n\rangle$ of $\mathcal{G}$ diagonalizing ρ it corresponds to a particular, diagonal type

$$\pi_d(A)=\sum_n\langle n|A|n\rangle\sigma_n\mu(n) \tag{3.2}$$

of the entangling map (2.6), and is discussed in this section.

Let us consider a finite or infinite input system indexed by the natural numbers $n\in\mathbf{N}$. The associated space $\mathcal{G}\subseteq\ell^2(\mathbf{N})$ is the Hilbert space of the input system described by a quantum projection-valued measure $n\mapsto|n\rangle\langle n|$ on $\mathbb{N}$, given an orthogonal partition of unity $I=\sum|n\rangle\langle n|\in\mathcal{A}$ of the finite or infinite dimensional input Hilbert space $\mathcal{G}$. Each input pure state, identified with the one-dimensional density operator $|n\rangle\langle n|\in\mathcal{A}$ corresponding to the elementary symbol $n\in\mathbb{N}$, defines the elementary output state ς_n on $\mathcal{B}$. If the elementary states ς_n are pure, they are described by output amplitudes $\eta_n\in\mathcal{H}$ satisfying $\eta_n^\dagger\eta_n=1=\mathrm{Tr}\sigma_n$, where $\sigma_n=\eta_n\eta_n^\dagger$ are the corresponding output one-dimensional density operators. If these amplitudes are non-orthogonal $\eta_m^\dagger\eta_n\neq\delta_n^m$, they cannot be identified with the input amplitudes $|n\rangle$.

The elementary joint input-output states are given by the density operators $|n\rangle\langle n| \otimes \eta_n \eta_n^\dagger$ in $\mathcal{G} \otimes \mathcal{H}$. Their mixtures define the compound states

$$\varpi(A \otimes B) = \sum_n \langle n|A|n\rangle \eta_n^\dagger B \eta_n \mu(n), \tag{3.3}$$

on $\mathcal{A} \otimes \mathcal{B}$, given by the classical-quantum correspondences, or *encodings* $n \mapsto |n\rangle\langle n|$ with the probabilities $\mu(n)$. Here we note that such quantum encoding is described by the simplest classical-quantum channel, and any d-compound state for a quantum-quantum channel in quantum communication can be obtained in this way due to the orthogonality of the decomposition (3.3), corresponding to the orthogonality of the Schatten decomposition $\rho = \sum_n |n\rangle \mu(n) \langle n|$ for $\rho = \mathrm{Tr}_{\mathcal{H}} \omega$.

The comparison of the general compound state (2.7) with (3.3) suggests that the encodings are described as the diagonalizing entanglements

$$\pi(B) = \sum_n |n\rangle \eta_n^\dagger B \eta_n \langle n| \mu(n). \tag{3.4}$$

Thus the encodings, which correspond to the stronger orthogonality

$$\psi(m)\psi(n)^\dagger = \mu(n)\delta_n^m, \tag{3.5}$$

for the amplitude operators $\psi(n) : \mathcal{F} \to \mathcal{H}$ of the decomposition of the amplitude operator $\upsilon = \sum |n\rangle \otimes \psi(n)$ in comparison with the orthogonality (2.8), are the particular cases of c-entanglements and are not true quantum entanglements. The strong orthogonality (3.5) can be achieved in the following way: Take in (2.6) $\mathcal{F} \subseteq \ell^2(\mathbb{N})$, $\chi(n) = |n\rangle \otimes \eta(n)$, where $\eta(n) = \mu(n)^{1/2} \eta_n$, with $\langle m|n\rangle = \delta_n^m$ so that

$$\chi(n)^\dagger (I \otimes B) \chi(m) = \mu(n) \eta_n^\dagger B \eta_n \delta_n^m$$

for any $B \in \mathcal{B}$. Then the strong orthogonality condition (3.5) is fulfilled by the amplitude operators $\psi(n) = \eta(n)\langle n| = \tilde{\chi}(n)$, and

$$\chi^\dagger \chi = \sum_n \mu(n) |n\rangle\langle n| = \rho, \ \chi\chi^\dagger = \sum_n \eta(n)\eta(n)^\dagger = \sigma.$$

It corresponds to the amplitude operator for the compound state (3.3) of the form

$$\upsilon = \sum_n |n\rangle \otimes \psi(n) U,$$

where U is arbitrary unitary operator from $\mathcal{F}$ onto $\mathcal{G}$, i.e. υ is unitary equivalent to the diagonal amplitude operator

$$\chi = \sum_n |n\rangle\langle n| \otimes \eta(n)$$

on $\mathcal{F} = \mathcal{G}$ into $\mathcal{G} \otimes \mathcal{H}$. Thus, we have proved the following theorem in the case of pure output states $\sigma_n = \eta_n \eta_n^\dagger$.

Theorem 2. *Let $\varpi : \mathcal{A} \otimes \mathcal{B} \to \mathbb{C}$ be a normal d-compound state*

$$\varpi(A \otimes B) = \sum_n \langle n|A|n\rangle \mathrm{Tr}_{\mathcal{F}_n} \psi_n^\dagger B \psi_n \mu(n). \tag{3.6}$$

Then it corresponds to an entanglement by operators $\chi(n) = \tilde{\psi}(n)$ with strongly orthogonal amplitudes $\psi(n) : \mathcal{F} \to \mathcal{G} \otimes \mathcal{H}$ corresponding to the orthogonal decomposition

$$\pi(B) = \sum_n |n\rangle\langle n| \mathrm{Tr}_{\mathcal{H}} B\sigma_n \mu(n), \tag{3.7}$$

which maps the algebra $\mathcal{B}$ into the diagonal subalgebra on $\mathcal{G} = \ell^2(\mathbb{N})$.

Proof. Let $\oplus_n \mathcal{F}_n$ be the Hilbert orthogonal sum of the domains $\mathcal{F}_n$ for the amplitude operators ψ_n in (3.6) with an isometric involution $\oplus C_n$. In the case $\mathcal{F}_n = \mathbb{C}$ of the amplitudes $\psi_n \in \mathcal{H}$ corresponding to pure states σ_n the involution $\oplus_n C_n$ is the componentwise complex conjugation in $\oplus_n \mathbb{C} \subseteq \ell^2(\mathbb{N})$; in the general case it is given by some isometric involutions C_n in the Hilbert spaces $\mathcal{F}_n$, which are equivalent to the ranges $\mathcal{H}_n = \sigma_n \mathcal{H}$ of the density operators $\sigma_n = \psi_n \psi_n^\dagger$ with the standard involutions in their eigen-representations, or contain these ranges. We can define the global output amplitude operator $\psi(n)$ on $\mathcal{F} = \oplus_n \mathcal{F}_n$ by

$$\psi(n) = \mu(n)^{1/2} \psi_n \epsilon_n^\dagger,$$

where $\epsilon_n : \mathcal{F}_n \to \mathcal{F}$ are the canonical orthogonal isometries, $\epsilon_m^\dagger \epsilon_n = I_n \delta_n^m$, and by $\upsilon = \sum_n |n\rangle \otimes \psi(n) U$ an amplitude operator $\upsilon : \mathcal{F} \to \mathcal{G} \otimes \mathcal{H}$ of the compound state (3.6) defining its density operator $\omega = \upsilon\upsilon^\dagger$ independently of the unitary transformation U of the Hilbert space onto $\oplus_n \mathcal{F}_n$.

The entangling operator $\chi = \sum_n \chi(n) \langle n|$ is then defined by its components $\chi(n) \in \mathcal{F} \otimes \mathcal{H}$ of the form

$$\chi(n) = (\epsilon_n \otimes I) \tilde{\psi}_n \mu(n)^{1/2} = \tilde{\psi}(n),$$

Here the $\tilde{\psi}_n$ are the amplitudes in $\mathcal{F}_n \otimes \mathcal{H}$ obtained from the operators $\psi_n : \mathcal{F}_n \to \mathcal{H}$ by

$$(\xi_n \otimes \eta)^\dagger \tilde{\psi}_n = \eta^\dagger \psi_n J_n \xi_n, \quad \forall \eta \in \mathcal{H}, \xi_n \in \mathcal{F}_n$$

In particular χ is the diagonal amplitude operator with the components $\chi(n) = \oplus_m \delta_n^m \tilde{\psi}(n)$ in $\oplus_m \mathcal{F}_m \otimes \mathcal{H}$:

$$\chi = \sum_n \chi(n) \langle n| = \oplus_m \tilde{\psi}(m) \langle m|. \tag{3.8}$$

Thus the entanglement (2.6) corresponding to (3.6) is given by the diagonal map (3.4) dual to (3.2) with the density operators $\sigma(n) = \psi(n)\psi(n)^\dagger = \mathrm{Tr}_{\mathcal{F}} \chi(n) \chi(n)^\dagger$ normalized to the probabilities $\mu(n) = \chi(n)^\dagger \chi(n)$. □

Note that (3.7) defines the general form of a positive map on $\mathcal{B}$ with values in the simultaneously diagonal trace-class operators in A.

Definition 2. *The convex combination (3.1) of the primitive CP maps $\sigma_n \varrho_n$ is called c-entanglement, and it is called d-entanglement, or quantum encoding if it has diagonalizing form (3.4) on $\mathcal{A}$. The d-entanglement is called o-entanglement and a compound state is called an o-compound if all density operators σ_n are orthogonal: $\sigma_m \sigma_n = 0$ for all m and n. These three types are generalized but not true quantum entanglements.*

Note that due to the commutativity of the operators $A \otimes I$ with $I \otimes B$ on $\mathcal{G} \otimes \mathcal{H}$, one can treat the correspondences as nondemolition measurements [8] in $\mathcal{A}$ with respect to $\mathcal{B}$. So the compound state is the state prepared for such measurements on the input $\mathcal{G}$. It coincides with the mixture of the states, corresponding to those after the measurement without reading the message sent. The set of all d-entanglements corresponding to a given Schatten decomposition of the input state ρ on $\mathcal{A}$ is obviously convex with the extreme points given by the pure output states σ_n on $\mathcal{B}$, corresponding to the not necessarily orthogonal decompositions $\sigma = \sum_n \sigma(n)$ into the one-dimensional density operators $\sigma(n) = \mu(n)\sigma_n$.

The Schatten decompositions $\sigma = \sum_n \nu(n)\sigma_n$ correspond to the extreme d-entanglements, $\sigma_n = \eta_n\eta_n^\dagger$, $\mu(n) = \nu(n)$ characterized by orthogonality $\sigma_m\sigma_n = 0$, $m \neq n$. They form a convex set of d-entanglements with mixed commuting σ_n for each Schatten decomposition of σ. The orthogonal d-entanglements were used in [16] to construct a particular type of Accardi's transitional expectations [15] and to define the entropy in a quantum dynamical system via such transitional expectations.

The established structure of the general q-compound states suggests also the general form

$$\Phi^*(A, \varsigma_0) = \mathrm{Tr}_{\mathcal{F}_1} X^\dagger (A \otimes \sigma_0) X = \mathrm{Tr}_{\mathcal{G}} \left(\tilde{A} \otimes I\right) Y (I \otimes \sigma_0) Y^\dagger$$

of transitional expectations $\Phi^* : \mathcal{A} \times \mathcal{B}^0_* \to \mathcal{B}_*$ describing the entanglements $\pi = \Phi^*(\varsigma_0)$ of the states $\varrho = \pi^*(I)$ to $\varsigma = \pi(I)$ for each initial state $\varsigma_0 \in \mathcal{B}^0_*$ with the density operator $\sigma_0 \in \mathcal{B}^0 \subseteq \mathcal{L}(\mathcal{H}_0)$ by $\pi(A) = \mathrm{Tr}_{\mathcal{F}} \chi (A \otimes I) \chi^\dagger$, where $\chi = X^\dagger (I \otimes \sigma_0)^{1/2}$. It is given by an entangling transition operator $X : \mathcal{F} \otimes \mathcal{H} \to \mathcal{G} \otimes \mathcal{H}_0$, which is defined by a transitional amplitude operator $Y : \mathcal{H}_0 \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{H}$ up to a unitary operator U in $\mathcal{F}$ as

$$(\zeta \otimes \eta_0)^\dagger X (U\xi \otimes \eta) = (\eta_0 \otimes J\xi)^\dagger Y^\dagger (J\zeta \otimes \eta).$$

The dual map $\Phi : \mathcal{B} \to \mathcal{A}_* \otimes \mathcal{B}^0$ is obviously normal and completely positive,

$$\Phi(B) = X (I \otimes B) X^\dagger \in \mathcal{A}_* \otimes \mathcal{B}^0, \ \forall B \in \mathcal{B}, \tag{3.9}$$

with $\mathrm{Tr}_{\mathcal{G}} \Phi(I) = I^0$, and is called a *filtering* map with the output states

$$\varrho = \mathrm{Tr}_{\mathcal{H}_0} \Phi(I) (I \otimes \sigma_0)$$

in the theory of CP flows [17] over $\mathcal{B} = \mathcal{B}^0$. The operators Y normalized as $\mathrm{Tr}_{\mathcal{F}} Y^\dagger Y = I^0$ describe $\mathcal{B}$-valued q-compound states

$$\mathrm{E}(A \otimes B) = \mathrm{Tr}_{\mathcal{F}} Y^\dagger (A \otimes B) Y = \mathrm{Tr}_{\mathcal{G}} \left(\tilde{A} \otimes I\right) \Phi(B),$$

defined as the normal completely positive maps $\mathcal{A} \otimes \mathcal{B} \to \mathcal{B}^0$ with $\mathrm{E}(I \otimes I) = I^0$.

If the $\mathcal{B}$-valued compound state has the diagonal form given by the orthogonal decomposition

$$\Phi(B) = \sum_n |n\rangle \mathrm{Tr}_{\mathcal{F}} \Psi(n)^\dagger B \Psi(n) \langle n|, \tag{3.10}$$

corresponding to $Y = \sum_n |n\rangle \otimes \Psi(n)$, where $\Psi(n) : \mathcal{H}_0 \otimes \mathcal{F} \to \mathcal{H}$, it is achieved by the d-transitional expectations

$$\Phi^*(A, \varsigma_0) = \sum_n \langle n|A|n\rangle \Psi(n) (\sigma_0 \otimes I) \Psi(n)^\dagger .$$

The d-transitional expectations correspond to the instruments [20] of the dynamical theory of quantum measurements. The elementary filters

$$\Theta_n(B) = \frac{1}{\mu(n)} \mathrm{Tr}_{\mathcal{F}} \Psi^\dagger(n) B \Psi(n), \quad \mu(n) = \mathrm{Tr}\, \Psi(n)(\sigma_0 \otimes I) \Psi^\dagger(n)$$

define posterior states $\varsigma_n = \varsigma_0 \Theta_n$ on $\mathcal{B}$ for quantum nondemolition measurements in $\mathcal{A}$, which are called *indirect* if the corresponding density operators σ_n are non-orthogonal. They describe the posterior states with orthogonal

$$\sigma_n = \Psi_n(\sigma_0 \otimes I) \Psi_n^\dagger, \quad \Psi_n = \Psi(n)/\mu(n)^{1/2}$$

for all σ_0 iff $\Psi(n)^\dagger \Psi(n) = \delta_n^m M(n)$.

4. Quantum Entropy via Entanglements

As was shown in the previous section, diagonalizing entanglements $\pi^d = \pi_d^*$ describe classical-quantum encodings $\pi_d : \mathcal{A} \to \mathcal{B}_*$, i.e. correspondences of classical symbols to quantum, in general, not orthogonal and pure, states. As we have seen in contrast to the classical case, not every entanglement can be achieved in this way. The general entangled states ϖ are described by density operators $\omega = vv^\dagger$ of the form (2.7) which are not necessarily block-diagonal in the eigen-representation of the density operator ρ, and they cannot be achieved even by the more general c-entanglement (3.1). The nonseparable, true entangled states are called in [13] *q-compound* states, so we can also call quantum-quantum nonseparable correspondences *q-encodings* in contrast to the d-encodings described by diagonal entanglements.

As we shall prove in this section, the most informative for a quantum system $(\mathcal{B}, \varsigma)$ is the self-dual standard entanglement $\pi^q = \pi_q^* = \pi_q$ to the probe system $\left(\mathcal{A}^0, \varrho_0\right) = (\mathcal{B}, \varsigma)$ described in (2.9). The other extreme cases of the self-dual input entanglements

$$\pi^0(B) = \sum_n \sigma(n)^{1/2} \tilde{B} \sigma(n)^{1/2},$$

are the pure c-entanglements given by the decompositions $\sigma = \sum \sigma(n)$ into pure states $\sigma(n) = \eta_n \eta_n^\dagger \mu(n)$. We shall see that these c-entanglements corresponding to the separable states

$$\omega_0 = \sum_n \eta_n \eta_n^\dagger \otimes \eta_n \eta_n^\dagger \mu(n), \tag{4.1}$$

are in general less informative then the pure d-entanglements (3.4).

Now, let us consider entangled mutual information and quantum entropies of states by means of the above three types of compound states. To define the quantum mutual entropy, we need to apply a quantum version of the relative entropy to compound state on the algebra $\mathcal{A} \otimes \mathcal{B}$, called also the information divergency of the state ϖ with respect to a reference state φ. It was defined in [21, 22, 23] even for more general von Neumann algebra $\mathcal{M}$ (not necessary decomposable as $\mathcal{A} \otimes \mathcal{B}$) with a trace Tr by the density operators ω, ϕ of these states as

$$\mathsf{R}(\varpi; \varphi) = \mathrm{Tr}\omega(\ln \omega - \ln \phi). \tag{4.2}$$

It has a positive value $\mathsf{R}(\varpi; \varphi) \in [0, \infty]$ if the states are equally normalized, say (as usually) Tr $\omega = 1 =$ Tr ϕ, and it can be finite only if the state ϖ is absolutely continuous with respect to the reference state φ, i.e. iff $\varpi(E) = 0$ for the maximal null-orthoprojector $E \in \mathcal{M}$, $E\phi = 0$. The most important property of the information divergence R is its monotonicity property [21, 24], i. e. nonincrease of the divergency $\mathsf{R}(\varpi_0; \varphi_0)$ after the application of the pre-dual of a normal completely positive unital map $\mathrm{K} : \mathcal{M} \to \mathcal{M}^0$ to the states ϖ_0 and φ_0 on a von Neumann algebra $\mathcal{M}^0$:

$$\varpi = \varpi_0 \mathrm{K}, \varphi = \varphi_0 \mathrm{K} \Rightarrow \mathsf{R}(\varpi; \varphi) \leq \mathsf{R}(\varpi_0; \varphi_0). \tag{4.3}$$

The *mutual information* $\mathsf{I}_{\mathcal{A};\mathcal{B}}(\pi) = \mathsf{I}_{\mathcal{B};\mathcal{A}}(\pi^*)$ in a compound state ϖ achieved by an entanglement $\pi : \mathcal{B} \to \mathcal{A}_*$, or by $\pi^* : \mathcal{A} \to \mathcal{B}_*$ with the marginals

$$\varrho(A) = \varpi(A \otimes I) = \mathrm{Tr}_{\mathcal{G}} A\rho, \ \varsigma(B) = \varpi(I \otimes B) = \mathrm{Tr}_{\mathcal{H}} B\sigma$$

is defined as the relative entropy (4.2) of the state ϖ on $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$ with respect to the product state $\varphi = \varrho \otimes \varsigma$:

$$\mathsf{I}_{\mathcal{A};\mathcal{B}}(\pi) = \mathrm{Tr}\,\omega\left(\ln\omega - \ln(\rho \otimes I) - \ln(I \otimes \sigma)\right). \tag{4.4}$$

This quantity, generalizing the classical mutual information corresponding to the case of Abelian $\mathcal{A}$, $\mathcal{B}$, describes an information gain in a quantum system $(\mathcal{A}, \varrho)$ via the entanglement π, or in $(\mathcal{B}, \varsigma)$ via an entanglement π^*. It is naturally treated as a measure of the strength of the generalized entanglement having zero value only for completely disentangled states $\varpi = \varrho \otimes \varsigma$.

Proposition 1. *Let $\pi : \mathcal{B} \to \mathcal{A}_*$ be an entanglement of $(\mathcal{B}, \varsigma)$ of a state $\varsigma(B) = \mathrm{Tr}\pi(B)$, $B \in \mathcal{B}$ to $(\mathcal{A}, \varrho)$ with the density operator $\rho = \pi(I)$, and $\pi_0 : \mathcal{A}^0 \to \mathcal{B}_*$ be an entanglement defining π by the composition $\pi^* = \pi_0 \mathrm{K}$ with a normal completely positive unital map $\mathrm{K} : \mathcal{A} \to \mathcal{A}^0$. Then $\mathsf{I}_{\mathcal{A};\mathcal{B}}(\pi) \leq \mathsf{I}_{\mathcal{A}^0;\mathcal{B}}(\pi^0)$, where $\pi^0 = \pi_0^*$. In particular, for any c-entanglement $\pi = \pi^c$ on $(\mathcal{B}, \varsigma)$ there exists a not less informative d-entanglement $\pi^0 = \pi^d$ on $\mathcal{B}$ with the same ς, and the standard entanglement $\pi^0 = \pi^q$ to $\varrho_0 = \varsigma$ with $\mathcal{A}^0 = \mathcal{B}$ is the maximal one in this sense.*

Proof. The first follows from the monotonicity property (4.3) applied to the ampliation $\mathrm{K}(A \otimes B) = \mathrm{K}(A) \otimes B$ of the CP map K from $\mathcal{A} \to \mathcal{A}^0$ to $\mathcal{A} \otimes \mathcal{B} \to \mathcal{A}^0 \otimes \mathcal{B}$, with the compound state $\mathrm{K}_*(\omega_0) = \varpi_0(\mathrm{K} \otimes \mathrm{I})$ (I denotes the identity map $\mathcal{B} \to \mathcal{B}$) corresponding to the entanglement $\pi^* = \pi_0 \mathrm{K}$ and $\mathrm{K}_*(\phi_0) = \varrho \otimes \varsigma$, $\varrho = \varrho_0 \mathrm{K}$ corresponding to $\varphi_0 = \varrho_0 \otimes \varsigma$.

This monotonicity property proves, in particular, that for any separable compound state on $\mathcal{A} \otimes \mathcal{B}$, which is prepared by the c-entanglement $\pi^c = \pi_c^*$ in (3.1), there exists a diagonal entanglement $\pi_0 = \pi_d$ with the system $(\mathcal{B}, \varsigma)$ having the same, or even larger information gain (4.4). One can take even a classical system $\left(\mathcal{A}^0, \varrho_0\right)$, say the diagonal sublagebra $\mathcal{A}^0$ on $\mathcal{G}_0 = \ell^2(\mathbf{N})$ with the state ϱ_0, induced by the measure γ, and consider the classical-quantum correspondence (encoding)

$$\pi_0(A) = \sum_n \alpha(n)\,\sigma_n \gamma(n), \quad A = \sum_n |n\rangle\alpha(n)\langle n|,$$

assigning the states $\varsigma_n(B) = \mathrm{Tr}\,B\sigma_n$ to the letters n with the probabilities $\gamma(n)$. The information gain

$$\mathsf{I}_{\mathcal{A}^0;\mathcal{B}}\left(\pi^0\right) = \sum_n \gamma(n)\,\mathrm{Tr}\,\sigma_n\left(\ln\sigma_n - \ln\sigma\right)$$

is equal or bigger than $\mathsf{I}_{\mathcal{A};\mathcal{B}}(\pi)$ corresponding to $\omega=\sum_n \rho_n\otimes\sigma_n\gamma(n)$ because the entanglement (3.1) is represented as the composition $\pi_0\mathrm{K}$ with the CP map

$$\mathrm{K}(A)=\sum_n |n\rangle\varrho_n(A)\langle n|,\quad A\in\mathcal{A}$$

into the diagonal algebra $\mathcal{A}^0$.

The inequality (4.3) can also be applied to the standard entanglement corresponding to the compound state (2.10) on $\mathcal{B}\otimes\mathcal{B}=\oplus_{i,j}\mathcal{B}(i)\otimes\mathcal{B}(j)$, where $\mathcal{B}(i)=\mathcal{L}(\mathcal{H}_i)$. It is described by the density operator

$$\omega_q=\oplus_{i,j}\mathrm{P}_{\mathcal{B}\otimes\mathcal{B}}(i,j)=\oplus_i\psi_i\psi_i^\dagger\varkappa(i)\,, \tag{4.5}$$

where $\mathrm{P}_{\mathcal{B}\otimes\mathcal{B}}(i,j)=\delta^i_j\omega_j\varkappa(j)$ is concentrated on the diagonal $\oplus_i\mathcal{B}(i)\otimes\mathcal{B}(i)$ of $\mathcal{B}\otimes\mathcal{B}$. The amplitudes $\psi_i\in\mathcal{H}_i\otimes\mathcal{H}_i$ are defined in (2.10) as $\psi_i=|\sigma_i^{1/2}\rangle$ by the components $\chi_0(i)=|i\rangle\otimes\sigma(i)^{1/2}$ of the standard entangling operator χ_0 on $\mathcal{G}_0=\mathcal{H}$ into $\ell^2(\mathbf{N})\otimes\mathcal{H}$. Indeed, any entanglement $\pi^*(A)=\mathrm{Tr}_{\mathcal{F}}\chi A\chi^\dagger$ as a normal CP map $\mathcal{A}\to\mathcal{B}$ normalized to the density operator $\sigma=\mathrm{Tr}_{\mathcal{F}}\chi\chi^\dagger$ can be represented as the composition $\pi_0\mathrm{K}$ of the standard entanglement $\pi^0=\pi^q$ on $\left(\mathcal{A}^0,\varrho_0\right)=(\mathcal{B},\varsigma)$ and a normal unital CP map $\mathrm{K}:\mathcal{A}\to\mathcal{B}$. The CP map K is defined by $\sigma^{1/2}\mathrm{K}(A)\sigma^{1/2}=\pi^*\left(\tilde{A}\right)$. It has the form

$$\mathrm{K}(A)=\mathrm{Tr}_{\mathcal{F}_-}X^\dagger AX,\quad A\in\mathcal{A},$$

where X is an operator $\mathcal{F}_-\otimes\mathcal{H}\to\mathcal{G}$, $\mathrm{Tr}_{\mathcal{F}_-}X^\dagger X=I$ defining the entangling operator $\chi=(I^-\otimes\chi_0)X^\dagger$ for π. Thus the standard entanglement $\pi^q(B)=\sigma^{1/2}\tilde{B}\sigma^{1/2}$ corresponds to the maximal mutual information. □

Note that any d-entanglement $\pi=\pi^*_d$ (3.2) has positive conditional entropy

$$\mathsf{S}_{\mathcal{B}}\left(\pi^d\right)=-\sum_n\mu(n)\,\mathrm{Tr}\sigma_n\ln\sigma_n,$$

and if π_d decomposes σ into pure normalized states $\sigma_n=\eta_n\eta_n^\dagger$, $\mathsf{S}_{\mathcal{B}}\left(\pi^d\right)=0$. Such extreme entanglements are maximal among all c-entanglements in the sense $\mathsf{I}_{\mathcal{A};\mathcal{B}}\left(\pi^d\right)\geq\mathsf{I}_{\mathcal{A};\mathcal{B}}(\pi^c)$ as

$$\mathsf{I}_{\mathcal{A};\mathcal{B}}\left(\pi^d\right)=\sum_n\mu(n)\,\mathrm{Tr}\sigma_n(\ln\sigma_n-\ln\sigma)$$

with a fixed $\mathrm{Tr}\pi^d(B)=\varsigma(B)$ achieves its supremum

$$\mathsf{S}_{\mathcal{B}}(\varsigma)=-\sum_n\mu(n)\,\mathrm{Tr}\sigma_n\ln\sigma=-\mathrm{Tr}_{\mathcal{H}}\sigma\ln\sigma. \tag{4.6}$$

Thus the supremum of the information gain (4.4) over all c-entanglements to the system $(\mathcal{B},\varsigma)$ is the von Neumann entropy

$$\mathsf{S}_{\mathcal{B}}(\varsigma)=-\sum_n\nu(n)\ln\nu(n)\,.$$

It is achieved on any extreme π^d, for example, given by the maximal Abelian subalgebra $\mathcal{A}^0\subseteq\mathcal{B}$, with the probability measure $\mu=\nu$ corresponding to a Schatten decomposition $\sigma=\sum_n|n\rangle\langle n|\nu(n)$. The maximal value $\ln\mathrm{rank}\,\mathcal{B}$ of the von Neumann entropy is defined by the dimensionality $\mathrm{rank}\,\mathcal{B}=\dim\mathcal{A}^0$ of the maximal Abelian subalgebra of the decomposable algebra $\mathcal{B}$, i.e. by $\dim\mathcal{H}$. However, if π

is not c-entanglement, the difference $\mathsf{S}_{\mathcal{B}}\left(\pi^{d}\right) = \mathsf{S}_{\mathcal{B}}\left(\varsigma\right) - \mathsf{I}_{\mathcal{A};\mathcal{B}}\left(\pi\right)$ can achieve the negative value, and cannot serve as a measure of conditional entropy in such case.

Definition 3. *The maximal mutual information*

$$\mathsf{H}_{\mathcal{B}}\left(\varsigma\right) = \sup_{\pi(I)=\sigma} \mathsf{I}_{\mathcal{A};\mathcal{B}}\left(\pi\right) = \mathsf{I}_{\mathcal{B},\mathcal{B}}\left(\pi^{q}\right) \tag{4.7}$$

achieved on $\mathcal{A} = \mathcal{B}$ by the standard q-entanglement $\pi^{q}\left(B\right) = \sigma^{1/2}\tilde{B}\sigma^{1/2}$ for a fixed state $\varsigma\left(B\right) = \mathrm{Tr}_{\mathcal{H}} B\sigma$ is called q-entropy *of the state ς. The differences*

$$\mathsf{H}_{\mathcal{B}|\mathcal{A}}\left(\pi\right) = \mathsf{H}_{\mathcal{B}}\left(\varsigma\right) - \mathsf{I}_{\mathcal{A};\mathcal{B}}\left(\pi\right)$$

$$\mathsf{D}_{\mathcal{B}|\mathcal{A}}\left(\pi\right) = \mathsf{S}_{\mathcal{B}}\left(\varsigma\right) - \mathsf{I}_{\mathcal{A};\mathcal{B}}\left(\pi\right)$$

are called respectively, the q-conditional entropy *on $\mathcal{B}$ with respect to $\mathcal{A}$ and the* (degree of) disentanglement *for the map $\pi : \mathcal{B} \to \mathcal{A}$. A compound state is said to be essentially entangled if $\mathsf{D}_{\mathcal{B}|\mathcal{A}}\left(\pi\right) < 0$.*

Obviously, $\mathsf{H}_{\mathcal{B}|\mathcal{A}}\left(\pi\right)$ is positive in contrast to the disentanglement $\mathsf{D}_{\mathcal{B}|\mathcal{A}}\left(\pi\right)$ having a positive value

$$\mathsf{D}_{\mathcal{B}|\mathcal{A}}\left(\pi\right) \geq \mathsf{S}_{\mathcal{B}|\mathcal{A}^{0}}\left(\pi^{0}\right) \geq 0$$

in the case of a c-entanglement $\pi = \pi^{c}$, with the corresponding $\pi^{0} = \pi^{d}$, but which can also achieve the negative value

$$\inf_{\pi:\pi^{*}(I)=\sigma} \mathsf{D}_{\mathcal{B}|\mathcal{A}}\left(\pi\right) = \mathsf{S}_{\mathcal{B}}\left(\varsigma\right) - \mathsf{H}_{\mathcal{B}}\left(\varsigma\right) = \sum_{i} \nu\left(i\right) \mathrm{Tr}_{\mathcal{H}_i}\sigma_i \ln \sigma_i \tag{4.8}$$

as the following theorem states. Here the $\sigma_i \in \mathcal{L}\left(\mathcal{H}_i\right)$ are the density operators of the normalized factor-states $\varsigma_i = \varkappa\left(i\right)^{-1} \varsigma|\mathcal{L}\left(\mathcal{H}_i\right)$ with $\varkappa\left(i\right) = \varsigma\left(I^{i}\right)$, where I^{i} are the orthoprojectors onto $\mathcal{H}_i$. Note that $\mathsf{H}_{\mathcal{B}}\left(\varsigma\right) = \mathsf{S}_{\mathcal{B}}\left(\varsigma\right)$ if the algebra $\mathcal{B}$ is completely decomposable, i.e. Abelian, and the maximal value $\ln \mathrm{rank}\, \mathcal{B}$ of $\mathsf{S}_{\mathcal{B}}\left(\varsigma\right)$ can be written as $\ln \dim \mathcal{B}$ in this case. The disentanglement $\mathsf{D}_{\mathcal{B}|\mathcal{A}}\left(\pi\right)$ is always positive in this case, and $\mathsf{D}_{\mathcal{B}|\mathcal{A}}\left(\pi\right) = \mathsf{H}_{\mathcal{B}|\mathcal{A}}\left(\pi\right)$ as in the case of Abelian $\mathcal{A}$.

Theorem 3. *Let $\mathcal{B}$ be a discrete decomposable algebra on $\mathcal{H} = \oplus_i \mathcal{H}_i$, with a normal state given by the density operator $\sigma = \oplus\sigma\left(i\right)$, and $\mathcal{C} \subseteq \mathcal{B}$ be its center with the state $\varkappa = \varsigma|\mathcal{C}$ induced by the probability distribution $\varkappa\left(i\right) = \mathrm{Tr}\, \sigma\left(i\right)$. Then the q-entropy is given by the formula*

$$\mathsf{H}_{\mathcal{B}}\left(\varsigma\right) = \sum_{i}\left(\varkappa\left(i\right) \ln \varkappa\left(i\right) - 2\mathrm{Tr}_{\mathcal{H}_i}\sigma\left(i\right) \ln \sigma\left(i\right)\right), \tag{4.9}$$

i.e. $\mathsf{H}_{\mathcal{B}}\left(\varsigma\right) = \mathsf{H}_{\mathcal{B}|\mathcal{C}}\left(\varsigma\right) + \mathsf{H}_{\mathcal{C}}\left(\varkappa\right)$, where $\mathsf{H}_{\mathcal{C}}\left(\varkappa\right) = -\sum_i \varkappa\left(i\right) \ln \varkappa\left(i\right) = \mathsf{S}_{\mathcal{C}}\left(\varkappa\right)$, and

$$\mathsf{H}_{\mathcal{B}|\mathcal{C}}\left(\varsigma\right) = -2\sum_{i} \varkappa\left(i\right) \mathrm{Tr}_{\mathcal{H}_i}\sigma_i \ln \sigma_i = 2\mathsf{S}_{\mathcal{B}|\mathcal{C}}\left(\varsigma\right),$$

with $\sigma_i = \sigma\left(i\right)/\varkappa\left(i\right)$. It is positive, $\mathsf{H}_{\mathcal{B}}\left(\varsigma\right) \in [0, \infty]$, and if $\mathcal{B}$ is finite-dimensional, it is bounded, with the maximal value $\mathsf{H}_{\mathcal{B}}\left(\varsigma^{\circ}\right) = \ln \dim \mathcal{B}$ which is achieved for $\sigma^{\circ} = \oplus\sigma_i^{\circ}\varkappa^{\circ}\left(i\right)$,

$$\sigma_i^{\circ} = \left(\dim \mathcal{H}_i\right)^{-1} I^{i}, \quad \varkappa^{\circ}\left(i\right) = \dim \mathcal{B}\left(i\right) / \dim \mathcal{B},$$

where $\dim \mathcal{B}\left(i\right) = \left(\dim \mathcal{H}_i\right)^{2}$, $\dim \mathcal{B} = \sum_i \dim \mathcal{B}\left(i\right)$.

Proof. The q-entropy $\mathsf{H}_{\mathcal{B}}(\varsigma)$ is the supremum (4.7) of the mutual information (4.4) which is achieved on the standard entanglement, corresponding to the density operator (4.5) of the standard compound state (2.10) with $\mathcal{A}=\mathcal{B}$, $\rho=\sigma$. Thus $\mathsf{H}_{\mathcal{B}}(\sigma)=\mathsf{I}_{\mathcal{B},\mathcal{B}}(\kappa)$, where

$$\begin{aligned}\mathsf{I}_{\mathcal{B},\mathcal{B}}(\kappa) &= \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}}\omega\left(\ln\omega-\ln(\sigma\otimes I)-\ln(I\otimes\sigma)\right)\\ &= \sum_i \varkappa(i)\ln\varkappa(i)-2\mathrm{Tr}\,\sigma\ln\sigma=-\sum_i\varkappa(i)\left(\ln\varkappa(i)+2\mathrm{Tr}_{\mathcal{H}_i}\sigma_i\ln\sigma_i\right).\end{aligned}$$

Here we used that $\mathrm{Tr}\ \omega\ln\omega=\sum_i\varkappa(i)\ln\varkappa(i)$ due to

$$\omega\ln\omega=\oplus_{i,k}\mathrm{P}_{\mathcal{B}\otimes\mathcal{B}}(i,k)\ln\mathrm{P}_{\mathcal{B}\otimes\mathcal{B}}(i,k)=\oplus_i\varkappa(i)\,\psi_i\psi_i^\dagger\ln\varkappa(i)$$

for the orthogonal diagonal decomposition (4.5) of ω into one-dimensional orthoprojectors $\psi_i\psi_i^\dagger=\mathrm{P}_{\mathcal{B}\otimes\mathcal{B}}(i,i)/\varkappa(i)$, and that $\mathrm{Tr}\ \sigma\ln\sigma=\sum_i\varkappa(i)\left(\ln\varkappa(i)-\mathsf{S}_{\mathcal{B}_i}(\varsigma_i)\right)$ due to

$$\sigma\ln\sigma=\oplus_i\mathrm{P}_{\mathcal{B}}(i)\ln\mathrm{P}_{\mathcal{B}}(i)=\oplus_i\varkappa(i)\,\sigma_i\left(\ln\varkappa(i)+\ln\sigma_i\right)$$

for the orthogonal decomposition $\sigma=\oplus_i\varkappa(i)\,\mathrm{P}_{\mathcal{B}(i)}$, where $\mathrm{P}_{\mathcal{B}(i)}=\mathrm{P}_{\mathcal{B}}(i)/\varkappa(i)=\sigma_i$, $\varkappa(i)=\mathrm{Tr}\ \mathrm{P}_{\mathcal{B}}(i)$, $\mathrm{P}_{\mathcal{B}}(i)=\sum_k\mathrm{Tr}_{\mathcal{H}}\mathrm{P}_{\mathcal{B}\otimes\mathcal{B}}(i,k)=\sigma(i)$.

Thus $\mathsf{H}_{\mathcal{B}}(\varsigma)=\mathsf{H}_{\mathcal{B}|\mathcal{C}}(\varsigma)+\mathsf{H}_{\mathcal{C}}(\varkappa)=2\mathsf{S}_{\mathcal{B}|\mathcal{C}}(\varsigma)+\mathsf{S}_{\mathcal{C}}(\varkappa)$ is positive, and it is bounded by

$$\begin{aligned}\mathsf{C}_{\mathcal{B}} &= \sup_{\varkappa}\sum_i\varkappa(i)\left(2\sup_{\varsigma_i}\mathsf{S}_{\mathcal{B}(i)}(\varsigma_i)-\ln\varkappa(i)\right)\\ &= -\inf_{\varkappa}\sum_i\varkappa(i)\left(\ln\varkappa(i)-2\ln\dim\mathcal{H}_i\right)=\ln\dim\mathcal{B}.\end{aligned}$$

Here we used the fact that the supremum of von Neumann entropies

$$\mathsf{S}_{\mathcal{B}(i)}(\varsigma_i)=-\sum_i\mathrm{Tr}_{\mathcal{H}_i}\sigma_i\ln\sigma_i$$

for the simple algebras $\mathcal{B}(i)=\mathcal{L}(\mathcal{H}_i)$ with $\dim\mathcal{B}(i)=(\dim\mathcal{H}_i)^2<\infty$ is achieved on the tracial density operators $\sigma_i=(\dim\mathcal{H}_i)^{-1}I^i\equiv\sigma_i^\circ$, and the infimum of the relative entropy

$$\mathsf{I}(\varkappa;\varkappa^\circ)=\sum_i\varkappa(i)\left(\ln\varkappa(i)-\ln\varkappa^\circ(i)\right),$$

where $\varkappa^\circ(i)=\dim\mathcal{B}(i)/\dim\mathcal{B}$, is zero, achieved at $\varkappa=\varkappa^\circ$. □

5. Quantum Channel and its Q-Capacity

Let $\mathcal{H}_1$ be a Hilbert space describing a quantum input system and $\mathcal{H}$ describe its output Hilbert space. A quantum channel is an affine operation sending each input state defined on $\mathcal{H}_1$ to an output state defined on $\mathcal{H}$ such that the mixtures of states are preserved. A deterministic quantum channel is given by a linear isometry $Y:\mathcal{H}_1\to\mathcal{H}$ with $Y^\dagger Y=I^1$ (I^1 is the identify operator in $\mathcal{H}_1$) such that each input state vector $\eta_1\in\mathcal{H}_1$, $\|\eta_1\|=1$, is transmitted into an output state vector $\eta=Y\eta_1\in\mathcal{H}$, $\|\eta\|=1$. The orthogonal mixtures $\sigma_1=\sum_n\sigma_1(n)$ of the pure input states $\sigma_1(n)=\eta_1(n)\eta_1(n)^\dagger$ are sent into the orthogonal mixtures $\sigma=\sum_n\sigma(n)$ of the corresponding pure states $\sigma(n)=Y\sigma_1(n)Y^\dagger$.

A noisy quantum channel sends pure input states ς_1 on an algebra $\mathcal{B}^1 \subseteq \mathcal{L}(\mathcal{H}_1)$ into mixed ones $\varsigma = \Lambda^*(\sigma_1)$ given by the predual $\Lambda_* = \Lambda^*|\mathcal{B}^1_*$ to a normal completely positive unital map $\Lambda : \mathcal{B} \to \mathcal{B}^1$,

$$\Lambda(B) = \mathrm{Tr}_{\mathcal{F}_1} Y^\dagger B Y, \quad B \in \mathcal{B}$$

where Y is a linear operator from $\mathcal{H}_1 \otimes \mathcal{F}_+$ to $\mathcal{H}$ with $\mathrm{Tr}_{\mathcal{F}_+} Y^\dagger Y = I^1$, and $\mathcal{F}_+$ is a separable Hilbert space of quantum noise in the channel. Each input mixed state ς_1 is transmitted into an output state $\varsigma = \varsigma_1 \Lambda$ given by the density operator

$$\Lambda^*(\sigma_1) = Y\left(\sigma_1 \otimes I^+\right) Y^\dagger \in \mathcal{B}_*$$

for each density operator $\sigma_1 \in \mathcal{B}^1_*$, where I^+ is the identity operator in $\mathcal{F}_+$. Without loss of generality, we can assume that the input algebra $\mathcal{B}^1$ is the smallest decomposable algebra generated by the range $\Lambda(\mathcal{B})$ of the given map Λ.

The input entanglements $\pi^1 : \mathcal{A} \to \mathcal{B}^1_*$ dual to $\pi_1 : \mathcal{B}^1 \to \mathcal{A}$ will be denoted as $\pi^1 = \kappa = \pi_1^*$. They define the quantum-quantum correspondences (q-encodings) of probe systems $(\mathcal{A}, \varrho)$ with the density operator $\rho = \kappa^*\left(I^1\right)$, to the input $\left(\mathcal{B}^1, \varsigma_1\right)$ of the channel Λ with $\sigma_1 = \kappa(I)$. As was proved in the previous section, the most informative is the standard entanglement $\kappa_0^* = \pi_1^q = \kappa_0$, at least in the case of the trivial channel $\Lambda = \mathrm{I}$. This extreme input q-entanglement $\pi_1^q(B) = \sigma_1^{1/2}\tilde{B}\sigma_1^{1/2}$, $B \in \mathcal{B}^1$, corresponding to the choice $\left(\mathcal{A}^0, \varrho_0\right) = \left(\mathcal{B}^1, \varsigma_1\right)$, defines the following density operator

$$\omega_q = (\mathrm{I} \otimes \Lambda^*)(\omega_{q1}), \quad \omega_{q1} = \sum_i \psi_1(i)\,\psi(i)^\dagger \tag{5.1}$$

of the input-output compound state $\varpi_{q1}\Lambda$ on $\mathcal{B}^1 \otimes \mathcal{B}$. It is given by the amplitude $\psi_1 = |\sigma_1^{1/2}(i)\rangle \in \mathcal{H}_1^{\otimes 2}$ defined by the components $\sigma_1^{1/2}(i) = \tilde{\psi}_1(i)$ of the central decomposition $\sigma_1 = \oplus \sigma_1(i)$.

The other extreme case of self-dual entanglements, the pure c-entanglements corresponding to (4.1), can be less informative then the d-entanglements, given by the decompositions $\sigma_1 = \sum \sigma_1(n)$ into pure states $\sigma_1(n) = \eta_n \eta_n^\dagger \mu(n)$. They define the density operators

$$\omega_d = (\mathrm{I} \otimes \Lambda^*)(\omega_{d1}), \quad \omega_{d1} = \sum_n |n\rangle\langle n| \otimes \eta_1(n)\,\eta_1(n)^\dagger, \tag{5.2}$$

of the $\mathcal{B}^1 \otimes \mathcal{B}$-compound state $\varpi_{d1}\Lambda$, which are known as the Ohya compound states $\varpi_o = \varpi_{o1}\Lambda$ [9] in the case

$$\sigma_1(n) = \eta_1^o(n)\,\eta_1^o(n)^\dagger, \quad \eta_1^o(n)^\dagger\,\eta_1^o(m) = \nu_1(n)\,\delta_n^m,$$

of orthogonality of the density operators $\sigma_1(n)$ normalized to the eigen-values $\nu_1(n)$ of σ_1. They are described by the input-output density operators

$$\omega_o = (\mathrm{I} \otimes \Lambda)^*\,\omega_{o1}, \quad \omega_{o1} = \sum_n |n\rangle\langle n| \otimes \eta_1^o(n)\,\eta_1^o(n)^\dagger. \tag{5.3}$$

coinciding with (5.1) in the case of Abelian $\mathcal{B}^1$. These input-output compound states ϖ are achieved by compositions $\lambda = \pi_0^1\Lambda$, corresponding to the entanglements of the output $(\mathcal{B}, \varsigma)$ of the channel to the extreme probe system $\left(\mathcal{A}^0, \varrho_0\right) = \left(\mathcal{B}^1, \varsigma_1\right)$.

If $\mathrm{K} : \mathcal{A} \to \mathcal{A}^0$ is a normal completely positive unital map

$$\mathrm{K}(A) = \mathrm{Tr}_{\mathcal{F}_-} X^\dagger A X, \quad A \in \mathcal{A},$$

where X is a bounded operator $\mathcal{F}_- \otimes \mathcal{G}_0 \to \mathcal{G}$ with $\mathrm{Tr}_{\mathcal{F}_-} X^\dagger X = I^0$, the compositions $\kappa = \pi_0^1 \mathrm{K}$, $\pi = \Lambda^* \kappa$ are the entanglements of the probe system $(\mathcal{A}, \varrho)$ with the channel input $\left(\mathcal{B}^1, \varsigma_1\right)$ and to the output $(\mathcal{B}, \varsigma)$ via this channel. The state $\varrho = \varrho_0 \mathrm{K}$ is given by

$$\mathrm{K}^* \left(\rho_0\right) = X \left(I^- \otimes \rho_0\right) X^\dagger \in \mathcal{A}_*$$

for each density operator $\rho_0 \in \mathcal{A}_*^0$, where I^- is the identity operator in $\mathcal{F}_-$. The resulting entanglement $\pi = \lambda^* \mathrm{K}$ defines the compound state $\varpi = \varpi_{01} (\mathrm{K} \otimes \Lambda)$ on $\mathcal{A} \otimes \mathcal{B}$ with

$$\varpi_{01} \left(A^0 \otimes B^1\right) = \mathrm{Tr}\ \tilde{A}^0 \pi_1^0 \left(B^1\right) = \mathrm{Tr}\ \upsilon_{01}^\dagger \left(A^0 \otimes B^1\right) \upsilon_{01}$$

on $\mathcal{A}^0 \otimes \mathcal{B}^1$. Here $\upsilon_{01} : \mathcal{F}_{01} \to \mathcal{G}_0 \otimes \mathcal{H}_1$ is the amplitude operator, uniquely defined by the input compound state $\varpi_{01} \in \mathcal{A}_*^0 \otimes \mathcal{B}_*^1$ up to a unitary operator U^0 on $\mathcal{F}_{01}$, and the effect of the input entanglement κ and the output channel Λ can be written in terms of the amplitude operator of the state ϖ as

$$\upsilon = (X \otimes Y) \left(I^- \otimes \upsilon_{01} \otimes I^+\right) U$$

up to a unitary operator U in $\mathcal{F} = \mathcal{F}_- \otimes \mathcal{F}_{01} \otimes \mathcal{F}_+$. Thus the density operator $\omega = \upsilon\upsilon^\dagger$ of the input-output compound state ϖ is given by $\varpi_{01} (\mathrm{K} \otimes \Lambda)$ with the density

$$(\mathrm{K} \otimes \Lambda)^* \left(\omega_{01}\right) = (X \otimes Y) \, \omega_{01} \, (X \otimes Y)^\dagger , \tag{5.4}$$

where $\omega_{01} = \upsilon_{01} \upsilon_{01}^\dagger$.

Let $\mathcal{K}_q^1$ be the set of all normal TCP maps $\kappa : \mathcal{A} \to \mathcal{B}_*^1$ with any probe algebra $\mathcal{A}$, normalized as $\mathrm{Tr}\kappa (I) = 1$, and $\mathcal{K}_q (\varsigma_1)$ be the subset of all $\kappa \in \mathcal{K}_q^1$ with $\kappa (I) = \varsigma_1$. Each $\kappa \in \mathcal{K}_q^1$ can be decomposed as $\kappa_0 \mathrm{K}$, where $\kappa_0^* = \pi_1^q = \kappa_0$ is the standard entanglement on $\left(\mathcal{A}^0, \varrho_0\right) = \left(\mathcal{B}^1, \varsigma_1\right)$, and K is a normal unital CP map $\mathcal{A} \to \mathcal{B}^1$. Further let $\mathcal{K}_c^1$ be the set of all c-entanglements κ, described by the combinations

$$\kappa (A) = \sum_n \varrho_n (A) \, \sigma_1 (n) \tag{5.5}$$

of the primitive maps $A \mapsto \varrho_n (A) \, \sigma_1 (n)$, and $\mathcal{K}_d^1$ be the subset of the diagonalizing entanglements κ, i. e. the decompositions

$$\kappa (A) = \sum_n \langle n|A|n\rangle \sigma_1 (n) \, . \tag{5.6}$$

As in the first case $\mathcal{K}_c (\varsigma_1)$ and $\mathcal{K}_d (\varsigma_1)$ denote the subsets corresponding to a fixed $\kappa (I) = \varsigma_1$, and each $\mathcal{K}_c (\varsigma_1)$ can be represented as $\kappa = \kappa_0 \mathrm{K}$, where κ_0 normalized as $\kappa_0 (I) = \sigma_1$ is given by a pure d-entanglement $\kappa_0^* = \pi_1^d$ on $\mathcal{B}^1$ and a proper choice of the CP map $\mathrm{K} : \mathcal{A} \to \mathcal{B}^1$. Furthermore let $\mathcal{K}_o^1$ and $\mathcal{K}_o (\varsigma_1)$ be the subset of all decompositions (5.5) with orthogonal $\sigma_1 (n)$ (and fixed $\sum_n \sigma_1 (n) = \sigma_1$):

$$\sigma_1 (m) \, \sigma_1 (n) = 0, \, m \neq n.$$

Each $\kappa \in \mathcal{K}_o (\varsigma_1)$ can also be represented as $\kappa = \kappa_0 \mathrm{K}$, with $\kappa_0^* = \pi_1^o$ given by a pure o-entanglement on $\mathcal{B}_1$ to $\mathcal{A}^0 = \mathcal{B}^1$ with $\varrho_0 = \varsigma_1$.

Now, let us maximize the entangled mutual entropy for a given quantum channel Λ (and a fixed input state ς_1) by means of the above four types entanglements κ. The mutual entropy (4.4) was defined in the previous section by the density operators ω of the corresponding compound state ϖ on $\mathcal{A}\otimes\mathcal{B}$, and the product-state $\varphi=\varrho\otimes\varsigma$ of the marginals ϱ, ς for ϖ. In each case

$$\varpi=\varpi_{01}\left(\mathrm{K}\otimes\Lambda\right),\quad \varphi=\varphi_{01}\left(\mathrm{K}\otimes\Lambda\right),$$

where K is a CP map $\mathcal{A}\to\mathcal{A}^{\circ}=\mathcal{B}^{1}$, ϖ_{01} is one of the corresponding extreme compound states ϖ_{q1}, $\varpi_{c1}=\varpi_{d1}$, ϖ_{o1} on $\mathcal{B}^{1}\otimes\mathcal{B}^{1}$, and $\varphi_{01}=\rho_{0}\otimes\varsigma_{1}$. The density operator $\omega=\left(\mathrm{K}\otimes\Lambda\right)^{*}\left(\omega_{01}\right)$ is written in (5.4), and $\phi=\rho\otimes\sigma$ can be written as

$$\phi=\kappa^{*}\left(I\right)\otimes\lambda^{*}\left(I\right),$$

where $\lambda^{*}=\Lambda^{*}\pi_{1}^{0}$.

Proposition 2. *The entangled mutual informations achieve the following maximal values*

$$\sup_{\kappa\in\mathcal{K}_{q}(\varsigma_{1})}\mathsf{I}_{\mathcal{A};\mathcal{B}}\left(\kappa^{*}\Lambda\right)=\mathsf{I}_{q}\left(\varsigma_{1},\Lambda\right):=\mathsf{I}_{\mathcal{B}^{1};\mathcal{B}}\left(\pi_{1}^{q}\Lambda\right),\tag{5.7}$$

$$\mathsf{I}_{c}\left(\varsigma_{1},\Lambda\right):=\sup_{\kappa\in\mathcal{K}_{c}(\varsigma_{1})}\mathsf{I}_{\mathcal{A};\mathcal{B}}\left(\kappa^{*}\Lambda\right)=\sup_{\pi_{1}^{d}}\mathsf{I}_{\mathcal{B}^{1};\mathcal{B}}\left(\pi_{1}^{d}\Lambda\right)\equiv\mathsf{I}_{d}\left(\varsigma_{1},\Lambda\right),$$

$$\sup_{\kappa\in\mathcal{K}_{o}(\varsigma_{1})}\mathsf{I}_{\mathcal{A};\mathcal{B}}\left(\kappa^{*}\Lambda\right)=\mathsf{I}_{o}\left(\varsigma_{1},\Lambda\right):=\sup_{\pi_{1}^{o}}\mathsf{I}_{\mathcal{B}^{1};\mathcal{B}}\left(\pi_{1}^{o}\Lambda\right),\tag{5.8}$$

where $\pi_{1}^{\cdot}$ *are the corresponding extremal input entanglements states on* $\mathcal{B}^{1}\to\mathcal{B}_{*}^{1}$ *with* $\mathrm{Tr}\pi_{1}^{\cdot}\left(B\right)=\varsigma_{1}\left(B\right)$ *for all* $B\in\mathcal{B}^{1}$. *They are ordered as*

$$\mathsf{I}_{q}\left(\varsigma_{1},\Lambda\right)\geq\mathsf{I}_{c}\left(\varsigma_{1},\Lambda\right)=\mathsf{I}_{d}\left(\varsigma_{1},\Lambda\right)\geq\mathsf{I}_{o}\left(\varsigma_{1},\Lambda\right).\tag{5.9}$$

Proof. Owing to the monotonicity

$$\mathsf{I}_{\mathcal{A};\mathcal{B}}\left(\kappa^{*}\Lambda\right)\leq\mathsf{I}_{\mathcal{B}^{1};\mathcal{B}}\left(\pi_{1}^{0}\Lambda\right),$$

the supremum over all c-entanglements $\kappa\in\mathcal{K}_{c}^{1}$ coincides with the supremum over $\mathcal{K}_{d}^{1}\subset\mathcal{K}_{c}^{1}$ which is achieved on the pure d-entanglements on $\left(\mathcal{B}^{1},\varsigma_{1}\right)$ corresponding to the extreme compound states ϖ_{d1}. By the same monotonicity arguments, we can get the equalities (5.7) and (**??**). Thus the entanglements $\kappa\in\mathcal{K}_{q}^{1}$, written as

$$\kappa\left(A\right)=\sum_{m,n}\langle m|A|n\rangle\psi_{1}\left(m\right)\psi_{1}\left(n\right)^{\dagger}=\kappa_{0}\left(\mathrm{K}\left(A\right)\right)$$

in terms of $\psi_{1}\left(n\right)=\left(\langle n|X\otimes I^{1}\right)\left(I^{-}\otimes\upsilon_{01}\right)$ with amplitude operators $\upsilon_{01}:\mathcal{F}_{0}\to\mathcal{G}_{0}\otimes\mathcal{H}_{1}$ on $\mathcal{G}_{0}=\mathcal{H}_{1}$, can be restricted to the case $\mathcal{A}^{0}=\mathcal{B}^{1}$ by taking $X=I^{0}=I^{1}$ corresponding to $\mathcal{F}_{-}=\mathbb{C}$, $\mathcal{G}=\mathcal{H}_{1}$ and sufficiently large $\mathcal{F}_{0}=\mathcal{F}$. Such amplitude operators $\upsilon_{01}=\sum\left(|n\rangle\otimes I^{1}\right)\psi_{01}\left(n\right)$ are weakly orthogonal

$$\mathrm{Tr}\ \psi_{01}\left(m\right)\psi_{01}\left(n\right)^{\dagger}=\mu\left(n\right)\delta_{n}^{m}.$$

in a basis $|n\rangle$ for the Schatten decomposition $\rho_{0}=\sum_{n}|n\rangle\mu\left(n\right)\langle n|$ of $\rho_{0}=\pi_{1}^{0}\left(I^{1}\right)$ for $\pi_{1}^{0}=\kappa_{0}^{*}$. If $\kappa_{0}\in\mathcal{K}_{d}^{1}$,

$$\kappa_{0}\left(A\right)=\sum_{n}\langle n|A|n\rangle\psi_{01}\left(n\right)\psi_{01}\left(n\right)^{\dagger}$$

corresponding to the stronger orthogonal amplitude operators

$$\psi_{01}\left(m\right)\psi_{01}\left(n\right)^{\dagger}=\sigma_{1}\left(n\right)\delta_{n}^{m}$$

defining not necessarily the orthogonal decompositions $\sigma_1 = \sum \sigma_1(n)$. Moreover, if $\kappa \in \mathcal{K}_1^o$, the amplitude operators $\psi_1(n)$ satisfying the second orthogonality condition

$$\psi_{01}(n)^\dagger \psi_{01}(m) = \tau_0(n)\,\delta_n^m,$$

where the $\tau_0(n)$ are density operators in $\mathcal{F} = \mathcal{F}_0$ with the traces $\mathrm{Tr}\,\tau_0(n) = \mu(n)$. Thus, the inequalities in (5.9) follow from $\mathcal{K}_q^1 \supseteq \mathcal{K}_c^1 \supseteq \mathcal{K}_d^1 \supseteq \mathcal{K}_o^1$. □

We shall denote the maximal informations $\mathsf{I}_c(\varsigma_1, \Lambda) = \mathsf{I}_d(\varsigma_1, \Lambda)$ simply as $\mathsf{I}_1(\varsigma_1, \Lambda)$.

Definition 4. *The suprema*

$$\mathsf{C}_q(\Lambda) = \sup_{\kappa \in \mathcal{K}_q^1} \mathsf{I}_{\mathcal{A};\mathcal{B}}(\kappa^*\Lambda) = \sup_{\varsigma_1} \mathsf{I}_q(\varsigma_1, \Lambda),$$

$$\sup_{\kappa \in \mathcal{K}_d^1} \mathsf{I}_{\mathcal{A};\mathcal{B}}(\kappa^*\Lambda) = \mathsf{C}_1(\Lambda) := \sup_{\varsigma_1} \mathsf{I}_1(\varsigma_1, \Lambda), \tag{5.10}$$

$$\mathsf{C}_o(\Lambda) = \sup_{\kappa \in \mathcal{K}_o^1} \mathsf{I}_{\mathcal{A};\mathcal{B}}(\kappa^*\Lambda) = \sup_{\varsigma_1} \mathsf{I}_o(\varsigma_1, \Lambda),$$

are called the q-, c- or d-, and o-capacities respectively for the quantum channel defined by a normal unital CP map $\Lambda : \mathcal{B} \to \mathcal{B}^1$.

Obviously, the capacities (5.10) satisfy the inequalities

$$\mathsf{C}_o(\Lambda) \le \mathsf{C}_1(\Lambda) \le \mathsf{C}_q(\Lambda).$$

Theorem 4. *Let $\Lambda(B) = Y^\dagger B Y$ be a unital CP map $\mathcal{B} \to \mathcal{B}^1$ describing a quantum deterministic channel. Then*

$$\mathsf{I}_1(\varsigma_1, \Lambda) = \mathsf{I}_o(\varsigma_1, \Lambda) = \mathsf{S}(\varsigma_1), \quad \mathsf{I}_q(\varsigma_1, \Lambda) = \mathsf{S}_q(\varsigma_1),$$

where $\mathsf{S}_q(\varsigma_1) = \mathsf{H}_{\mathcal{B}^1}(\varsigma_1)$, and thus in this case

$$\mathsf{C}_1(\Lambda) = \mathsf{C}_o(\Lambda) = \ln \operatorname{rank} \mathcal{B}^1, \quad \mathsf{C}_q(\Lambda) = \ln \dim \mathcal{B}^1.$$

Proof. It was proved in the previous section for the case of the identity channel $\Lambda = \mathrm{I}$, and thus it is also valid for any isomorphism Λ described by a unitary operator Y. In the case of non-unitary Y we can use the identity

$$\mathrm{Tr}\;Y\left(\sigma_1 \otimes I^+\right)Y^\dagger \ln Y\left(\sigma_1 \otimes I^+\right)Y^\dagger = \mathrm{Tr}\;S\left(\sigma_1 \otimes I^+\right)\ln S\left(\sigma_1 \otimes I^+\right),$$

where $S = Y^\dagger Y$. Due to this $\mathsf{S}(\varsigma_1\Lambda) = -\mathrm{Tr}\;S(\sigma_1 \otimes I^+)\ln S(\sigma_1 \otimes I^+)$, and $\mathsf{S}(\varpi_{01}(\mathrm{K} \otimes \Lambda)) =$

$$-\mathrm{Tr}\;(R \otimes S)\left(I^- \otimes \omega_{01} \otimes I^+\right)\ln(R \otimes S)\left(I^- \otimes \omega_{01} \otimes I^+\right),$$

where $R = X^\dagger X$. Thus $\mathsf{S}(\varsigma_1\Lambda) = \mathsf{S}(\varsigma_1)$, $\mathsf{S}(\varpi_{01}(\mathrm{K} \otimes \Lambda)) = \mathsf{S}(\varpi_{01}(\mathrm{K} \otimes \mathrm{I}))$ if $Y^\dagger Y = I$, and

$$\begin{aligned} \mathsf{I}_{\mathcal{A};\mathcal{B}}((\pi_1\Lambda)) &= \mathsf{S}(\varrho_0\mathrm{K}) + \mathsf{S}(\varsigma_1) - \mathsf{S}(\varpi_{01}(\mathrm{K} \otimes \mathrm{I})) \\ &\le \mathsf{S}(\varrho_0) + \mathsf{S}(\varsigma_1) - \mathsf{S}(\varpi_{01}) = \mathsf{I}_{\mathcal{A}^0;\mathcal{B}^1}(\varpi_{01}) \end{aligned}$$

for $\pi_1^* = \pi_0^1\mathrm{K}$ with any normal unital CP map $\mathrm{K} : \mathcal{A} \to \mathcal{A}^0$ and a compound state ϖ_{01} on $\mathcal{A}^0 \otimes \mathcal{B}^1$. The supremum (5.7), which is achieved at the standard entanglement, corresponding to $\varpi_{01} = \varpi_{q1}$, coincides with q-entropy $\mathsf{H}_{\mathcal{B}^1}(\varsigma_1)$, and the supremum (**??**), coinciding with $\mathsf{S}_{\mathcal{B}^1}(\varsigma_1)$, is achieved for a pure o-entanglement, corresponding to $\varpi_{01} = \varpi_{o1}$ given by any Schatten decomposition for σ_1. Moreover, the entropy $\mathsf{H}_{\mathcal{B}^1}(\varsigma_1)$ is also achieved by any pure d-entanglement, corresponding to $\varpi_{01} = \varpi_{d1}$ given by any extreme decomposition for σ_1, and thus is the maximal mutual information $\mathsf{I}_1(\varsigma_1, \Lambda)$ in the case of deterministic Λ. Thus the capacity $\mathsf{C}_1(\Lambda)$ of the deterministic channel is given by the maximum $\mathsf{C}_o = \ln \dim \mathcal{H}_1$ of the von Neumann entropy $\mathsf{S}_{\mathcal{B}^1}$, and the q-capacity $\mathsf{C}_q(\Lambda)$ is equal $\mathsf{C}_{\mathcal{B}^1} = \ln \dim \mathcal{B}^1$. □

In the general case, d-entanglements can be more informative than o-entanglements as can be shown by an example of a quantum noisy channel for which

$$\mathsf{I}_1(\varsigma_1, \Lambda) > \mathsf{I}_o(\varsigma_1, \Lambda), \quad \mathsf{C}_1(\Lambda) > \mathsf{C}_o(\Lambda).$$

The last equalities of the above theorem will be related to the work on entropy by Voiculescu [25].

References

[1] Bennett, C.H. and G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W.K. Wootters, Phys. Rev. Lett., **70**, pp.1895-1899, 1993.
[2] Ekert, A., Phys. Rev. Lett, **67**, pp.661-663.
[3] Jozsa, R. and B. Schumacher, J. Mod. Opt., **41**, pp.2343-2350, 1994.
[4] Schumacher, B., Phy. Rev. A, **51**, pp.2614-2628, 1993; Phy. Rev. A, **51**, pp.2738-2747, 1993.
[5] Bennett, C.H. and G. Brassard, S. Popescu, B. Schumacher, J.A. Smolin, W.K. Wootters, Phys. Rev. Lett., **76**, pp.722-725, 1996.
[6] Majewski, A.W., Separable and entangled states of composite quantum systems; Rigorous description, Preprint.
[7] Belavkin, V. P., Radio Eng. Electron. Phys., **25**, pp. 1445–1453 (1980).
[8] Belavkin, V. P., Found. of Phys., **24**, pp. 685–714 (1994).
[9] Ohya, M., IEEE Information Theory, **29**, pp. 770–774 (1983).
[10] Ohya, M., L. Nuovo Cimento, **38**, pp. 402–406, 1983.
[11] Accardi, L. and M. Ohya., "Compound Chanels, Transition Expectations and Liftings". to appear in Journal of Applied Mathematics and Optimization.
[12] Belavkin, V. P. and M. Ohya, "Quantum Entanglements and Entangled Mutual Entropy", to be published.
[13] Ohya, M., Rep. Math. Phys.,**27**, pp.19–47, 1989.
[14] Ohya, M. and D.Petz, "Quantum Entropy and Its Use"**,** Springer, 1993.
[15] Accardi, L., "Noncommutative Markov Chain". International School of Mathematical Physics, Camerino, pp.268–295 (1974).
[16] Accardi, L., Ohya M. and N. Watanabe., "Note on Quantum Dynamical Entropy." Reports on Mathematical Physics, **38**, 3, pp. 457–469 (1996).
[17] Belavkin, V. P., "Quantum Stochastic Positive Evolutions: Characterization, Construction, Dilation." Commun. Math. Phys., **184**, pp. 533–566 (1997).
[18] Stinespring, W. F., Proc. Amer. Math. Soc. **6**, p. 211 (1955).
[19] Kraus, K., Ann. Phys. **64**, p. 311 (1971).
[20] Davies E. B. and J. Lewis, Commun. Math. Phys., **17**, pp. 239–260 (1971).
[21] Lindblad, G., "Entropy, Information and Quantum Measurement". Comm. in Math. Phys. **33**, p305–322 (1973).
[22] Araki, H., "Relative Entropy of states of von Neumann Algebras", Publications RIMS, Kyoto University, **11**, pp. 809–833, (1976).
[23] Umegaki, H., Kodai Math. Sem. Rep., **14**, pp59-85, 1962.
[24] Uhlmann, A., Commun. Math. Phys., **54**, pp.21–32, 1977.
[25] Voiculescu, D.,Commun. Math. Phys., **170**, pp249-281, 1995

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