CONTINUOUS NON-DEMOLITION OBSERVATION, QUANTUM FILTERING AND OPTIMAL ESTIMATION

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Abstract. A quantum stochastic model for an open dynamical system (quantum receiver) and output multi-channel of observation with an additive nonvacuum quantum noise is given. A Master equation for the moment generating operator of the corresponding instrument is derived and quantum stochastic filtering equations both for the Heisenberg operators and the reduced density matrix of the system under the nondemolition observation are found. Thus the dynamical problem of quantum filtering is generalized for a noncommutative output process, and a quantum stochastic model and optimal filtering equation for the dynamical estimation of an input Markovian process is found. The results are illustrated on an example of optimal estimation of an input Gaussian diffusion signal, an unknown gravitational force say in a quantum optical or Weber's antenna for detection and filtering a gravitational waves.

Introduction. The time evolution of quantum system under a continuous observation can be obtained in the frame work of quantum stochastic (QS) calculus of output nondemolition processes, firstly introduced in [2] and recently developed in a quite general form in [11, 3, 4, 1, 5]. A stochastic posterior Schrödinger wave equation for an observed spinless particle derived in [4] by using the quantum filtering method [5], provided an explanation of the quantum Zeno paradox [7, 9]. In this paper we give a derivation of the reduced wave equation for a Markovian open system described by Heisenberg quantum stochastic operators $X(t)$ with respect to noncommuting Bose output fields $Y(s), s \in \mathbb{R}_+$ which are assumed to be nondemolition in the sense [4, 1, 5] of the commutativity $[X(t),\ Y(s)] = 0$ at each time $t \geq s$. We shall obtain it by a non–unitary dilation of the characteristic operator of an instrument for the observable output process, but in contrast to [6] we restrict ourselves to the diffusion observation, i.e. to a continuous nondemolition measurement of a quantum Brownian motion. This gives the possibility to solve the dynamical problems of quantum detection and estimation theory [12] as demonstrated in an example.

1. The dynamical model

We are going to describe a dynamical model for continuous in time indirect nondemolition observation of an arbitrary family $\boldsymbol{Q} = (Q_1, \dots, Q_n)$ of Hermitian

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operators $Q_j = Q_j^\dagger$, acting in initial Hilbert space $\mathcal{H}_0$ of an open quantum system (antenna), with additive δ–correlated Bose type quantum error-noises in linear n-dimensional output channel of observation $\mathrm{e}_j(t),\ j = 1, \dots, n$. We shall describe the output quantum error-noise $\mathbf{e}(t) = (\mathrm{e}_1, \dots, \mathrm{e}_n)(t)$ by the components $\widetilde{\mathrm{v}}_i(t) = \mathrm{e}_i(t)$ of a quantum stochastic process which is opposite, or inverse to an *input quantum noise* $\mathrm{v}_\cdot = (\mathrm{v}_i)$ of the same or higher dimensionality $m \geq n$. The total *quantum output noise* $\widetilde{\mathrm{v}}_\cdot = (\widetilde{\mathrm{v}}_i)$ as the opposite to $\mathrm{v}_\cdot$ can be defined as a compatible (commuting with $\mathrm{v}_\cdot$) but maximaly closed (maximally entangled with $\mathrm{v}_\cdot$) quantum stochastic process $\widetilde{\mathrm{v}}_\cdot$ which simply coincides with the input $\mathrm{v}_\cdot$ if it is is self-compatible, i.e. is classical, having all commutig components v_i.

Let us describe the quantum input noise $\mathrm{v}_\cdot$ by the classical white noise components $\mathrm{v}_i(t)$ represented by noncommuting Hermitian operator-valued distributions $\mathrm{v}_i = \mathrm{v}_i{}^\dagger$. They are completely determined in a quantum Gaussian state by the first and second moments

$$\langle \mathrm{v}_i(t)\rangle = 0,\ \ \langle \mathrm{v}_i(t)\mathrm{v}_k(t')\rangle = \kappa_{ik}\delta(t'-t). \tag{1.1}$$

Here κ_{ik} are complex elements of a Hermitian-positve matrix $\kappa = [\kappa_{ik}]$ of the same or higher dimensionality $m \geq n$, with imaginary part $\operatorname{Im}\kappa$ defining the Bose commutation relations

$$[\mathrm{v}_i(t),\ \mathrm{v}_k(t')] = 2\mathrm{i}\operatorname{Im}\kappa_{ik}\delta(t-t')\,1\ ,\ 2\mathrm{i}\operatorname{Im}\kappa_{ik} = \kappa_{ik} - \overline{\kappa}_{ik},$$

such that complex conjugate components $\overline{\kappa_{ij}} = \kappa_{ji} \equiv \widetilde{\kappa}_{ij}$ define the intensity covariance matrix $\widetilde{\kappa} = [\widetilde{\kappa}_{ij}]$ of the output noise $\widetilde{\mathrm{v}}_\cdot = (\widetilde{\mathrm{v}}_i)$:

$$\langle \widetilde{\mathrm{v}}_i(t)\widetilde{\mathrm{v}}_k(t')\rangle = \widetilde{\kappa}_{ik}\delta(t-t'),\ \ [\widetilde{\mathrm{v}}_i(t'),\ \widetilde{\mathrm{v}}_k(t)] = 2\mathrm{i}\operatorname{Im}\widetilde{\kappa}_{ik'}\delta(t-t')\,1$$

as transposed (or complex conjugate, $\overline{\kappa} = \widetilde{\kappa}$) to κ. Thus all output components $\widetilde{\mathrm{v}}_i$ commute with all input components v_k, and $\widetilde{\mathrm{v}}$ must also be maximally correlated with v in the sense that the intensities γ_{jk} of real covariances

$$\langle \widetilde{\mathrm{v}}_i(t)\mathrm{v}_k(t')\rangle = \gamma_{ik}\delta(t-t') = \gamma_{ki}\delta(t'-t) = \langle \mathrm{v}_k(t)\widetilde{\mathrm{v}}_i(t')\rangle$$

are the elements of a symmetric $m \times m$-matrix $\gamma = [\gamma_{ik}]$ as the *geometric mean* $\gamma = (\kappa \cdot \widetilde{\kappa})^{1/2}$. The geometric mean with $\widetilde{\kappa}$ for an invertible κ is defined as a Hermitian-positive matrix γ such that $\widetilde{\kappa} = \gamma\kappa^{-1}\gamma$. The matix γ is symmetric and invertible, with the inverse $\gamma^{-1} = \left[\gamma^{ik}\right]$ determining the inverse matrix κ^{-1} as the intensity matrix $\kappa^{-1} = \gamma^{-1}\widetilde{\kappa}\gamma^{-1} \equiv \left[\widetilde{\kappa}^{ik}\right]$ for the covariances

$$\langle \widetilde{\mathrm{v}}^i(t)\widetilde{\mathrm{v}}^k(t')\rangle = \widetilde{\kappa}^{ik}\delta(t-t'),\ \ [\widetilde{\mathrm{v}}^i(t),\ \widetilde{\mathrm{v}}^k(t')] = 2\mathrm{i}\operatorname{Im}\widetilde{\kappa}^{ik}\delta(t-t')\,1$$

of the *contravariant components* $\widetilde{\mathrm{v}}^i(t) = \gamma^{i,\cdot}\widetilde{\mathrm{v}}_\cdot(t)$ for the output noise $\widetilde{\mathrm{v}}_\cdot(t)$. (We assume that Hermitian matrix $\kappa = [\kappa_{ik}]$ is strictly positive, with the inverse $\kappa^{-1} = \left[\widetilde{\kappa}^{ik}\right]$ corresponding to a finite temperature of the output quantum noise $\widetilde{\mathrm{v}}^\cdot = \left(\widetilde{\mathrm{v}}^i\right)$).

As usual the operator-valued distributions $\mathrm{v}_i(t)$ can be described as generalized derivatives $\mathrm{v}_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}_i^t$ of quantum Wiener vector-process $\mathrm{v}_\cdot^t$ represented by selfadjoint quantum stochastic integrators $\mathrm{v}_i^t = \mathrm{v}_j^{t\dagger}$ with $\mathrm{v}_j^0 = 0$ and independent increments $\mathrm{d}\mathrm{v}_j^t = \mathrm{v}_j^{t+\mathrm{d}t} - \mathrm{v}_j^t$ satisfying the multiplication table $\mathrm{d}\mathrm{v}_i^t\mathrm{d}\mathrm{v}_k^t = \kappa_{ik}\mathrm{d}t$ which is noncommutative if $\operatorname{Im}\kappa^{ik} \neq 0$. Assuming for simplicity that matrix κ commutes with the transposed $\widetilde{\kappa}$ one can realize such quantum Wiener noise with respect to the faithful state given by the vacuum vector $\delta_\emptyset$ in a Fock space $\mathcal{F}$ as $\mathrm{v}_j^t = \breve{\mathrm{a}}_j^t + \breve{\mathrm{a}}_j^{t\dagger} \equiv 2\Re\breve{\mathrm{a}}_j^t$, the doubled Hermitian parts of the linear combinations

$\breve{\mathrm{a}}^t_{\cdot} = \kappa^{1/2}\mathrm{a}^{\cdot}_t$. Here a^k_t are the canonical annihilation integrators which are adjoint to the creation operators $\mathrm{a}^{t\dagger}_j$ in the intervals $[0, t)$ defined on the symmetrical tensors over the complex vector-functions $\alpha.(t) = [\alpha_1, \alpha_2, \ldots](t)$ with

$$(\alpha.|\alpha.) = \sum_{ik}\int \overline{\alpha}(t)\alpha(t)\mathrm{d}t \equiv \| \alpha \|^2 < \infty$$

as the symmetric tensor multiplication by the indicator function $1_{[0,t)}$. The canonical commutation relations

$$[\mathrm{a}(\alpha^*_{\cdot}),\ \mathrm{a}^\dagger(\alpha'_{\cdot})] = (\alpha.|\alpha'_{\cdot}),\ [\mathrm{a}(\alpha.),\ \mathrm{a}(\alpha'_{\cdot})] = 0 \tag{1.2}$$

then are realized by the quantum stochastic integrals $\mathrm{a}(\alpha^*_{\cdot}) = \int \bar{\alpha}_k(t)\mathrm{d}\mathrm{a}^k_t$, $\mathrm{a}^\dagger(\alpha.) = \mathrm{a}(\alpha^*_{\cdot})^\dagger$. (We use Einstein notations for the convolution $\bar{\alpha}_k\beta^k = \sum \bar{\alpha}_k\beta^k$ over the indices $k = 1, 2, \ldots$ in contrast to the scalar product notations $\beta \cdot \boldsymbol{\alpha}^*$ for the finite sums $\sum_{j=1}^n \beta^j\bar{\alpha}_j$, and omit the identity operator 1).

The output vector-process $\dot{\boldsymbol{Y}}(t) = \boldsymbol{Q}(t) + I_0 \otimes \mathbf{e}(t)$, defined by the integrals

$$Y_j(t) = \int_0^t Q_j(r)\mathrm{d}r + I_0 \otimes \mathrm{e}^t_j,\ \ j = 1, \ldots, n \tag{1.3}$$

of the Heisenberg operators $Q_j(t) = U(t)^\dagger (Q_j \otimes 1)\, U(t)$, can be realized for a singular coupling of the system with Bose fields a^k_t by the output observables $\mathrm{e}^t_j = 2\Re\hat{\mathrm{a}}^t_j$, $\hat{\mathrm{a}}^t_{\cdot} = \widetilde{\kappa}^{1/2}\mathrm{a}^{\cdot}_t$ in the interaction picture

$$Y_j(t) = U(t)^\dagger \left(I_0 \otimes \mathrm{e}^t_j\right) U(t) = 2\Re B_j(t), \tag{1.4}$$

where $B_j(t) = U(t)^\dagger(I \otimes \mathrm{a}^t_j)U(t)$ are the annihilation output processes, introduced in [11, 3, 4, 1], and I_0 is the identity operator in $\mathcal{H}_0$. The unitary evolution $U(t)$ will be described on the tensor product $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{F}_\kappa$ by a Schrödinger-Itô quantum stochastic equation [14]

$$\mathrm{d}U(t) + KU(t)\mathrm{d}t = \mathrm{i}\left(\frac{1}{\hbar}\ Q \otimes \mathrm{d}f(\vartheta_t) - 2\Im(L^\dagger_k \otimes \mathrm{d}\breve{\mathrm{a}}^k_t)\right) U(t), \tag{1.5}$$

in terms of the input integrators $\breve{\mathrm{a}}^{\cdot}_t = \widetilde{\kappa}^{-1/2}\mathrm{a}^{\cdot}_t = \gamma^{-1}\breve{\mathrm{a}}^t_{\cdot}$, where $L_k,\ \ k = 1, 2, \ldots$ are the operators in $\mathcal{H}_0$ with $L_j + L^\dagger_j = Q_j,\ \ j = 1, \ldots, n,$

$$K = \frac{1}{2}\ (\frac{\sigma^2}{\hbar^2}Q^2 f'(\vartheta_t)^2 + L^\dagger_i\kappa^{ik}L_k) + \frac{\mathrm{i}}{\hbar}H,$$

$f'(\vartheta) = \frac{\mathrm{d}}{\mathrm{d}\vartheta}\, f(\vartheta)$, $H = H^\dagger$ is a Hamiltonian of the system, and $Q = Q^\dagger$ is an operator in $\mathcal{H}_0$ of a generalized coordinate conjugate to the generalized force $f(t) = \frac{\mathrm{d}}{\mathrm{d}t}f(\vartheta_t)$ depending on an independent input diffusive signal ϑ_t, the random position of a gravitational source say, with $(\mathrm{d}\vartheta_t)^2 = \sigma^2\mathrm{d}t$. Note that in the case $L_k = L^\dagger_k$ this equation can be written as

$$\mathrm{d}U(t) + KU(t)\mathrm{d}t = \frac{\mathrm{i}}{\hbar}\left(\ Q \otimes \mathrm{d}f(\vartheta_t) + Q_k \otimes \mathrm{d}\mathrm{f}^k_t\right) U(t)$$

in terms of the integrators $\mathrm{f}^k_t = -\hbar\Im\breve{\mathrm{a}}^k_t$ of quantum Langevin forces $\mathrm{f}^k(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{f}^k_t$ satisfying the canonical commutation relations

$$\left[\mathrm{e}_j(t'), \mathrm{f}^i(t)\right] = \frac{\hbar}{\mathrm{i}}\delta^i_j\delta(t - t'),\quad [\mathrm{f}^i(t),\ \mathrm{f}^k(t')] = \mathrm{i}\frac{\hbar^2}{2}\,\mathrm{Im}\,\kappa^{ik}\delta(t - t')$$

corresponding to noncommutative multiplication tables

$$\mathrm{de}_j^t \mathrm{df}_t^k = \frac{\hbar}{2\mathrm{i}} \delta_j^k \mathrm{d}t, \quad \mathrm{df}_t^i \mathrm{df}_t^k = \left(\frac{\hbar}{2}\right)^2 \kappa^{ik} \mathrm{d}t,$$

including $\widetilde{\kappa}^{i,0} = (2\sigma f'/\hbar)^2 \delta_0^i$ for $\mathrm{f}_t^0 = f(\vartheta_t)$.

The solution $t \mapsto U(t)$ of the equation (1.4) for $U(0) = I$ is adaptive $U(s) = U^s \otimes I_{[s}$ with respect to the tensor decomposition $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{F}_{[s}$, where $\mathcal{H}_s = \mathcal{H}_0 \otimes \mathcal{F}_s$ and $\mathcal{F}_s$, $\mathcal{F}_{[s}$ are the Fock spaces, generated by the vector–function with $\alpha_s(t) = 0$, $\forall t \geq s$ and $\alpha_{[s}(t) = 0$, $t < s$ respectively. Hence $(I_0 \otimes \mathrm{e}_j^s)U(t) = U(t)Y_j(s)$ for any $s \leq t$, and

$$\begin{aligned} [Y_j(s),\ Y_k(t)] &= U(t)^\dagger (I_0 \otimes [2\Re \hat{\mathrm{a}}_i^s,\ 2\Re \hat{\mathrm{a}}_k^t]) U(t) = 2\mathrm{i}s \operatorname{Im} \widetilde{\kappa}_{jk} I \\ [Y_j(s),\ X(t)] &= U(t)^\dagger [I_0 \otimes \mathrm{e}_j^s,\ X \otimes 1] U(t) = 0, \quad \forall s \leq t \end{aligned} \tag{1.6}$$

for any operator $X(t) = U(t)^\dagger (X \otimes 1) U(t)$ of the system in the Heisenberg picture. This proves the nondemolition property of any observable process $Y_j(t)$ with respect to the system. It was introduced as the nondemolition causality principle for the output quantum processes in [2, 11]. Using the quantum Itô formula (1.4) for [4] with $\mathrm{de}_j = 2\Re \mathrm{d}\hat{\mathrm{a}}_j$ and multiplication table

$$\mathrm{d}\check{\mathrm{a}}_t^i \mathrm{d}\check{\mathrm{a}}_t^{k\dagger} = \kappa^{ik} \mathrm{d}t, \quad \mathrm{d}\check{\mathrm{a}}_t^i \mathrm{d}\hat{\mathrm{a}}_k^{t\dagger} = \delta_k^i \mathrm{d}t, \quad \mathrm{d}\hat{\mathrm{a}}_i^t \mathrm{d}\hat{\mathrm{a}}_k^{t\dagger} = \widetilde{\kappa}_{ik} \mathrm{d}t \tag{1.7}$$

with all other products being zero, one can derive the equation (1.3): $\mathrm{d}\boldsymbol{Y}(t) = \boldsymbol{Q}(t)\mathrm{d}t + I_0 \otimes \mathrm{d}\mathbf{e}(t)$, where $\boldsymbol{Q}(t) = [Q_1, \ldots, Q_n](t)$, $Q_j(t) = U(t)^\dagger (L_j + L_j^\dagger) U(t)$.

Theorem 1. *Let us suppose that the input real–valued signal ϑ_t satisfies the stochastic differential equation*

$$\mathrm{d}\vartheta_t + \upsilon(\vartheta_t)\mathrm{d}t = \sigma \mathrm{d}\mathrm{w}_t, \tag{1.8}$$

where w_t is an independent standard Wiener process, defined by the moments: $\langle \mathrm{w}_t \rangle = 0$, $\langle \mathrm{w}_s \mathrm{w}_t \rangle = s$ for any $s \leq t$ with respect to the vacuum state as $\mathrm{w}_t = 2\Re \mathrm{a}_t^0$ for the canonical annihilation integrator a_t^0 in the Fock space $\mathcal{F}$. Then any twice differentiable in the strong operator topology function $X : \vartheta \in \mathbf{R} \mapsto X(\vartheta) \in \mathcal{B}(\mathcal{H}_0)$ in the Heisenberg picture $X(t) = X^t(\vartheta_t)$, where $X^t = U(t)^\dagger X U(t)$, satisfies the following quantum stochastic Langevin equation

$$\begin{aligned} &\mathrm{d}X(t) + \left(\upsilon \delta X(t) + \frac{\mathrm{i}}{\hbar} \left[X(t), H(t) \right] - \frac{1}{2} \sigma^2 \delta^2 X(t) - \Lambda_L^{*t} \left[X(t) \right] \right) \mathrm{d}t \\ &= [X(t),\ 2\Re L_k^t] \otimes \frac{\mathrm{i}}{\hbar} \mathrm{df}_t^k + [X(t),\ \mathrm{i}\Im L_k^t] \otimes \mathrm{dv}_t^k + \sigma \delta X(t) \otimes \mathrm{dw}_t. \end{aligned} \tag{1.9}$$

Here $\mathrm{v}_t^i = 2\Re \check{\mathrm{a}}_t^i = \gamma^{ik} \mathrm{v}_k^t$, $\delta X(\vartheta) = X'(\vartheta) + f'(\vartheta) [X(\vartheta), \frac{\mathrm{i}}{\hbar} Q]$, $X'(\vartheta) = \frac{\mathrm{d}}{\mathrm{d}\vartheta} X(\vartheta)$ and

$$\Lambda_L^{*t}[X^t(\vartheta)] = \frac{1}{2} \sum_{i,k \geq 1}^{m} \kappa^{ik} (L_i^{t\dagger} [X^t(\vartheta),\ L_k^t] + [L_i^{t\dagger},\ X^t(\vartheta)] L_k^t). \tag{1.10}$$

2. The reduced evolution

Let $\mathcal{A}$ denote the input-system algebra which is assumed to be the von Neumann algebra of all essentially bounded operator-valued functions $X :\ \vartheta \in \mathbf{R} \mapsto X(\vartheta) \in \mathcal{B}(\mathcal{H}_0)$, $\hat{\mathfrak{b}}_t \subseteq \mathcal{B}(\mathcal{F}_t)$ be von Neumann subalgebra generated by the error-noises $\{\mathrm{e}_1^s, \ldots, \mathrm{e}_n^s\}$ for all $s \in [0, t)$, and $\mathcal{G}_t \subseteq \mathcal{F}_t$ be subspace generated by $\hat{\mathfrak{b}}_t$ on the vacuum $\delta_\emptyset \in \mathcal{F}_t$ for each $t > 0$, where $\mathcal{F}_t$ are Fock subspaces generated on $\delta_\emptyset$ by all input processes v_s^i, w_s or, equivalently, by forces $\left(\mathrm{f}_s^0, \mathrm{f}_s^1, \ldots\right)$ up to time t. The increasing family $\left\{\hat{\mathfrak{b}}_t : t > 0\right\}$ with respect to the output states induced by $\psi(t) = U(t)(\psi_0 \otimes \delta_\emptyset) \in \mathcal{F}_t$, $\psi_0 \in \mathcal{H}_0$ will be called the output filtration in the Schrödinger picture; It is equivalent to the Heisenberg filtration $\{\mathcal{B}_t, t > 0\}$ with $\mathcal{B}_t \subseteq \mathcal{A} \otimes \mathcal{B}(\mathcal{F}_t)$ generated by the output family $\{Y_1(s), \ldots, Y_n(s)\}$ for all $s \in [0, t)$ with respect to the initial states induced by $\psi(0) = \psi_0 \otimes \delta_\emptyset$, since $U_t^\dagger Y_j(s) U_t = I_0 \otimes \mathrm{e}_j^s$ with $U_t^\dagger = U(t) | \mathcal{H}_t$ for any $s \leq t$. The *filtered dynamics* of a quantum stochastic system, described in the Heisenberg picture by homomorphisms $X(t) = U_t X U_t^\dagger$ of $\mathcal{A} \ni X$ into $\mathcal{A} \otimes \mathcal{B}(\mathcal{F}_t)$, is defined by the cocycle of CP maps $\check{\Phi}_t : \mathcal{A} \to \mathcal{A} \otimes \check{\ell}_t$, where $\check{\ell}_t$ are (unbounded) commutants of $\hat{\mathfrak{b}}_t$ on $\mathcal{G}_t$, such that these dynamics induce the same input-output states on $\mathcal{A} \otimes \hat{\mathfrak{b}}_t$ with respect to the initial vacuum state:

$$\epsilon_\emptyset \left[\check{\Phi}_t[X] \left(I_0 \otimes \hat{b}\right)\right] = \epsilon_\emptyset \left[U_t \left(X(\vartheta_t) \otimes \hat{b}\right) U_t^\dagger\right] \tag{2.1}$$

for all $X \in \mathcal{A}$ and $\hat{b} \in \hat{\mathfrak{b}}_t$. Here $\epsilon_\emptyset[\cdot] = (I_0 \otimes \delta_\emptyset^*)[\cdot](I_0 \otimes \delta_\emptyset)$ is the vacuum (conditional) expectation $\mathcal{A} \otimes \mathcal{B}(\mathcal{F}) \to \mathcal{A}$ such that the composition $\rho \circ \epsilon_\emptyset \equiv \epsilon_\emptyset \circ \rho$ with any normal state ρ on $\mathcal{A}$ is the product state $\rho \otimes \epsilon_\emptyset$.

Since each $\check{\Phi}_t$ is normalized as $\check{\Phi}_t(I_0) = \check{P}_t$ to a positive element $\check{P}_t \in \mathcal{A} \otimes \check{\ell}_t$ defining typically unbounded density operator $\check{p}_t = \rho\left[\check{P}_t\right] \in \check{\ell}_t$ for the output state

$$\varsigma_t\left(\hat{b}\right) = \rho\left(\epsilon_\emptyset\left[U_t\left(I_0 \otimes \hat{b}\right) U_t^\dagger\right]\right) = \epsilon_\emptyset\left[\hat{b}\check{p}_t\right]$$

with respect to the input state $\epsilon_\emptyset\left[\hat{b}\right]$ on the algebra $\hat{\mathfrak{b}}_t$, it is necessary to give a more precise meaning of the unbounded commutant $\check{\ell}_t$ of $\hat{\mathfrak{b}}_t$.

First of all let us note that due to the nondegeneracy of the covariance matrix-function $[\widetilde{\kappa}_{jj'} \min\{s, s'\}]$ of $\mathbf{e}^s : s < t$ for each $t > 0$, the cyclic representation $\hat{\mathfrak{b}}_t | \mathcal{G}_t$ on the minimal invariant subspace $\mathcal{G}_t$ containing $\delta_\emptyset$ is faithful in the sense that $\hat{b} = 0$ in $\hat{\mathfrak{b}}_t$ if $\hat{b}\delta_\emptyset = 0$ in $\mathcal{F}_t$. Therefore $\hat{\mathfrak{b}}_t$ is transposed to its bounded commutant $\hat{\mathfrak{b}}_t'$, coinciding with $\check{\mathfrak{b}}_t = J\hat{\mathfrak{b}}_t J$, where J is a standard isometric involution defining the transposition $\hat{\mathfrak{b}}_t \to \check{\mathfrak{b}}_t$ by $\hat{b}_t' = J\check{b}_t^\dagger J$ common for all subspaces $\mathcal{G}_t$. Moreover, the von Neumann algebras $\hat{\mathfrak{b}}_t$ and $\hat{\mathfrak{b}}_t'$ are in one-to-one correspondence with the achieved Tomita algebras $\mathfrak{b}_t = \check{\mathfrak{b}}_t \delta_\emptyset$, $\mathfrak{b}_t' = \hat{\mathfrak{b}}_t \delta_\emptyset$ [15] as dense subspaces of $b = \hat{b}'\delta_\emptyset$, $b' = \hat{b}\delta_\emptyset$ in $\mathcal{G}_t$, where $\check{b} = J\hat{b}^\dagger J$, $\hat{b} \in \hat{\mathfrak{b}}_t$, with common identity $1 := \delta_\emptyset$, involutions $b^\sharp := \check{b}^\dagger \delta_\emptyset$, $b'^\flat := \hat{b}^\dagger \delta_\emptyset$ such that $b^{\sharp\prime} = b'^\flat$ and the norm $\| b \|_\infty := \left\|\check{b}\right\| = \left\|\hat{b}\right\| \equiv \| b' \|_\infty$.

Let us define a dual space ℓ_t to the Banach algebra $\mathfrak{b}_t'$ as the completion of $\mathfrak{b}_t \subseteq \mathcal{G}_t$ with respect to the dual norm

$$\| a \|_1 = \sup\{\left|(b', a)_\emptyset\right| : b \in \mathfrak{b}_t', \| b' \|_\infty \leq 1\} \equiv \|\check{a}\|_*,$$

where the bilinear form is the standard pairing of $\mathfrak{b}_t$ and $\mathfrak{b}_t'$,

$$(b', a)_\emptyset := \epsilon_\emptyset\left[\hat{b}\check{a}\right] \equiv \left\langle \hat{b}, \check{a} \right\rangle_\emptyset, \tag{2.2}$$

which we will extend by continuity on all $a \in \ell_t \supseteq \mathfrak{b}_t$. Thus the co-algebra $\check{\ell}_t$ of the operator algebra $\hat{\mathfrak{b}}_t$ can be described as the Banach space of operators $\check{a} : b' \mapsto \hat{b}a$, mapping $\mathfrak{b}_t' \subseteq \mathcal{G}_t$ into $\ell_t \supseteq \mathcal{G}_t$, bounded with respect to the predual norm $\|\check{a}\|_* = \| a \|_1$. This space is a linear span of positive cone $\left\{\check{p} \in \check{\ell}_t : \check{p} \geq 0\right\}$ such that $\left\langle \hat{b}^{\dagger}\hat{b}, \check{p}\right\rangle_{\emptyset} \geq 0$ for all $\hat{b} \in \hat{\mathfrak{b}}_t$, and therefore is invariant under the right and left action $\check{p} \mapsto \check{b}^{\dagger}\check{p}\check{b}$ of $\check{\mathfrak{b}}_t$ which is defined as the dual to the selfaction $\hat{q} \mapsto \hat{b}^{\dagger}\hat{q}\hat{b}$ on $\hat{\mathfrak{b}}_t$,

$$\left\langle \hat{q}, \check{b}^{\dagger}\check{p}\check{b}\right\rangle_{\emptyset} = \left\langle \hat{b}^{\dagger}\hat{q}\hat{b}, \check{p}\right\rangle_{\emptyset} \quad \forall \hat{q} \in \hat{\mathfrak{b}}_t, \check{p} \in \check{\mathfrak{b}}_t \tag{2.3}$$

extending the transposed selfaction on $\check{\mathfrak{b}}_t$ by $\check{b} = J\hat{b}^{\dagger}J$ for all $\hat{b} \in \hat{\mathfrak{b}}_t$. The coalgebra $\check{\ell}_t$ is also equipped with involution $\check{a} \mapsto \check{a}^{\dagger}$ defined by $\check{a}^{\dagger}b' = \hat{b}a^{\sharp}, \forall \check{a} \in \check{\ell}_t$ such that $\langle b',\ a^{\sharp}\rangle = \langle \hat{b},\ \check{a}^{\dagger}\rangle_{\emptyset}$, and with the identity $\check{1} = \hat{1}$, corresponding to the vacuum state $\epsilon_{\emptyset}\left(\hat{b}\right) = \langle \hat{b},\ \check{1}\rangle_{\emptyset}$. Note that the elements $\check{a} \in \check{\ell}$ obviously commute with $\hat{b} \in \hat{\mathfrak{b}}_t$:

$$\check{a}\hat{b}c' = \hat{b}\hat{c}a = \hat{b}\check{a}c' \quad \forall \check{a} \in \ell_t, \hat{b}, \hat{c} \in \hat{\mathfrak{b}}_t,$$

but are unbounded in the Hilbert space $\mathcal{G}_t$ if $a \notin \mathfrak{b}_t$. However they are densely defined as the bounded kernels in the Gelfand triple $\mathfrak{b}_t \subseteq \mathcal{G}_t \subseteq \ell_t$, and $\|\check{p}\|_* = \left\langle \hat{1}, \check{p}\right\rangle_{\emptyset}$ for any positive element $\check{p} \in \check{\ell}_t$ which means that it is density operator of a normal state on $\hat{\mathfrak{b}}_t$ if $\left\langle \hat{1}, \check{p}\right\rangle_{\emptyset} = 1$. Moreover, every normal state is uniquely given by such density, i.e. that the space $\check{\ell}_t$ is predual to $\hat{\mathfrak{b}}_t$ as is preadjoint to $\check{\mathfrak{b}}_t$ which we denote as $\check{\ell}_t^{\intercal} = \hat{\mathfrak{b}}_t$, $\check{\ell}_t^{*} = \check{\mathfrak{b}}_t$. In most cases the density operator $\check{p} \in \check{\ell}$ of an output state $\varsigma_t\left(\hat{b}\right) = \left\langle \hat{b}, \check{p}\right\rangle_{\emptyset}$ has the range $\check{p}\ell$ in $\mathcal{F}_t$, as it is in the case of a majorized positive form $\varsigma(\hat{b}^{\dagger}\hat{b}) \leq \lambda\epsilon_{\emptyset}(\hat{b}^{\dagger}\hat{b})$ for a $\lambda > 0$, corresponding to the bounded $\check{p}$ on $\mathcal{F}_t$: $\|\check{p}\| \leq \lambda$; moreover, any operator $\check{p} \in \check{\ell}$ can be approximated by the density operators from the bounded commutant $\check{\mathfrak{b}}_t$ in the norm $\|\check{p}\|_* = \| p \|_1$, where $p = \check{p}\delta_{\emptyset}$.

In order to derive a quantum stochastic equation for the reduced dynamics $\check{\Phi}_t$, let us find the differential evolution for the factorial generating map $\Phi_t^{(\beta)} :\ \mathcal{A} \to \mathcal{A}$,

$$\Phi_t^{(\beta)}[X] = \epsilon_{\emptyset}\left[\check{\Phi}_t[X](I \otimes \hat{z}_t^{(\beta)})\right] = (\delta_{\emptyset}|X^{(\beta)}(t)|\delta_{\emptyset}). \tag{2.4}$$

Here $\hat{b}_t = \hat{z}_t^{(\beta)}$ are exponential elements, defined by the Wick (normal) exponent

$$\hat{z}_t^{(\beta)} = e^{\hat{\mathrm{a}}^{\dagger}(\beta_t)}e^{\hat{\mathrm{a}}(\beta_t)} \equiv: \exp\left[\mathrm{e}_t(\beta)\right] : \tag{2.5}$$

of the observable $\mathrm{e}(\beta_t) = \hat{\mathrm{a}}\left(\beta_t\right) + \hat{\mathrm{a}}^{\dagger}\left(\gamma\beta_t\right) \equiv \int_0^t \beta(r) \cdot \mathrm{d}\mathbf{e}^r$, where $\beta = \left(\beta^i\right)$ is a column of locally square-integrable functions with $\beta_s^j(t) = 0$ for $t \geq s, j > n$, $\beta_s^j(r) = \beta^j(r)$ for $r < s$, and $X^{(\beta)}(t) = U_t(X(\vartheta_t) \otimes \hat{z}_t^{(\beta)})U_t^{\dagger} = X(t)Z(t)$. Taking into account the equation (1.9) and

$$\mathrm{d}Z(t) = Z(t)\beta^j(t)(I \otimes 2\mathrm{d}\Re\hat{\mathrm{a}}_j^t + U_t Q_j U_t^{\dagger}\mathrm{d}t)$$

for $Z(t) = U_t\left(I_0 \otimes \hat{z}_t^{(\beta)}\right)U_t^{\dagger}$, one can obtain by quantum Itô formula [14]

$$\begin{aligned} \mathrm{d}(X(t)Z(t)) \quad = \quad & U_t(\Lambda^*[X](\vartheta_t) + (\boldsymbol{L}^{\dagger}X_t^{(\beta)} + X_t^{(\beta)}\boldsymbol{L})\beta(t))U_t^{\dagger}\mathrm{d}t \\ & +\sigma U_t\delta X_t^{(\beta)}U_t^{\dagger} \otimes \mathrm{dw}_t + U_t X_t^{(\beta)}U_t^{\dagger}\beta^j(t) \otimes \mathrm{d}2\Re\hat{\mathrm{a}}_j^t \\ & +U_t[L_k^{\dagger}, X_t^{(\beta)}]U_t^{\dagger} \otimes \mathrm{d}\check{\mathrm{a}}_t^k + \mathrm{d}\check{\mathrm{a}}_t^{k\dagger} \otimes U_t[X_t^{(\beta)},\ L_k]U_t^{\dagger}, \end{aligned}$$

where $X_t^{(\beta)} = X(\vartheta_t) \otimes \hat{z}_t^{(\beta)}$, $\delta = \partial/\partial\vartheta_t + f'\ [\cdot, \frac{\mathrm{i}}{\hbar}Q]$. Hence the map $\Phi_t^{(\beta)} :\ X \in \mathcal{A} \mapsto (\delta_{\emptyset}|U_t X_t^{(\beta)}U_t^{\dagger}|\delta_{\emptyset})$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\ \Phi_t^{(\beta)}[X] = \Phi_t^{(\beta)}[\Lambda^*[X] + (\boldsymbol{L}^{\dagger}X + X\boldsymbol{L})\boldsymbol{\beta}(t)], \quad \Phi_0^{(\beta)}[X] = X, \tag{2.6}$$

where $\Lambda^*[X](\vartheta) = -\upsilon(\vartheta)\,\delta X(\vartheta) - \frac{\mathrm{i}}{\hbar}\,[X(\vartheta), H] + \frac{1}{2}\sigma^2\,\delta^2 X(\vartheta) + \Lambda_L^*[X(\vartheta)]$.

In the same way, using the Itô formula for the product $\check{G}_t\hat{Z}_t$ of $\check{G}_t = \check{\Phi}_t[X]$ and the Wick exponent (2.5), one can obtain the equation for (2.4) if

$$\mathrm{d}\check{\Phi}_t[X] - \check{\Phi}_t[\Lambda^*[X]]\mathrm{d}t = \sum_{j=1}^{n} \check{\Phi}_t[L_j^\dagger X + XL_j] \otimes \mathrm{dv}_t^j, \tag{2.7}$$

where the operators $\mathrm{v}_t^j = 2\Re\check{\mathrm{a}}_t^j$, satisfying the canonical commutation relations

$$[\mathrm{v}_t^{j'},\ \mathrm{v}_{t'}^j] = 2\mathrm{i}\,\mathrm{Im}\widetilde{\kappa}^{j'j}\min\{t,\ t'\},\quad \left[\mathrm{v}_{t'}^j, \mathrm{e}_{j'}^t\right] = 0,$$

generate the predual coalgebra $\check{\ell}_t$ as the unbounded commutant of $\hat{\mathfrak{b}}_t = \{\mathrm{e}_j^s : s < t\}''$ since $\mathrm{v}_s^j, j = 1, \dots, n$ leave the subspaces $\mathcal{G}_t$ invariant for any $t \geq s$, and $\{\mathrm{e}_j^s : s < t, j \leq n\}' = \check{\mathfrak{b}}_t = \{\mathrm{v}_s^j : s < t, j \leq n\}''$ on $\mathcal{G}_t$. This proves the following theorem.

Theorem 2. *Let the initial state $\rho : \mathcal{A} \to \mathbf{C}$ be a normal one, described by a density $\varrho :\ \vartheta \mapsto \varrho(\vartheta)$ with values in trace–class operators on $\mathcal{H}_0$ such that$\rho(X) = \int \mathrm{Tr}\varrho(\vartheta)X(\vartheta)d\vartheta$. Then the conditional state*

$$\check{\phi}_t[X] = \int \mathrm{Tr}_{\mathcal{H}_0}[\varrho(\vartheta)\check{\Phi}_t[X](\vartheta)]\mathrm{d}\vartheta \tag{2.8}$$

is described in the standard representation by the operator-function $\check{\varphi}_t(\vartheta)$ as $\check{\phi}_t[X] = \int \mathrm{Tr}_{\mathcal{H}_0}[\check{\varphi}_t(\vartheta)X(\vartheta)]d\vartheta$ satisfying the quantum stochastic equation

$$\mathrm{d}\check{\varphi}_t(\vartheta) = \Lambda[\check{\varphi}_t](\vartheta)\mathrm{d}t + \sum_{j=1}^{n}(L_j\check{\varphi}_t(\vartheta) + \check{\varphi}_t(\vartheta)L_j^\dagger) \otimes \mathrm{dv}_t^j. \tag{2.9}$$

with $\check{\varphi}_0(\vartheta) = \varrho(\vartheta)$. Here the quantum stochastic differentials dv_t^j together with de_j^t have the canonical multiplication table

$$\mathrm{dv}_t^j\mathrm{dv}_t^{j'} = \widetilde{\kappa}^{jj'}\mathrm{d}t,\ \ \mathrm{dv}_t^j\mathrm{de}_{j'}^t = \delta_{j'}^j\mathrm{d}t,\ \ \mathrm{de}_j^t\mathrm{de}_{j'}^t = \widetilde{\kappa}_{jj'}\mathrm{d}t,$$

and $\Lambda[\varphi] = \delta(\upsilon\varphi) + \frac{\mathrm{i}}{\hbar}\,[\varphi, H] + \frac{1}{2}\sigma^2\,\delta^2\varphi + \Lambda_L[\varphi]$ is preadjoint generator defined on $\mathcal{A}_*$ by $\delta\varphi(\vartheta) = \varphi'(\vartheta) + f'(\vartheta)\left[\varphi(\vartheta), \frac{\mathrm{i}}{\hbar}Q\right]$,

$$\delta^2\varphi(\vartheta) \quad = \quad \varphi''(\vartheta) + (2f'(\vartheta)\ + f''(\vartheta))\,[\varphi'(\vartheta), \frac{\mathrm{i}}{\hbar}Q] + f'(\vartheta)^2\ \ [[\varphi(\vartheta), \frac{\mathrm{i}}{\hbar}Q], \frac{\mathrm{i}}{\hbar}Q],$$

$$\Lambda_L[\varphi](\vartheta) = \frac{1}{2}\sum_{i,k\geq 0}\kappa^{ki}([L_i,\ \varphi(\vartheta)L_k^\dagger] + [L_i\varphi(\vartheta),\ L_k^\dagger]), \tag{2.10}$$

It is normalized to a positive martingale $\check{p}_t = \int \mathrm{Tr}_{\mathcal{H}_0}\check{\varphi}_t(\vartheta)d\vartheta \in \check{\ell}_t$ as the density operator defining by (2.1) the output state $\langle\hat{b}(t)\rangle = \rho(\epsilon_\emptyset[B(t)])$ on $\ell' \ni z$ for $B(t) = U(t)^\dagger(I \otimes \hat{b})U(t)$ with respect to the vacuum state vector $\delta_\emptyset \in \mathcal{F}$ simply as $\langle B(t)\rangle = \langle\check{p}_t, \hat{b}\rangle_\emptyset$.

Note that since all the input components $\mathrm{v}_t^i, i = 1, \dots, n$ commute with the output components e_j^t and have with them zero correlations and thus are independent of e_j^t unless $i = j$, they generate the same subspaces $\mathcal{G}_t$ as $\mathrm{e}_j^t, j = 1, \dots, n$. On these subspaces they simply coincide with the transposed $\mathrm{e}_t^{i\prime} = J\mathrm{e}_t^iJ = \mathrm{e}_t^{i\prime\dagger}$ to the contravariant components $\mathrm{e}_t^i = 2\Re\hat{\mathrm{a}}_t^i$ of the output noises e_k^t, having the same autocorrelation and mutual correlation matrices with e_k^t as v_t^i and being given by

$\hat{\mathrm{a}}^{\cdot}_{t} = \kappa^{-1/2}\mathrm{a}^{\cdot}_{t} = \gamma^{-1}\hat{\mathrm{a}}^{t}_{\cdot}$ such that $\mathrm{e}^{i}_{t} = \sum_{k=1}^{m}\gamma^{ik}\mathrm{e}^{t}_{k}$. Thus in the filtering equation (2.9) the input noises v^{j}_{t} on $\mathcal{G}_t$ can be replaced by $\mathrm{e}^{j\prime}_{t}$ which can be given in the Gaussian case as linear combinations of the transposed noises $\mathrm{e}'_j = J\mathrm{e}_j J = \mathrm{e}'^{\dagger}_j$.

3. The optimal measurement.

The quantum filter defines a quantum measurement on the output of the system at a time instant $t \in \mathbf{R}_+$. In general it is described by a $\mathcal{B}_t$-valued positive measure $M(t, \mathrm{d}x)$ on a Borel space $\mathcal{X}$, normalized to the identity operator in $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{F}$: $\int M(t, \mathrm{d}x) = I$. The problem of optimal quantum observation is the problem of finding the optimal measurement $M^{\circ}(t)$, giving the minimal value of the mean

$$\int \langle S(t,x)M(t,\mathrm{d}x)\rangle = \int (m'_t(\mathrm{d}x), c_t(x))_{\emptyset} = \int \langle \hat{m}_t(\mathrm{d}x), \check{c}_t(x)\rangle_{\emptyset} \tag{3.1}$$

for an $\mathcal{A}$-valued measurable function $S : x \in \mathcal{X} \mapsto S(x) = S(x)^{\dagger}$ in the Heisenberg picture $S(t,x) = U_t S(x,\vartheta_t)U_t^{\dagger}$ with respect to an initial state $\langle\cdot\rangle = \rho \circ \epsilon_{\emptyset}[\cdot]$ on $\mathcal{A} \otimes \mathcal{B}_t$. Here the mean (3.1) is given in the standard representation $\mathcal{B}_t \to \hat{\mathfrak{b}}_t$ according to (2.1) as the integral of the pairing (2.2) for

$$m'_t(\mathrm{d}x) = \hat{m}_t(\mathrm{d}x)\delta_{\emptyset}, \;\; c_t(x) = \check{c}_t(x)\delta_{\emptyset}, \tag{3.2}$$

where $\hat{m}_t$ defines the measure $M(t) = U_t(I \otimes \hat{m}_t)U_t^{\dagger}$ in the Schrödinger picture, and

$$\check{c}_t(x) = \int \mathrm{Tr}_{\mathcal{H}_0}[\check{\varphi}_t(\vartheta)S(x,\vartheta)]\mathrm{d}\vartheta = \check{\phi}_t[S_t(x)]. \tag{3.3}$$

The duality principle gives the necessary and sufficient conditions [13] of optimality

$$\int \langle \hat{m}^{\circ}_t(\mathrm{d}x), \check{c}_t(x)\rangle = \inf_{\hat{m}\geq 0}\{\int \langle \hat{m}_t(\mathrm{d}x), \check{c}_t(x)\rangle : \int \hat{m}^{\circ}_t(\mathrm{d}x) = \hat{1}\} \tag{3.4}$$

of a positive $\hat{\mathfrak{b}}_t$-valued measure $\hat{m}^{\circ}_t \geq 0$ with $\int \hat{m}^{\circ}_t(\mathrm{d}x) = \hat{1}$ as the conditions for the dual problem

$$\sup\{\langle \hat{1}, \check{l}\rangle_{\emptyset} : \check{l}_t \leq \check{c}_t(x), x \in \mathcal{X}\} = \langle \hat{1}, \check{l}^{\circ}_t\rangle_{\emptyset},$$

where $\check{l}_t = \check{l}^{\circ}_t$ defined by $\check{l}_t b' = \hat{b} l_t$ is the operator $\check{l}^{\circ}_t \leq \check{c}_t(x)$, $\forall x \in \mathcal{X}$, for which $\int \langle \hat{m}^{\circ}_t(\mathrm{d}x), \check{c}_t(x)\rangle_{\emptyset} = \langle \hat{1}, \check{l}^{\circ}_t\rangle_{\emptyset}$. The last condition of optimality can be written in the form of the equation

$$(\check{c}_t(x) - \check{l}_t)m'_t(\mathrm{d}x), \text{ or } \hat{m}_t(\mathrm{d}x)(c_t(x) - l_t) = 0.$$

Let us consider now the problem of optimal estimation [12] of a selfadjoint operator-process $X(t) = X^t(\vartheta_t)$, given in the Schrödinger picture by a $\mathcal{B}(\mathcal{H}_0)$-valued function $X(\vartheta) = X(\vartheta)^{\dagger}$, with real $x = \lambda \in \mathbf{R}$ and $S(x,\vartheta) = (X(\vartheta) - xI)^2$. One can treat in such a way the problem of filtering of a real measurable function $x(\vartheta_t)$ of the input diffusion signal ϑ_t taking $X(\vartheta) = x(\vartheta)I$. In order to formulate the optimal estimate in terms of the measurement of the optimal output observable $\hat{x}^{\circ}_t \in \hat{\mathfrak{b}}_t$ as an appropriate posterior mean of $X(t)$ we need the quantum stochastic equation for $\hat{\phi}_t[X] = J\check{\phi}\left[X^{\dagger}\right]J$ in terms of the output noise e^{t}_{i}. It can be obtained by complex conjugation

$$\mathrm{d}\hat{\phi}_t[X] = \hat{\phi}_t \circ \Lambda^*[X](\vartheta)\mathrm{d}t + \sum_{j=1}^{n}\hat{\phi}_t\left[L_j X + X L_j^{\dagger}\right] \otimes \mathrm{d}\mathrm{e}^{j}_{t} \tag{3.5}$$

of the equation for the CP map $\check{\phi}_t : \mathcal{A} \to \check{\mathfrak{b}}_t$ described in (2.8) by the density operator $\check{\varphi}_t$ satisfying the equation (2.9). Here $\mathrm{e}_t^j = \sum_{j=1}^n \theta^{jj'} \mathrm{e}_{j'}^t$, where $\left[\theta^{jj'}\right] \equiv \boldsymbol{\theta}^{-1}$ is real symmetric $n\times n$-matrix with the inverse $\boldsymbol{\theta} = [\theta_{jj'}]$ as an intensity matrix of the standard covariaces

$$\langle \mathrm{e}_{j'}(t')\mathrm{e}_j'(t)\rangle = \theta_{j'j}\delta(t'-t) = \theta_{jj'}\delta\left(t-t'\right) = \left\langle \mathrm{e}_j'\left(t\right)\mathrm{e}_{j'}\left(t'\right)\right\rangle$$

of the output noises $\mathrm{e}_{j'}$ with the transposed components $\mathrm{e}_j' = J\mathrm{e}_j J = \mathrm{e}_j'^{\dagger}$ such that $\boldsymbol{\theta}\boldsymbol{\kappa}^{-1}\boldsymbol{\theta}^{\intercal} = \overline{\boldsymbol{\kappa}}$ for the $n\times n$-submatrix $\boldsymbol{\kappa}$ of κ. (It is the geometric mean of $\boldsymbol{\kappa}$ and $\overline{\boldsymbol{\kappa}}$, e.g. $\boldsymbol{\theta} = \boldsymbol{\kappa}^{\frac{1}{2}}\overline{\boldsymbol{\kappa}}^{\frac{1}{2}}$ if $\boldsymbol{\kappa}$ and $\overline{\boldsymbol{\kappa}}$ commute.) Note that since in general $\boldsymbol{\theta}$ is smaller than the $n\times n$-submatrix $\boldsymbol{\gamma} = \left[\gamma_{jj'}\right]$ of the mutual covariance matrix γ for $\mathrm{e}_i = \widetilde{\mathrm{v}}_i$ and v_k, $\mathrm{e}_j' \neq \mathrm{v}_j$ unless $m=n$, and similar $\mathrm{e}_t^j \neq \widetilde{\mathrm{v}}_t^j$ since $\mathrm{e}_t' = \mathrm{v}_t$ on $\mathcal{G}_t$ and $\boldsymbol{\gamma}^{-1} \leq \boldsymbol{\theta}^{-1}$.

Theorem 3. *The solution of the optimization problem (3.4) for the quadratic cost function*

$$\check{c}_t(\lambda) = \lambda^2\check{p}_t - 2\lambda\check{\phi}_t[X] + \check{\phi}_t[X^2]$$

is given by the spectral measure $\hat{m}_t^\circ$ of a selfadjoint operator $\hat{x}_t^\circ \in \hat{\mathfrak{b}}_t$ defined by $\hat{x}_t^\circ = J\check{x}_t J$ as transposed to the operator $\check{x}_t = \check{x}_t^\dagger$ resolving in $\check{\mathfrak{b}}_t$ the equation

$$\check{x}_t\check{p}_t + \check{p}_t\check{x}_t = 2\int \mathrm{Tr}_{\mathcal{H}_0}[X\left(\vartheta\right)\check{\varphi}_t(\vartheta)]\mathrm{d}\vartheta. \tag{3.6}$$

It is given as the symmetric posterior expectation $\hat{x}^\circ = \hat{\rho}_t\left[X\right]$ by an operator-function $\vartheta \mapsto \hat{\varrho}_t\left(\vartheta\right) \in \mathcal{B}_*\left(\mathcal{H}\right)\otimes\hat{\ell}_t$ with $\hat{\ell}_t = J\check{\ell}_t J$ as the density of the solution to the equation

$$\hat{p}_t\hat{\rho}_t\left[X\right] + \hat{\varrho}_t\left[X\right]\hat{p}_t = 2\hat{\phi}_t\left[X\right] \quad \forall X \in \mathcal{A}, \tag{3.7}$$

where $\hat{p}_t = J\check{p}_t J$. The symmetric posterior density $\hat{\varrho}_t = \hat{\varrho}_t^\dagger$ satisfies the nonlinear quantum stochastic equation

$$\mathrm{d}\hat{\varrho}_t(\vartheta) = \Gamma_t[\hat{\varrho}_t](\vartheta)\mathrm{d}t + \sum_{j=1}^n \Xi_t^j[\hat{\varrho}_t]\otimes \mathrm{de}_j^t, \;\; \hat{\varrho}_0(\vartheta) = \varrho(\vartheta), \tag{3.8}$$

where $\Gamma_t(\vartheta)$ and $\Xi_t^j(\vartheta)$ are defined by the solutions to the equations

$$\begin{aligned} \Re\hat{P}_t\Xi^j[\hat{\varrho}](\vartheta) &= \Re\left(\hat{P}_t(L^j\hat{\varrho}(\vartheta) + \hat{\varrho}(\vartheta)L^{j\dagger}) - P_t^j\varrho(\vartheta)\right), \\ \Re\hat{P}_t\Gamma_t[\hat{\varrho}](\vartheta) &= \Re(\hat{P}_t\Lambda[\hat{\varrho}](\vartheta) - \sum_{i,j=1}^n \kappa^{ji}\hat{P}_t^i\Xi_t^j[\hat{\varrho}]), \end{aligned} \tag{3.9}$$

with $L^j = \sum_{i=1}^n L_i\theta^{ij}$, $\hat{P}_t = I_0\otimes\hat{p}_t$ and $\hat{P}_t^j = I_0\otimes\hat{p}_t^j$ defined by $Q^j = L^j + L^{j\dagger}$ and $\hat{\varrho}_t$ as

$$\hat{p}_t^j = \Re[\hat{p}_t\hat{q}_t^j],\; \hat{q}_t^j = \int \mathrm{Tr}_{\mathcal{H}_0}[\hat{\varrho}_t(\vartheta)Q^j]\mathrm{d}\vartheta. \tag{3.10}$$

Proof. Denoting $\hat{u}_t = \int \lambda\hat{m}_t(\mathrm{d}\lambda)$ for an orthogonal projective-valued measure $\hat{m}_t(\mathrm{d}\lambda) \in \hat{\mathfrak{b}}_t$, one obtains

$$\begin{aligned} \int\langle\hat{m}_t(\mathrm{d}\lambda), \check{c}_t(\lambda)\rangle &= \langle\hat{u}_t^2, \check{p}_t\rangle - 2\langle\hat{u}_t, \check{\phi}_t[X]\rangle + \langle 1, \check{\phi}_t[X^2]\rangle \\ &= \langle\hat{u}_t^2, \check{p}_t\rangle - 2\langle\Re\left(\hat{x}_t\hat{u}_t\right), \check{p}_t\rangle + \langle 1, \check{\phi}_t[X^2]\rangle \geq \langle\hat{1}, \check{l}_t^\circ\rangle, \end{aligned}$$

where $\hat{x}_t$ is defined by the duality (2.3) as the transposed to the solution $\check{x}$ of the equation (3.6) written as $\Re\check{p}_t\check{x}_t = \check{\phi}_t[X]$, and $\check{l}_t^\circ \in \check{\ell}_t$ is defined as

$$\check{l}_t^\circ = \check{\phi}_t[X^2] - \check{x}_t\check{p}_t\check{x}_t.$$

The inequality is due to positivity of $\check{p}_t = \check{\phi}\,[I_0] \in \check{\ell}_t$ and $(\hat{u} - \hat{x})^2 \in \hat{\mathfrak{b}}_t$:

$$\int \langle \hat{m}_t(\mathrm{d}\lambda), \check{c}_t(\lambda)\rangle - \langle \hat{1}, \check{l}_t^\circ\rangle = \langle (\hat{u}_t - \hat{x}_t)^2,\ \check{p}_t\rangle \geq 0,$$

and the equality is achieved at the spectral measure $\hat{m}_t^\circ$ of the selfadjoint operator $\hat{x}_t \in \hat{\mathfrak{b}}_t$ defining $\hat{u}_t$ as $\hat{x}_t$.

Representing the solution of the equation (3.6) as

$$\check{x}_t = \int \mathrm{Tr}_{\mathcal{H}_0} X\,(\vartheta)\,\check{\varrho}_t\,(\vartheta)\,\mathrm{d}\vartheta \equiv \check{\rho}_t\,[X]$$

in terms of the density function $\check{\varrho}_t \in \mathcal{A}_* \otimes \check{\ell}_t$ we note that $\check{\rho}_t$ satisfies the equation transposed to (3.7), and therefore $\hat{\rho}_t\,[X] = J\check{\rho}_t\,\left[X^\dagger\right] J$ satisfies the equation (3.7).

Looking for the operator $\hat{x}_t = \hat{\rho}_t\,[X]$ as the solution of a quantum stochastic equation

$$\mathrm{d}\hat{x}_t = \hat{g}_t\mathrm{d}t + \sum_{j=1}^n \hat{c}_t^j \otimes \mathrm{de}_j^t$$

with some $\hat{g}_t = \hat{g}_t^\dagger,\ \hat{c}_t^j = \hat{c}_t^{j\dagger}$, we should compare the quantum stochastic differential for $\hat{\phi}_t\,[X] = J\check{\phi}_t\,\left[X^\dagger\right] J$, with the differential

$$\begin{aligned}\mathrm{d}\Re\hat{p}_t\hat{x}_t \quad &= \quad \Re(\mathrm{d}\hat{p}_t\mathrm{d}\hat{x}_t + \mathrm{d}\hat{p}_t\hat{x}_t + \hat{p}_t\mathrm{d}\hat{x}_t) = \Re\sum_{i,j=}^n \kappa_{ji}\hat{\phi}_t[Q^i]\hat{c}_t^j\mathrm{d}t \\ &\quad +\Re\sum_{i=1}^n \hat{\phi}_t[Q^i]\hat{x}_t \otimes \mathrm{de}_i^t + \Re\hat{p}_t(\sum_{j=1}^n \hat{c}_t^j \otimes \mathrm{de}_j^t + \hat{g}_t\mathrm{d}t),\end{aligned}$$

for $\Re\hat{p}_t\hat{x}_t = \hat{\phi}_t[X]$ obtained applying the quantum Itô formula, where we took

$$\mathrm{d}\hat{p}_t = \sum_{j=1}^n \hat{\phi}_t[Q^j] \otimes \mathrm{de}_j^t,\ \ Q^j = \sum_{i=i}^n Q_i\theta^{ij}$$

for the martingale $\hat{p}_t = \hat{\phi}_t\,[I_0]$. Here $\mathrm{e}_i^t = \sum_{j=1}^n \theta_{ij}\mathrm{e}_t^j$ as we have expressed the differential $\mathrm{d}\hat{p}_t = \sum_{j=1}^n \hat{\phi}_t[Q_j] \otimes \mathrm{de}_t^j$ for the martingale $\hat{p}_t = J\check{p}_tJ$ in terms of the driving output error noises e_j^t by the real linear transformation $\boldsymbol{\theta}^{-1}$. Comparing $\mathrm{d}\Re\hat{p}_t\hat{x}_t$ with the equation (3.5) for $\mathrm{d}\hat{\phi}_t\,[X]$

$$\mathrm{d}\hat{\phi}_t\,[X] = \hat{\phi}_t \circ \Lambda^*[X](\vartheta)\mathrm{d}t + \sum_{j=1}^n \hat{\phi}_t\,\left[L^jX + XL^{j\dagger}\right] \otimes \mathrm{de}_j^t$$

written in terms of e_j^t, we derive

$$\begin{aligned}\Re[\hat{\phi}_t[Q^j]\hat{x} + \hat{p}_t\hat{c}_t^k] \quad &= \quad \hat{\phi}_t[L^{j\dagger}X + XL^j] = \Re\hat{p}_t\hat{\rho}_t[L^{j\dagger}X + XL^j], \\ \Re[\kappa_{ji}\hat{\phi}_t[Q^i]\hat{c}_t^j + \hat{p}_t\hat{g}_t] \quad &= \quad \hat{\phi}_t[\Lambda^*[X]] = \Re\hat{p}_t\hat{\rho}_t[\Lambda^*[X]],\end{aligned}$$

where $\hat{\rho}_t[X] = \hat{x}_t$. This gives (3.8)–(3.10) in terms of $\Xi_t^j,\ \Gamma_t$, defining

$$\hat{c}_t^j = \int \mathrm{Tr}_{\mathcal{H}_0}[\Xi_t^j[\hat{\varrho}_t](\vartheta)X(\vartheta)]\mathrm{d}\vartheta,\ \hat{g}_t = \int \mathrm{Tr}_{\mathcal{H}_0}[\Gamma_t[\hat{\varrho}_t](\vartheta)X(\vartheta)]\mathrm{d}\vartheta.$$

Note that the unnormalized posterior expectation $\hat{\phi}_t : X \mapsto \hat{\phi}_t[X]$, as well as its normalized version $\hat{\rho}_t[X] = \int \mathrm{Tr}_{\mathcal{H}_0}[X(\vartheta)\,\hat{\varrho}_t(\vartheta)]\mathrm{d}\vartheta$ is not CP map but transpose-CP, in the contrast to the CP map $\check{\phi}_t : \mathcal{A} \to \check{\mathfrak{b}}_t$ and its normalized version $\check{\rho}_t[X] = J\hat{\rho}\left[X^\dagger\right]J$.

Example. Let us consider the case $n = 1$ with $L_1 = \frac{1}{2}\,Q = L_1^\dagger$, $\gamma^{1/2}\check{\mathrm{a}}_t = \mathrm{a}_t^1 = \gamma^{-1/2}\hat{\mathrm{a}}^t$, where a_t^1 is the standard annihilation integrator $[\mathrm{a}_s^1, \mathrm{a}_t^{1\dagger}] = \min\{s,t\}$ defining the input noise $\mathrm{v}_t = 2\Re\check{\mathrm{a}}_t$ and nondemolition observation of the commutative output process

$$\mathrm{d}Y(t) = Q(t)\mathrm{d}t + I_0 \otimes \mathrm{de}^t,\ \ Q(t) = U(t)^\dagger QU(t)$$

by $\mathrm{e}^t = 2\Re\hat{\mathrm{a}}^t = \gamma\mathrm{v}_t$. In this case $\mathrm{e}_t = \mathrm{v}_t$, $y^t := \mathrm{e}^t\delta_\emptyset$ is $v_t' = \gamma v_t$, where $v_t = \mathrm{v}_t\delta_\emptyset$ is given as $\gamma^{-1/2}w_t^1$ by the standard Wiener process w_t^1 identified with the vector process $\mathrm{a}_t^{1\dagger}\delta_\emptyset$ in Fock space, and the output process $Y(t)$ on the initial state vector $\psi_0\otimes\delta_\emptyset$ is identified with the classical output process $y^t = \gamma^{1/2}w_t^1 = \gamma v_t$ relatively to the probability density $p_t(v) \equiv \hat{p}_t\delta_\emptyset$ with respect to the Wiener probability measure P_γ of the input Wiener process $\{v_t\}$ with the intensity γ^{-1}. The equations (3.5) and (3.8) are classical stochastic equations in the linear (for the nonnormalized $\varphi_t(\vartheta, v) \equiv \hat{\varphi}_t(\vartheta)\delta_\emptyset$) and nonlinear (for $\varrho_t(\vartheta, v) = \varphi_t(\vartheta, v)/p_t(v)$) posterior density operators with respect to the output states

$$\langle\psi \otimes \delta_\emptyset|B_t\psi \otimes \delta_\emptyset\rangle = \int b_t(v)\,p_t(v)\,\mathrm{P}_\gamma(dv) \equiv \langle b_t(\mathrm{v}), p_t(\mathrm{v})\rangle_\emptyset$$

on $B(t) = b_t\left(\gamma^{-1}Y\right)$ given by any adapted functional b_t of v.

Suppose that the quantum receiver is an open oscillator (e.g. Weber's antenna), described by the Hamiltonian $H = \frac{1}{2}\,A^\dagger A$, where $A = \mathrm{i}P + \omega Q$ and Q, P are the canonical coordinate and momentum operators: $[Q,P] = \mathrm{i}\hbar I_0$. Then the quantum Langevin equation (1.9) for $A(t) = U_tAU_t^\dagger$ is the linear one

$$\mathrm{d}A(t) + \mathrm{i}\omega A(t)\mathrm{d}t = \mathrm{i}I \otimes (\mathrm{d}\vartheta_t + \mathrm{df}_t),$$

where $\mathrm{f}_t = \hbar\Im\check{\mathrm{a}}_t^\dagger$ is the Langevin force (thermal noise) as a classical Wiener process of the intensity $\sigma_\gamma^2 = \hbar^2/4\gamma$ acting on the coordinate Q, defining the total force in the right hand side of the equation as the sum $\vartheta_t + \mathrm{f}_t$ of the unknown gravitational force $f(\vartheta_t) = \vartheta_t$ and the thermal noise through the additive channel $\vartheta_t(\mathrm{w}) + \mathrm{f}_t$. In the case of Gaussian input process ϑ_t, corresponding to the linear $v(\vartheta) = \upsilon\vartheta$ and Gaussian initial state ϕ the optimal estimate of $X(\vartheta_t) = \vartheta_tI \equiv X(t)$ is given by the linear posterior mean value $\hat{\vartheta}_t = \hat{x}_t$ with respect to the output coordinate process $Y(t)$. In the standard Fock representation it is given as the last component of the stochastic row $\hat{\boldsymbol{x}}_t = (\hat{q}_t, \hat{p}_t, \hat{x}_t)$ of the posterior mean values for $\boldsymbol{X}(t) = (Q(t), P(t), X(t))$, satisfying the Kalman equation

$$\mathrm{d}\hat{\boldsymbol{x}}_t + \hat{\boldsymbol{x}}_t\boldsymbol{\Lambda}\mathrm{d}t = \boldsymbol{k}_t\mathrm{d}\tilde{\mathrm{v}}_t,\ \ \boldsymbol{\Lambda} = \begin{pmatrix} 0 & \omega^2 & 0 \\ -1 & 0 & 0 \\ 0 & -\upsilon & \upsilon \end{pmatrix},$$

where $\boldsymbol{k}_t = (k_t^{11}, k_t^{12}, k_t^{13})$ is the first row of the symmetric 3×3–matrix $\boldsymbol{K}_t^{\mathrm{s}} = (k_t^{ij})$ satisfying the Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\boldsymbol{K}_t + \boldsymbol{K}_t\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^\top\boldsymbol{K}_t + \frac{1}{\gamma}\boldsymbol{k}_t^\top\boldsymbol{k}_t = \boldsymbol{\Upsilon},\ \ \boldsymbol{\Upsilon} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma^2 + \sigma_\gamma^2 & \sigma^2 \\ 0 & \sigma^2 & \sigma^2 \end{pmatrix},$$

with an initial symmetric covariance matrix $\boldsymbol{K}_0^{\mathrm{s}}$ of (Q, P, ϑ_0) and $\tilde{\mathrm{v}}_t = \mathrm{v}_t - \gamma^{-1}\hat{q}_t$. The pair $(\hat{\boldsymbol{x}}_t, \boldsymbol{K}_t)$ defines the posterior (normalized) Gaussian state of the quantum system with input signal $x_t = \vartheta_t$ and the mean square error $\langle \overset{\sim}{\vartheta}{}_t^2 \rangle$, $\overset{\sim}{\vartheta}_t = \vartheta_t - \hat{\vartheta}_t$ is given by the component k_t^{33} of the posterior correlation matrix $\boldsymbol{K}_t$.

A posterior dynamics of the quantum system under another nondemolition measurement of the received electromagnetic field by a photon counter is considered in [8].

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