POSITIVE DEFINITE GERMS OF QUANTUM STOCHASTIC PROCESSES

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Abstract. A new notion of stochastic germs for quantum processes is introduced and a characterisation of the stochastic differentials for positive definite (PD) processes is found for an arbitrary Itô algebra. A representation theorem, giving the pseudo-Hilbert dilation for the germ-matrix of the differential, is proved. This suggests the general form of quantum stochastic evolution equations with respect to the Poisson (jumps), Wiener (diffusion) or general quantum noise.

Germes positivement definis de processus quantiques stochastiques

Résumé. On trouve une characterisation des différentielles stochastiques des processus quantiques positifs définis (PD) pour une algèbre de Itô arbitraire. On démontre un théorème de représentation qui donne la dilatation pseudo-Hilbertienne de la matrice des germes de la différentielle. Ceci suggère une forme générale des équations d'évolution quantiques stochastiques par rapport aux sauts de Poisson, à la diffusion de Wiener ou aux bruits quantiques généralisés.

Version française abrégée.

1. Introduction. Le but du présent article est de généraliser le théorème de Evans-Lewis [1] sur la construction différentielle de la dilatation de Stinespring [2] au cas des différentielles stochastiques parametrisées par une $\star$-algèbre quelconque. Comme j'ai démontré dans [3], tout processus stochastique stationaire $\Lambda(a)$ à incréments indépendents peut être représenté dans un espace de Fock $\mathcal{F}$ sur l'espace $\mathcal{K}\otimes L^2(\mathbb{R}_+)$ des fonctions au carré intégrable sur $\mathbb{R}_+$ à valeurs dans $\mathcal{K}$ sous la forme $\mathrm{A}(\mathbf{a}) = a^\mu_\nu \mathrm{A}^\nu_\mu$, où

$$a^\mu_\nu \mathrm{A}^\nu_\mu(t) = a^\bullet_\bullet \mathrm{A}^\bullet_\bullet(t) + a^\bullet_+ \mathrm{A}^+_\bullet(t) + a^-_\bullet \mathrm{A}^\bullet_-(t) + a^-_+ \mathrm{A}^+_-(t),$$

est la décomposition canonique de Λ dans l'échange des processus $\mathrm{A}^\bullet_\bullet$ de création $\mathrm{A}^+_\bullet$, d'annihilation $\mathrm{A}^\bullet_-$ et temps $\mathrm{A}^+_- = t\mathrm{I}$ par rapport à l'état vide dans $\mathcal{F}$. Ainsi l'algèbre de parametrisation $\mathfrak{a}$ peut être identifiée à l'algèbre des quadruplets GNS $\boldsymbol{a} = (a^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$, associés à la $*$ -functionelle linéaire positive $l(a) = \langle\Lambda(t,a)\rangle/t$.

Les principaux résultats de cet article sont des théorèmes de characterisation et de représentation des germes stochastiques quantiques $\boldsymbol{\lambda} = \boldsymbol{i}+\boldsymbol{\alpha}$ des $*$-représentations

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unitaires i qui définissent les différentielles de Itô $\mathrm{d}\phi = \alpha^\mu_\nu \mathrm{dA}^\nu_\mu$ à $t = 0$ pour les processus complètement positifs ϕ_t avec $\phi_0 = i$.

2. Notations. Pour tout espace pré-Hilbertien $\mathcal{D}$, le symbole $\mathcal{D}'$ désigne l'espace dual de toutes les fonctionelles antilinéaires $\eta' : \eta \mapsto \langle \eta | \eta' \rangle$ avec le plongement naturel $\mathcal{D} \subseteq \mathcal{D}'$, donné par le produit intérieur $\langle \eta' | \eta \rangle = \langle \eta | \eta' \rangle^*$ pour $\eta' \in \mathcal{D}$. Toute forme sesquilinéaire $\langle \eta | Q\eta \rangle$ sur $\mathcal{D}$ s'identifie à l'opérateur linéaire $Q : \mathcal{D} \to \mathcal{D}'$. L'espace linéaire $\mathcal{Q}(\mathcal{D})$ de tous ces opérateurs est muni d'une involution $Q \mapsto Q^\dagger$, où $Q^\dagger : \mathcal{D} \to \mathcal{D}'$ est l'opérateur Hermitien adjoint: $\langle \eta | Q^\dagger \eta \rangle = \langle \eta | Q\eta \rangle^*$. Tout opérateur linéaire $A : \mathcal{D} \to \mathcal{D}$ avec son adjoint $A^\dagger : \eta \mapsto A^\dagger \eta \in \mathcal{D}$ pour tout $\eta \in \mathcal{D}$ peut être continué à $\mathcal{D}'$ comme $A' = A^{\dagger *}$, où $A^* : \mathcal{D}' \to \mathcal{D}'$ est l'opérateur dual defini par $\langle \eta | A^* \eta' \rangle = \langle A\eta | \eta' \rangle$. La $\star$-algèbre de tous ces opérateurs dans $\mathcal{D}$ est notée $\mathcal{A}(\mathcal{D})$.

Soient $\mathfrak{D}$ la $\mathcal{D}$-envelope des vecteurs cohérents (exponentiels) de l'espace de Fock $\mathcal{F}$, et $(\mathfrak{D}_t)_{t>0}$ la filtration naturelle des sous-espaces $\mathfrak{D}_t$ engendré par l'espace pré-Hilbertien $\mathcal{D} = \mathfrak{D}_0$ et les vecteurs cohérents $f^\otimes$ sur les fonctions au carré integrable $f^\bullet : [0, t) \to \mathcal{K}$, où $\mathcal{K}$ est l'espace pré-Hilbertien associé à l'algèbre de Itô $\mathfrak{a}$. Soit B un semigroupe à involution $x \mapsto x^\star, (x^\star y)^\star = y^\star x$. Le processus PD quantique stochastique adapté sur un $*$-semigroup B à unité est décrit par une famille de paramétres $\phi = (\phi_t)_{t>0}$ d'applications positives définies $\phi_t : B \to \mathcal{Q}(\mathfrak{D})$,

$$\sum_{x,y \in B} \langle \mathfrak{h}_y | \phi_t (y^\star x) \mathfrak{h}_x \rangle := \sum_{k,l} \langle \mathfrak{h}^l | \phi_t (x_l^\star x_k) \mathfrak{h}^k \rangle \geq 0 \tag{0.1}$$

pour toute suite $\mathfrak{h}^k \in \mathfrak{D}$ et $x_k \in B$, $k = 1, 2, ...,$ satisfaisant la propriété de causalité

$$\phi_t (x) \left(\eta \otimes f^\otimes \right) = \eta' \otimes f^\otimes \quad \forall x \in B, \eta \in \mathfrak{D}_t, f^\bullet \in \mathcal{K} \otimes L^2[t, \infty),$$

où $\eta' \in \mathfrak{D}'_t$. La famille stochastistiquement différentiable ϕ par rapport à un processus quantique stationaire Λ à increments indépendents engendré par l'algèbre de Itô $\mathfrak{a}$ est donné par l'intégrale quantique stochastique [4],[3],

$$\phi_t (x) - \phi_0 (x) = \int_0^t \boldsymbol{\alpha} (r, x) \mathrm{d}\mathbf{A} := \sum_{\mu,\nu} \int_0^t \alpha^\mu_\nu (r, x) \mathrm{d}A^\nu_\mu, \qquad x \in B \tag{0.2}$$

où $\boldsymbol{\alpha} (t, x)$ pour tout $x \in B$ est un processus quantique stochastique adapté à valeurs $\boldsymbol{a} (t) = \langle \mathfrak{h} | \boldsymbol{\alpha} (t, x) \mathfrak{h} \rangle$ dans les quadruplets $(a^\mu_\nu)(t)$ qui représentent l'algèbre $\mathfrak{a}$. Les intégrateurs quantiques stochastiques $\mathbf{A} (t) = \left(\mathrm{A}^\nu_\mu \right)^{\nu=+,\bullet}_{\mu=-,\bullet} (t)$ sont formellement définis par la $\flat$-propriété [3] $(\boldsymbol{a}\mathbf{A})^\dagger = \boldsymbol{a}^\flat \mathbf{A}$ et la table de multiplication de Hudson-Parthasarathy [4]

$$(\boldsymbol{a}\mathrm{d}\mathbf{A}) (\boldsymbol{b}\mathrm{d}\mathbf{A}) = (\boldsymbol{a}\boldsymbol{b}) \mathrm{d}\mathbf{A}, \tag{0.3}$$

où $a^{\flat\mu}_{-\nu} = a^{\nu\dagger}_{-\mu}$ par rapport à la symétrie $-(-) = +$, $-(+) = -$ des indices $(-, +)$ seulement, et $(\boldsymbol{b}\boldsymbol{a})^\mu_\nu = b^\mu_\bullet a^\bullet_\nu$.

3. Résultats.

Theorem 0.1. *Supposons que le PD processus ϕ_t possède la différentielle quantique stochastique $\boldsymbol{\alpha} (t) \mathrm{d}\mathbf{A}$ à $t \geq 0$. Alors la matrice des germes $\boldsymbol{\lambda} = \boldsymbol{\phi}_t + \boldsymbol{\alpha} (t)$ pour $\boldsymbol{\phi}_t (x) = (\phi_t (x) \delta^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ est conditionnellement positive définite*

$$\sum_{k,l} \langle \boldsymbol{\eta}^l | \boldsymbol{\iota} (x_l^\star x_k) \boldsymbol{\eta}^k \rangle = 0 \Rightarrow \sum_{k,l} \langle \boldsymbol{\eta}^l | \boldsymbol{\lambda} (x_l^\star x_k) \boldsymbol{\eta}^k \rangle \geq 0,$$

où $\boldsymbol{\eta}^k \in \mathfrak{D}_t \oplus \mathfrak{D}_t^\bullet$, $\mathfrak{D}_t^\bullet = \mathcal{K} \otimes \mathfrak{D}_t$, *par rapport à l'application dégénerée* $\boldsymbol{\iota} = (\iota_\nu^\mu)_{\nu \neq -}^{\mu \neq +}$, $\iota_\nu^\mu(x) = \phi_t(x)\,\delta_\nu^+ \delta_-^\mu$, *toutes les deux écrites sous la forme matricielle suivante*

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda & \lambda_\bullet \\ \lambda^\bullet & \lambda_\bullet^\bullet \end{pmatrix}, \qquad \boldsymbol{\iota} = \begin{pmatrix} \iota & 0 \\ 0 & 0 \end{pmatrix} \tag{0.4}$$

avec $\lambda = \alpha_+^-(t)$, $\lambda^\bullet = \alpha_+^\bullet(t)$, $\lambda_\bullet = \alpha_\bullet^-(t)$, $\lambda_\bullet^\bullet = \phi_t \delta_\bullet^\bullet + \alpha_\bullet^\bullet(t)$, *et* $\iota = \phi_t$.

Theorem 0.2. *Les conditions suivantes sont équivalentes:*

(i) *L'application de germes* $\boldsymbol{\lambda} = \boldsymbol{i} + \boldsymbol{\alpha}$ *à* $t = 0$ *est conditionellemnt positive définie par rapport à la représentation matricielle* $\boldsymbol{\iota}$ *in (3.1), où* $\iota = i$.

(ii) *Il éxiste un espace pré-Hilbertien* $\mathcal{D}^\circ$, *une* $*$*- représentation à unité* j *de* B *dans* $\mathcal{A}(\mathcal{D}^\circ)$,

$$j(y^\star x) = j(y)^* j(x), \quad j(1) = I, \tag{0.6}$$

une (j,i)*-dérivation de* B,

$$k(y^\star x) = j(y)^* k(x) + k(y^\star) i(x), \tag{0.7}$$

à valeurs dans les opérateurs $\mathcal{D} \to \mathcal{D}^\circ$, *et une application* $l : B \to \mathcal{Q}(\mathcal{D})$ *telle que*

$$l(y^\star x) = i(y)^* l(x) + l(y^\star) i(x) + k(y)^\dagger k(x), \tag{0.8}$$

avec l'application adjointe $l(x)^\dagger = l(x^\star) + Di(x^\star) - i(x)^* D$ *telle que* $l(x) + Di(x) = \lambda(x) = l^*(x) + i(x)' D$,

$$L_\circ^\bullet k(x) + L_+^\bullet i(x) = \lambda^\bullet(x), \quad \lambda_\bullet(x) = k^*(x) L_\bullet^\circ + i(x)' L_\bullet^-,$$

et $\lambda_\bullet^\bullet(x) = L_\circ^\bullet j(x) L_\bullet^\circ$ *pour certains opérateurs* $L_\circ^\bullet : \mathcal{D}^\circ \to \mathcal{D}^\bullet$ *avec les adjoints* $L_\bullet^\circ = L_\circ^{\bullet\dagger}$ *and* $L_+^\bullet : \mathcal{D} \to \mathcal{D}^\bullet$ *où* $L_\bullet^- = L_+^{\bullet\dagger}$

1. Introduction.

The purpose of this paper is to extend the Evans-Lewis theorem [1] for the differential construction of the Stinespring dilation [2] to the stochastic differentials, generated by an Itô $\star$–algebra

$$\mathrm{d}\Lambda(a)^\dagger \mathrm{d}\Lambda(a) = \mathrm{d}\Lambda(a^\star a), \quad \sum \lambda_i \mathrm{d}\Lambda(a_i) = \mathrm{d}\Lambda\left(\sum \lambda_i a_i\right), \quad \mathrm{d}\Lambda(a)^\dagger = \mathrm{d}\Lambda(a^\star),$$

of independent increments $\mathrm{d}\Lambda(t,a) = \Lambda(t + \mathrm{d}t, a) - \Lambda(t,a)$, with given mean values $\langle \mathrm{d}\Lambda(t,a)\rangle = l(a)\,\mathrm{d}t$, $a \in \mathfrak{a}$. Here $l : \mathfrak{a} \to \mathbb{C}$ is a positive linear functional on the parametrizing $\star$-algebra $\mathfrak{a}$, defining the GNS representation $a \mapsto \boldsymbol{a} = (a_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$ of $\mathfrak{a}$ in terms of the quadruples

$$a_\bullet^\bullet = j(a), \quad a_+^\bullet = k(a), \quad a_\bullet^- = k^*(a), \quad a_+^- = l(a), \tag{1.1}$$

where $j(a^\star a) = j(a)^* j(a)$ is the operator representation $j(a)^* k(a) = k(a^\star a)$ on the pre-Hilbert space $\mathcal{K}$ of the Kolmogorov decomposition $l(a^* a) = k(a)^* k(a)$, and $k^*(a) = k(a^\star)^*$.

As was proved in [3], the stochastic process $\Lambda(a)$ with independent increments can be represented in the Fock space $\mathfrak{F}$ over the space of $\mathcal{K}$-valued square-integrable functions on $\mathbb{R}_+$ as $\mathrm{A}(\mathbf{a}) = a_\nu^\mu \mathrm{A}_\mu^\nu$, where

$$a_\nu^\mu \mathrm{A}_\mu^\nu(t) = a_\bullet^\bullet \mathrm{A}_\bullet^\bullet(t) + a_+^\bullet \mathrm{A}_\bullet^+(t) + a_\bullet^- \mathrm{A}_-^\bullet(t) + a_+^- \mathrm{A}_-^+(t), \tag{1.2}$$

is the canonical decomposition of Λ into the exchange $\mathrm{A}^{\bullet}_{\bullet}$, creation $\mathrm{A}^{+}_{\bullet}$, annihilation $\mathrm{A}^{\bullet}_{-}$ and time $\mathrm{A}^{+}_{-} = t\mathrm{I}$ processes with respect to the vacuum state in $\mathfrak{F}$. Thus the parametrizing algebra $\mathfrak{a}$ can be identified with the $\flat$-algebra of quadruples $\boldsymbol{a}$ with respect to the product $(\boldsymbol{ba})^{\mu}_{\nu} = b^{\mu}_{\bullet} a^{\bullet}_{\nu}$ and the involution $a^{\flat\mu}_{-\nu} = a^{\nu\dagger}_{-\mu}$, where $-(-) = +$, $-\bullet = \bullet$, $-(+) = -$.

The main results of this paper are characterization and representation theorems for the quantum stochastic germs $\boldsymbol{\lambda} = \boldsymbol{i} + \boldsymbol{\alpha}$ of unital $*$-representations i defining the Itô differentials $\mathrm{d}\phi = \alpha^{\mu}_{\nu} \mathrm{dA}^{\nu}_{\mu}$ at $t = 0$ for completely positive processes ϕ_t with $\phi_0 = i$. The Evans-Lewis case $\Lambda(t, a) = at\mathrm{I}$ is described by the simplest one-dimensional algebra $\mathfrak{a}$ with $l(a) = a$ and the nilpotent multiplication $a^{\star} a = 0$ corresponding to the non-stochastic (Newton) calculus $(\mathrm{d}t)^2 = 0$. The unital $\star$-algebra $\mathfrak{a} = \mathbb{C}$ with the commutative multiplication $a^{\star} a = |a|^2$ has the GNS representation $a^{\mu}_{\nu} = a$, corresponding to $\Lambda(t, a) = a\mathrm{P}(t)$, where P is the standard Poisson process $\mathrm{P} = \sum \mathrm{A}^{\nu}_{\mu}$. The standard Wiener process $\mathrm{Q} = \mathrm{A}^{\bullet}_{-} + \mathrm{A}^{+}_{\bullet}$ is described by the second order nilpotent algebra $\mathfrak{a}$ of quadruples $a^{-}_{+} = a, \quad a^{-}_{\bullet} = b = a^{\bullet}_{+}, \quad a^{\bullet}_{\bullet} = 0$, corresponding to $\Lambda(t, a) = at\mathrm{I} + b\mathrm{Q}(t)$. Thus our results are applicable also to the classical stochastic differentials of completely positive processes, corresponding to the commutative Itô algebras, which are decomposable into the Wiener, Poisson and Newton orthogonal components.

2. Notation.

Throughout the pre-Hilbert space $\mathcal{D}$ is complex, $\mathcal{D}'$ denotes the dual space of all antilinear functionals $\eta' : \eta \mapsto \langle \eta | \eta' \rangle$ with the natural embedding $\mathcal{D} \subseteq \mathcal{D}'$, $\langle \eta | \eta' \rangle = \|\eta\|^2$ if $\eta' = \eta \in \mathcal{D}$, $\mathcal{Q}(\mathcal{D})$ denotes the space of all linear operator $Q : \mathcal{D} \to \mathcal{D}'$ identified with the sesquilinear forms $\langle \eta | Q\eta \rangle$ on $\mathcal{D}$ and $\mathcal{A}(\mathcal{D})$ denotes the unital $\star$-algebra of all operators $Q \in \mathcal{Q}(\mathcal{D})$ with $Q\mathcal{D} \subseteq \mathcal{D}$, $Q^{\dagger}\mathcal{D} \subseteq \mathcal{D}$, where $\langle \eta | Q^{\dagger} \eta \rangle = \langle \eta | Q\eta \rangle^*$. Any operator $A \in \mathcal{A}(\mathcal{D})$ can be extended onto $\mathcal{D}'$ as $A' = A^{\dagger *}$, where $A^* : \mathcal{D}' \to \mathcal{D}'$ is the dual operator, defined by $\langle \eta | A^* \eta' \rangle = \langle A\eta | \eta' \rangle$.

Let $\mathfrak{D}$ denote the $\mathcal{D}$-span of the coherent (exponential) vectors $f^{\otimes} : \tau \mapsto \otimes_{t \in \tau} f^{\bullet}(t)$, $f^{\bullet} \in \mathfrak{K}$ of the Fock space $\mathfrak{F}$ over $\mathfrak{K} = \mathcal{K} \otimes L^2(\mathbb{R}_+)$, where $\mathcal{K}$ is the pre-Hilbert space, associated with the Itô algebra $\mathfrak{a}$, and $(\mathfrak{D}_t)_{t>0}$ be the natural filtration of the subspaces $\mathfrak{D}_t$ generated by $\eta \otimes f^{\otimes}$, $\eta \in \mathcal{D}$ and $f^{\otimes}$, $f^{\bullet} \in \mathcal{K} \otimes L^2[0, t)$, embedded into $\mathfrak{F}$ as $f^{\otimes}(\tau) = 0$ for any finite $\tau \subset \mathbb{R}_+$ if $\tau \cap [t, \infty) \neq \emptyset$. Let B be a unital semigroup with involution $x \mapsto x^{\star}$, $(x^{\star} y)^{\star} = y^{\star} x$. Say $B = \mathfrak{b}$ is the $\star$-semigroup $1 + \mathfrak{a}$ of the Itô algebra $\mathfrak{a}$ with $(1 + a)^{\star}(1 + b) = 1 + a \star b$, where $a \star b = b + a^{\star} b + a^{\star}$, or $B = \mathcal{B}$ is an operator algebra $\mathcal{B} \subseteq \mathcal{A}(\mathcal{D})$. The quantum stochastic adapted PD process over B is described by a one parameter family $\phi = (\phi_t)_{t>0}$ of positive definite maps $\phi_t : B \to \mathcal{Q}(\mathfrak{D})$,

$$\sum_{x,y \in B} \langle \mathfrak{h}_y | \phi_t(y^{\star} x) \mathfrak{h}_x \rangle := \sum_{k,l} \langle \mathfrak{h}^l | \phi_t(x_l^{\star} x_k) \mathfrak{h}^k \rangle \geq 0 \tag{2.1}$$

for any sequence $\mathfrak{h}^k \in \mathfrak{D}$ and $x_k \in B$, $k = 1, 2, ...$, satisfying the causality property

$$\phi_t(x)\left(\eta \otimes f^{\otimes}\right) = \eta' \otimes f^{\otimes} \quad \forall x \in B, \eta \in \mathfrak{D}_t, f^{\bullet} \in \mathcal{K} \otimes L^2[t, \infty),$$

where $\eta' \in \mathfrak{D}'_t$. The positive definiteness (2.1) obviously implies the $*$-property $\phi^*_t = \phi_t$, where $\phi^*_t(x) = \phi_t(x^{\star})^{\dagger}$. The stochastically differentiable family ϕ with respect to a quantum stationary process Λ with independent increments generated by the Itô algebra $\mathfrak{a}$ is given by the quantum stochastic integral [4],[3],

$$\phi_t(x) - \phi_0(x) = \int_0^t \boldsymbol{\alpha}(r,x)\,\mathrm{d}\mathbf{A} := \sum_{\mu,\nu}\int_0^t \alpha^\mu_\nu(r,x)\,\mathrm{d}\mathrm{A}^\nu_\mu, \qquad x \in B \tag{2.2}$$

where $\boldsymbol{\alpha}(t,x)$ for each $x \in B$ is a quantum stochastic adapted process with the values $\boldsymbol{a}(t) = \langle \mathfrak{h}|\boldsymbol{\alpha}(t,x)\,\mathfrak{h}\rangle$ into the quadruples $(a^\mu_\nu)(t)$, representing the algebra $\mathfrak{a}$. The quantum stochastic integrators $\mathbf{A}(t) = \left(\mathrm{A}^\nu_\mu\right)^{\nu=+,\bullet}_{\mu=-,\bullet}(t)$ are formally defined by the $\flat$-property [3] $(\boldsymbol{a}\mathbf{A})^\dagger = \boldsymbol{a}^\flat\mathbf{A}$ and the Hudson-Parthasarathy multiplication table [4]

$$(\boldsymbol{a}\mathrm{d}\mathbf{A})(\boldsymbol{b}\mathrm{d}\mathbf{A}) = (\boldsymbol{a}\boldsymbol{b})\,\mathrm{d}\mathbf{A}. \tag{2.3}$$

The quantum stochastic derivatives α^μ_ν for the PD processes $\phi^*_t = \phi_t$ should obviously satisfy the $\flat$-property $\boldsymbol{\alpha}^\flat = \boldsymbol{\alpha}$ where $\alpha^{\flat\nu}_{-\mu} = \alpha^{\mu *}_{-\nu}$, $\alpha^{\mu *}_\nu(t,x) = \alpha^\mu_\nu(t,x^\star)^\dagger$.

In order to make the formulation of the dilation theorem as concise as possible, we need the notion of the $\flat$-representation of B in the operator algebra $\mathcal{A}(\mathcal{E})$ of a pseudo-Hilbert space $\mathcal{E} = \mathcal{D}' \oplus \mathcal{D}^\circ \oplus \mathcal{D}$ with respect to the indefinite metric

$$(\xi|\,\xi) = 2\,\mathrm{Re}\left(\xi^-\middle|\,\xi^+\right) + \|\xi^\circ\|^2 + \left\|\xi^+\right\|^2_D \tag{2.4}$$

for the triples $\xi^\mu, \mu = -,\circ,+$, where $\xi^+ \in \mathcal{D}$, $\xi^\circ \in \mathcal{D}^\circ$, $\xi^- \in \mathcal{D}'$, $\mathcal{D}^\circ$ is a pre-Hilbert space, and $\|\eta\|^2_D = \langle\eta|D\eta\rangle$. The operators $L \in \mathcal{A}(\mathcal{E})$ given by 3×3-block-matrices $\mathbf{L} = [L^\mu_\nu]^{\mu=-,\circ,+}_{\nu=-,\circ,+}$ have the Pseudo-Hermitian adjoints $\left(\xi|L^\flat\xi\right) = (L\xi|\xi)$, which are defined by the Hermitian adjoints $L^{\dagger\mu}_\nu = L^{\nu *}_\mu$ as $\mathbf{L}^\flat = \mathbf{G}^{-1}\mathbf{L}^\dagger\mathbf{G}$ respectively to the indefinite metric tensor $\mathbf{G} = [G_{\mu\nu}]$ and its inverse $\mathbf{G}^{-1} = [G^{\mu\nu}]$, given by

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & I \\ 0 & I^\circ_\circ & 0 \\ I & 0 & D \end{bmatrix}, \qquad \mathbf{G}^{-1} = \begin{bmatrix} -D & 0 & I \\ 0 & I^\circ_\circ & 0 \\ I & 0 & 0 \end{bmatrix} \tag{2.5}$$

with Hermitian D, where $I^\circ_\circ$ is the identity operator in $\mathcal{D}^\circ$, being equal $I^\bullet_\bullet = \mathbf{1}\otimes I$ in the case $\mathcal{D}^\circ = \mathcal{D}^\bullet$, where $\mathcal{D}^\bullet = \mathcal{K}\otimes\mathcal{D}$.

3. The Results

1. The following theorem in particular proves that the quantum stochastic germ $\boldsymbol{\lambda} = (\lambda^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$, $\lambda^\mu_\nu(x) = i(x)\,\delta^\mu_\nu + \alpha^\mu_\nu(x)$ of a unital $*$-representation $\phi_0 = i$ of B on $\mathcal{D}$, defined by the quantum stochastic derivatives $\boldsymbol{\alpha} = (\alpha^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ for a PD adapted process ϕ at $t = 0$, must be conditionally PD with respect to the embedded representation $\boldsymbol{\iota}(x) = \left(i(x)\,\delta^+_\nu\delta^\mu_-\right)^{\mu=-,\bullet}_{\nu=+,\bullet}$ on $\mathcal{D}\oplus\mathcal{D}^\bullet$. Here δ^μ_ν is the Kronecker delta with $\delta^\bullet_\bullet = \mathbf{1}$, where $\mathbf{1}$ is the identity operator on $\mathcal{K}$, and $X\delta^\bullet_\bullet = X\otimes\mathbf{1}$.

Theorem 3.1. *Suppose that the PD process ϕ_t has the quantum stochastic differential $\boldsymbol{\alpha}(t)\,\mathrm{d}\mathbf{A}$ at a $t \geq 0$. Then the germ-map $\boldsymbol{\lambda}{=}\boldsymbol{\phi}_t + \boldsymbol{\alpha}(t)$ for $\boldsymbol{\phi}_t(x) = (\phi_t(x)\,\delta^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ is conditionally positive definite*

$$\sum_{k,l}\langle\boldsymbol{\eta}^l|\boldsymbol{\iota}(x^\star_l x_k)\,\boldsymbol{\eta}^k\rangle = 0 \Rightarrow \sum_{k,l}\langle\boldsymbol{\eta}^l|\boldsymbol{\lambda}(x^\star_l x_k)\,\boldsymbol{\eta}^k\rangle \geq 0$$

where $\boldsymbol{\eta}^k \in \mathfrak{D}_t \oplus \mathfrak{D}_t^\bullet, \mathfrak{D}_t^\bullet = \mathcal{K} \otimes \mathfrak{D}_t$*, with respect to the degenerate map* $\boldsymbol{\iota} = (\iota_\nu^\mu)_{\nu=\bullet,+}^{\mu=-,\bullet}$*,* $\iota_\nu^\mu(x) = \phi_t(x)\,\delta_\nu^+\delta_-^\mu$*, both written in the matrix form as*

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda & \lambda_\bullet \\ \lambda^\bullet & \lambda_\bullet^\bullet \end{pmatrix}, \qquad \boldsymbol{\iota} = \begin{pmatrix} \iota & 0 \\ 0 & 0 \end{pmatrix} \tag{3.1}$$

with $\lambda = \alpha_+^-(t)$*,* $\quad \lambda^\bullet = \alpha_+^\bullet(t)$*,* $\quad \lambda_\bullet = \alpha_\bullet^-(t)$*,* $\quad \lambda_\bullet^\bullet = \phi_t\delta_\bullet^\bullet + \alpha_\bullet^\bullet(t)$ *,and* $\iota = \phi_t$ *such that* $\lambda(x^\star) = \lambda(x)^\dagger$*,* $\quad \lambda^\bullet(x^\star) = \lambda_\bullet(x)^\dagger$*,* $\quad \lambda_\bullet^\bullet(x^\star) = \lambda_\bullet^\bullet(x)^\dagger$

Proof. Let us represent the pre-Hilbert space $\mathfrak{D}$ as the $\mathfrak{D}_t$-span

$$\left\{ \sum_f \zeta^f \otimes f^\otimes \,\middle|\, \zeta^f \in \mathfrak{D}_t, \quad f^\bullet \in \mathcal{K} \otimes L^2[t,\infty) \right\}$$

of coherent vectors $f^\otimes(\tau) = \bigotimes_{t\in\tau} f^\bullet(t)$, given for each finite subset $\tau = \{t_1, ..., t_n\} \subseteq [t,\infty)$ by tensor products $f^\bullet(t_1) \otimes ... \otimes f^\bullet(t_n)$, with $\zeta^f = 0$ for almost all $f^\bullet$. Then the positivity (2.1) of the quantum stochastic adapted maps $\phi_s, s > t$ into the $\mathfrak{D}$-forms $\langle h|\,\phi_t(x)\,h\rangle$, for $h \in \mathfrak{D}$ can be obviously written as

$$\sum_{x,y}\sum_{f,g}\left\langle \zeta_y^g\right|\phi_s(g^\bullet, y^\star x, f^\bullet)\,\zeta_x^f\Big\rangle \geq 0, \tag{3.2}$$

the positive definiteness of the $\mathfrak{D}_t$-forms

$$\langle\eta|\,\phi_s(g^\bullet, x, f^\bullet)\,\zeta\rangle = \left\langle \eta \otimes g^\otimes\right|\phi_s(x)\,\zeta \otimes f^\otimes\big\rangle\, e^{-\int_s^\infty g^\bullet(r)^* f^\bullet(r)\mathrm{d}r},$$

with $\zeta_x^f \neq 0$ for a finite sequence of $x_k \in B$, and for a finite sequence of $f_k^\bullet \in \mathcal{K} \otimes L^2[t,\infty)$. If the $\mathfrak{D}$-form $\phi_t(x)$ has the stochastic differential (2.2), the $\mathfrak{D}_t$-form $\phi_s(g^\bullet, x, f^\bullet)$ has the derivative

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi_t(g^\bullet, x, f^\bullet) \quad &= \quad g^\bullet(t)^* f^\bullet(t)\,\phi_t(x) + g^\bullet(t)^*\,\alpha_\bullet^\bullet(t,x)\,f^\bullet(t) \\ &\quad + g^*(t)^*\,\alpha_+^\bullet(t,x) + \alpha_\bullet^-(t,x)\,f^\bullet(t) + \alpha_+^-(t,x) \end{aligned} \tag{3.3}$$

at $s = t$. The positive definiteness, (3.2), ensures the conditional positivity

$$\sum_{x,y}\sum_{f,g}\langle\zeta_y^g|\phi_t(y^\star x)\,\zeta_x^f\rangle = 0 \Rightarrow \sum_{x,y}\sum_{f,g}\left\langle\zeta_y^g\right|\lambda_s^t(g^\bullet, y^\star x, f^\bullet)\,\zeta_x^f\Big\rangle \geq 0 \tag{3.4}$$

of the form $\lambda_s^t(g^\bullet, x, f^\bullet) = \frac{1}{s-t}(\phi_s(g^\bullet, x, f^\bullet) - \phi_t(x))$ for each $s > t$ and of the limit (3.3) at $s \searrow t$, given by the quadratic $\mathcal{K}$- form

$$\lambda(g^\bullet, x, f^\bullet) = b_\bullet\lambda_\bullet^\bullet(x)\,a^\bullet + b_\bullet\lambda^\bullet(x) + \lambda_\bullet(x)\,a^\bullet + \lambda(x), \tag{3.5}$$

where $a^\bullet = f^\bullet(t)$, $\quad b_\bullet^* = g^\bullet(t)$, and the λ's are defined in (3.1). Hence the form

$$\sum_{x,y}\sum_{\mu,\nu}\left\langle\eta_y^\mu\right|\lambda_\nu^\mu(y^\star x)\,\eta_x^\nu\rangle := \sum_{x,y}\left\langle\eta_y\right|\lambda(y^\star x)\,|\eta_x\rangle$$

$$+\sum_{x,y}\left(\left\langle\eta_y\right|\lambda_\bullet(y^\star x)\,\eta_x^\bullet\rangle + \left\langle\eta_y^\bullet\right|\lambda^\bullet(y^\star x)\,\eta_x\rangle + \left\langle\eta_y^\bullet\right|\lambda_\bullet^\bullet(y^\star x)\,\eta_x^\bullet\rangle\right)$$

with $\eta = \sum_f \zeta^f$, $\quad \eta^\bullet = \sum_f \zeta^f \otimes f^\bullet(t)$ is positive if $\sum_{x,y}\langle\eta_y|\phi_t(y^\star x)\,\eta_x\rangle = 0$. The components $\eta \in \mathfrak{D}_t$ and $\eta^\bullet \in \mathfrak{D}_t^\bullet$ are independent because for any $\boldsymbol{\eta} \in \mathfrak{D}_t \oplus \mathfrak{D}_t^\bullet$ there exists such a function $a^\bullet \mapsto \zeta^a$ on $\mathcal{K}$ with a finite support, that $\sum_a \zeta^a = \eta$, $\quad \sum_a \zeta^a \otimes a^\bullet = \eta^\bullet$, namely, $\zeta^a = 0$ for all $a^\bullet \in \mathcal{K}$ except $a^\bullet = a_n^\bullet$, the n-th $\mathcal{K}$-coefficient of the span $\eta^\bullet = \sum_n \eta_n \otimes a_n^\bullet$, for which $\zeta^a = \eta_n$, and $a^\bullet = 0$, for which $\zeta^a = \eta - \sum_n \eta_n$. This proves the complete positivity of the matrix form $\boldsymbol{\lambda}$, with respect to the matrix representation $\boldsymbol{\iota}$ defined in (3.1) on the column-vectors $\boldsymbol{\eta}$. □

2. The conditional positivity of the germ-matrix $\boldsymbol{\lambda}$ at $t = 0$ with respect to the representation $i : B \to \mathcal{A}(\mathcal{D})$ embedded into the matrix form as in (3.1), obviously implies the positivity of the dissipation form

$$\sum_{x,y} \langle \boldsymbol{\eta}_y | \boldsymbol{\Delta}(x,y)\, \boldsymbol{\eta}_x \rangle := \sum_{x,y} \sum_{\mu,\nu} \langle \eta^\mu_y | \Delta^\mu_\nu(y,x)\, \eta^\nu_x \rangle, \tag{3.6}$$

with $\eta = h_0 \in \mathcal{D}$, $\eta^\bullet = h^\bullet_0 \in \mathcal{D}^\bullet$, $\eta^- = \eta = \eta^+$, corresponding to non-zero $\boldsymbol{\eta}_x \in \mathcal{D} \oplus \mathcal{D}^\bullet$. Here $\boldsymbol{\Delta} = (\Delta^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ is the stochastic dissipator, given by the blocks

$$\begin{aligned} \Delta^\bullet_\bullet(y,x) &= \alpha^\bullet_\bullet(y^\star x) + i(y^\star x) \otimes \mathbf{1}, \\ \Delta^-_\bullet(y,x) &= \alpha^-_\bullet(y^\star x) - i(y)^* \alpha^-_\bullet(x) = \Delta^\bullet_+(x,y)^* \\ \Delta^-_+(y,x) &= \alpha^-_+(y^\star x) - i(y)^* \alpha^-_+(x) - \alpha^-_+(y^\star)\, i(x) + i(y)^* D i(x), \end{aligned} \tag{3.7}$$

where $D = \lambda(1)$. This means that the map $\lambda^\bullet_\bullet$ is positive definite, and the conditions of the next theorem are fulfilled if the maps λ, $\lambda^\bullet$, $\lambda_\bullet$ have the following form

$$\begin{aligned} \lambda^\bullet(x) &= \varphi^\bullet(x) - K^\dagger_\bullet i(x), \qquad \lambda_\bullet(x) = \varphi_\bullet(x) - i(x) K_\bullet \\ \lambda(x) &= \varphi(x) - K^\dagger i(x) - i(x) K, \qquad \lambda^\bullet_\bullet(x) = \varphi^\bullet_\bullet(x) \end{aligned} \tag{3.8}$$

where $\boldsymbol{\varphi} = (\varphi^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ is a positive definite map from B into the sesquilinear forms on the pre-Hilbert space $\mathcal{D} \oplus \mathcal{D}^\bullet$.

Theorem 3.2. *The following are equivalent:*

(i) *The dissipation form (3.6), defined by the* ♭*-map* α *with* $\alpha^-_+(1) = D$*, is positive definite:* $\sum_{x,y} \langle \boldsymbol{\eta}_y | \boldsymbol{\Delta}(y,x)\, \boldsymbol{\eta}_x \rangle \geq 0$.

(ii) *There exists a pre-Hilbert space* $\mathcal{D}^\circ$*, a unital* $*$*- representation* j *of* B *in* $\mathcal{A}(\mathcal{D}^\circ)$*,*

$$j(y^\star x) = j(y)^* j(x), \quad j(1) = I, \tag{3.9}$$

a (i,j)*-derivation of* B*,*

$$k(y^\star x) = j(y)^* k(x) + k(y^\star) i(x), \tag{3.10}$$

having values in the operators $\mathcal{D} \to \mathcal{D}^\circ$ *, the adjoint map* $k^*(x) = k(x^\star)^\dagger$*,*

$$k^*(y^\star x) = i(y)^* k^*(x) + k^*(y^\star) j(x)$$

into the operators $\mathcal{D}^\circ \to \mathcal{D}'$*, and a map* $l : B \to \mathcal{Q}(\mathcal{D})$*, having the coboundary property*

$$l(y^\star x) = i(y)^* l(x) + l(y^\star) i(x) + k^*(y^\star) k(x), \tag{3.11}$$

such that $l(x) + Di(x) = \lambda(x) = l^*(x) + i(x)' D$*, where* $l^*(x) = l(x^\star)^\dagger$*,*

$$L^\bullet_\circ k(x) + L^\bullet_+ i(x) = \lambda^\bullet(x), \quad \lambda_\bullet(x) = k^*(x) L^\circ_\bullet + i(x)' L^-_\bullet,$$

and $\lambda^\bullet_\bullet(x) = L^\bullet_\circ j(x) L^\circ_\bullet$ *for some operators* $L^\bullet_+ : \mathcal{D} \to \mathcal{D}^\bullet$*,* $L^\bullet_\circ : \mathcal{D}^\circ \to \mathcal{D}^\bullet$*, with the adjoints* $L^-_\bullet = L^{\bullet\dagger}_+$*,* $L^\circ_\bullet = L^{\bullet\dagger}_\circ$*.*

(iii) *There exists a pseudo-Hilbert space,* $\mathcal{E}$*, a unital* ♭*-representation* $\jmath : B \to \mathcal{A}(\mathcal{E})$*, and a linear operator* $L : \mathcal{D} \oplus \mathcal{D}^\bullet \to \mathcal{E}$*, where* $\mathcal{D}^\bullet = \mathcal{K} \otimes \mathcal{D}$ *such that*

$$\boldsymbol{\lambda}(x) = L^\flat \jmath(x) L, \qquad \forall x \in B. \tag{3.12}$$

(iv) *The germ-mapping* $\boldsymbol{\lambda} = \boldsymbol{i} + \boldsymbol{\alpha}$ *is conditionally positive definite with respect to the matrix representation* $\boldsymbol{\iota}$ *in (3.1), where* $\iota = i$.

Proof. The proof of the implication (i)⇒(ii), generalizing the Evans-Lewis Theorem, is similar to the proof of the dilation theorem in [3].The proof of the implication (ii)⇒(iii) can be also obtained as in [5] by the explicit construction of $\mathcal{E}$ as $\oplus_{\mu=-}^{+}\mathcal{D}^{\mu}$, where $\mathcal{D}^{+} = \mathcal{D}$, $\mathcal{D}^{-} = \mathcal{D}'$, with the indefinite metric tensor $\mathbf{G} = [G_{\mu\nu}]$ given above for $\mu, \nu = -, \circ, +$, and $D = \lambda(1)$, of the unital $\flat$-representation $\boldsymbol{\ae} = [\jmath_{\nu}^{\mu}]_{\nu=-,\circ,+}^{\mu=-,\circ,+}$ of B on $\mathcal{E}$:

$$\boldsymbol{\ae}(y^{\star}x) = \boldsymbol{\ae}(y)^{\flat}\,\boldsymbol{\ae}(x), \quad \boldsymbol{\ae}(1) = \mathbf{I}$$

with $\boldsymbol{\ae}(x)^{\flat} = \mathbf{G}^{-1}\boldsymbol{\ae}(x)^{\dagger}\,\mathbf{G}$, given by the components

$$\jmath_{+}^{+} = i, \quad \jmath_{\circ}^{\circ} = j, \quad \jmath_{-}^{-} = i' \quad \jmath_{+}^{\circ} = k, \quad \jmath_{\circ}^{-} = k^{*}, \quad \jmath_{+}^{-} = l, \tag{3.13}$$

where $i'(x) = i(x)'$ and all other $\jmath_{\nu}^{\mu} = 0$. The linear operator L is given by $\mathbf{L} = [L^{\mu}, L_{\bullet}^{\mu}]$ with the components

$$L^{+} = I, \quad L^{\circ} = 0, \quad L^{-} = 0, \quad L_{\bullet}^{+} = 0, \quad L_{\bullet}^{\circ} = L_{\circ}^{\bullet\dagger}, \quad L_{\bullet}^{-} = L_{+}^{\bullet\dagger},$$

and $\mathbf{L}^{\flat} = \begin{pmatrix} I & 0 & D \\ 0 & L_{\circ}^{\bullet} & L_{+}^{\bullet} \end{pmatrix} = \mathbf{L}^{\dagger}\mathbf{G}$ such that $\mathbf{L}^{\flat}\boldsymbol{\ae}(x)\,\mathbf{L} = \boldsymbol{\lambda}(x)$. The implication (iii)⇒(iv) is a straight forward consequence of this construction, and the implication (iv)⇒(i) is similar to the non-stochastic case [6] . □

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