

A Stochastic Hamiltonian Approach for Quantum Jumps, Spontaneous Localizations, and Continuous Trajectories.

V.P.Belavkin

University of Nottingham, NG7 2RD, UK

O. Melsheimer

Fachbereich Physik, Renthof 7, Marburg, W-3550, Germany

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Abstract

We give an explicit stochastic Hamiltonian model of discontinuous unitary evolution for quantum spontaneous jumps like in a system of atoms in quantum optics, or in a system of quantum particles that interacts singularly with "bubbles" which admit a continual counting observation. This model allows to watch a quantum trajectory in a photodetector or in a cloud chamber by spontaneous localisations of the momentums of the scattered photons or bubbles. Thus, the continuous reduction and spontaneous localization theory is obtained from a Hamiltonian singular interaction as a result of quantum filtering, i.e., a sequential time continuous conditioning of an input quantum process by the output measurement data. We show that in the case of indistinguishable particles or atoms the a posteriori dynamics is mixing, giving rise to an irreversible Boltzmann-type reduction equation. The latter coincides with the nonstochastic Schrödinger equation only in the mean field approximation, whereas the central limit yields Gaussian mixing fluctuations described by a quantum state reduction equation of diffusive type.

Keywords :

Quantum Jumps, Counting Processes, Continuous Reduction, State Diffusion, Quantum Trajectories, Spontaneous Localisation.

Introduction

This paper gives a microscopic theory of quantum jumps, inducing the spontaneous localizations. It is based on a new type of quantum dynamics, described by the generalized (singular) stochastic Schrödinger equation, and on the quantum filtering process, amounting to a modification of quantum evolution along the quantum measurement trajectories. We do not assume the existence of a unique universal mechanism for continuous reduction and spontaneous localization during the measurement process. There are many such mechanisms, as many as the considered kinds of the singular interactions, amounting to completely unequivalent modifications of quantum evolution (e.g. quantum jumps and quantum diffusion) along lines, formally similar to different types of macro-objectification of pointer positions.

The quantum measurement theory based on the ordinary von Neumann reduction postulate applies neither to instantaneous observations with continuous spectra nor to continual (continuous in time) measurements. Although such phenomena can be described in the more general framework of Ludwig's Davies-Lewies operational approach [1]–[5] there is a particular interest in describing quantum measurements by concrete Hamiltonian models from which the operational description can be derived by an averaging procedure. Perhaps the first model of such kind for instantaneous unsharp measurement of particle localization was given by von Neumann [6]. He considered the singular interaction Hamiltonian

$$h_x(t) = x\delta(t)\frac{\hbar}{i}\frac{d}{d\lambda}, \quad \delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \quad (0.1)$$

with Dirac δ -function, producing the translation operator

$$s_x = \exp\left\{-\frac{i}{\hbar}\int_{-\infty}^{\infty} h_x(t)dt\right\} = e^{-xd/d\lambda} \quad (0.2)$$

at time $t = 0$ of the measurement. Here x is the position of the particle and λ is the pointer position, defining, say, the momentum p of a quantum

meter. The particle scattering operator $S = \{s_x\}$ applied to the (generalized) position eigen-kets $|x\rangle$ as $S|x\rangle = |x\rangle s_x$ does not affect the position x of the particle but changes the meter momentum p to

$$y = s_x^\dagger p s_x = x + p, \quad (0.3)$$

which implies that the initial wave function $\psi_0(x, \lambda)$ of the system “particle plus meter” is transformed into

$$\psi(x, \lambda) = s_x \psi_0(x, \lambda) = \psi_0(x, \lambda - x). \quad (0.4)$$

If in the initial state the particle and the meter were not coherent, $\psi_0(x, \lambda) = \eta(x)f_0(\lambda)$, and the wave function f_0 of the meter was fixed, then one can obtain the unitary transformation $S : \psi_0 \mapsto s_\bullet \psi_0$ via a family $\{F(\lambda)\}$ of reduction transformations $F(\lambda) : \eta \mapsto f_\bullet(\lambda)\eta$ for the prior particle vector-states η . Specifically, Eq. (0.4) can be defined as

$$\psi(x, \lambda) = f_0(\lambda - x)\eta(x) \equiv f_x(\lambda)\eta(x). \quad (0.5)$$

The linear nonunitary operators $F(\lambda) = s_\bullet f_0(\lambda)$ act on $|x\rangle$ as the multiplication $F(\lambda)|x\rangle = |x\rangle f_x(\lambda)$ by $f_x(\lambda) = f_0(\lambda - x) = s_x f_0(\lambda)$ and would give a sharp localization of any particle wave function $\eta(x)$ at the point $x = \lambda$ of the pointer position provided that the wave function $f_0(\lambda)$ could be initially localized at $\lambda = 0$. But there are no such sharply localized quantum states for the continuous pointer, and the best that one can do is to take an approximately sharp wave packet $f_0(\lambda)$. Say, f_0 is the Gaussian packet rescaled to the standard form

$$f_0(\lambda) = \exp\left\{-\frac{\pi}{2}\lambda^2\right\}. \quad (0.6)$$

This results in the unsharp localization $[F(\lambda)\eta](x) = \psi(x, \lambda)$,

$$\psi(x, \lambda) = \exp\left\{-\frac{\pi}{2}(x - \lambda)^2\right\}\eta(x) \quad (0.7)$$

of any wave function $\eta(x)$ about the observed value λ of the pointer. The localized wave function $\psi(x, \lambda)$ defines for each measurement result λ the posterior particle vector-state

$$\eta_\lambda = c_\lambda^{-1}\psi(\lambda), \quad |c_\lambda|^2 = \|\psi(\lambda)\|^2 \quad (0.8)$$

up to the normalization

$$\|\psi(\lambda)\|^2 = \int |\psi(x, \lambda)|^2 dx = p(\lambda) \quad (0.9)$$

to the probability density

$$p(\lambda) = \int |f_0(\lambda - x)|^2 |\eta(x)|^2 dx. \quad (0.10)$$

For commuting operators x and y , this is equivalent to the classical measurement model $y = x + p$ for unsharp measurement of an unknown signal x via the sharp measurement of the signal plus Gaussian noise p with given probability density $|f_0(\lambda)|^2$ of the random values $p = \lambda$.

These arguments illustrate how to interpret the reduction model involving the continuous spectrum of a quantum measurement as a Hamiltonian interaction model with nondemolition observation for a quantum object using the measurement of the pointer coordinate of the quantum meter. Since von Neumann introduced this approach, they were used by numerous other authors [7, 8] for the derivation of a generalized reduction $\eta \mapsto F(\lambda)\eta$ that would replace the von Neumann postulate $\eta \mapsto E(\lambda)\eta$ given by orthoprojections $\{E(\lambda)\}$ in the case of discrete values λ .

As extended to nondemolition observations continual in time [9]–[15], this idea constitute an essence of the quantum filtering method for the derivation of nonlinear stochastic wave equations describing the quantum dynamics under the observation. Since a particular type of such equations has been taken as a postulate in the phenomenological theory of permanent reduction, quantum jumps and spontaneous localization [16]–[20], the question arises whether it is possible to obtain this equation from an appropriate Schrödinger equation. Here we shall show how this can be done by second quantisation of the singular interaction Hamiltonian considered by von Neumann obtaining a stochastic model of continual nondemolition observation for the position of a quantum particle by counting some other quanta.

First we show in the Sec.1 that even the projection postulate can be derived in the framework of this approach with the suggested Hamiltonian interaction and a proper nondemolition observation with a discrete spectrum and sharp initial state of the meter. Then we define a single-kick stochastic wave equation for the reduced state-vector, corresponding to the unsharp $f_0(\lambda)$.

In the Sec.2 we develop the differential treatment of discontinuous unitary evolution in terms of generalized Schrödinger equation corresponding to the scatterings at given or randomly distributed time instants. Then we show how it can be reduced to the many-kick stochastic wave equation, describing spontaneous localization of a quantum particle under the continual observation of its trajectory in a bubble chamber.

The quantum system of many similar interacting particles in a bubble chamber is treated in the Sec.3. We prove that the reduced dynamics of the particles is mixing under the continual observation of positions of the scattered bubbles. It can not be described by any reduction equation for the posterior wave function but can be described by an irreversible stochastic equation for the density matrix of the particles.

Finally we obtain in the Sec.4 the macroscopic and diffusion limits of the generalized Schrödinger and reduction equations under the limit of weak coupling constant and higher frequency of the interaction with the apparatus. In the macroscopic limit these two equations coincide with the ordinary Schrödinger equation, corresponding to the mean field approximation, while in the diffusion approximation, which describes the fluctuations, they lead to the essentially different kind of stochastic wave equations [23, 24].

These results are presented by the equations (1.10), (2.10), (3.10) and (4.10).

1 Hamiltonian Model for a Generalized Reduction

Let \mathcal{H} be a Hilbert space called the state space of a particle, and let R be a selfadjoint operator in \mathcal{H} with either integer or continuous spectrum \mathbf{Z} or \mathbf{R} . Let $\kappa > 0$ be a scaling parameter. One can regard the scaled operator κR as the position x of the particle in \mathbf{R} or in the lattice $\kappa\mathbf{Z}$ if it is quantised so that κR is given in the x -representation, the operator acts as the multiplication $\kappa R|x\rangle = |x\rangle\lambda(x)$ by $\lambda(x) = \kappa[x/\kappa]$, where $[x] \in \mathbf{Z}$ denotes the integer part of x .

A quantum meter with continuous ($\Lambda = \mathbf{R}$) or discrete ($\Lambda = \varepsilon\mathbf{Z}$) pointer scale is described by the Hilbert space $L^2(\Lambda)$ of complex-valued functions $f : \Lambda \rightarrow \mathbf{C}$ square integrable in the sense that $\|f\|^2 = \int |f(y)|^2 dy < \infty$.

The last integral coincides with $\sum_{y \in \Lambda} |f(y)|^2 \varepsilon$ if $f(y) = \sum_{l \in \mathbf{Z}} \delta_l^\varepsilon(y) f(\varepsilon l) \varepsilon$ is the isometric interpolation of the discrete wave-function $f(\varepsilon l)$, $l \in \mathbf{Z}$, given by

$$\delta_l^\varepsilon(y) = \int_0^{1/\varepsilon} e^{2\pi i k(\varepsilon l - y)} dk, \quad y \in \mathbf{R}.$$

Consider a moving particle with Hamiltonian H in \mathcal{H} . Its singular evolution corresponding to the position measurement at time $t = 0$ is described in the product space $\mathcal{H}_1 = \mathcal{H} \otimes L^2(\Lambda)$ by time-dependent Hamiltonian

$$H_1(t) = H \otimes \mathbf{1} - \kappa R \otimes \delta(t)Q, \quad (1.1)$$

in the interaction picture corresponding to the free translation evolution of the pointer coordinate $p \in \Lambda$. Here $\mathbf{1}$ is the identity operator in $L^2(\Lambda)$, defining the particle Hamiltonian $H_0 = H \otimes \mathbf{1}$ in \mathcal{H}_1 , and Q is treated as a coordinate operator of the meter, $Q = i\hbar \partial / \partial y$, having the bound spectrum $2\pi\hbar[0, 1/\varepsilon)$ for $\Lambda = \varepsilon\mathbf{Z}$. This generates the shift operator (0.2) in $L^2(\Lambda)$ with $x \in \Lambda$ and the scattering operator

$$S_t = \exp\left(\frac{i}{\hbar} \kappa R \otimes Q_t\right) = \begin{cases} S, & t > 0 \\ I, & t \leq 0 \end{cases} \quad (1.2)$$

in \mathcal{H}_1 , where $Q_t = 1_t Q$, $1_t = 1$ if $t > 0$, and $1_t = 0$ otherwise.

The singular time dependence of the Hamiltonian (1.1) makes it impossible to define the Schrödinger equation $i\hbar d\psi/dt = H_1(t)\psi(t)$ at $t = 0$ in the usual sense. But one can define a discontinuous unitary evolution $U_1(t) : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ for some $t_0 < 0$ as the single-jump unitary process

$$U_1(t) = \exp\left(-\frac{i}{\hbar} \int_{t_0}^t H_1(s) ds\right) = e^{iH_0(t_0-t)/\hbar} S_t$$

provided that $[R, H] = 0$. If R and H do not commute, the usual method which a physicist would follow at facing this difficulty is to consider the free evolution $\psi_-(t)$ for $t < 0$ and $\psi_+(t)$ for $t > 0$, supplemented by the boundary condition $\psi_+(0) = S\psi_-(0)$. Although this method is hardly applicable in the case of spontaneous interaction at a random instant $t_1 > t_0$, let us show that the discontinuous unitary evolution corresponding to the fixed $t_1 = 0$ can be defined as the fundamental solution $\psi(t) = U_1(t)\psi_0$ of a generalized

Schrödinger equation. Such regularized equation can be written in terms of the forward differentials

$$d\psi(t) = \psi(t + dt) - \psi(t), \quad d1_t = 1_{t+dt} - 1_t,$$

as the Ito integral of

$$d\psi(t) + \frac{i}{\hbar} H_0 \psi(t) dt = (S - I) \psi(t) d1_t, \quad \psi(t_0) = \psi_0. \quad (1.3)$$

For every initial $\psi_0 \in \mathcal{H}$ the equation (1.3) has a unique solution at $t_0 \leq 0$; this solution is given by the unitary operator $U_1(t) = U_0(t - t_0) S_t(-t_0)$, where $U_0(t) = \exp\{-iH_0 t/\hbar\}$, $S_t(r) = U_0^\dagger(r) S_t U_0(r)$.

To prove this let us rewrite the generalized Schrödinger equation in the integral form

$$\psi(t) = e^{-iH_0 t/\hbar} \left(e^{iH_0 t_0/\hbar} \psi_0 + \int_{t_0}^t e^{iH_0 r/\hbar} (S - I) \psi(r) d1_r \right) \quad (1.4)$$

(the equivalence of (1.4) to (1.3) can be shown by straightforward differentiation). We can write $\psi(0) = U_0(-t_0) \psi_0$ for the solution (1.4) when $t = 0$ since $\int_{t_0}^0 \varphi(r) d1_r = 0$ if $t_0 \leq 0$; hence $\psi(t)$ can be rewritten as

$$\psi(t) = U_0(t) (\psi(0) + (S - I) 1_t \psi(0)) = U_0(t) S_t \psi(0).$$

It follows from $\int_{t_0}^t \varphi(r) d1_r = 1_t \varphi(0)$ for any function φ and any $t_0 < 0$, and $(S - I) 1_t = S_t - I$ by the definition of the scattering operator (1.2). Thus, $U_1(t)$ is the unitary operator

$$U_0(t) S_t U_0(-t_0) = U_0(t - t_0) S_t(-t_0).$$

This gives the solution to equation (1.3) for $t_0 = 0$ as well, since $\psi(t) = U_0(t) S_t \psi(0)$ is equal to ψ_0 for $t = 0$.

The rescaled pointer operator $P = \kappa^{-1} I \otimes p$, switched on at the instant $t = 0$ of the scattering, is described in the space $\mathcal{H} \otimes L^2(\Lambda)$ by the operator $P_t = 1_t P$, having in the Heisenberg picture $Y_t = U_1^\dagger(t) P_t U_1(t)$ the constant form $Y_t = Y(-t_0)$ for all $t > 0$. Here $Y(r) = S^\dagger(r) P S(r)$ taken at $r = -t_0$, $t_0 \leq 0$ is the shifted operator

$$Y(r) = R(r) \otimes \mathbf{1} + \frac{1}{\kappa} I \otimes p, \quad (1.5)$$

where $R(r) = U^\dagger(r)RU(r)$ and $S(r) = \exp(\kappa R(r) \otimes iQ/\hbar)$. Obviously the observables $\{Y_t\}$ are self-nondemolition in the sense of their joint measurability. It follows from the trivial commutativity condition

$$[Y_s, Y_t] = 0, \quad \forall s, t. \quad (1.6)$$

And they are nondemolition with respect to an arbitrary particle operator $X_t(r) = U_1(t)^\dagger(X \otimes \mathbf{1})U_1(t)$ in the Heisenberg picture at $t_0 = -r$ in the sense of their predictability [9, 13]

$$[X_s, Y_t] = 0, \quad \forall s \geq t, \quad (1.7)$$

indeed, $[X_s, Y_s] = 0$ and $Y_s = Y_t$ for $s, t > 0$ since

$$U_1^\dagger(s)P_tU_1(s) = S^\dagger(-t_0)P_tS(-t_0), \quad \forall s \geq t.$$

Let us fix a state vector $f_0 \in L^2(\Lambda)$, $\|f_0\| = 1$, given by a wave function $f_0(y)$ on Λ localized at $y = 0$. Let $|y\rangle$ denote the (generalized) eigen function in the spectral representation $p = \int y|y\rangle\langle y|d\lambda$ of the pointer coordinate $(pf)(y) = yf(y)$, where $d\lambda = dy$ if $y \in \mathbf{R}$ and $d\lambda = \varepsilon$ if $y \in \varepsilon\mathbf{Z}$. This yields the localizing transformations $F_t(y) = \langle y|S_t f_0$ in the form

$$F_t(y) = f_0(yI - \kappa R_t) = \begin{cases} F(y), & t > 0 \\ f_0(y)I, & t \leq 0 \end{cases} \quad (1.8)$$

where $F(y) = f_0(yI - \kappa R)$. The reduction transformations $\eta \mapsto \psi(t, y)$, defined on the particle space \mathcal{H} by the formula

$$\psi(t, y) = U(t - t_0)F_t(-t_0, y)\eta, \quad y \in \Lambda,$$

with $U(t) = \exp(-iHt/\hbar)$ and $F_t(r) = U^\dagger(r)F_tU(r)$ reproduce the unitary evolution $U_1(t)$ in the y -representation on $\eta \otimes f_0 \in \mathcal{H}_1$ similarly to Eq. (0.4) and (0.5), namely,

$$U(t - t_0)F_t(-t_0, y)\eta = U(t)(y|S_t f_0 U(-t_0)\eta = (y|U_1(t)(\eta \otimes f_0).$$

The operator $F(r, y) = f_0(yI - \kappa R(r))$ at $r = -t_0$ with a given initial vector-state η of the particle before the scattering ($t_0 \leq 0$) define the probability measure

$$\mu(\Delta) = \int_{\Delta} \|F(r, y)\eta\|^2 d\lambda = \langle \eta, \Pi_t(\Delta)\eta \rangle, \quad \Delta \subseteq \Lambda,$$

for the statistics of the nondemolition measurement of κR via the observation of the pointer position $y \in \Delta$ after the scattering. It is given by a positive operator-valued measure $\Pi_t(\Delta) = \int_{\Delta} |F_t(y, -t_0)|^2 d\lambda$ at $t > 0$. (Before the scattering it defines the initial probability measure

$$\mu_0(\Delta) = \langle \eta, \Pi_t(\Delta) \eta \rangle = \int_{\Delta} |f_0(y)|^2 d\lambda, \quad t \leq 0$$

which is independent of η .)

In the case of discrete scale the eigen functions $|y\rangle$ are normalisable: $\langle y|y\rangle = 1/\varepsilon$, and one can take a sharply localized $f_0 = \varepsilon^{d/2}|0\rangle$, given by the eigen function $f_0(y) = e(y)/\varepsilon^{d/2}$, where $e(y) = 1$ if $y = 0$, $e(y) = 0$ if $y \neq 0$. By renormalising the operator $F(y)$ as $E(y) = \varepsilon^{d/2}F(y)$ with step $\varepsilon = \kappa$ of the position quantisation, one obtain the orthogonal projections

$$e(\kappa R - yI) = \int_{x:|x/\kappa|=y/\kappa} |x\rangle\langle x| dx = E(y), \quad y \in \kappa\mathbf{Z}$$

of $\kappa R = \int \lambda(x)|x\rangle\langle x| dx = \sum_{y \in \kappa\mathbf{Z}} y E(y)$ corresponding to eigen values $\lambda(x) = \kappa|x/\kappa|$. Thus, $\Pi_t(\Delta)$, $\Delta \subseteq \Lambda$ is the spectral measure $\sum_{y \in \Delta} E(t_0, y)$ of the quantised position $\kappa R(-t_0)$ of the particle for a $t > 0$ given by the eigen orthoprojections $E(r, y) = U^\dagger(r)E(y)U(r)$ of the operator $R(r)$, corresponding to the rescaled pointer integer values y/κ . Thus, the projection reduction postulate has been deduced from the Hamiltonian interaction (1.1) and the nondemolition measurement for the sharply localized initial state f_0 . But there is no continuous limit as $\kappa \rightarrow 0$ of such sharp reduction with non-trivial $e \neq 0$, since $\|e\|^2 = \int |e(y)|^2 d\lambda = \kappa \rightarrow 0$ and the sharp function $e(y) = \delta_0^\kappa(y)\kappa$ on $\Lambda = \kappa\mathbf{Z}$ disappears as an element of the state space $L^2(\mathbf{R})$ of the continuous meter.

In the continuous case, one can take only an approximately sharp $f_0 \in L^2(\mathbf{R})$, say $f_0(y) = \delta_0^\varepsilon(y)\varepsilon$, $y \in \mathbf{R}$, and renormalise the operator (1.8) as $G_t(y) = f_0(yI - \kappa R_t)/f_0(y)$ if $f_0(y) \neq 0$, as in the Gaussian case (0.6). They define $G_t(y)$ as the identity operator for $t \leq 0$ or $r > 0$, whereas for $t > 0$ and $r \leq 0$ the operator

$$G(y) = \langle y|Sf_0 = (y|G, \quad G = f_0^{-1}Sf_0, \quad (1.9)$$

say, of the Gaussian form

$$G(y) = \exp\left\{\pi\kappa R(yI - \frac{1}{2}\kappa R)\right\}$$

given by the generalized eigen functions $|y\rangle = |y\rangle/f_0(y)$ for the spectral representation $p = \int y|y\rangle\langle y|d\mu_0$ with respect to the initial probability measure $d\mu_0 = |f_0(y)|^2d\lambda$ with density $|f_0(y)|^2 = \exp\{-\pi y^2\}$ in the Gaussian case.

The corresponding propagators

$$T(t, y) = U(t - t_0)G_t(-t_0, y), \quad y \in \Lambda, \quad t_0 \leq 0,$$

where $G_t(r) = U^\dagger(r)G_tU(r)$, define the operator-valued measure $\Pi_t(\Delta)$ as

$$\Pi_t(\Delta) = \int_\Delta T^\dagger(t, y)T(t, y)d\mu_0 = \int_\Delta |G_t(-t_0, y)|^2d\mu_0$$

so that the output probability measure μ is absolutely continuous with respect to μ_0 for any $\eta \in \mathcal{H}$: $\mu(\Delta) = \int_\Delta \|G(-t_0, y)\eta\|^2d\mu_0$. Hence, the reduced state vector $\chi(t, y) = T(t, y)\eta$ is normalized to 1 as a stochastic vector process $\chi(t) : y \mapsto \chi(t, y) \in \mathcal{H}$ in the mean square sense with respect to the input probability measure μ_0 ,

$$\|\chi(t)\|_0^2 = \int \langle \chi(t, y), \chi(t, y) \rangle d\mu_0 = \langle \eta, \eta \rangle = 1.$$

This model of nondemolition observation with continuous data $y \in \mathbf{R}$ also applies to unsharp measurement of operator R with discrete spectrum. In contrast to sharp measurement, unsharp measurement is not sensitive to the continuous spectrum limit as $\kappa \rightarrow 0$ of $R = [x/\kappa]$, corresponding to the replacement of κR by $x \in \mathbf{R}$.

For any initial $t_0 \leq 0$ the stochastic vector process $\chi(t) = T(t)\eta$ satisfies the single-kick equation

$$d\chi(t) + \frac{i}{\hbar}H\chi(t)dt = d1_t[G - I]\chi(t), \quad \chi(t_0) = \eta \quad (1.10)$$

generated by the random differential $d1_t[G - I](y) = (G(y) - I)d1_t$ on $y \in \Lambda$ with respect to the initial probability measure μ_0 . This simplest reduction equation is written in terms of the forward differentials $d\chi(t, y) = \chi(t + dt, y) - \chi(t, y)$, that is, is understood in the sense of Ito.

Indeed, by representing G_t as $G_t(y) = (G(y) - I)1_t + I = I + 1_t[G - I](y)$, one can describe $\chi(t) = U(t - t_0)G_t(-t_0)\eta$ by the integral equation

$$\begin{aligned} \chi(t) &= U(t - t_0)\eta + U(t)1_t[G - I]U(-t_0)\eta = \\ &= e^{-iHt/\hbar} \left(e^{iHt_0/\hbar}\eta + \int_{t_0}^t e^{iHr/\hbar}d1_r[G - I]\chi(r) \right). \end{aligned}$$

But this equation is equivalent to the differential equation (1.10), a fact that can be proved by straightforward differentiation taking into account the Ito multiplication table

$$(dt)^2 = 0, \quad dt d1_t = 0 = d1_t dt, \quad (d1_t)^2 = d1_t.$$

Similarly, one can the simplest nonlinear stochastic equation for the normalized reduced state vector $\chi_y(t) = \chi(t, y) / \|\chi(t, y)\|$:

$$d\chi_y(t) + \frac{i}{\hbar} H \chi_y(t) dt = (G_y(t) - I) \chi_y(t) d1_t, \quad \chi_y(0) = \eta$$

where $G_y(t) = G(y) / \|G(y) \chi_y(t)\|$, $\eta \in \mathcal{H}$. This equation is an equivalent differential form the nonlinear integral stochastic equation

$$\begin{aligned} \chi_y(t) &= e^{-iHt/\hbar} \left(e^{iHt_0/\hbar} \eta + \int_{t_0}^t e^{iHr/\hbar} (G_y(r) - I) \chi_y(r) d1_r \right) = \\ &= U(t - t_0) \eta + U(t) (G_y - I) 1_t \chi_y(0) = U(t) G_t(y) \chi_y(0) / \|G_t(y) \chi_y(0)\|. \end{aligned}$$

This yields $\chi_y(t) = T(t, y) \eta / \|T(t, y) \eta\|$ for a $t_0 \leq 0$ because of $\|G_t(y) \eta\| = \|T(t, y) \eta\|$.

Note that the random state vector $\chi_y(t)$ is obtained by conditioning with respect to the output (rather than input) probability measure $d\mu = \|\chi(t, y)\|^2 d\mu_0$.

2 Spontaneous Localization of a Single Particle

Let us consider a spontaneous process of scattering interactions (1.1) of a quantum particle (or an atom) at random time instants $t_n > 0, t_1 < t_2 < \dots$, with a renewable meter in an apparatus of the cloud chamber (photodetector) type with bubbles (photons) serving as the meter. We consider the increasing sequences (t_1, t_2, \dots) as countable subsets $\tau \subset \mathbf{R}_+$ such that $\tau_t = \tau \cap [0, t)$ is finite for any $t \geq 0$ in accordance with the finiteness of the number of scattered bubbles on the finite observation interval $[0, t)$. The set of all such infinite τ will be denoted by Γ_∞ , and Γ is the inductive limit $\cup \Gamma_t$ as $t \rightarrow \infty$ of $\Gamma_t = \{\tau_t | \tau \in \Gamma_\infty\}$, which is the disjoint union $\Gamma_t = \sum_{n=0}^\infty \Gamma_t(n)$ of n -simplex $\Gamma_t(n) = \{t_1 < \dots < t_n\} \subset [0, t)^n$.

The measurement apparatus is assumed to be a quantum system of infinitely many bubbles each of which is identical to the single meter described in the previous section. The pointer coordinate is attached to the momentum p_n of a bubble labeled by the scattering number $n \in \mathbf{N}$, such that it shows the momentum $\lambda_n \in \Lambda$ of the scattered bubble at each time $t_n \in \tau$.

The corresponding Hamiltonian of the moving particle is given by the series

$$H(t, \tau) = H_0 - \kappa R \otimes \sum_{n=1}^{\infty} \delta(t - t_n) Q(n) \quad (2.1)$$

having at most two nonzero terms if $t \in \tau$. Here $H_0 = H \otimes \mathbf{1}^{\otimes \infty}$ is the Hamiltonian describing the time evolution on the intervals between the scatterings $t \in \tau$ and $Q(n)$ is the coordinate of the n th scattered bubble, given as the operator $Q(n) = i\hbar d/d\lambda_n$.

The generalized Schrödinger equation corresponding to the Hamiltonian (1.1) can be written for fixed $\tau \in \Gamma_\infty$ by analogy with the single-kick case

$$d\psi(t) + \frac{i}{\hbar} H_0 \psi(t) dt = (S(n_t) - I) \psi(t) dn_t, \quad \psi(0, \tau) = \psi_0. \quad (2.2)$$

Here $S(n) = \exp\{\frac{i}{\hbar} \kappa R \otimes Q(n)\}$ and $n_t(t) = |\tau_t|$ is the numerical process that gives the cardinality $|\tau_t| = \sum_{r \in \tau} 1_{t-r}$ of the localized subset $\tau_t = \{t_n < t\}$, so that $dn_t(\tau)$ is equal to 1 for $t \in \tau$, and zero otherwise.

The solution to this equation is uniquely determined for every $\tau \in \Gamma_\infty$ by the initial state ψ_0 of the system. Namely,

$$\psi(t, \tau) = U(t, \tau) \psi_0,$$

where $U(t, \tau) = U_0(t) V_t^\dagger(\tau)$, $V_t^\dagger(\tau)$ is the chronological product $\prod_{r \in \tau}^{\leftarrow} S_t(r) = S(t_{n_t}) \dots S(t_1)$, and

$$V_t(\tau) = S_t^\dagger(t_1) S_t^\dagger(t_2) \dots = \left(\prod_{r \in \tau}^{\leftarrow} S_t(r) \right)^\dagger. \quad (2.3)$$

Here $S_t(t_n) = U_0^\dagger(t_n) S_t(n) U_0(t_n)$ for $t_n < t$, where $S(n) = \exp\{-\frac{i}{\hbar} \kappa R \otimes Q(n)\}$, and $S_t(t_n) = I$ if $t_n \geq t$ so that the infinite product (2.3) contains only a finite number $n_t = \sum_{r \in \tau} 1_{t-r}$ of factors different from the identity operator I .

Recall that the differential equation (2.2) is equivalent to the integral equation given by the recurrence relation

$$\psi(t, \tau) = e^{-iH_0 t/\hbar} \left(\psi_0 + \sum_{r \in \tau}^{r < t} e^{iH_0 r/\hbar} (S(n_r) - I) \psi(r, \tau) \right) \quad (2.4)$$

for every $\tau \in \Gamma_\infty$. Hence, $\psi(t, \tau) = U_0(t) V_t^\dagger(\tau) \psi_0$, where $U_0(t) = e^{-iH_0 t/\hbar}$ and $V_t(\tau)$ is a solution to the operator equation

$$V_t(\tau) = I + \sum_{r \in \tau}^{r < t} V_r(t) (S_t^\dagger(r, \tau) - I), \quad V_0(\tau) = I,$$

where $S^\dagger(t, \tau) = U_0(t)^\dagger S(n_t(\tau))^\dagger U_0(t)$. But this equation has a unique solution (2.2), which can be written as the binomial sum

$$[L_t(t_1) + I][L_t(t_2) + I] \dots = \sum_{\sigma \subseteq \tau_t} L(s_1, \tau) \dots L(s_n, \tau)$$

in terms of $\sigma = \{s_1, \dots, s_n\}$, $s_1 < \dots < s_n$, $n \leq n_t$, $L_t(r) = S_t^\dagger(r) - I$ ($= 0$ if $r \geq t$) and $L(r, \tau) = S^\dagger(r, \tau) - I$. Indeed, this sum contains I as the null product corresponding to $\sigma = \emptyset$, and the sum of the other terms is equal to

$$\begin{aligned} V_t(\tau) - I &= \sum_{r \in \tau}^{r < t} \sum_{\sigma \subseteq \tau_r} L(s_1, \tau) \dots L(s_m, \tau) L(r, \tau) = \\ &= \sum_{r \in \tau}^{r < t} V_r(\tau) L(r, \tau) = \sum_{r \in \tau}^{r < t} V_r(\tau) (S^\dagger(r, \tau) - I), \end{aligned}$$

where $m \leq n_t - 1$.

Note that the generalized differential equation (2.2) depending on $\tau \in \Gamma_\infty$ via $n_t = n_t(\tau)$ is not necessarily stochastic as long as we have not fixed a probability distribution for the instants $\tau = (t_1, t_2, \dots)$ of the singular interactions. In order to obtain a continuous (at least in the mean) dynamics for such a quantum jump process it is necessary to assume the interactions are spontaneous with a continuous probability distribution of random instants τ . One can assume that the number process $n_t(\tau)$ is stochastic, given by the Poisson law $\pi_0(d\tau)$ on Γ_∞ presented as the projective limit as $t \rightarrow \infty$ of the probability measures

$$\pi_0(d\tau_t) = e^{-\nu t} \nu^{|\tau_t|} d\tau_t, \quad \nu > 0. \quad (2.5)$$

Here $\tau_t = \tau$ is a finite time-ordered sequence $\tau(n) = (t_1, \dots, t_n) \in \Gamma_t$ with $n = n_t$, $d\tau_t = \prod_{k=1}^{n_t} dt_k$ is the measure on Γ_t given by the sum of product measures $dt_1, \dots, dt_n = d\tau(n)$ on the simplices $\Gamma_t(n)$, $d\tau(0) = 1$ on $\Gamma_t(0) = \{\emptyset\}$ such that

$$\int_{\Gamma_t} \nu^{|\tau|} d\tau := \sum_{n=0}^{\infty} \nu^n \int \dots \int_{0 \leq t_1 < \dots < t_n < t} dt_1 \dots dt_n = e^{\nu t}.$$

Note that any other numerical process can be described by a positive density function $f(\tau)$ with respect to the Poissonian measure, that is, has the form $f(\tau)\pi(d\tau)$.

Let us fix an initial state $\varphi_0 = f_0^\infty$ of the bubbles as the infinite product $f_0^\infty = \otimes_{k=1}^{\infty} f_k$ of the identical state vectors $f_k = f_0$ of the bubbles given by a normalized element $f_0 \in L^2(\Lambda)$. This defines the solutions $\psi(t, \tau)$ of the stochastic equation (2.2) with the initial data $\psi_0 = \eta \otimes \varphi_0$ given by state vectors $\eta \in \mathcal{H}$ in the particle space \mathcal{H} as the state vectors in the product space $\mathcal{H}_\infty = \mathcal{H} \otimes \mathcal{E}$, where \mathcal{E} is the Hilbert space generated by the infinite-product functions $\varphi(v) = \prod_{k=1}^{\infty} f_k(\lambda_k)$ with the equal elements $f_k = f_0$ for almost all k . We suppose, as in Sec.1, that $\varphi_0(v) \neq 0$ for almost all $v = (\lambda_1, \lambda_2, \dots)$ such that one can identify the space \mathcal{E} with the space $L_0^2(\Lambda^\infty)$ of all square-integrable functions $f = \varphi/\varphi_0$ with respect to the product measure $\mu_0^\infty(d\boldsymbol{\lambda}) = \mu_0(d\lambda_1) \cdot \mu_0(d\lambda_2) \cdot \dots$ on the space $\Lambda^\infty = \Lambda \times \Lambda \times \dots$ of the sequences $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$: $\|f\|_0^2 = \int |f(\boldsymbol{\lambda})|^2 \mu_0^\infty(d\boldsymbol{\lambda}) = \|\varphi\|^2$. The generalized product-vectors $|\boldsymbol{\lambda}\rangle = |\lambda_1\rangle \otimes |\lambda_2\rangle \otimes \dots$ of this space are defined by tensor-product of the δ -functions $\langle \lambda' | \lambda \rangle$, respectively to μ_0 such that $\int |\boldsymbol{\lambda}\rangle \langle \boldsymbol{\lambda} | \mu_0^\infty(d\boldsymbol{\lambda}) = \mathbf{1}^{\otimes \infty}$.

Consider the sequence (p_1, p_2, \dots) of momentums p_n , of the scattered bubbles at the time instants $\{t_1, t_2, \dots\}$. The commuting operators p_n , $n \in \mathbf{N}$, described in \mathcal{E} by the multiplications $p_n |\boldsymbol{\lambda}\rangle = |\boldsymbol{\lambda}\rangle \lambda_n$, are assumed to be measured at the random time instants t_n , $n \in \mathbf{N}$. The point trajectories of such measurements are given by the sequences $y_n = (y_1, y_2, \dots)$ of pairs $y_n = (t_n, \lambda_n)$ with $t_1 < t_2, \dots$ and $\lambda_n \in \Lambda$, identified with countable subsets $\{y_1, y_2, \dots\} \subset \mathbf{R}_+ \times \Lambda$. As elements $v = (\tau, \boldsymbol{\lambda})$ of the Cartesian product $\Upsilon_\infty = \Gamma_\infty \times \Lambda^\infty$, they have the probability distribution $P_0(dv) = \pi_0(d\tau) \mu_0^\infty(d\boldsymbol{\lambda})$, where Λ^∞ is the space of all sequences $\boldsymbol{\lambda} \in \Lambda^\infty$, $\lambda_n \in \Lambda$, equipped with the probability product-measure $\mu_0^\infty(d\boldsymbol{\lambda})$.

The measurement data of the observable process up to a given time instant $t > 0$ is described by a finite sequence $v_t = (y_1, \dots, y_n)$ with $n = n_t(v)$ given by the numerical process $n_t(\tau)$ for the component τ of v .

Let us introduce the counting distribution $n_t(\Delta) = |v_t \cap (\mathbf{R}_+ \times \Delta)|$ as the number $n_t(\Delta, v)$ of scatterings in the time-space region $[0, t) \times \Delta$ and define the counting integral $\int_0^\infty \int_\Lambda L(r, \lambda) dn_r(d\lambda)$ over $y \in \mathbf{R}_+ \times \Lambda$ as the series

$$n[L](v) = \sum_{y \in v} L(y), \quad \forall v \in \Upsilon_\infty. \quad (2.6)$$

Having fixed an integer-valued distribution $n_t(\Delta) \in \{0, 1, \dots\}$ as a function of $t \geq 0$ and of measurable sets $\Delta \subseteq \Lambda$, one can obtain the corresponding trajectory v as a sequence of the counts of the jumps of $n_t(\Delta)$ in the time-space $\mathbf{R}_+ \times \Lambda$.

Given an initial state vector in \mathcal{H}_∞ of the form $\psi_0 = \eta \otimes \varphi_0$ with fixed $\varphi_0 = f_0^\infty$, one can define a nonunitary stochastic evolution $\eta \mapsto T(t, v)\eta$ by setting

$$T(t, v) = \langle \boldsymbol{\lambda} | U(t, \tau) \varphi_0, \quad v = (\tau, \boldsymbol{\lambda}),$$

which reproduces the unitary evolution $U(t, \tau) = U_0(t) V_t^\dagger(\tau)$ defined by (2.2). This can also be written as $T(t, v) = U(t) F_t^\dagger(v)$, since $U_0(t) = U(t) \otimes \mathbf{1}^{\otimes \infty}$ commutes with the (generalized) eigen-bras $\langle \boldsymbol{\lambda} | = \langle \lambda_1, \lambda_2, \dots |$ of the bubble coordinates $(p_1, p_2, \dots) : \langle \boldsymbol{\lambda} | U_0(t) = U(t) \langle \boldsymbol{\lambda} |$. The reduction transformations $F_t(v)$, $v \in \Upsilon_\infty$, are given by the chronological products

$$F_t(v) = G_t^\dagger(y_1) G_t^\dagger(y_2) \dots \equiv \prod_{y \in v}^{\rightarrow} G_t^\dagger(y) \quad (2.7)$$

of $G_t(t_n, \lambda_n) = U^\dagger(t_n) G(\lambda_n) U(t_n)$ for $t_n < t$, where $G(\lambda) = \langle \lambda | S f_0$, owing to the product form (2.3) of the unitary transformations $V_t(\tau)$, $\tau \in \Gamma$, and $\langle \boldsymbol{\lambda} | = \otimes_{k=1}^\infty \langle \lambda_k |$, $\varphi_0(\boldsymbol{\lambda}) = \prod_{k=1}^\infty f_0(\lambda_k)$, $\langle \boldsymbol{\lambda} | \varphi_0 \equiv 1$ for $\boldsymbol{\lambda} \in \Lambda^\infty$. The stochastic operator (2.7) defined by the single-point reductions

$$G_t(r, \lambda) = \langle \lambda | S_t(r) f_0 = \begin{cases} G(r, \lambda), & r < t \\ I, & r \geq t \end{cases} \quad (2.8)$$

is normalized with respect to the initial probability $P_0(dv)$, $\Pi_{\Upsilon_\infty}[I](t) = I$, where

$$\Pi_A[X](t) = \int_A T^\dagger(t, v) X T(t, v) P_0(dv) = \int_A F_t(v) X F_t^\dagger(v) P_0(dv), \quad (2.9)$$

is a continual operational-valued measure [3]–[5] defined on measurable sets $A \subseteq \Upsilon_\infty$ of the point trajectories $v_t = \{(r, \lambda) \in v | r < t\}$ given by the operations $\Phi_t(v) : X \mapsto F_t(v) X F_t^\dagger(v)$ for particle operators $X : \mathcal{H} \rightarrow \mathcal{H}$.

The positive operator-valued measure $\Pi_t(A) = \Pi_A[I](t)$ gives the statistics

$$P_t(A) = \int_A \|\langle \boldsymbol{\lambda} | U(t, \tau) \psi_0 \rangle\|^2 \mu_0^\infty(d\boldsymbol{\lambda}) \pi_0(d\tau_t)$$

for the continual observation with respect to an arbitrary initial wave-function $\psi_0 = \eta \otimes f_0^\infty$, $\eta \in \mathcal{H}$ in the form

$$P_t(A) = \langle \eta, \Pi_t(A) \eta \rangle.$$

The output probability measure $P(A)$, $A \subseteq \Upsilon_\infty$, is defined by the marginales $P_t(A)$, $A \subseteq \Upsilon_t$, as $t \rightarrow \infty$.

The reduced wave function $\chi(t, v) = T(t, v)\eta$ is normalized

$$\|\chi(t)\|^2 = \int \|\chi(t, v)\|^2 P_0(dv) = 1$$

as a stochastic vector process $\chi(t) : \Upsilon_\infty \rightarrow \mathcal{H}$ with respect to the initial probability distribution P_0 on Υ_∞ .

It satisfies the stochastic wave equation

$$d\chi(t) + \frac{i}{\hbar} H\chi(t)dt = dn_t[G - I]\chi(t), \quad \chi(0) = \eta, \quad (2.10)$$

expressed in terms of the random differential $dn_t[G - I](v) = (G(\lambda_{n_t}(v)) - 1)dn_t(v)$, $n_t(v) = n_t(\tau)$ for the point distribution $n_t[L] = \int_\Lambda L(\lambda)n_t(d\lambda) = n[L_t]$ over $\lambda \in \Lambda$ defined in (2.6) for $L_t(r, \lambda) = 1_t(r)L(\lambda)$.

To prove Eq.(2.10), discovered for the first time in [13], we rewrite it in the following integral form

$$\chi(t) = e^{-iHt/\hbar} \left(\eta + \int_0^t \int_\Lambda e^{iHr/\hbar} (G(\lambda) - I) \chi(r) dn_r(d\lambda) \right),$$

given for each $v \in \Upsilon_\infty$ by the finite sum

$$n[1_t U^\dagger(r)(G - I)\chi](v) = \sum_{(r, \lambda) \in v}^{r < t} U^\dagger(r)(G(\lambda) - I)\chi(r).$$

We express the solution to this equation in the form $\chi(t, v) = U(t)F_t^\dagger(v)\eta$ via the solution (2.7) to the recursion equation

$$F_t(v) = I + \sum_{(r, \lambda) \in v}^{r < t} F_r(v)(G(r, \lambda) - I), \quad F_0(v) = I,$$

with $G(t, \lambda) = U^\dagger(t)G(\lambda)U(t)$, as was done for the unitary case.

Let us also write on F the nonlinear equation

$$d\chi_v(t) + \frac{i}{\hbar} H\chi_v(t)dt = (G_v(t) - I)\chi_v(t)dn_t(v), \quad \chi_v(0) = \eta,$$

with $G_v(t) = G(\lambda_{n_t(v)})/\|G(\lambda_{n_t(v)})\chi_v(t)\|$. Its solutions define the normalized reduction $\chi_v(t) = \chi(t, v)/\|\chi(t, v)\|$ for continual counting measurements as a stochastic vector process $\chi_v(t) \in \mathcal{H}$ with respect to the output probability measure P of the point process $t \mapsto v_t$. This can easily be obtained, as in [14] by applying the Ito multiplication table

$$(dt)^2 = 0, \quad dt dn_t = 0 = dn_t dt, \quad (dn_t)^2 = dn_t.$$

3 Mixing Reduction for Many Identical Particles

We now consider M identical particles (atoms) interacting independently with the bubbles (photons) in accordance with the scattering term in the Hamiltonian (2.1). The spontaneous process of scatterings is described by the time-ordered sequences of pairs (k_n, t_n) , $t_1 < t_2 < \dots$, where $k_n \in \{1, \dots, M\}$ is the number of the particle labeled by the scattering number $n \in \mathbf{N}$ at time instant $t_n > 0$. We have excluded possibility of two or more scatterings of the bubbles at the same instant of time, as was done for a single particle in Sec.2. The sequence $(k_1, t_1), (k_2, t_2), \dots$ of the scatterings can be represented by the occupational subsets $\tau_k = \{t_n \in \tau | k_n = k\}$ of the time set $\tau = \{t_1, t_2, \dots\}$, which are disjoint, $\tau_k \cap \tau_l = \emptyset$ if $k \neq l$ since the scatterings for different particles are independent. We shall consider the M -tuples $\tau_\bullet = (\tau_1, \dots, \tau_M)$ of these countable subsets $\tau_k \subset \mathbf{R}_+$ as elements $\tau_\bullet \in \Gamma_\infty^M$ of the Cartesian M -product of Γ_∞ , given by the partition $\tau = \sqcup \tau_k := \cup \tau_k$, $\tau_k \cap \tau_l = \emptyset$ if $k \neq l$ of a $\tau \in \Gamma_\infty$.

The Hamiltonian of the interacting particles for fixed $\tau_\bullet = \{(k_1, t_1), (k_2, t_2), \dots\}$ reads

$$H(t, \tau_\bullet) = H_0^M - \kappa \sum_{n=1}^{\infty} R(k_n) \otimes \delta(t - t_n) Q(n). \quad (3.1)$$

Here $H_0^M = H^M \otimes \mathbf{1}$ is the Hamiltonian of the particles describing the time evolution on the intervals between the scatterings with the bubbles:

$$H^M = \sum_{k=1}^M H(k) + \sum_{k=1}^M \sum_{l>k}^M W(k, l),$$

where $H(k) = I^{\otimes(k-1)} \otimes H \otimes I^{\otimes(M-k)}$ is the Hamiltonian of the k th particle and $W(k, l)$ is the interaction potential in $\mathcal{H}^{\otimes M}$ of the k th and l th particle, $1 \leq k < l \leq M$.

Let \mathcal{H}^M denote the M -particle Hilbert space, which is an invariant subspace $\mathcal{H}^M \subseteq \mathcal{H}^{\otimes M}$ of symmetric (bosons) or antisymmetric (fermions) M -tensors $\eta^M \in \mathcal{H}^{\otimes M}$ generated by the product-vectors $\otimes_{k=1}^M \eta_k \in \mathcal{H}^{\otimes M}$ with $\eta_k \in \mathcal{H}$. The correspondent Ito-Schrödinger equation for the stochastic state vector $\psi^M(t) : \tau_\bullet \mapsto \psi(t, \tau_\bullet)$ with values $\psi(t, \tau_\bullet) \in \mathcal{H}_\infty^M$ in the product space $\mathcal{H}_\infty^M = \mathcal{H}^M \otimes \mathcal{E}$ of the M -particle space \mathcal{H}^M by $\mathcal{E} = \lim_{n \rightarrow \infty} L^2(\Lambda^n)$ reads

$$d\psi^M(t) + \frac{i}{\hbar} H_0^M \psi^M(t) dt = (S(k_t, n_t) - I) \psi^M(t) dn_t. \quad (3.2)$$

Here $S(k, n) = \exp\{\frac{i}{\hbar} \kappa R(k) \otimes Q(n)\}$, $R(k) = I^{\otimes(k-1)} \otimes R \otimes I^{\otimes(M-k)}$,

$$k_t(\tau_\bullet) = \sum_{k=1}^M k 1_{\tau_k}(t), \quad \text{where } 1_\tau(t) = \begin{cases} 1, & t \in \tau \\ 0, & t \notin \tau \end{cases}$$

is the random number $k_t : \Gamma_\infty^M \rightarrow \{1, \dots, M\}$, labeling a particle by k at any instant $t \in \tau_k$ of its collision with a bubble labeled by $n_t(\tau) = \sum_{k=1}^M n_{k,t} = |\tau \cap [0, t]|$, where $n_{k,t} = |\tau_k \cap [0, t]|$, $\tau = \cup \tau_k$.

The solutions $\psi(t, \tau_\bullet) = U(t, \tau_\bullet) \psi_0^M$, $\psi_0^M \in \mathcal{H}_\infty^M$, of equation (3.2) can be written as $U(t, \tau_\bullet) = U_0^M(t) V_t^\dagger(\tau_\bullet)$ in terms of the finite chronological product

$$V_t(\tau_\bullet) = S_t^\dagger(k_1, t_1) S_t^\dagger(k_2, t_2) \dots, \quad \tau_\bullet \in \Gamma_\infty^M, \quad (3.3)$$

where $S_t(k, t_n) = I$ if $t_n \geq t$, $S_t(k, t_n) = U_0^{M\dagger}(t_n) S(k, n) U_0^M(t_n)$ if $t_n < t$, and $U_0^M(t) = \exp\{-\frac{i}{\hbar} H_0^M(t)\}$. The proof is exactly the same as for the case of a single particle $M = 1$.

Let $\omega = (w_1, w_2, \dots)$ denote a chronologically ordered sequence of triples $w_n = (k_n, t_n, \lambda_n)$ and Ω the space of all such sequences with $\{t_1, t_2, \dots\} \in \Gamma_\infty$. Every sequence $\omega \in \Omega$ can be represented as a pair $\omega = (\tau_\bullet, \boldsymbol{\lambda})$, where $\tau_\bullet =$

(τ_1, \dots, τ_M) is a partition of the corresponding sequence $\tau = \{t_1, t_2, \dots\}$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$, so that Ω can be identified with the product $\Gamma_\infty^M \times \Upsilon_\infty$. The space Ω is equipped with the probability measure $P_0(d\omega) = \pi_0(d\tau_\bullet)\mu_0^\infty(d\boldsymbol{\lambda})$, where $\pi_0(d\tau_1, \dots, d\tau_M) = \prod_{k=1}^M \pi_0(d\tau_k)$ is the product of the identical Poisson measures (2.5), in accordance with the independence of the spontaneous interactions of each particle with the bubbles.

Given an initial state vector $\psi^M = \eta^M \otimes \varphi_0$, where $\varphi_0 = f_0^\infty$, one can easily prove that the nonunitary stochastic evolution

$$T(t, \omega) = \langle \boldsymbol{\lambda} | U(t, \tau_\bullet) \varphi_0, \quad \omega = (\tau_\bullet, \boldsymbol{\lambda})$$

is also a finite chronological product

$$T(t, \omega) = U^M(t) F_t^\dagger(\omega), \quad F_t(\omega) := G_t^\dagger(w_1) G_t^\dagger(w_2) \dots$$

Here $U^M(t) = \exp\{-\frac{i}{\hbar} H^M t\}$, $G_t(k, t_n, \lambda) = I$ for $t_n \geq t$, and $G_t(k, t_n, \lambda) = U^{M\dagger}(t_n) G(k, \lambda) U^M(t_n)$, $t_n < t$, is defined by the reduced scattering operator

$$G(k, \lambda) = I^{\otimes(k-1)} \otimes G(\lambda) \otimes I^{\otimes(M-k)}, \quad G(\lambda) = \langle \lambda | S f_0 \quad (3.4)$$

for $f_0 \in L^2(\Lambda)$, $\|f_0\| = 1$, applied to the k th particle only in \mathcal{H}^M . The stochastic operator $T(t)$ defines the solutions $\chi^M(t, \omega) = T(t, \omega) \eta^M$ to the Ito differential equation

$$d\chi^M(t) + \frac{i}{\hbar} H^M \chi^M(t) dt = \sum_{k=1}^M dn_{k,t} [G(k) - I] \chi^M(t), \quad \chi^M(0) = \eta^M$$

for the stochastic vector states $\chi^M(t) : \Omega \rightarrow \mathcal{H}^M$ of the M -particle system, corresponding to an initial $\eta^M \in \mathcal{H}^M$. The right-hand side of this equation is written as the forward increment of the point integral

$$n[L_t] = \sum_{k=1}^M \int_0^t \int_\Lambda L(k, r, \lambda) dn_{k,r}(d\lambda)$$

given by the stochastic distribution $n[L](\omega) = \sum_{w \in \omega} L(w)$ for $L_t(k, r, \lambda) = 1_t(r) L(k, r, \lambda)$. The vector $\chi^M(t, \omega) \in \mathcal{H}^M$ as well as $\chi(t, v)$ in (2.10), is no longer normalized ($\|\chi^M(t, \omega)\| \neq 1$) for an initial state-vector $\eta^M \in \mathcal{H}^M$, $\|\eta^M\| = 1$, but it is normalized with respect to the probability measure P_0 on Ω in the mean square sense. But, in contrast to $\chi(t, v)$, $\chi^M(t, \omega)$ is not

yet the reduced description of the M -particle system under the observation of the scattering process $v_t = \{(r, \lambda) \in v | r \leq t\}$, given by the registration of the pointer positions $\lambda_n \in \Lambda$ at random time instants t_n .

The reduced dynamics corresponding to the observation is described by a stochastic operational process $X \mapsto \Theta[X](t)$,

$$\Theta[X](t, v) = \Phi_t[U^{M\dagger}(t)XU^M(t)](v) \quad (3.5)$$

for the M -particle operators $X : \mathcal{H}^M \rightarrow \mathcal{H}^M$ given by the conditional expectation

$$\Phi_t[X](\tau, \boldsymbol{\lambda}) = \frac{1}{M^{|\tau_t|}} \sum_{\sqcup \sigma_k = \tau} F_t(\sigma_{\bullet}, \boldsymbol{\lambda}) X F_t^\dagger(\sigma_{\bullet}, \boldsymbol{\lambda}) \quad (3.6)$$

where the sum is taken over all partitions $\sigma_{\bullet} = (\sigma_1, \dots, \sigma_M)$ of a finite subset $\tau_t = \tau \cap [0, t)$. This averaging is due to the impossibility to detect the individuality of the identical particles producing the indistinguishable effects on the bubbles by measuring the scatterings of the bubbles.

To prove Eq. (3.6), we need to compare the correlations of $F_t(\omega)X F_t^\dagger(\omega)$ and of an arbitrary functional $g(v_t)$ of the observable point process v_t with the correlations of (3.6) and of $g(v_t)$. But by applying the well known formula [21] one can easily find

$$\int_{\Gamma_t^M} x(\sigma_1, \dots, \sigma_M) \prod_{k=1}^M d\sigma_k = \int_{\Gamma_t} \sum_{\sigma_k : \sqcup \sigma_k = \sigma} x(\sigma_1, \dots, \sigma_M) d\sigma$$

for the multiple point integration that these correlations with respect to the probability measure P_0 on Ω given by the Poisson law (2.5) simply coincide:

$$\begin{aligned} & \int_{\Gamma_t^M} \pi_t(d\sigma_{\bullet}) \int_{\Lambda^\infty} g(\sqcup_{k=1}^M \sigma_k, \boldsymbol{\lambda}) F(\sigma_{\bullet}, \boldsymbol{\lambda}) X F_t^\dagger(\sigma_{\bullet}, \boldsymbol{\lambda}) \mu_0^\infty(d\boldsymbol{\lambda}) \\ &= \int_{\Gamma_t^M} \langle g(\sqcup_{k=1}^M \sigma_k), X(\sigma_1, \dots, \sigma_M) \rangle_0 \prod_{k=1}^M e^{-\nu t} \nu^{|\sigma_k|} d\sigma_k \\ &= \int_{\Gamma_t} \langle g(\sigma), \sum_{\sigma_k : \sqcup \sigma_k = \sigma} X(\sigma_1, \dots, \sigma_M) \rangle_0 e^{-M\nu t} \nu^{|\sigma|} d\sigma \\ &= \int_{\Gamma_t} \pi_t^M(d\sigma) \int_{\Lambda^\infty} g(\sigma, \boldsymbol{\lambda}) \frac{1}{M^{|\sigma|}} \sum_{\tau_k : \sqcup \tau_k = \sigma} F(\omega) X F^\dagger(\omega) \mu_0^\infty(d\boldsymbol{\lambda}). \end{aligned}$$

Here $\langle \cdot, \cdot \rangle_0$ is the abbreviation for the inner product in \mathcal{E} of the test function $v \mapsto g(\tau, \boldsymbol{\lambda})$ with fixed $\tau \in \Gamma_t$ and the operator function $X_t(\tau_{\bullet}, \boldsymbol{\lambda}) =$

$F_t(\tau_\bullet, \lambda)X F_t^\dagger(\tau_\bullet, \lambda)$ with fixed $\tau_\bullet \in \Gamma_t^M$. The probability measure

$$\pi_0^M(d\tau_t) = \sum_{\sqcup \sigma_k = \tau_t} \prod_{k=1}^M \pi_0(d\sigma_k) = e^{-M\nu_t} |M\nu|^{| \tau_t |} d\tau_t \quad (3.7)$$

on Γ_t has the intensity $M\nu$. It is induced by the measure $\pi_0(d\tau_\bullet)$ on Γ_∞^M with respect to the particle identification map $\tau_\bullet \in \Gamma_\infty^M \mapsto \tau = \cup_{k=1}^M \tau_k$, defining the observable data $v = (\tau, \lambda)$ by the stochastic map $\omega = (\tau_\bullet, \lambda) \mapsto v \in \Upsilon_\infty$ on $\omega \in \Omega$.

By the coincidence of the correlations proved above, the stochastic operator (3.5) is indeed the conditional expectation of the stochastic operators $X_t(\omega)$ with respect to the observable process v_t .

In contrast to the pure operations $X \mapsto X_t(\omega)$ the reduction operation $X \mapsto \Phi_t[X](v)$, preserves the symmetry of the M -particle operators X with respect to particle permutation. It is the least mixing operation which preserves the indistinguishability of the particles respectively to the observations of the bubble scatterings, corresponding to the complete nondemolition measurement of the particles.

Indeed, the reduction operation (3.6) can be simply written as the finite iteration

$$\begin{aligned} \frac{1}{M^n} \sum_{k_1, \dots, k_n=1}^M G_t^\dagger(k_1, y_1) \dots G_t^\dagger(k_n, y_n) X G_t(k_n, y_n) \dots G_t(k_1, y_1) \\ = \Psi_t[\dots \Psi_t[X](y_1) \dots](y_n) = \Phi_t[X](y_1, \dots, y_n) \end{aligned}$$

with $n = |\tau_t|$ single mixing reductions

$$\Psi[X](y) = \frac{1}{M} \sum_{k=1}^M G_t^\dagger(k, y) X G_t(k, y). \quad (3.8)$$

Given as the arithmetic mean value of the permutations for the pure operations $X \mapsto G_t^\dagger(k, y) X G_t(k, y)$, corresponding to the identical operators (3.4), the reductions $X \mapsto \Psi_t[X](y)$ are permutationally symmetric, and are not mixing only if the pure operations if they do not break this symmetry.

The derived mixing property of the reduced stochastic dynamics $t \mapsto \rho[X](t, v)$ obtained above for the corresponding statistical states

$$\rho^M[X](t, v) = \langle \eta^M, \Theta[X](t, v) \eta^M \rangle = \text{Tr}\{X \varrho^M(t, v)\} \quad (3.9)$$

gives an increase in the entropy

$$\sigma^M(t, v) = -\text{Tr}\{\varrho^M(t, v) \ln \varrho^M(t, v)\}$$

for an ensemble of identical particles even under the condition of complete nondemolition observation. According to (3.9), the reduced density operators $\varrho^M(t, v)$ for the system of M identical particles gives the probability density

$$P^M(t, v_t) = \text{Tr}\{\varrho^M(t, v_t)\} = P^M(dv_t)/P_0^M(dv_t)$$

of the output process v_t . Here $P_0^M = \pi^M \otimes \mu_0^\infty$ is the probability measure on $\Upsilon_\infty = \Gamma_\infty \times \Lambda^\infty$ defined by the Poisson measure (3.7). This means that the a posteriori density operator $\varrho^M(t)$ is defined as a stochastic positive trace class operator normalized in the mean sense

$$\|\varrho^M(t)\|_1 = \int \text{Tr}\{\varrho^M(t, v)\} P_0^M(dv) = 1.$$

The density $\varrho^M(t)$ satisfies the stochastic operator equation

$$d\varrho^M(t) + \frac{i}{\hbar} [H, \varrho^M(t)]dt = dn_t \left[\frac{1}{M} \sum_{k=1}^M G(k) \varrho^M(t) G^\dagger(k) - \varrho^M(t) \right], \quad (3.10)$$

which has a unique solution for every initial condition $\varrho^M(0, v) = \varrho_0^M$ given by the density operator ϱ_0^M for the M -particle states $\rho_0^M[X] = \text{Tr}\{X \varrho_0^M\}$.

Let derive the differential equation (3.10) in the equivalent integral form

$$\rho^M[X](t) = \rho^M[X(t)](t) + \int_0^t \int_\Lambda \rho^M[(\Psi[X(t-r)](\lambda)) - X(t-r)](r) dn_t(d\lambda)$$

for an $\rho_0^M[X] = \langle \eta^M, X \eta^M \rangle$, $\eta^M \in \mathcal{H}^M$, where $X(t) = U^M(t)^\dagger X U^M(t)$,

$$U^M(t) = e^{-iH^M t/\hbar}, \quad \Psi[X](\lambda) = \frac{1}{M} \sum_{k=1}^M G^\dagger(k, \lambda) X G(k, \lambda).$$

Taking into account the fact that the stochastic integral (2.6) in this equation is simply a finite sum for every $v \in \Upsilon_\infty$ and t , one can write it as recursive operator equation

$$\Phi_t[X](v) = X + \sum_{(r,y) \in v}^{r < t} \Phi_r[\Psi[X](r, \lambda) - X](v),$$

for a stochastic operation Φ_t , defining the solutions to Eq. (3.10), as in (3.9), in terms of the composition (3.5).

But such recurrence has the unique solution $\Phi_t(v) = \Psi_t(y_1) \circ \Psi_t(y_2) \circ \dots$, defined for a $v = (\tau, \boldsymbol{\lambda}) \in \Upsilon_\infty$ as the chronological composition of the maps $\Psi_t(r, \lambda) : X \mapsto \Psi[X](r, \lambda)$ if $r < t$ and $\Psi_t[X](r, \lambda) = X$ if $r \geq t$. This solution can be found by the iterations

$$\begin{aligned}\Phi_t(v) &= \text{I} + \sum_{(r,\lambda) \in v}^{r < t} \Phi_r(v) \circ \Lambda(r, \lambda) \\ &= \text{I} + \sum_{(r,\lambda) \in v}^{r < t} \left(\text{I} + \sum_{(s,\lambda) \in v}^{s < r} \Phi_s(v) \circ \Lambda(s, \lambda) \right) \circ \Lambda(r, \lambda) \\ &= \dots = \sum_{\sigma \subseteq v_t} \Lambda(z_1) \circ \dots \circ \Lambda(z_n),\end{aligned}$$

where $\Lambda(y) = \Psi(y) - \text{I}$, $\Phi_s(v)$ is the identical map $\text{I} : X \mapsto X$ if $s = t_1$ and $\sum_{\sigma \subseteq v_t} \Lambda(z_1) \circ \dots \circ \Lambda(z_n) = \Psi_t(y_1) \circ \Psi_t(y_2) \circ \dots$ in terms of $\sigma = \{z_1, \dots, z_n\}$, $z = (r, \lambda)$, $s_1 < \dots < s_n$, $n \leq n_t$.

Let us also write the nonlinear stochastic equation

$$d\rho_v^M(t) + \frac{i}{\hbar} [H, \rho_v^M(t)]dt = \rho_v^M(t) \circ (\Psi_v(t) - \text{I})dn_t(v),$$

where $\Psi_v(t) = \Psi(\lambda_{n_t(v)})/\text{Tr}\{E(\lambda_{n_t(v)})\rho\}$,

$$\rho \circ \Psi(\lambda) = \frac{1}{M} \sum_{k=1}^M G(k, \lambda) \rho G^\dagger(k, \lambda), \quad E(\lambda) = \frac{1}{M} \sum_{k=1}^M G^\dagger(k, \lambda) G(k, \lambda)$$

for the normalized density operator

$$\rho_v^M(t) = \varrho^M(t, v)/\text{p}^M(t, v).$$

This describes the conditional expectations $\rho_v^M[X](t) = \text{Tr}\{X \rho_v^M(t)\}$ of the M -particle operators with respect to the output probability measure

$$\text{P}^M(dv) = \text{p}^M(t, v) \text{P}_0^M(dv),$$

where $\text{p}^M(t, v) = \text{Tr}\{\rho^M(t, v)\}$.

4 Macroscopic and Continuous Reduction Limits

We now consider the mean field approximation of the measurement apparatus fixing its total effect $\nu\kappa = \gamma$ given by the mean number ν of scattered bubbles (photons) per second and an interaction constant κ coupling each bubble to a particle (atom) in the Hamiltonian (2.1). We look for the limits of the unitary and reduced evolutions (2.2) and (2.10) as $\nu \rightarrow \infty$ and $\kappa \rightarrow 0$ such that γ is a real constant. To perform these limits we need the expansions

$$\begin{aligned} S(n) &= I \otimes \mathbf{1} + i\frac{\kappa}{\hbar} R \otimes Q(n) - \left(\frac{1}{2}\right) \left(\frac{\kappa}{\hbar}\right)^2 (R \otimes Q(n))^2 + \dots \quad (4.1) \\ G(\lambda) &= I - \kappa \frac{f'_0(\lambda)}{f_0(\lambda)} R + \frac{1}{2} \kappa^2 R \frac{f''_0(\lambda)}{f_0(\lambda)} R + \dots \end{aligned}$$

of the scattering operator $S(n) = \exp\{i\frac{\kappa}{\hbar} R \otimes Q(n)\}$ and the reduced operator $G(\lambda) = f_0(\lambda I - \kappa R)/f_0(\lambda)$ with respect to the coupling constant κ . The first term of the expansion for $S(n)$ is disappearing in the r.h.s. of Eq.(2.2) while the second and third terms are appearing as the differentials of the operator-valued stochastic integrals

$$\hat{n}_t[Q] = \int_0^t Q(n_r) dn_r, \quad \hat{n}_t[Q^2] = \int_0^t Q(n_r)^2 dn_r.$$

The corresponding terms

$$n_t \left[\frac{f'_0}{-f_0} \right] = \int_0^t \int_{\Lambda} \frac{f'_0(\lambda)}{-f_0(\lambda)} dn_r(d\lambda), \quad n_t \left[\frac{f''_0}{f_0} \right] = \int_0^t \int_{\Lambda} \frac{f''_0(\lambda)}{f_0(\lambda)} dn_r(d\lambda)$$

in the right-hand side. of eq.(2.10) can also be written as the integrals $\hat{n}_t[L] = \int_0^t L(n_r) dn_r$ with values in operator functions of $\tau \in \Gamma_{\infty}$

$$\hat{n}_t[L](\tau) = \sum_{n=1}^{n_t(\tau)} L(n), \quad L(n) = \mathbf{1}^{\otimes(n-1)} \otimes L \otimes \mathbf{1}^{\otimes\infty}, \quad (4.2)$$

where $L = [l(\lambda)]$ is one of the multiplication operators

$$L' = -[f'_0(\lambda)/f_0(\lambda)], \quad L'' = [f''_0(\lambda)/f_0(\lambda)].$$

Here $f'_0(\lambda)$ denotes the derivative $\partial f_0(\lambda) = \partial f_0(\lambda)/\partial \lambda$ and $f''(\lambda)$ denotes $\partial \partial f(\lambda)$. The stochastic integral $\hat{n}_t[L](\tau)$ corresponding to the pointwise multiplication $L : f \mapsto lf$ of $f \in L^2(\Lambda)$ by a function l of the bubble coordinate λ acts in \mathcal{E} as the multiplication operator

$$[\hat{n}_t[L](\tau)\varphi](\boldsymbol{\lambda}) = \sum_{n=1}^{n_t(\tau)} l(\lambda_n)\varphi(\boldsymbol{\lambda}) \equiv n_t[l](\tau, \boldsymbol{\lambda})\varphi(\boldsymbol{\lambda}).$$

Hence, the main terms on the right-hand side of Eq.(2.2) and (2.10) for $\kappa \rightarrow 0$ are given by the renormalised stochastic integrals

$$\hat{\lambda}(t) = \frac{1}{\nu t} \int_0^t L(n_r)dn_r = \frac{1}{\nu t} \hat{n}_t[L] \quad (4.3)$$

of the operator-valued stochastic functions $L(t, \tau) = L(n_t(\tau))$ with respect to the numerical process $n_t(\tau)$ that has the Poisson probability distribution (2.5) on Γ_∞ .

To pass to the large number limit $\nu \rightarrow \infty$ in (4.3) for an arbitrary operator L in $L^2(\Lambda)$, we need to use the quantum stochastic representation [22] of the integral (4.3) in the Fock space \mathcal{F} over $L^2(\mathbf{R}_+ \times \Lambda)$. The space \mathcal{F} can be defined as the $L^2(\Upsilon)$ -space of all square integrable functions $\varphi : \Upsilon \rightarrow \mathbf{C}$, $\|\varphi\|^2 = \int_\Upsilon |\varphi(v)|^2 \lambda(dv) < \infty$ of time ordered finite sequences $v = (y_1, \dots, y_n)$, $y = (t, \lambda)$ identified with subsets $v \subset \mathbf{R}_+ \times \Lambda$ of cardinality $|v| = 0, 1, 2, \dots$. The measure $\lambda(dv)$ on the union $\Upsilon = \sum_{n=0}^\infty \Upsilon(n)$ of the disjoint subsets $\Upsilon(n) = \{v \in \Upsilon : |v| = n\}$ is given as the sum

$$\lambda(A) = \sum_{n=0}^\infty \lambda(\Upsilon(n) \cap A)$$

of the product $\lambda(dv) = \prod_{y \in v} dy$ of measures $dy = dt d\lambda$ on $\mathbf{R}_+ \times \Lambda$ such that

$$\|\varphi\|^2 = \sum_{n=0}^\infty \iint_{0 \leq t_1 < \dots < t_n < \infty} |\varphi(y_1, \dots, y_n)|^2 \prod_{i=1}^n dy_i.$$

Let define $N_t[L]$ define the numerical integral in \mathcal{F} by the action of the operator (4.2) in each Fock component $\mathcal{F}(\tau) = L^2(\Lambda^{|\tau|})$ for all $\tau \in \Gamma$:

$$[N_t[L]\varphi](\tau) = \sum_{n=1}^{n_t(\tau)} L(n)\varphi(\tau) = \hat{n}_t[L](\tau)\varphi(\tau). \quad (4.4)$$

Here $\varphi(\tau)$ is the function $\varphi(\tau, \boldsymbol{\lambda}) = \varphi(v)$ of $\boldsymbol{\lambda} \in \Lambda^{|\tau|}$, corresponding to a $\varphi \in L^2(\Upsilon)$ with a fixed time component of $v = (\tau, \boldsymbol{\lambda}) \in \Upsilon$.

In order to obtain the initial probability measure $P_0(dv) = \pi(d\tau)\mu_0^\infty(d\boldsymbol{\lambda})$ on Υ_∞ induced by an initial Fock vector $\varphi_0 \in L^2(\Upsilon)$, we need an isomorphic transformation of (4.4)

$$\hat{N}_t[L] = N_t[L] + \sqrt{\nu}(A_t[f_0^\dagger L] + A_t^\dagger[L f_0]) + \nu t f_0^\dagger L f_0 \quad (4.5)$$

which can be locally performed by a unitary transformation

$$\hat{N}_t[L] = U_s^\dagger N_t[L] U_s, \quad U_s = \exp\{\sqrt{\nu}(A_s^\dagger[f_0] - A_s[f_0^\dagger])\}$$

for every $t < s$. Here $A_t^\dagger[f]$ and $A_t[f^\dagger]$ are the creation and annihilation integrals of $f \in L^2(\Lambda)$, $f^\dagger \in L^2(\Lambda)^*$, given by the operators

$$\begin{aligned} [A_t^\dagger[f]\varphi](v) &= \sum_{y \in v_t} f(\lambda)\varphi(v \setminus y) \\ [A_t[f^\dagger]\varphi](v) &= \int_{([0,t) \times \Lambda)} f(\lambda)^* \varphi(v \sqcup y) dy \end{aligned}$$

in the Fock space $L^2(\Upsilon)$, where $v \setminus y$ means the sequence $v \in \Upsilon$ with deleted $y = (r, \lambda)$, $r < t$, and $v \sqcup y$ means the sequence $v \in \Upsilon$ with an additional element $y \notin v$. The characteristic functional of the stochastic operators $\hat{n}_t[L]$ with respect to the initial state-vector $f_0^\infty \in \mathcal{E}$ and the Poisson probability measure (2.5) is now given simply by the vacuum expectation

$$\int_{\Gamma^\infty} (f^\infty, e^{i\hat{n}_t[L]} f^\infty) \pi_0(d\tau) = \langle \delta_\phi, e^{i\hat{N}_t[L]} \delta_\phi \rangle,$$

where $\delta_\phi(v) = 1$ if $v = \emptyset$; otherwise, $\delta_\phi(v) = 0$.

The corresponding representation $\hat{l}(t) = \frac{1}{\nu t} \hat{N}_t[L]$ for (4.3) helps us immediately obtain the quantum large number limit

$$\lim_{\nu \rightarrow \infty} \frac{1}{\nu t} \hat{N}_t[L] = f_0^\dagger L f_0 \hat{1}$$

as the mean value $l_0 = (f_0, L f_0) \equiv f_0^\dagger L f_0$ of a single-bubble operator with respect to an initial wave packet $f_0 \in L^2(\Lambda)$. This gives the following macroscopic limit

$$d\psi(t) + \frac{i}{\hbar} H_0 \psi(t) dt = \frac{i}{\hbar} \gamma(R \otimes q_0 \hat{1}) \psi(t) dt$$

of the generalized Schrödinger equation (2.2) which turns out to be a nonsingular one with an additional potential $-\gamma q_0 R$ corresponding to the mean momentum $q_0 = (f_0, Q f_0)$ of a bubble in the initial state f_0 . As one could expect, the mean field dynamics preserves the product structure $\psi(t, v) = \eta(t)\varphi_0(v)$ of an initial product-vector $\psi_0 = \eta \otimes \varphi_0$ being trivial on the Fock component $\varphi_0 \in \mathcal{F}$ because of $H_0 = H \otimes \hat{1}$. But unexpectedly (compare with [19]) the macroscopic limit

$$d\chi(t) + \frac{i}{\hbar} H\chi(t)dt = \frac{i}{\hbar} \gamma q_0 R\chi(t)dt \quad (4.6)$$

of the reduction equation (2.10) corresponds to the same unitary dynamics $\eta(t) = U(t)\eta = \chi(t)$ of the particle state-vector if $\chi(0) = \eta$, because of

$$\frac{1}{\nu t} n_t \left[\frac{f'_0}{-f_0} \right] \rightarrow (f_0, L' f_0) = \int f_0(\lambda)^* f'_0(\lambda) d\lambda = \frac{i}{\hbar} q_0.$$

The macroscopic limits for the M -particle system also give essentially the same continuous unitary evolutions in the large space $\mathcal{H}^M \otimes \mathcal{F}$ and in the reduced space \mathcal{H}^M . To get this correspondence, one has only to replace the measure (2.5) on Γ_∞ for Eq.(3.2) by the product measure $\pi_0^{\otimes M}$ on Γ_∞^M and for Eq.(3.10) by the induced measure (3.7) on Γ_∞ , taking $M\nu$ instead of ν in (4.5). This means that the mixing property of the reduced equation (3.10) vanishes in the mean field approximation for the bubble system.

Let us now pay attention to the fluctuations with respect to the obtained large number limits. Such fluctuations might appear for $\kappa = \gamma/\nu \rightarrow 0$ in the large time scale $t \sim 1/\kappa$. We can get these fluctuations without rescaling the time t if we assume that $q_0 = 0$ and $\kappa = \gamma/\sqrt{\nu}$, so that we have to take into account also the κ^2 -terms in (4.1).

It follows from the Fock space representation (4.5) that the quantum central limit

$$\lim_{\nu \rightarrow \infty} \frac{l}{\sqrt{\nu}} \hat{N}_t[L] = A_t[f_0^\dagger L] + A_t^\dagger[L f_0]$$

exists for any single-bubble operator L with zero mean value $(f_0, L f_0) = 0$. We first apply this central limit theorem to the right-hand side in (2.2):

$$d\psi(t) + K_0\psi(t)dt = \frac{i}{\hbar} \gamma (R \otimes d\hat{u}_t)\psi(t). \quad (4.7)$$

Here $K_0 = K \otimes \hat{1}$, $K = \frac{i}{\hbar} H + \frac{1}{2} \left(\frac{\gamma}{\hbar} \right)^2 R \sigma^2 R$,

$$\hat{u}_t = A_t[f_0^\dagger Q] + A_t^\dagger[Q f_0] = 2\Re A_t^\dagger[Q f_0]$$

is the Fock space representation of the Wiener process u_t with the dispersion $\sigma^2 = \hbar^2(\partial f_0, \partial f_0)$, defined by the quantum stochastic multiplication formula [21, 22]

$$d\hat{u}d\hat{u} = dA_t[f_0^\dagger Q]dA_t^\dagger[Q f_0] = f_0^\dagger Q Q f_0 dt.$$

The central limit equation (4.7) for the unitary evolution of the coupled system turns out to be a stochastic Schrödinger-Ito equation of diffusive type driven by the Wiener process u_t . The same conclusion obviously holds for the M -particle system driven by M independent Wiener processes $\hat{u}_t(k)$ identical to \hat{u}_t , $k = 1, \dots, M$:

$$d\psi^M(t) + K_0^M \psi(t) dt = \frac{i}{\hbar} \gamma \left(\sum_{k=1}^M R(k) \otimes d\hat{u}_t(k) \right) \psi^M(t),$$

where $K^M = \frac{i}{\hbar} H^M + \frac{1}{2} \left(\frac{\gamma}{\hbar} \right)^2 \sum_{k=1}^M R(k) \sigma^2 R(k)$, $K_0^M = K^M \otimes \hat{1}$.

The application of the central limit theorem to the r.h.s. of the reduction equation (2.10) yields an essentially different type of the stochastic evolution,

$$d\chi(t) + K\chi(t)dt = \gamma R\chi(t)d\hat{v}_t, \quad (4.8)$$

originally derived in [21] by quantum calculus method. Here v_t is a complex Wiener vector-process, with the Fock-space representation

$$\hat{v}_t = A_t[f_0^\dagger L'] + A_t^\dagger[L' f_0] = \Re A_t^\dagger[(w_0 + \bar{w}_0)f_0] + i\Im A_t^\dagger[(w_0 - \bar{w}_0)f_0] \quad (4.9)$$

given by the complex osmotic velocity $w_0(\lambda) = -\partial \ln f_0(\lambda)$ of a single bubble, and the operator K is essentially the same as in (4.7). The linear stochastic equation (4.8) has a unique solution $\chi(t, v) = T(t, v)\eta$ for a given $\chi(0, v) = \eta \in \mathcal{H}$ which is not normalized $\|\chi(t, v)\| \neq 0$ for every Wiener trajectory $t \mapsto v_t$ but is normalized in the mean square sense $\int \|\chi(t, v)\|^2 P_0(dv) = 1$ with respect to the Gaussian probability measure P_0 of $v = \{v_t | t > 0\}$. The measure P_0 is defined by the zero mean values of v_t and by the table

$$\begin{aligned} d\hat{v}d\hat{v} &= dA_t[f_0^\dagger L']dA_t^\dagger[L' f_0] = f_0^\dagger L' L' f_0 dt \\ d\hat{v}^*d\hat{v} &= dA_t[f_0^\dagger L'^\dagger]dA_t^\dagger[L' f_0] = f_0^\dagger L'^\dagger L' f_0 dt \end{aligned}$$

of commuting multiplication

$$d\hat{v}d\hat{v}^* = d\hat{v}^*d\hat{v}.$$

But the reduction noise \hat{v}_t obtained in the Fock space representation does not commute with the real Wiener process $\hat{u}_t = \hat{u}_t^*$ in (4.7),

$$\begin{aligned} d\hat{v}d\hat{u} &= dA_t[f_0^\dagger L']dA_t^\dagger[Qf_0] = f_0^\dagger L'Qf_0 dt \\ d\hat{u}d\hat{v} &= dA_t[f_0^\dagger Q]dA_t^\dagger[L'f_0] = f_0^\dagger QL'f_0 dt \end{aligned}$$

if $g(\lambda) \equiv \partial^2 \ln f_0(\lambda) \neq 0$, since

$$[d\hat{v}, d\hat{u}] = f_0^\dagger [L', Q] f_0 dt = \frac{\hbar}{i} (f_0, g f_0) dt.$$

In the same way one can obtain the continuous reduction equation

$$\begin{aligned} d\rho^M(t) + (K\rho^M(t) + \rho^M(t)K^\dagger)dt = \\ \left(\frac{\gamma}{\hbar}\right)^2 \sum_{k=1}^M R(k)\sigma^2 \rho^M(t)R(k)dt + \gamma(dw_t R\rho^M(t) + \rho^M(t)Rdw_t^*) \end{aligned} \quad (4.10)$$

where $R = \frac{1}{M} \sum_{k=1}^M R(k)$ and the operator $K = K^M$ is the same as in the equation for the unitary evolution of the M -particle system coupled with the bubbles. The derived stochastic equation for the M -particle density operator $\rho^M(t, w)$ normalized in the mean is driven by the complex Wiener process $w_t = v_t^M$ having the same multiplication table as the process $\sqrt{M}v_t$ in (4.8). The diffusive type equation (4.10), (3.10) also has the mixing property. It was also derived in [5] by operational method.

5 Conclusion

The projection postulate for the reduction process follows as a result of conditioning by the results of a nondemolition measurement in the von Neumann Hamiltonian model [6] for the particle-meter interaction. This model gives also all unsharp reductions for the measurements with continuous spectrum.

The singular interaction, corresponding to the collision model of scatterings can be treated in quantum mechanics also in terms of Schrödinger equation in the generalized sense. The spontaneous localization [19, 20] of

the particle under the continual observation is explained as the result of the nondemolition countings of the scattered bubbles in a measurement apparatus like cloud chamber.

The spontaneous localization of a system of the identical particles (atoms) in a bubble chamber (photodetector) is mixing due to the indistinguishability of the particles via the measurements of the bubbles. It is described in terms of the filtering equation for the density matrix of the particles, driven by the Poisson process of the total scatterings of the bubbles. This equation was derived in [5] within the operational approach to the quantum continual measurements.

The macroscopic limit of the spontaneous localization equation is described in terms of the Schrödinger one, corresponding to the mean field approximation. The central limit of the spontaneous localization equation proves the diffusive reduction equation, which appeared in [17, 18] for some cases of R and H . Our theory shows the origin of the Wiener processes generating the Ito diffusion equations for quantum states [23, 24], and also derives a diffusive mixing equation for the reduced density matrix of the system of particles under the continual observation.

The quantum jump model, based on Poisson distributed impulsive measurements, is not new, see for example [25, 26], and it has been described on the statistical (a priori) level by the master equation which also results from the quantum diffusion models. However the treatment of these processes on the individual (a posteriori) level gains different types of quantum dynamics which can be described only in terms of the stochastic differential equations. The first such treatment of the counting measurements and the a posteriori dynamics was suggested in [13, 14] in terms of quantum stochastic calculus method [22]. The present work puts this method on the solid mathematical basis by treatment of δ -function couplings with the classical Ito calculus applied to the quantum jumps and spontaneous localizations.

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