

Quantum Probabilities and Paradoxes of the Quantum Century

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Dedicated to Robin Hudson on his 60th birthday

Abstract

A history and drama of the development of quantum probability theory is outlined starting from the discovery of the Plank's constant exactly a 100 years ago. It is shown that before the rise of quantum mechanics 75 years ago, the quantum theory had appeared first in the form of the statistics of quantum thermal noise and quantum spontaneous jumps which have never been explained by quantum mechanics. Moreover, the only reasonable probabilistic interpretation of quantum theory put forward by Max Born was in fact in irreconcilable contradiction with traditional mechanical reality and classical probabilistic causality. This led to numerous quantum paradoxes, some of them due to the great inventors of quantum theory such as Einstein and Schroedinger. They are reconsidered in this paper from the modern quantum probabilistic point of view.

Contents

1 Introduction: The Common Thread of Mathematical Sciences	2
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2	The Quantum Century Begins	4
2.1	The Discovery of Matrix Mechanics	4
2.2	The Discovery of Wave Mechanics	6
2.3	Interpretations of Quantum Mechanics	7
3	Quantum Probabilities and Paradoxes	10
3.1	Uncertainties and new logics	10
3.1.1	Heisenberg uncertainty relations	10
3.1.2	Nonexistence of hidden variables	11
3.1.3	Complementarity and common sense	14
3.2	Quantum measurement and decoherence	16
3.2.1	Spooky action at distance	16
3.2.2	Releasing Schrödinger’s cat	18
3.2.3	Von Neumann’s projection postulate	20
4	Causality and Prediction of Future	23
4.1	Reality and nondemolition principle	23
4.1.1	Compatibility and time arrow	23
4.1.2	Transition from possible to actual	25
4.1.3	The true Heisenberg principle	28
4.2	Consistent Histories and Filtering	30
4.2.1	Symbolic Stochastic Calculus	31
4.2.2	Stochastic decoherence equations	33
4.2.3	Quantum trajectories and filtering	36
4.2.4	A quantum message from the future	38
5	Conclusion: The Greatest Form of Beauty	40

1 Introduction: The Common Thread of Mathematical Sciences

The whole is more than the sum of its parts – Aristotle.

This is the famous superadditivity law from Aristotle’s *Metaphysics* which studies ‘the most general or abstract features of reality and the principles that have universal validity’. Certainly in this broad definition mathematics and physics are parts of metaphysics.

The aim of this lecture is to demonstrate, by the example of the history and drama of the development of Quantum Theory during last century, that this law is also applicable to metaphysics itself, as the unification of different branches of mathematics and physics, which is more than just the sum of pure and applied mathematics, statistics and mathematical physics. Quantum theory is a mathematical theory which studies the most fundamental features of reality in a unified form of waves and matter it raises and solves the most fundamental riddles of Nature by developing and utilizing mathematical concepts and methods of all branches of modern mathematics, including statistics.

Indeed, as we shall see, it began with the discovery of new laws for “quantum” numbers, the natural objects which are the foundation of pure mathematics. (God made the integers; the rest is man’s work). Next it invented new applied mathematical methods for solving quantum mechanical matrix and partial differential equations. Next it married probability with algebra to obtain unified treatment of waves and particles in nature, giving birth to quantum probability and creating new branches of mathematics such as quantum logics, quantum topologies, quantum geometries, quantum groups. It inspired the recent creation of quantum analysis and quantum calculus, as well as quantum statistics and quantum stochastics.

Specialists in different narrow branches of mathematics rarely understand quantum theory as a common thread which runs through everything. The creators of quantum mechanics, the theory invented for interpretation of the dynamical laws of fundamental particles, were unable to find a consistent interpretation of it since they were physicists with a classical mathematical education. After inventing quantum mechanics they spent much of their lives trying to tackle the Problem of Quantum Measurement, the greatest problem of quantum theory, not just of quantum mechanics, or even of unified quantum field theory, which would be the same “thing in itself” as quantum mechanics of closed systems without such interpretation. As we shall see, the solution to this problem can be found in the framework of Quantum Probability as a part of a unified mathematics rather than physics. Most modern mathematical physicists have a broad mathematical education, but it ignores just two crucial aspects – information theory and statistical conditioning. So they gave up this problem as an unsolvable – and it is indeed unsolvable in the traditional framework of mathematical physics.

In order to appreciate the quantum drama which has been developing through the whole century, and to estimate possible consequences of it in the

new quantum technological age, it seems useful to give a brief account of the discovery of quantum theory at the beginning of the 20th century.

2 The Quantum Century Begins

In science one tries to tell people, in such a way as to be understood by everyone, something that no one even knew before. But in poetry, it's quite opposite – Paul Dirac.

Quantum Theory is the greatest intellectual achievement of the past century. Since the discovery of quanta by Max Planck exactly 100 years ago [1] on the basis of spectral analysis of quantum thermal noise it has produced numerous paradoxes and confusions even in the greatest scientific minds such as Einstein, de Broglie, Schrödinger, Bell, and it still confuses many contemporary philosophers and scientists. The rapid development of a beautiful and sophisticated mathematics for quantum mechanics and the development of its interpretation by Bohr, Born, Heisenberg, Dirac and many others who abandoned traditional causality, were little help in resolving these paradoxes despite the astonishing success in the prediction of quantum phenomena. Both the implication and consequences of the quantum theory of light and matter, as well as its profound mathematical, conceptual and philosophical foundations are not yet understood completely. As Planck stated later:-

If anyone says he can think about quantum problems without getting giddy, that only shows he has not understood the first thing about them

2.1 The Discovery of Matrix Mechanics

In 1905 Einstein, examining the photoelectric effect, proposed a quantum theory of light, only later realizing that Planck's theory made implicit use of this quantum light hypothesis. Einstein saw that the energy changes in a quantum material oscillator occur in jumps which are multiples of ω .

In 1912 Niels Bohr worked in the Rutherford group in Manchester on his theory of the electron in an atom. He was puzzled by the discrete spectra of light which is emitted by atoms when they are subjected to an excitation. He was influenced by the ideas of Planck and Einstein and addressed a certain

paradox in his work. How can energy be conserved when some energy changes are continuous and some are discontinuous, i.e. change by quantum amounts? Bohr conjectured that an atom could exist only in a discrete set of stable energy states, the differences of which amount to the observed energy quanta. Bohr returned to Copenhagen and published a revolutionary paper on the hydrogen atom in the next year. He suggested his famous formula

$$E_m - E_n = \hbar\omega_{mn}$$

from which he derived the major laws which describe physically observed spectral lines. This work earned Niels Bohr the 1922 Nobel Prize about 10^5 Swedish Kroner. This was a time when it was still possible to get such a sum for so simple an equation!

In 1925 a young German theoretical physicist, Heisenberg, gave a preliminary account of a new and highly original approach to the mechanics of the atom [2]. He was influenced by Niels Bohr and proposed to substitute for the position coordinate of an electron an array

$$q_{mn}(t) = q_{mn}e^{i\omega_{mn}t}.$$

His Professor, Max Born, was a mathematician who immediately recognized an infinite matrix algebra in Heisenberg's arrays $Q = [q_{mn}]$. The classical momentum was also replaced by a similar matrix,

$$P(t) = [p_{mn}e^{i\omega_{mn}t}]$$

and P and Q matrices were postulated to follow a commutation law:

$$[Q(t), P(t)] = i\hbar I,$$

where I is the unit matrix. The classical Hamiltonian equations of dynamical evolution were now replaced by

$$\frac{d}{dt}Q(t) = \frac{i}{\hbar} [H, Q(t)], \quad \frac{d}{dt}P(t) = \frac{i}{\hbar} [H, P(t)],$$

where $H = [E_n\delta_{mn}]$ is the diagonal Hamilton matrix [3]. Thus quantum mechanics was first invented in the form of *matrix mechanics*, emphasizing the possibilities of quantum transitions, or jumps between the stable energy states E_n of an electron. In 1932 Heisenberg was awarded the Nobel Prize for his work in mathematical physics.

Conceptually, the new atomic theory was based on the positivism of Mach as it operated not with real space-time but with only observable quantities like atomic transitions. However many leading physicists were greatly troubled by the prospect of losing reality and deterministic causality in the emerging quantum physics. Einstein, in particular, worried about the element of ‘chance’ which had entered physics. In fact, this worry came rather late since Rutherford had introduced a spontaneous effect when discussing radio-active decay in 1900.

Thus, quantum theory first emerged as the result of experimental data not in the form of quantum mechanics but in the form of statistical observations of quantum noise, the basic concept of quantum probability and quantum stochastic processes.

2.2 The Discovery of Wave Mechanics

The corpuscular nature of light seemed to contradict the Maxwell electromagnetic wave theory of light. In 1924 Einstein wrote:

There are therefore now two theories of light, both indispensable, and - as one must admit today, despite twenty years of tremendous effort on the part of theoretical physics - without any logical connection.

The solution to the paradox of the wave/corpuscular duality of light came unexpectedly when de Broglie made in 1923 the even more bizarre conjecture of extending this duality also to material particles. He used the Hamilton-Jacobi theory which had been applied both to particles and waves.

In 1925, Schrödinger gave a seminar on de Broglie’s material waves, and a member of the audience suggested that there should be a wave equation. Within a few weeks Schrödinger found his celebrated wave equation, first in a relativistic, and then in the non-relativistic form [4]. Instead of seeking the classical solutions to the Hamilton-Jacobi equation

$$H\left(q, \frac{\hbar}{i} \frac{\partial}{\partial q} \ln \psi\right) = E$$

he suggested finding those wave functions ψ which satisfy the equation

$$H\left(q, \frac{\hbar}{i} \frac{\partial}{\partial q}\right) \psi = E\psi$$

(It coincides with the former equation only if H is linear with respect to p). He also obtained the non-stationary wave equation written in terms of the Hamiltonian operator H as

$$i\hbar\frac{\partial}{\partial t}\psi(t) = H\psi(t).$$

Schrödinger published his revolutionary *wave mechanics* in a series of six papers [5] in 1926 during a short period of sustained creative activity that is without parallel in the history of science. Like Shakespeare, whose sonnets were inspired by a dark lady, Schrödinger was inspired by a mysterious young lady of Arosa where he took ski holidays during the Christmas 1925 but “had been distracted by a few calculations”. This was the second formulation of quantum theory, which he successfully applied to the Hydrogen atom, oscillator and other quantum mechanical systems. The mathematical equivalence between the two formulations of quantum mechanics was understood by Schrödinger immediately, and he also introduced operators associated with each dynamical variable.

As a young girl said later to Schrödinger, who discovered the quantum mechanics of wave matter:

Hey, you never even thought when you began that so much sensible stuff would come out of it.

Unlike Heisenberg and Born’s matrix mechanics, the general reaction towards wave mechanics was immediately enthusiastic. Plank described Schrödinger’s wave mechanics as “epoch-making work”. Einstein wrote:- “the idea of your work springs from true genius...”. Next year Schrödinger was nominated for the Nobel Prize, but he failed to receive it in this and five further consecutive years of his nominations by most distinguished physicists of the world, the reason behind his rejection being “the highly mathematical character of his work”. Only in 1933 did he receive his prize, this time jointly with Dirac, and this was the first, and perhaps the last, time when the Nobel Prize for physics was given to true mathematical physicist.

2.3 Interpretations of Quantum Mechanics

The creators of the rival matrix quantum mechanics were forced to accept the simplicity and beauty of Schrödinger’s approach. In 1926 Max Born put

forward the statistical interpretation of the wave function by introducing the statistical mean

$$\langle H \rangle = \int \bar{\psi}(x) H \psi(x) dx$$

for a dynamical variable H in the physical state, described by ψ . This was developed in Copenhagen and gradually was accepted by almost all physicists as the “Copenhagen interpretation”. Born by education was a mathematician, and he would be the only mathematician ever to receive the Nobel Prize, and then only in 1953, for his statistical studies of wave functions, if he did not become a physicist, later Professor of Natural Philosophy at Edinburgh. Bohr, Born and Heisenberg considered electrons and quanta as unpredictable particles which cannot be visualized in the real space and time.

Schrödinger was a champion of the idea that the most fundamental laws of the microscopic world are absolutely random before he discovered wave mechanics. While he was preparing his Inaugural lecture for December 9, 1922 at the University of Zürich, he wrote to Pauli:

I for my part believe, horrible dictu, that the energy – momentum law is violated in the process of radiation.

(quoted on page 152 in [6])

Following de Broglie, Schrödinger initially thought that the wave function corresponds to a physical vibration process in a real continuous space-time because it was not stochastic, but he was puzzled by the failure to explain the blackbody radiation and photoelectric effect from this wave point of view. In fact the wave interpretation applied to light quanta leads back to classical electrodynamics, as his relativistic wave equation for a single photon coincides mathematically with the classical wave equation. However after realizing that the time-dependent ψ is a complex function in his fourth 1926 paper [5], submitted just a few days before Born’s, he admitted that the wave function ψ cannot be given a direct interpretation, and described the wave density $\psi\bar{\psi}$ as a sort of weight function for superposition of point-mechanical configurations.

Bohr invited Schrödinger to Copenhagen and tried to convince him of the particle-probabilistic interpretation of quantum mechanics. The discussion between them went on day and night, without reaching any agreement. The conversation, however deeply affected both men. Schrödinger recognized the necessity of admitting both wave and particles, but he never devised a

comprehensive interpretation rival to Copenhagen orthodoxy. Bohr ventured more deeply into philosophical waters and emerged with his concept of complementarity:

Evidence obtained under different experimental conditions cannot be comprehended within a single picture, but must be regarded as complementary in the sense that only the totality of the phenomena exhausts the possible information about the objects.

Later Schrödinger accepted the probabilistic interpretation of $\psi\bar{\psi}$, although he did not consider these probabilities classically, but as the strength of our belief or anticipation of an experimental result. In this sense the probabilities are closer to propensities than to the frequencies of the statistical interpretation of Born and Heisenberg. Schrödinger had never accepted the subjective positivism of Bohr and Heisenberg, and his philosophy is closer to that called representational realism. He was content to remain a critical unbeliever. For a deeper analysis of probabilistic roots in interpretation of quantum mechanics see [7].

The most outspoken opponent of a/the probabilistic interpretation was Einstein. Albert Einstein admired the new development of quantum theory but with suspicion, and rejected its acausality and probabilistic interpretation. It was against his scientific instinct to accept quantum mechanics with its statistical interpretation as a complete description of physical reality. There are famous sayings of him on that account:

“■ *God is subtle but he is not malicious*” ■ “*God doesn't play dice*” ■

In the famous debates on the probabilistic interpretation of quantum mechanics of Einstein and Niels Bohr, Schrödinger was often taking the side of his friend Einstein, and this may explain why he was distancing himself from the statistical interpretation of his wave function.

There have been many other attempts to retain the deterministic realism in the quantum world, the most extravagant among these being the many world interpretation. Certainly there are some advantages living in many worlds: One can give, for example, many inaugural lectures if allowed to jump from one world into another. From the philosophical and practical point of view, however, to have an infinite (continuum?) number of real worlds at the same time seems not better than to have none.

3 Quantum Probabilities and Paradoxes

How wonderful we have met with a paradox, now we have some hope of making progress - Niels Bohr.

In 1932 von Neumann put quantum theory on firm theoretical basis by setting the mathematical foundation for new, quantum, probability theory, the quantitative theory for counting non commuting events of quantum logics. This noncommutative probability theory is based on essentially more general axioms than the classical (Kolmogorovian) probability of commuting events, which form common sense Boolean logic, first formalized by Aristotle. It has been under extensive development during the last 30 years since the introduction of algebraic and operational approaches for treatment of non-commutative probabilities, and currently serves as the mathematical basis for quantum information and measurement theory.

3.1 Uncertainties and new logics

Bohr was concerned with the paradox of spontaneous emission. He addressed the question: How does the electron know when to emit radiation? Bohr, Born and Heisenberg abandoned causality of traditional physics in the most positivistic way. Max Born said:-

If God made the world a perfect mechanism, ... we need not solve innumerable differential equations, but can use dice with fair success.

3.1.1 Heisenberg uncertainty relations

In 1927 Heisenberg derived [8] his famous uncertainty relation

$$\Delta Q \Delta P \geq \hbar/2, \quad \Delta T \Delta E \geq \hbar/2$$

which gave mathematical support to the revolutionary complementary principle of Bohr. The first relation was easily proved in the Schrödinger representations $Q = x$, $P = \frac{\hbar}{i} \frac{\partial}{\partial x}$ in terms of the standard deviations

$$\Delta Q = (\langle Q^2 \rangle - \langle Q \rangle^2)^{1/2}, \quad \Delta P = (\langle P^2 \rangle - \langle P \rangle^2)^{1/2}.$$

The second relation, which was first stated by analogy of t with x and of E with P , can be proved [9, 10] in terms of the optimal measurement of the initial time as an unknown parameter τ of the Schrödinger's state $\psi(t - \tau)$ which is realized by the measurement of self-adjoint operator $T = t$ in an extended representation where the Hamiltonian H is represented by the operator $E = i\hbar \frac{\partial}{\partial t}$. As Dirac stated:

Now when Heisenberg noticed that, he was really scared.

Einstein launched an attack on the uncertainty relation at the Solvay Congress in 1927, and then again in 1930, by proposing cleverly devised thought experiments which would violate this relation. Most of these imaginary experiments were designed to show that interaction between the microphysical object and the measuring instrument is not so inscrutable as Heisenberg and Bohr maintained. He suggested, for example, a box filled with radiation with a clock. The clock is designed to open a shutter and allow one photon to escape. By weighing the box the photon energy and the time of escape can both be measured with arbitrary accuracy.

After proposing this argument Einstein is reported to have spent a happy evening, and Niels Bohr an unhappy one. After a sleepless night he showed next morning that Einstein was not right. Mathematically his solution can be expressed by the following formula

$$X = t + Q, \quad Y = i\hbar \frac{\partial}{\partial t} + P$$

for the measuring quantity X , the pointer coordinate of the clock, and the observable Y for indirect measurement of photon energy $E = i\hbar \frac{\partial}{\partial t}$ in the Einstein experiment, where Q and P are the position and momentum operators of the compensation weight under the box. Due to the initial independence of the weight, the commuting observables X and Y have even greater uncertainty

$$\Delta X \Delta Y \geq \Delta T \Delta E + \Delta Q \Delta P \geq \hbar$$

than that predicted by Heisenberg uncertainty $\Delta T \Delta E \geq \hbar/2$.

3.1.2 Nonexistence of hidden variables

Einstein hoped that eventually it would be possible to explain the uncertainty relations by expressing quantum mechanical observables as functions

of some hidden variables λ in deterministic physical states such that the statistical aspect will arise as in classical statistical mechanics by averaging these observables over λ .

Von Neumann's monumental book [11] on the mathematical foundations of quantum theory was therefore a timely contribution, clarifying, as it did, this point. Inspired by Lev Landau, he introduced, for the unique characterization of the statistics of a quantum ensemble, the statistical density operator ρ which eventually, under the name state, became a major tool in quantum statistics. He considered the linear space \mathcal{L} of all bounded observables in a quantum system described by the Hermitian operators $L^\dagger = L$ in a Hilbert space \mathfrak{h} , and defined the expectation $\langle L \rangle$ of each $L \in \mathcal{L}$ in a state ρ by ultra-weakly continuous functional $\langle L \rangle = \text{Tr}L\rho$. Then he noted that any *physically continuous* additive functional $L \mapsto \langle L \rangle$ has such trace form, and proved that in order to have positive probabilities $\text{Pr}(E)$ for all quantum mechanical events as expectations $\langle E \rangle$ of yes-no observables $E \in \mathcal{L}$ described by Hermitian projectors $E^2 = E$ and probability one for the identity event described by identity operator I , i.e.

$$\text{Pr}(E) \geq 0, \quad \text{Pr}(I) = 1,$$

the statistical operator ρ must be positive-definite and have trace one. He applied this technique to the analysis of the completeness problem of quantum theory, i.e. whether it constitutes a logically closed theory or whether it could be reformulated as an entirely deterministic theory through the introduction of hidden parameters (additional variables which, unlike ordinary observables, are inaccessible to measurements). He came to the conclusion that

the present system of quantum mechanics would have to be objectively false, in order that another description of the elementary process than the statistical one may be possible

(quoted on page 325 in [11])

To prove this theorem, von Neumann showed that there is no such state which would be dispersion-free simultaneously for all quantum events $E \in \mathcal{L}$. For each such state, he argued,

$$\langle E^2 \rangle = \langle E \rangle = \langle E \rangle^2$$

for all events E would imply that $\rho = 0$, which cannot be statistical operators as 0 has trace 0. Thus no state can be considered as a mixture of dispersion-free states, each of them associated with a definite value of hidden parameters. There are simply no such states, and thus, no hidden parameters. In particular this implies that the statistical nature of pure states, which are described by one-dimensional projectors $\rho = P_\psi$ corresponding to wave functions ψ , cannot be removed by supposing them to be a mixture of dispersion-free substates.

It is widely believed that in 1966 John Bell showed that von Neuman's proof was in error, or at least his analysis left the real question untouched [12]. To discredit the von Neumann's proof he constructed an example of dispersion-free states parametrized for each quantum state ρ by a real parameter λ for the case of two dimensional \mathfrak{h} . He succeeded to do this by weakening the assumption of the additivity for such states, requiring it only for the commuting observables in \mathcal{L} , and by abandoning the linearity of the constructed expectations in ρ described by spin polarization vector \mathbf{r} . There is no reason, he argued, to keep the linearity in ρ for the observable eigenvalues determined by λ and ρ , and to demand the additivity for non-commuting observables as they are not simultaneously measurable: The measured eigen-values of a sum of noncommuting observables are not the sums of the eigen-values of this observables measured separately. Bell found for each spin-projection $L = \sigma(\mathbf{e})$ a family $s_\lambda(\mathbf{e})$, $|\lambda| \leq 1/2$ of dispersion free values ± 1 , reproducing the expectation $\langle \sigma(\mathbf{e}) \rangle = \mathbf{e} \cdot \mathbf{r}$ in the pure quantum state when uniformly averaged over the λ . However his example does not contradict the von Neumann theorem even if the latter is strengthened by the restriction of the additivity only to the commuting observables: The constructed dispersion-free expectation function $L \mapsto \langle L \rangle_\lambda$ is not *physically continuous* on \mathcal{L} because the value $\langle L \rangle_\lambda = s_\lambda(\mathbf{e})$ is one of the eigen-values ± 1 for each λ , and it covers both values when the directional vector \mathbf{e} rotates over the three-dimensional sphere. A function $\mathbf{e} \mapsto \langle \sigma(\mathbf{e}) \rangle_\lambda$ on the continuous manifold (sphere) with discontinuous values can be continuous only if it is constant, but this is ruled out by the demand to reproduce the variable in \mathbf{e} expectation $\langle \sigma(\mathbf{e}) \rangle = \mathbf{e} \cdot \mathbf{r}$ by averaging the constant $\langle \sigma(\mathbf{e}) \rangle_\lambda$ over all λ . Measurements of the projections of spin on the physically close directions should be described by close expected values in any physical state specified by λ , otherwise it cannot have physical meaning!

Since then there were innumerable attempts to introduce hidden variables in ever more sophisticated forms, perhaps not yet discovered, which would

determine the complementary variables if the hidden variables were measured precisely. In higher dimensions of \mathfrak{h} all these attempts are ruled out by Gleason's theorem [13] who proved that there is not even one additive zero-one probability function if $\dim \mathfrak{h} > 2$.

3.1.3 Complementarity and common sense

In view of the decisive importance of this analysis for the foundations of quantum theory, Birkhoff and von Neumann [14] setup a system of formal axioms for the calculus of logico-theoretical propositions concerning results of possible measurements in a quantum system. They started with formalizing the calculus of quantum events $E \in \mathcal{L}$ described by orthoprojectors E , i.e. projective operators $E = E^2$ which are orthogonal to the complements $E^\perp = I - E$ in the sense $E^\dagger E^\perp = O$, that is $E^\dagger = E^\dagger E = E$ on the Hilbert space \mathfrak{h} . These are the Hermitian projectors $E \in \mathcal{E}$ which have only two eigenvalues $\{0, 1\}$ (“no” and “yes”). Such calculus coincides with the calculus of the linear subspaces $\mathfrak{e} \subset \mathfrak{h}$ given by the ranges $\mathfrak{e} = E\mathfrak{h}$ of the orthoprojectors E , in the same sense as the propositional calculus of classical events coincides with the calculus of subsets of a Boolean algebra. In this calculus the logical ordering $E \leq F$ implemented by the algebraic relation $EF = E$ coincides with

$$\text{range}(E) \subseteq \text{range}(F),$$

the conjunction $E \wedge F$ corresponds to the intersection,

$$\text{range}(E \wedge F) = \text{range}(E) \cap \text{range}(F),$$

however the disjunction $E \vee F$ is represented by the linear sum $\mathfrak{e} + \mathfrak{f}$ of the corresponding subspaces but not their union,

$$\text{range}(E \vee F) \neq \text{range}(E) \cup \text{range}(F).$$

Note that although $\mathfrak{e} + \mathfrak{f} = \text{range}(E + F)$, the operator $E + F$ does not coincide with orthoprojector $E \vee F$ onto $\mathfrak{e} + \mathfrak{f}$ unless $EF = 0$. This implies that the distributive law characteristic for propositional calculus of classical logics no longer holds, but it still holds for compatible events, described by commutative orthoprojectors due to the orthomodularity

$$E \leq I - F \leq G \implies (E \vee F) \wedge G = E \vee (F \wedge G).$$

Two events E, F are called complementary if $E \vee F = I$, orthocomplementary if $E + F = I$, incompatible or disjunctive if $E \wedge F = 0$, and contradictory or orthogonal if $EF = 0$. As in the classical, common sense, logic contradictory events are incompatible. However *incompatible propositions or events are not necessary contradictory* as can be easily seen for any two nonorthogonal but not coinciding one-dimensional subspaces. In particular, in quantum logics there exist complementary incompatible pairs E, F , $E \vee F = I$, $E \wedge F = 0$ which are not ortho-complementary in the sense $E + F \neq I$, i.e. $EF \neq 0$ (it would be impossible in the classical case). This is a rigorous logico-mathematical proof of Bohr's complementarity.

As an example, we can consider the statement that a quantum system is in a stable energy state E , and an incompatible proposition F , that it collapses at a given time t , say. The incompatibility $E \wedge F = 0$ follows from the fact that there is no state in which the system would collapse preserving its energy, however these two propositions are not contradictory (i.e. not orthogonal, $EF \neq 0$): the system might not collapse if it is in other than E stable state (remember the Schrödinger's earlier belief that the energy law is valid only on average, and is violated in the process of radiation).

In 1952 Wick, Wightman, and Wigner [15] showed that there are physical systems for which not every orthoprojector corresponds to an observable event, so that not every orthoprojector P_ψ corresponding to a wave function ψ is a pure state. This is equivalent to the admission of some selective events which are dispersion-free in all pure states. Jauch and Piron [16] incorporated this situation into quantum logics and proved in the context of this most general approach that the hidden variable interpretation is only possible if the theory is observably wrong, i.e. if incompatible events are in fact compatible or contradictory.

Bell criticized this also, as well as he criticized the Gleason's theorem, but this time his arguments were not based even on the classical ground of usual probability theory. Although he explicitly used the additivity of the probability on the orthogonal events in his counterexample for $\mathfrak{h} = \mathbb{C}^2$, but he questioned : 'That so much follows from such apparently innocent assumptions leads us to question their innocence'. [12]. In fact this was equivalent to questioning of the additivity of the classical probability on the corresponding disjoint subsets, but he didn't suggest any other complete system of physically reasonable axioms for introducing such peculiar "nonclassical" hidden variables, not even a single counterexample to the orthogonal nonadditivity even for the simplest case $\dim \mathfrak{h} = 2$. Thus Bell implicitly rejected classical

probability theory in the quantum world, but he didn't want to accept quantum probability as the only possible theory for explaining the microworld. Obviously that even if such attempt was successful for a single quantum system (as he thought his unphysical discontinuous construction in the case $\mathfrak{h} = \mathbb{C}^2$ was), the classical composition law of two such systems would not allow to extend the dispersion-free product-states to the whole composed quantum system because of their nonadditivity on the whole observables space \mathcal{L} . The quantum composition law together with the orthoadditivity excludes the hidden variable possibility even for $\mathfrak{h} = \mathbb{C}^2$, otherwise it would contradict to Gleason's theorem for the case $\dim(\mathfrak{h} \otimes \mathfrak{h}) = 4$, not even to mention a hidden variable reproduction of nonseparable, entangled states). In fact it justifies the von Neumann's additivity assumption for the states on the whole operator algebra $\mathcal{B}(\mathfrak{h})$.

3.2 Quantum measurement and decoherence

Heisenberg derived from the uncertainty relation that 'the nonvalidity of rigorous causality is necessary and not just consistently possible'. Max Born even stated:-

One does not get an answer to the question, what is the state after collision? but only to the question, how probable is a given effect of the collision?

3.2.1 Spooky action at distance

After his defeat on uncertainty relations Einstein seemed to have become resigned to statistical interpretation of quantum theory, and at the 1933 Solvay Congress he listened to Bohr's paper on completeness of quantum theory without objections. Then, in 1935, he launched a brilliant and subtle new attack in a paper [17] with two young co-authors, Podolski and Rosen, which is known as the EPR paradox that has become of major importance to the world view of physics. They stated the following requirement for a complete theory as a seemingly necessary one:

Every element of physical reality must have a counterpart in the physical theory.

The question of completeness is thus easily answered as soon as soon as we are able to decide what are the elements of the physical reality. EPR then proposed a sufficient condition for an element of physical reality:

If, without in any way disturbing the system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this quantity.

Then they designed a thought experiment the essence of which is that two quantum “bits”, particles spins of two electrons say, are brought together to interact, and after separation an experiment is made to measure the spin orientation of one of them. The state after interaction is such that the measurement result $\tau = \pm\frac{1}{2}$ of one particle uniquely determination the spin z -orientation $\sigma = \mp\frac{1}{2}$ of the other particle. EPR apply their criterion of local reality: since the value of σ can be predicted by measuring τ without in any way disturbing σ , it must correspond to an existing element of physical reality. Yet the conclusion contradicts a fundamental postulate of quantum mechanics, according to which the sign of spin is not an intrinsic property of a complete description of the spin but is evoked only by a process of measurement. Therefore, EPR conclude, quantum mechanics must be incomplete, there must be hidden variables not yet discovered, which determine the spin as an intrinsic property. It seems Einstein was unaware of the von Neumann’s theorem, although they both had the positions at the same Institute for Advanced Studies at Princeton being among the original six mathematics professors appointed there in 1933.

Bohr carefully replied to this challenge by rejecting the assumption of local physical realism as stated by EPR [18]: ‘There is no question of a mechanical disturbance of the system under investigation during the last critical stage of the measuring procedure. But even at this stage there is essentially a question of *an influence on the very conditions which define the possible types of predictions regarding the future behavior of the system*’. This influence became notoriously famous as Bohr’s *spooky action at a distance*. He had obviously meant the semi-classical model of measurement, when one can statistically infer the state of one (quantum) part of a system immediately after observing the other (classical) part, whatever the distance between them. In fact, there is no paradox of “spooky action at distance” in the classical case, the statistical inference, playing the role of such immediate action, is simply based on the Bayesian selection rule of a posterior state from the

prior mixture of all such states, corresponding to the possible results of the measurement. Bohr always emphasized that one must treat the measuring instrument classically (the measured spin, or another bit interacting with this spin, as a classical bit), although the classical-quantum interaction should be regarded as purely quantum. The latter follows from non-existence of semi-classical Poisson bracket (i.e. classical-quantum potential interaction). Schrödinger clarified this point more precisely than Bohr, and he followed in fact the mathematical pattern of von Neumann measurement theory. EPR paradox is related to so called Bell inequality the probabilistic roots of which was evidenced in [19].

3.2.2 Releasing Schrödinger's cat

Motivated by EPR paper, in 1935 Schrödinger published a three part essay [20] on 'The Present Situation in Quantum mechanics'. He turns to EPR paradox and analyses completeness of the description by the wave function for the entangled parts of the system. (The word *entangled* was introduced by Schrödinger for the description of nonseparable states.) He notes that if one has pure states $\psi(\sigma)$ and $\varphi(\tau)$ for each of two completely separated bodies, one has maximal knowledge, $\chi(\sigma, \tau) = \psi(\sigma)\varphi(\tau)$, for two taken together. But the converse is not true for the entangled bodies, described by a non-separable wave function $\chi(\sigma, \tau) \neq \psi(\sigma)\varphi(\tau)$:

Maximal knowledge of a total system does not necessary imply maximal knowledge of all its parts, not even when these are completely separated one from another, and at the time can not influence one another at all.

To make absurdity of the EPR argument even more evident he constructed his famous burlesque example in quite a sardonic style. A cat is shut up in a steel chamber equipped with a camera, with an atomic mechanism in a pure state $\rho_0 = P_\psi$ which triggers the release of a phial of cyanide if an atom disintegrates spontaneously (it is assumed that it might not disintegrate in a course of an hour with probability $\text{Tr}(EP_\psi) = 1/2$). If the cyanide is released, the cat dies, if not, the cat lives. Because the entire system is regarded as quantum and closed, after one hour, without looking into the camera, one can say that the entire system is still in a pure state in which the living and the dead cat are smeared out in equal parts.

Schrödinger resolves this paradox by noting that the cat is a macroscopic object, the states of which (alive or dead) could be told apart by a macroscopic observation are distinct from each other whether observed or not. He calls this ‘the principle of state distinction’ for macroscopic objects, which is in fact the postulate that the directly measurable system (consisting of cat) must be classical:

It is typical in such a case that an uncertainty initially restricted to an atomic domain has become transformed into a macroscopic uncertainty which can be resolved through direct observation.

The dynamical problem of the transformation of the atomic, or “coherent” uncertainty, corresponding to a probability amplitude $\psi(\sigma)$, into a macroscopic uncertainty, corresponding to a mixed state ρ , is called quantum *decoherence* problem. In order to make this idea clear, let us give the solution of the Schrödinger’s elementary decoherence problem in the purely mathematical way. Instead of the values $\pm 1/2$ for the spin-variables σ and τ we shall use the values $\{0, 1\}$ corresponding to the states of a classical “bit”, the simplest nontrivial system in classical probability or information theory.

Consider the atomic mechanism as a quantum “bit”, $\mathfrak{h} = \mathbb{C}^2$, the pure states of which are described by ψ -functions of the variable $\sigma \in \{0, 1\}$ (if atom is disintegrated, $\sigma = 1$, if not, $\sigma = 0$) with scalar (complex) values $\psi(\sigma)$ defining the probabilities $|\psi(\sigma)|^2$ of the quantum elementary propositions corresponding to $\sigma = 0, 1$. The Schrödinger’s cat is a classical bit with only two pure states $\tau \in \{0, 1\}$ which can be identified with the probability distributions $\delta_0(\tau)$ when alive ($\tau = 0$) and $\delta_1(\tau)$ when dead ($\tau = 1$). These and other (mixed) states can also be described by the complex amplitudes $\varphi(\tau)$, however they are uniquely defined by the probabilities $|\varphi(\tau)|^2$ up to a phase function of τ , the phase multiplier of φ commuting with all cat observables $c(\tau)$, not just up to a phase constant as in the case of the atom (only constants commute with all atomic observables $A \in \mathcal{L}(\mathfrak{h})$). Initially the cat is alive, so its amplitude (uniquely defined up to the phase factor by the square root of probability distribution δ_0 on $\{0, 1\}$) is $\delta(\tau)$ that is 1 if $\tau = 0$, and 0 if $\tau = 1$.

The only meaningful classical-quantum reversible interaction affecting not atom but cat as it is said after the hour, is described as both bits were quantum, by

$$U[\psi \otimes \varphi](\sigma, \tau) = \psi(\sigma) \varphi((\tau \Delta \sigma)),$$

where $\tau \Delta \sigma = |\tau - \sigma| = \sigma \Delta \tau$ is the difference (mod 2) on $\{0, 1\}$. Applied to the initial product-state $\psi \otimes \delta$ it has the resulting probability amplitude

$$\chi(\sigma, \tau) = \psi(\sigma) \delta(\tau \Delta \sigma) = 0 \quad \text{if} \quad \sigma \neq \tau.$$

Despite the fact that the initial state was pure, $\chi_0 = \psi \otimes \delta$ corresponding to the Cartesian product $(\psi, 0)$ of the initial pure states $\psi \in \mathfrak{h}$ and $\tau = 0$, the reversible unitary evolution U induces the mixed state for the quantum-classical system “atom+cat” described by the wave function $\chi \in \mathfrak{h} \otimes \mathbb{C}^2$.

Indeed, the observables of such a system are operator-functions X of τ with values $X(\tau)$ in σ -matrices, represented as block-diagonal (σ, τ) -matrices $\hat{X} = [X(\tau) \delta_{\tau'}^{\tau}]$ of the multiplication $X(\tau) \chi(\cdot, \tau)$ at each point $\tau \in \{0, 1\}$. This means that the amplitude χ induces the same expectations

$$\langle \hat{X} \rangle = \sum_{\tau} \chi(\tau)^{\dagger} X(\tau) \chi(\tau) = \sum_{\tau} \text{Tr} X(\tau) \varrho(\tau) = \text{Tr} \hat{X} \hat{\varrho}$$

as the block-diagonal density matrix $\hat{\varrho} = [\varrho(\tau) \delta_{\tau'}^{\tau}]$ of the multiplication by

$$\varrho(\tau) = E(\tau) P_{\psi} E(\tau) = \text{Pr}(\tau) P_{E(\tau)\psi}$$

where $\text{Pr}(\tau) = |\psi(\tau)|^2$, $E(\tau) = P_{\delta_{\tau}}$ is the projection operator

$$[E(\tau) \psi](\sigma) = \delta(\tau \Delta \sigma) \psi(\sigma) = \psi(\tau) \delta_{\tau}(\sigma),$$

and $P_{E(\tau)\psi} = P_{\delta_{\tau}}$ is also the projector onto $\delta_{\tau}(\cdot) = \delta(\cdot \Delta \tau)$. The 4×4 -matrix $\hat{\varrho}$ is a mixture of two orthogonal projectors $P_{\tau} \otimes P_{\tau}$, $\tau = 0, 1$:

$$\hat{\varrho} = [P_{\delta_{\tau}} \delta_{\tau'}^{\tau} \text{Pr}(\tau)] = \sum_{\tau=0}^1 \text{Pr}(\tau) P_{\delta_{\tau}} \otimes P_{\delta_{\tau}}.$$

3.2.3 Von Neumann’s projection postulate

Inspired by Bohr’s complementarity principle, von Neumann proposed even earlier the idea that every quantum measuring process involves an unanalysable element. He postulated [11] that, in addition to the continuous causal propagation of the wave function generated by the Schrödinger equation, during a measurement, due to an action of the observer on the object, the function undergoes a discontinuous, irreversible instantaneous change. Thus, just prior

to a measurement of the event F , disintegration of atom, say, the quantum pure state P_ψ changes to the mixed one

$$\rho = \lambda P_{E\psi} + \mu P_{F\psi} = E\rho E + F\rho F,$$

where $E = I - F$ is the orthocomplement event, and $\lambda = \text{Tr}E\rho$, $\mu = \text{Tr}F\rho$ are the probabilities of E and F . Such change is projective as it shows the second part of this equation, and it is called the von Neumann projection postulate.

This linear irreversible decoherence process should be completed by the nonlinear, acausal random jump to one of the pure states

$$\rho \mapsto P_{E\psi}, \text{ or } \rho \mapsto P_{F\psi}$$

depending on whether the tested event F is false (the cat is alive, $\psi_0 = \lambda^{-1/2}E\psi$), or true (the cat is dead, $\psi_1 = \mu^{-1/2}F\psi$). Such last step is the posterior prediction, called *filtering* of the decoherent mixture of ψ_0 and ψ_1 by selection of only one result of the measurement, is an unavoidable element in every measurement process relating the state of the pointer of the measurement (in this case the cat) to the state of the whole system. This assures that the same result would be obtained in case of immediate subsequent measurement of the same event F . The resulting change of the initial wave-function ψ is described up to normalization by one of the projections

$$\psi \mapsto E\psi, \quad \psi \mapsto F\psi$$

and is sometimes called the Lüders projection postulate [21].

Although unobjectionable from the purely logical point of view the von Neumann theory of measurement soon became the target of severe criticisms. Firstly it seems radically subjective, postulating the spooky action at distance (the filtering) in a purely quantum system instead of deriving it. Secondly the analysis is applicable to only the idealized situation of discrete instantaneous measurements.

However as we already mentioned when discussing the EPR paradox, the process of filtering is free from conceptual difficulty if it is understood as the statistical inference about a mixed state in an extended stochastic representation of the quantum system as a part of a semiclassical one, based upon the results of observation in its classical part. In order to demonstrate this, we can return to the dynamical model of Schrödinger's cat, identifying

the quantum system in question with the Schrödinger's atom. The event E (the atom exists) corresponds then to $\tau = 0$ (the cat is alive), $E = E(0)$, and the complementary event is $F = E(1)$. This model explains that the origin of the von Neumann irreversible decoherence $P_\psi \mapsto \rho$ of the atomic state is in the ignorance of the result of the measurement described by the partial tracing over the cat's Hilbert space \mathbb{C}^2 :

$$\rho = \text{Tr}_{\mathbb{C}^2} \hat{\varrho} = \sum_{\tau=0}^1 \text{Pr}(\tau) P_{\delta_\tau} = \varrho(0) + \varrho(1),$$

where $\varrho(\tau) = |\psi(\tau)|^2 P_{\delta_\tau}$. It has the entropy $S(\rho) = \text{Tr} \rho \log \rho^{-1}$ of the compound state $\hat{\varrho}$ of the combined semi-classical system prepared for the indirect measurement of the disintegration of atom by means of cat's death:

$$S(\rho) = - \sum_{\tau=0}^1 |\psi(\tau)|^2 \log |\psi(\tau)|^2 = S(\hat{\varrho})$$

It is the initial coherent uncertainty in the pure quantum state of the atom described by the wave-function ψ which is equal to one bit if initially $|\psi(0)|^2 = 1/2 = |\psi(1)|^2$.

This dynamical model of the measurement which is due to von Neumann, also interprets the filtering $\rho \mapsto \rho_\tau$ simply as the conditioning

$$\rho_\tau = \varrho(\tau) / \text{Pr}(\tau) = P_{\delta_\tau}$$

of the joint classical-quantum state $\varrho(\cdot)$ by the Bayes formula which is applicable due to the commutativity of actually measured observable (the life of cat) with any other observable of the combined semi-classical system.

Thus the atomic decoherence is derived from the unitary interaction of the quantum atom with the classical cat. The spooky action at distance, affecting the atomic state by measuring τ , is simply the result of the statistical inference (prediction after the measurement) of the atomic posterior state $\rho_\tau = P_{\delta_\tau}$: the atom disintegrated if and only if the cat is dead. A formal derivation of the von-Neumann-Lüders projection postulate and the decoherence in the general case by explicit construction of unitary transformation in the extended semi-classical system is given in [22, 23].

4 Causality and Prediction of Future

In mathematics you don't understand things. You just get used to them - John von Neumann.

The quantum probability approach resolves the famous paradoxes of quantum measurement theory in a constructive way by giving exact nontrivial models for the statistical analysis of the quantum observation processes underlying these paradoxes. Conceptually it is based upon a new idea of quantum causality called the Nondemolition Principle which divides the world into the classical past, forming the consistent histories, and the quantum future, the state of which is predictable for each such history. The differential analysis of these models is based on Itô stochastic calculus. The most recent mathematical development of these methods leads to a profound quantum filtering and control theory in quantum open systems which has found numerous applications in quantum statistics, optics and spectroscopy, and is an appropriate tool for the solution of the dynamical decoherence problem for quantum communications and computations.

4.1 Reality and nondemolition principle

Schrödinger like Einstein was deeply concerned with the loss of reality and causality in the positivistic treatment of quantum measuring process by Heisenberg and Born. Schrödinger's remained unhappy with Bohr's reply to the EPR paradox, Schrödinger's own analysis was:

It is pretty clear, if reality does not determine the measured value, at least the measurable value determines reality.

4.1.1 Compatibility and time arrow

Von Neumann's projection postulate and its dynamical realization can be generalized to include cases with continuous spectrum of values. In fact there many such developments, we will only mention here the most general operational approach to quantum measurements of Ludwig [24], and its mathematical implementation by Davies and Lewis [25] in the "instrumental" form. The stochastic realization of the corresponding completely positive reduction map $\rho \mapsto \varrho(\cdot)$, resolving the corresponding instantaneous quantum measurement problem, can be found in [26, 22]. Because of the

crucial importance of these realizations for developing understanding of the mathematical structure and interpretation of modern quantum theory, analyze the mathematical consequences which can be drawn from such schemes we need to.

The generalized reduction of the wave function $\psi(x)$, corresponding to a complete measurement with discrete or continuous data y , is described by a function $V(y)$ whose values are linear operators $\psi \mapsto V(y)\psi$ but not isometric, $V(y)^\dagger V(y) \neq I$, with the following normalization condition. The resulting wave-function

$$\chi(x, y) = [V(y)\psi](x)$$

as a is normalized with respect to a given measure μ on y in the sense

$$\iint |\chi(x, y)|^2 d\mu d\lambda = \int |\psi(x)|^2 d\lambda$$

for any probability amplitude ψ (normalized with respect to a measure λ). In the discrete case such as the case of two-point variables $y = \tau$ (EPR paradox, or Schrödinger cat with the projection-valued $V(\tau) = E(\tau)$) this is usually satisfied by taking μ to be the counting measure:

$$\int_y V(y)^\dagger V(y) d\mu = I, \quad \text{or} \quad \sum_y V(y)^\dagger V(y) = I.$$

As in that simple case the realization can be always constructed [22] in terms of a unitary transformation U and a normalized wave function φ_0 such that

$$U[\psi \otimes \varphi_0](x, y) = \chi(x, y)$$

for any ψ . The additional system described by “the pointer coordinate y of the measurement apparatus” can be regarded as classical (like the cat) as the actual observables in question are the measurable functions $g(y)$ represented by commuting operators \hat{g} of multiplication by these functions. Note that such observables, extended to the quantum part as $I \otimes \hat{g}$, are compatible with any possible (future) event, represented by an orthoprojector $F \otimes I$. The probabilities (or, it is better to say, the propensities) of all such events are the same in all states whether an observable \hat{g} was measured but the result is not read, or it was not measured at all. In this sense the measurements of \hat{g} are called *nondemolition* with respect to the future observables F , they do

not demolish the picture of the possibilities, or propensities of F . But they are not necessary compatible with of the initial operators $F \otimes I$ of the quantum system under the question in the present representation $U(F \otimes I)U^*$ corresponding to the actual states $\chi = U(\psi \otimes \varphi_0)$.

Indeed, the Heisenberg operators

$$G = U^*(I \otimes \hat{g})U$$

of the nondemolition observables in general do not commute with the past operators $F \otimes I$ on the initial states $\chi_0 = \psi \otimes \varphi_0$. One can see this from the example of the Schrödinger cat. The “cat observables” in Heisenberg picture are represented by commuting operators $G = [g(\sigma + \tau) \delta_\sigma^\sigma, \delta_\tau^\tau]$ of multiplication by $g(\sigma + \tau)$, where the sum $\sigma + \tau = |\sigma - \tau|$ is modulo 2. They do not commute with $F \otimes I$ unless F is also a diagonal operator \hat{f} of multiplication by a function $f(\sigma)$ in which case

$$[F, G] \chi_0(\sigma, \tau) = [f(\sigma), g(\sigma + \tau)] \chi_0(\sigma, \tau) = 0, \quad \forall \chi_0.$$

However the restriction of the possibilities in a quantum system to only the diagonal operators $F = \hat{f}$ which would eliminate the time arrow in the nondemolition condition amounts to the redundancy of the quantum consideration as all such (possible and actual) observables can be simultaneously represented as the functions of (σ, τ) as in the classical case.

4.1.2 Transition from possible to actual

The analysis above shows that as soon as dynamics is taking into consideration even in the form of just a single unitary transformation, the measurement process needs to specify the arrow of time, what is the predictable future and what is the reduced past, what is possible and what is actual with respect to this measurement. As soon as a measured observable Y is specified, all other operators which do not commute with Y become redundant as possible in future observables. The algebra of such potential observables (not the state which stays invariant in the Schrödinger picture unless the selection due to an inference has taken place!) reduces to the subalgebra commuting with Y , and *this reduction doesn't change the reality* (the wave function remains the same and induces the same but now mixed state on the smaller - reduced algebra!). Possible observables in an individual system are only those which are compatible with the actual observables. This is another formulation of Bohr's complementarity. More specifically this can be rephrased in

the form of a dynamical postulate of quantum causality called the Nondemolition Principle [22] which we first formulate for a single instant of time t in a quite obvious form:

In the interaction representation of a quantum system by an algebra \mathcal{A} of (necessarily not all) operators on a Hilbert space \mathfrak{h} of the system with a measurement apparatus, causal, or nondemolition observables are represented only by those operators Y on \mathfrak{h} which are compatible with \mathcal{A} :

$$[X, Y] := XY - YX = 0, \quad \forall X \in \mathcal{A}$$

(this is usually written as $Y \in \mathcal{A}'$, where \mathcal{A}' , called the commutant of \mathcal{A} , in this formulation is not necessarily contained in \mathcal{A}). Each measurement process of the history for a quantum system \mathcal{A} can be represented as nondemolition by the causal observables in the appropriate representation of \mathcal{A} on a Hilbert space \mathfrak{h} .

Note that the space of interaction representation \mathfrak{h} plays here the crucial role: the reduced operators

$$X_0 = (I \otimes \varphi_0)^* X (I \otimes \varphi_0), \quad Y_0 = (I \otimes \varphi_0)^* Y (I \otimes \varphi_0)$$

for commuting X and Y might not commute on the smaller space \mathfrak{h}_0 of the initial states $\psi \otimes \varphi_0$ with a fixed $\varphi_0 \in \mathfrak{f}$. Even if the nondemolition observables Y is faithfully represented by Y_0 on initial space \mathfrak{h}_0 , as it is in the case $Y = G$ of the Schrödinger's cat with $\varphi_0(\tau) = \delta(\tau)$ where Y_0 is the multiplication operator $G_0 = \hat{g}$ for ψ :

$$G(\psi \otimes \delta)(\sigma, \tau) = g(\sigma + \tau) \psi(\sigma) \delta(\tau) = g(\sigma) \psi(\sigma) \delta(\tau) = (G_0 \psi \otimes \delta)(\sigma, \tau),$$

there is no usually room in \mathfrak{h}_0 to represent all Heisenberg operators $X \in \mathcal{A}$ commuting with Y on \mathfrak{h} . The induced operators Y_0 do not commute with all operators F of the system initially represented on \mathfrak{h}_0 , and this is why the measurement of Y_0 is thought to be demolition on \mathfrak{h}_0 . However in all such cases the future operators X reduced to X_0 on \mathfrak{h}_0 , commute with Y_0 as they are decomposable with respect to Y_0 , although the reduction $X \mapsto X_0$ is not the Heisenberg but irreversible dynamical map. It can be explicitly seen for

the atom described by the Heisenberg operators $X = U^* (F \otimes I) U$ in the interaction representation with the cat:

$$X_0 = \sum_{\tau} E(\tau) F E(\tau), \quad Y_0 = \sum_{\tau} g(\tau) E(\tau).$$

Thus the nondemolition principle should not be considered as a restriction on the possible observations for a given dynamics but rather a condition for the causal dynamics to be compatible with the given observations. As was proved in [27], the causality condition is necessary and sufficient for the existence of a conditional expectation for any state on the total algebra $\mathcal{A} \vee \mathcal{B}$ with respect to a commutative subalgebra \mathcal{B} of nondemolition observables Y . Thus the nondemolition causality condition amounts exactly to the existence of the conditional states, i.e. to the predictability of the states on the algebra \mathcal{A} upon the measurement results of the observables in \mathcal{B} . Then the transition from a prior ρ to a posterior state $\rho_y = P_{V(y)\psi}$ is simply the result of gaining a knowledge y defining the actual state in the decoherent mixture

$$\rho = \int V(y) P_{\psi} V(y)^* d\mu = \int P_{V(y)\psi} f(y) d\mu,$$

of all possible states, where $f(y) = \|V(y)\psi\|^2$ is the probability density of all possible y defining the output measure $d\nu = f d\mu$. As Heisenberg always emphasized, “quantum jump” is contained in the transitions from possible to actual.

If an algebra \mathcal{B} of actual observables is specified at a time t , there must be a causal representation \mathcal{B}_t of \mathcal{B} on \mathfrak{h} with respect to the present \mathcal{A}_t and all future possible representations \mathcal{A}_s , $s > t$ of the quantum system on the same Hilbert space \mathfrak{h} (they might not coincide with \mathcal{A}_t if the system is open [28]). The past representations \mathcal{A}_r , $r < t$ which are incompatible with a $G \in \mathcal{B}_t$ are meaningless as noncausal for the observation at the time t , they should be replaced by the causal histories \mathcal{B}_r , $r < t$ of the actual observables on \mathfrak{h} which must be consistent in the sense of compatibility of all \mathcal{B}_t . Thus the dynamical formulation of the nondemolition principle of quantum causality and the consistency of histories reads as

$$\mathcal{A}_s \subseteq \mathcal{B}'_r, \quad \mathcal{B}_s \subseteq \mathcal{B}'_r, \quad \forall r \leq s.$$

These are the only possible conditions when the posterior states always exist as results of inference (filtering and prediction) of future quantum states

upon the measurement results of the classical (i.e. commutative) past of process of observation. The act of measurement transforms quantum propensities into classical realities. As Lawrence Bragg, another Nobel prize winner, once said, everything in the future is a wave, everything in the past is a particle.

4.1.3 The true Heisenberg principle

The time continuous solution of the quantum measurement problem was motivated by analogy with the classical stochastic filtering problem which obtains the prediction of future for an unobservable dynamical process $x(t)$ by time-continuous measuring of another, observable process $y(t)$. Such a problem was first considered by Wiener and Kolmogorov who found its solution in the form of causal spectral filter but only for the stationary Gaussian case. The differential solution in the form of a stochastic filtering equation was then obtained by Stratonovich [32] in 1958 for an arbitrary Markovian pair (x, y) . This was really a break through in the statistics of stochastic processes which soon found many applications, in particular for solving the problems of stochastic control under incomplete information (it is possible that this was one of the reasons why the Russians were so successful in launching the rockets to the Moon and other planets of the Solar system in 60s).

If $X(t)$ is the unobservable process, a Heisenberg coordinate process of a quantum particle, say, and $Y(t)$ is an observable quantum process, describing the trajectories $y(t)$ of the particle in a cloud chamber, say, why don't we find a filtering equation for the a posteriori expectation $q(t)$ of $X(t)$ in the same way as we do it in the classical case if we know a history, i.e. a particular trajectory $y(r)$ up to the time t ? This problem was first considered and solved for the case of quantum Markovian Gaussian pair (X, Y) corresponding to a quantum open linear system with linear output channel, in particular for a quantum oscillator matched to a quantum transmission line [33]. By studying this example, the nondemolition condition

$$[X(s), Y(r)] = 0, \quad [Y(s), Y(r)] = 0 \quad \forall r \leq s$$

was first found, and this allowed to get the solution in the form of the causal equation for $q(t) = \langle X(t) \rangle_y$.

Let us describe this exact dynamical model of the causal nondemolition measurement first in terms of quantum white noise analysis for a one-

dimensional quantum nonrelativistic particle of mass m which is conservative if not observed, in a potential field ϕ . But we shall assume that the particle is under indirect observation by measuring of its Heisenberg position operator $X(t)$ with an additive random error $e(t)$:

$$Y(t) = X(t) + e(t).$$

We take the simplest statistical model for the error process $e(t)$, the white noise model (the worst, completely chaotic error), assuming that it is a classical (i.e. commutative) Gaussian white noise given by the first momenta

$$\langle e(t) \rangle = 0, \quad \langle e(s) e(r) \rangle = \sigma^2 \delta(s - r).$$

The measurement process $Y(t)$ should be commutative, satisfying the causal nondemolition condition with respect to the noncommutative process $X(t)$ (and any other Heisenberg operator-process of the particle), what can be achieved by perturbing the particle Newton-Erenfest equation:

$$m \frac{d^2}{dt^2} X(t) + \nabla \phi(X(t)) = f(t).$$

Here $f(t)$ is a Langevin force perturbing the dynamics due to the measurement, which is assumed to be another classical (commutative) white noise.

$$\langle f(t) \rangle = 0, \quad \langle f(s) f(r) \rangle = \tau^2 \delta(s - r).$$

In classical measurement and filtering theory the white noises $e(t)$, $f(t)$ are usually considered independent, and the intensities σ^2 and τ^2 can be arbitrary, even zeros, corresponding to the ideal case of the direct unperturbing observation of the particle trajectory $X(t)$. However in quantum theory corresponding to the standard commutation relations

$$X(0) = \hat{x}, \quad \frac{d}{dt} X(0) = \frac{1}{m} \hat{p}, \quad [\hat{x}, \hat{p}] = i\hbar \hat{1}$$

the particle trajectories do not exist, and it was always understood that the measurement error $e(t)$ and perturbation force $f(t)$ should satisfy a sort of uncertainty relation. This "true Heisenberg principle" had never been mathematically formulated and proved before the discovery [33] of quantum causality and nondemolition condition in the above form of commutativity of $X(s)$ and $Y(r)$ for $r \leq s$. As we showed first in the linear case [33], and

later even in the most general case [27], these conditions are fulfilled if and only if $e(t)$ and $f(t)$ satisfy the canonical commutation relations

$$[e(r), e(s)] = 0, [e(r), f(s)] = \frac{\hbar}{i} \delta(r - s), [f(r), f(s)] = 0.$$

This proves that the pair (e, f) must satisfy the uncertainty relation $\sigma\tau \geq \hbar/2$, i.e.

$$\Delta e_t \Delta f_t \geq \hbar t / 2,$$

terms of the standard deviations of the integrated processes

$$e_t = \int_0^t e(r) dr, \quad f_t = \int_0^t f(s) ds.$$

This inequality constitutes the precise formulation of the true Heisenberg principle for the square roots σ and τ of the intensities of error e and perturbation f : they are inversely proportional with the same coefficient proportionality $\hbar/2$ as for the pair (\hat{x}, \hat{p}) . The canonical pair (e, f) called quantum white noise cannot be considered classically despite of the possibility of the classical realizations of each process e and f separately due to the self-commutativity of the families e and f . Thus, a generalized matrix mechanics for the treatment of quantum open systems under continuous nondemolition observation and the true Heisenberg principle was invented exactly 20 years ago in [33].

4.2 Consistent Histories and Filtering

Schrödinger believed that all quantum problems including the interpretation of measurement should be formulated in continuous time in the form of differential equations. He thought that the measurement problem would have been resolved if quantum mechanics had been made consistent with relativity theory and the time had been treated appropriately. However Einstein and Heisenberg did not believe this, each for to his own reasons. While Einstein thought that the probabilistic interpretation of quantum mechanics was wrong, Heisenberg simply stated:-

Quantum mechanics itself, whatever its interpretation, does not account for the transition from ‘possible to the actual’

Perhaps the closest to the truth was Bohr when he said that it ‘must be possible so to describe the extraphysical process of the subjective perception as if it were in reality in the physical world’, extending the reality beyond the closed quantum mechanical form by including a subjective observer into a semiclassical world. He regarded the measurement apparatus, or meter, as a semiclassical object which interacts with the world in a quantum mechanical way but has only commuting observables - pointers. Thus Bohr accepted that *not all world is quantum mechanical, there is a classical part of the physical world, and we belong partly to this classical world.*

4.2.1 Symbolic Stochastic Calculus

In order to formulate the differential nondemolition causality condition and to derive a filtering equation for the posterior states in the time-continuous case we need quantum stochastic calculus.

The classical differential calculus for the infinitesimal increments

$$dx = x(t + dt) - x(t)$$

became generally accepted only after Newton gave a simple algebraic rule $(dt)^2 = 0$ for the formal computations of the differentials dx for smooth trajectories $t \mapsto x(t)$. In the complex plane \mathbb{C} of phase space it can be represented by a one-dimensional algebra $\mathfrak{a} = \mathbb{C}\dot{d}$ of the elements $a = \alpha\dot{d}$ with involution $a^* = \bar{\alpha}\dot{d}$. Here

$$\dot{d} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(\sigma_1 + i\sigma_2)$$

for dt is the nilpotent matrix, which can be regarded as Hermitian $\dot{d}^* = \dot{d}$ with respect to the Minkowski metrics $(\mathbf{z}|\mathbf{z}) = 2 \operatorname{Re} z_- \bar{z}_+$ in \mathbb{C}^2 .

This formal rule was generalized to non-smooth paths early in the last century in order to include the calculus of forward differentials $dw \simeq (dt)^{1/2}$ for continuous diffusions w_t which have no derivative at any t , and the forward differentials $dn \in \{0, 1\}$ for left continuous counting trajectories n_t which have zero derivative for almost all t (except the points of discontinuity where $dn = 1$). The first is usually done by adding the rules

$$(dw)^2 = dt, \quad dw dt = 0 = dt dw$$

in formal computations of continuous trajectories having the first order forward differentials $dx = \alpha dt + \beta dw$ with the diffusive part given by the increments of standard Brownian paths w . The second can be done by adding the rules

$$(dn)^2 = dn, \quad dndt = 0 = dtdn$$

in formal computations of left continuous and smooth for almost all t trajectories having the forward differentials $dx = \alpha dt + \gamma dm$ with jumping part given by the increments of standard compensated Poisson paths $m_t = n_t - t$. These rules were developed by Itô [29] into the form of a stochastic calculus.

The linear span of dt and dw forms the Wiener-Itô algebra $\mathfrak{b} = \mathbb{C}\bar{d} + \mathbb{C}d_w$, while the linear span of dt and dn forms the Poisson-Itô algebra $\mathfrak{c} = \mathbb{C}\bar{d} + \mathbb{C}d_m$, with the second order nilpotent $d_w = d_w^*$ and the idempotent $d_m = d_m^*$. They are represented together with \bar{d} by the triangular Hermitian matrices

$$\bar{d} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad d_w = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad d_m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

on the Minkowski space \mathbb{C}^3 with respect to the inner Minkowski product $(\mathbf{z}|\mathbf{z}) = z_-z^- + z_0z^0 + z_+z^+$, where $z^\mu = \bar{z}_{-\mu}$, $-(-, \circ, +) = (+, \circ, -)$.

Although both algebras \mathfrak{b} and \mathfrak{c} are commutative, the matrix algebra \mathfrak{a} generated by \mathfrak{b} and \mathfrak{c} on \mathbb{C}^3 is not:

$$d_w d_m = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = d_m d_w.$$

The four-dimensional \star -algebra $\mathfrak{a} = \mathbb{C}\bar{d} + \mathbb{C}e_- + \mathbb{C}e^+ + \mathbb{C}e$ of triangular matrices with the canonical basis

$$e_- = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

given by the algebraic combinations

$$e_- = d_w d_m - \bar{d}, \quad e^+ = d_m d_w - \bar{d}, \quad e = e - d_w - \bar{d}$$

is the canonical representation of the differential \star -algebra for one-dimensional vacuum noise in quantum stochastic calculus [?, 30]. It realizes the HP (Hudson – Parthasarathy) table [31]

$$d\Lambda_- d\Lambda^+ = dt, \quad d\Lambda_- d\Lambda = d\Lambda_-, \quad d\Lambda d\Lambda^+ = d\Lambda^+, \quad (d\Lambda)^2 = d\Lambda,$$

with zero products for all other pairs, for the multiplication of the canonical counting $d\Lambda = \Lambda(e)$, creation $d\Lambda^+ = \Lambda(e^+)$, annihilation $d\Lambda_- = \Lambda(e^-)$, and preservation $dt = \Lambda(\vec{d})$ quantum stochastic integrators in Fock space over $L^2(\mathbb{R}_+)$. As was proved in [30], any generalized Itô algebra can be represented as a \star -subalgebra of a multi-dimensional quantum vacuum algebra

$$d\Lambda_\mu^\iota d\Lambda_\kappa^\nu = \delta_{\kappa\mu}^\iota d\Lambda_\mu^\nu, \quad \kappa, \mu \in \{-, 1, \dots, d\}; \quad \iota, \nu \in \{1, \dots, d, +\},$$

where $d\Lambda_\pm^+ = dt$ and d is the dimensionality of quantum noise (could be infinite), similar to the representation of every semi-classical system with a given state as a subsystem of quantum system with a pure state. In particular, any real-valued process y_t with zero mean value $\langle y_t \rangle = 0$ and independent increments generating a two-dimensional Itô algebra has the differential dy in the form of a commutative combination of $d\Lambda, d\Lambda_-, d\Lambda^+$. We shall call it standard if it has the stationary increments with the standard variance $\langle y_t^2 \rangle = t$. In this case

$$y_t = (\Lambda^+ + \Lambda_- + \varepsilon\Lambda)_t = \varepsilon m_t + (1 - \varepsilon) w_t,$$

where $\varepsilon \geq 0$ is defined by the equation $(dy)^2 - dt = \varepsilon dy$.

4.2.2 Stochastic decoherence equations

The generalized wave mechanics which enables us to treat the quantum processes of continual in time observation, or in other words, quantum mechanics with trajectories $\omega = (y_t)$, was discovered only quite recently, in [34, 35, 37]. The basic idea of the theory is to replace the deterministic unitary Schrödinger propagation $\psi \mapsto \psi(t)$ by a linear causal stochastic one $\psi \mapsto \chi(t, \omega)$ which is not necessarily unitary for each history ω , but unitary in the mean square sense with respect to a standard probability measure $\mu(d\omega)$ for $d\omega$. Due to this positive measure

$$\Pr(t, d\omega) = \|\chi(t, \omega)\|^2 \mu(d\omega)$$

is normalized (if $\|\psi\| = 1$) for each t , and is interpreted as the probability measure for the histories $\omega_t = (y_r)_{r < t}$ of an output stochastic process y_t . In the same way as the abstract Schrödinger equation can be derived from only unitarity of propagation, the abstract decoherence wave equation can be derived from the mean square unitarity in the form of a linear stochastic

differential equation. The reason that Bohr and Schrödinger didn't derive such equation despite their firm belief that the measurement process can be described 'as if it were in reality in the physical world' is that the appropriate (stochastic) differential calculus had not been yet developed early in that century. As Newton had to invent the differential calculus in order to formulate the equations of classical dynamics, we had to develop the quantum stochastic calculus for nondemolition processes [34, ?] in order to derive the generalized wave equation for quantum dynamics with continual observation.

For the notational simplicity we shall consider here the one dimensional case, $d = 1$, the multi-dimensional case can be found elsewhere (e.g. in [34, ?]). The abstract stochastic wave equation can be written in this case as [35, 36]

$$d\chi(t) + K\chi(t) dt = L\chi(t) dy_t, \quad \chi(0) = \psi$$

where $y_t(\omega)$ is the output process, which is assumed to be a martingale (e.g. the independent increment process with zero expectations) representing a measurement noise with respect to the input probability measure $\mu(d\omega) = \Pr(0, d\omega)$ (but not with respect to the output probability measure $\Pr(\infty, d\omega)$). The stochastic process $\chi(t, \omega)$ normalized in the mean square sense for each t , is a wave function $\chi(t)$ in an extended Hilbert space, describing the process of continual decoherence of the initial pure state $\rho(0) = P_\psi$ into the mixture

$$\rho(t) = \int P_{\psi_\omega(t)} \Pr(t, d\omega)$$

of the posterior states corresponding to $\psi_\omega(t) = \chi(t, \omega) / \|\chi(t, \omega)\|$. Assuming that the conditional expectation $\langle dy_t dy_t \rangle_t$ in

$$\begin{aligned} \langle d(\chi^\dagger \chi) \rangle_t &= \langle d\chi^\dagger d\chi + \chi^\dagger d\chi + d\chi^\dagger \chi \rangle_t \\ &= \chi^\dagger (L^* \langle dy_t dy_t \rangle_t L - (K + K^*) dt) \chi \end{aligned}$$

is dt (e.g. $(dy_t)^2 = dt + \varepsilon dy_t$), the mean square normalization in its differential form $\langle d(\chi^\dagger \chi) \rangle_t = 0$ can be expressed as $K + K^* = L^* L$, i.e.

$$K = \frac{1}{2} L^* L + \frac{i}{\hbar} H,$$

where $H = H^*$ is the Schrödinger Hamiltonian such that this is the Schrödinger equation if $L = 0$. One can also derive the corresponding Master equation

$$\frac{d}{dt} \rho(t) + K\rho(t) + \rho(t) K^* = L\rho(t) L^*$$

for mixing the initially pure state $\rho(0) = \psi\psi^\dagger$, as well as a stochastic nonlinear wave equation for the dynamical prediction of the posterior state vector $\psi_\omega(t)$ which is the normalization of $\chi(t, \omega)$ at each ω .

Actually, there are two basic standard forms [37, 38] of such stochastic wave equations, corresponding to two basic types of stochastic integrators: the Brownian standard type, $\varepsilon = 0$, $y_t \simeq w_t$, and the Poisson standard type $\varepsilon = 1$, $y_t \simeq n_t - t$ with respect to the measure μ . (For the most general form see [40].) To get these forms we shall assume that y_t is standard with respect to the input measure μ , given by the multiplication table

$$(dy)^2 = dt + \nu^{-1/2}dy, \quad dydt = 0 = dt dy,$$

where $\nu > 0$ is the intensity of the Poisson process $n_t = \nu^{1/2}y_t + \nu t$ with respect to the standard measure μ , and

$$L = \nu^{1/2}(C - I), \quad H = E + i\frac{\nu}{2}(C - C^\dagger).$$

This corresponds to the stochastic decoherence equation for the counting observation initially derived in the form [35, 39]

$$d\chi(t) + \left(\frac{\nu}{2}(C^\dagger C - I) + \frac{i}{\hbar}E \right) \chi(t) dt = (C - I) \chi(t) dn_t,$$

where C and E are called collapse and energy operators respectively. The nonlinear filtering equation for $\psi_\omega(t)$ in this case has the form [38]

$$d\psi_\omega + \left(\frac{\nu}{2}(C^\dagger C - \|C\psi_\omega\|^2) + \frac{i}{\hbar}E \right) \psi_\omega dt = (C / \|C\psi_\omega\| - I) \psi_\omega dn_t^{\psi_\omega},$$

where n_t^ψ is the output counting process which is described by the probability measure $\text{Pr}(\infty, d\omega)$ with the increment dn_t^ψ independent of n_t^ψ under the condition $\psi_\omega(t) = \psi$, having the conditional expectation $\nu \|C\psi\|^2 dt$. This can be written also in the quasi-linear form [37, 38]

$$d\psi(t) + \tilde{K}(t) \psi_\omega(t) dt = \tilde{L}(t) \psi(t) d\tilde{y}_t^{\psi(t)},$$

where \tilde{y}_t^ψ is the innovating martingale with respect to the output measure which is described by the differential

$$d\tilde{y}_t^\psi = \nu^{-1/2} \|C\psi\|^{-1} dn_t^\psi - \nu^{1/2} \|C\psi\| dt$$

with the initial $\tilde{y}_0^\psi = 0$, the operator $\tilde{K}(t)$ similar to K has the form

$$\tilde{K}(t) = \frac{1}{2} \tilde{L}(t)^* \tilde{L}(t) + \frac{i}{\hbar} \tilde{H}(t),$$

and $\tilde{H}(t), \tilde{L}(t)$ depend on t and ω through the dependence on $\psi = \psi_\omega(t)$:

$$\tilde{L} = \nu^{1/2} (C - \|C\psi\|), \quad \tilde{H} = E + i \frac{\nu}{2} (C - C^\dagger) \|C\psi\|.$$

The last form of the nonlinear filtering equation admits the central limit $\nu \rightarrow \infty$ corresponding to the standard Wiener case $\varepsilon = 0$ when $y_t = w_t$ with respect to the input Wiener measure μ . If L and H do not depend on ν , i.e. C and E depend on ν as

$$C = I + \nu^{-1/2} L, \quad E = H + \frac{\nu^{1/2}}{2i} (L - L^\dagger),$$

then $\tilde{y}_t^\psi \rightarrow \tilde{y}_t$ as $\varepsilon^2 = \nu^{-1} \rightarrow 0$, where \tilde{y}_t with $d\tilde{y}_t = dy_t - 2 \operatorname{Re} \langle \psi | L \psi \rangle dt$ with respect to the output measure is another standard Wiener process \tilde{w}_t which does not depend on ψ . If $\|\psi_\omega(t)\| = 1$ (which follows from the initial condition $\|\psi\| = 1$), the operator-functions $\tilde{L}(t), \tilde{H}(t)$ defining the nonlinear filtering equation have the limits

$$\tilde{L} = L - \operatorname{Re} \langle \psi | L \psi \rangle, \quad \tilde{H} = H + \frac{i}{2} (L - L^\dagger) \operatorname{Re} \langle \psi | L \psi \rangle.$$

4.2.3 Quantum trajectories and filtering

Let us give the exact model based on quantum stochastic calculus for a quantum particle of mass m in a potential ϕ under indirect observation of its position given by the equation

$$dY_t = (2\lambda)^{1/2} X(t) dt + dy_t,$$

where $y_t = w_t$ is the standard Wiener process. This model coincides with the signal plus noise model $Y(t) = X(t) + y(t)$ in terms of the generalized derivatives

$$y(t) = (2\lambda)^{-1/2} dy_t/dt \equiv e(t),$$

where $e(t)$ is quantum white noise of intensity $\sigma^2 = (2\lambda)^{-1}$. Here $\lambda > 0$ is the accuracy of the measurement of $X(t)$ with respect to the standard Wiener

process represented as $w_t = (\Lambda_- + \Lambda^+)_t$ on the vacuum in Fock space. It was proved in [34, ?] that Y_t is a commutative nondemolition process with respect to the coordinate and momentum Heisenberg processes if they are perturbed by a Langevin force $f(t)$ of intensity $\tau^2 = \lambda\hbar^2/2$, the generalized derivative of $(\lambda/2)^{1/2} f_t$, where $f_t = i\hbar(\Lambda_- - \Lambda^+)_t$:

$$dP(t) + \nabla\phi(X(t)) dt = \left(\frac{\lambda}{2}\right)^{1/2} df_t, \quad P(t) = m\frac{d}{dt}X(t).$$

Note that the quantum error process $e_t = w_t$ does not commute with the perturbing quantum process f_t in Fock space due to the multiplication table

$$dfdw = i\hbar dt, \quad dwdf = -i\hbar dt.$$

This corresponds to the canonical commutation relations for the normalized derivatives $e(t)$ and $f(t)$, so that the true Heisenberg principle is fulfilled at the boundary $\sigma\tau = \hbar/2$. Thus our quantum stochastic model of nondemolition observation is the minimal perturbation model for the given accuracy λ of the continual indirect measurement of the position $X(t)$.

The stochastic decoherence equation for this model with $y_t = w_t$,

$$L = \left(\frac{\lambda}{2}\right)^{1/2} \hat{x}, \quad H = \frac{\hat{p}^2}{2m} + \phi(\hat{x})$$

was derived and solved in [34, 41, 37, 42] for the case of linear and quadratic potential ϕ . Here \hat{x} is multiplication by x , and $\hat{p} = \frac{\hbar}{i} \frac{d}{dx}$.

The stochastic posterior nonlinear equation in this case is

$$d\psi(t) + \left(\frac{i}{\hbar}H + \frac{\lambda}{4}\tilde{x}(t)^2\right)\psi(t) dt = \left(\frac{\lambda}{2}\right)^{1/2} \tilde{x}(t)\psi(t) d\tilde{y}_t,$$

where $\tilde{x}(t) = \hat{x} - q(t)\hat{1}$, $q(t, \omega) = \text{Tr}(\hat{x}P_{\psi_w(t)})$ is the posterior expectation (statistical prediction) of \hat{x} , and $d\tilde{y}_t = dy_t - (2\lambda)^{1/2}q(t)dt$ defines the innovating process \tilde{y}_t which is equivalent to the standard Brownian motion \tilde{w}_t with respect to the output (not input!) probability measure $\text{Pr}(\infty, d\omega)$.

Let us give the explicit solution of this stochastic wave equation for the free particle ($\phi = 0$) and the stationary Gaussian initial wave packet. One can show [43, 44] that the nondemolition observation of such particle is described by filtering of quantum noise which results in the continual collapse of any wave packet to the Gaussian stationary one centered

at the posterior expectation $q(t) = \langle \psi(t) | \hat{x} \psi(t) \rangle$ with finite dispersion $\| (q(t) - \hat{x}) \psi(t) \|^2 \rightarrow (\hbar/2\lambda m)^{1/2}$. The center can be found from the linear Newton equation

$$\frac{d^2}{dt^2} z(t) + 2\kappa \frac{d}{dt} z(t) + 2\kappa^2 z(t) = -g(t),$$

for the deviation process $z(t) = q(t) - y(t)$, with $z(0) = q_0 - y(0)$, $z'(0) = v_0 - y'(0)$. Here $\kappa = (\lambda\hbar/2m)^{1/2}$ is the decay rate which is also the frequency of effective oscillations, $q_0 = \langle \hat{x} \rangle$, $v_0 = \langle \hat{p}/m \rangle$ are the initial expectations, $y(t)$ is a (generalized) trajectory of the observable process $Y(t)$, and $g(t) = y''(t)$ is the effective gravitation for the particle in the moving framework of $y(t)$. The continuous collapse $z(t) \rightarrow 0$ of the posterior expectation $q(t)$ towards a linear trajectory $y(t)$ is illustrated in the Appendix.

4.2.4 A quantum message from the future

Although Schrödinger didn't derive the stochastic filtering equation for the continuously decohering wave function $\chi(t)$, describing the state of the semi-classical system including the observable nondemolition process y_t in continuous time in the same way as we did it for his cat just in one step, he did envisage a possibility of how to get it 'if one introduces two symmetric systems of waves, which are traveling in opposite directions; one of them presumably has something to do with the known (or supposed to be known) state of the system at a later point in time' [45]. This desire coincides with the "transactional" attempt of interpretation of quantum mechanics suggested in [46] on the basis that the relativistic wave equation yields in the nonrelativistic limit two Schrödinger type equations, one of which is the time reversed version of the usual equation: 'The state vector ψ of the quantum mechanical formalism is a real physical wave with spatial extend and it is identical with the initial "offer wave" of the transaction. The particle (photon, electron, etc.) and the collapsed state vector are identical with the completed transaction.' There was no mathematical proof of this statement in [46], and it is obviously not true for the deterministic state vector $\psi(t)$ satisfying the conventional Schrödinger equation, but we are going to show that this interpretation is true for the stochastic wave $\chi(t)$ satisfying our decoherence equation.

First let us note that the stochastic equation for the offer wave $\chi(t)$ and the standard input probability measure μ can be represented in Fock space

as

$$d\chi(t) + K\chi(t) dt = Ldy_t\chi(t), \quad \chi(0) = \psi \otimes \delta_0,$$

where $y_t = \Lambda^+ + \Lambda_- + \varepsilon\Lambda$. It coincides on the noise vacuum state δ_0 with the quantum stochastic Schrödinger equation

$$d\varphi(t) + K\varphi(t) dt = (Ld\Lambda^+ - L^\dagger d\Lambda_-) \varphi(t), \quad \varphi(0) = \psi \otimes \delta_0$$

corresponding to the generalized Heisenberg equation with the Langevin force, $if_t = \hbar(\Lambda^+ - \Lambda_-)_t$, if $L^\dagger = L$. Indeed, due to adaptedness both Ldy and $Ld\Lambda^+ - L^\dagger d\Lambda_-$ act on the tensor product states with future vacuum δ_0 on which they have the same action since $\Lambda_- \delta_0 = 0$, $\Lambda \delta_0 = 0$ (the annihilation process Λ_- is zero on the vacuum δ_0 , as well as the number process Λ). Thus when extended from δ_0 to any initial Fock vector φ_0 , quantum stochastic evolution is the HP unitary propagation [31] which is a unitary cocycle on Fock space over $L^2(\mathbb{R}_+)$ with respect to the free time-shift evolution $\varphi(t, s) = \varphi(0, s + t)$ in the Fock space. This free plain wave evolution in the half of space $s > 0$ in the extra dimension is the input, or offer wave evolution for our three dimensional (or more?) world located at the boundary of \mathbb{R}_+ . The single offer waves do not interact in the Fock space until they reach the boundary $s = 0$ where they produce the quantum jumps described by the stochastic differential equation.

As has been recently shown in [47], by doubling the Fock space it is possible to extend the cocycle to a unitary group evolution which will also include the free propagation of the output waves in the opposite direction. The conservative boundary condition corresponding to the interaction with our world at the boundary, is including the creation, annihilation and exchange of the input-output waves. The corresponding ‘‘Schrödinger’’ boundary value problem is the second quantization of the Dirac wave equation on the half line, with a boundary condition in Fock space which is responsible for the stochastic interaction of quantum noise with our world in the course of the transaction of the input-output waves. The nondemolition continual observations are represented in this picture by the measuring at the boundary of the arrival times and positions of the particles corresponding to the quantized waves in Fock space with respect to an ‘‘offer state’’ the input vacuum, dressed into the output wave. The continual reduction process for our world wave function then is simply represented as the decohering input wave function in the extended space, which is filtered from the corresponding mixture of pure states by the process of innovation of the initial knowledge during

the continual measurement. The result of this filtering gives the best possible prediction of future states which is allowed by the quantum causality. As was shown on the example of a free quantum particle under observation, the filtering appear as a dissipation, oscillation and gravitation as a result of nondemolition observation.

5 Conclusion: The Greatest Form of Beauty

My friend Robin Hudson wrote in his Lecture Notes on Quantum Theory:

Quantum theory is a beautiful mathematical theory. If only it didn't have to mean something, to be interpreted.

Obviously here he used beautiful in the sense of simple: Everything that is simple is indeed beautiful. However Nature is beautiful but not simple: we live at the edge of two worlds, one is quantum, the other one is classical, everything in the future is quantized waves, everything in the past is trajectories of recorded particles.

In my philosophy I am a follower of those about whom Aristotle wrote in his *Metaphysica*: *'they fancied that the principles of mathematics are the principles of all things'*, i.e. the things of Nature, and I agree that *'...these are the greatest forms of beauty'*.

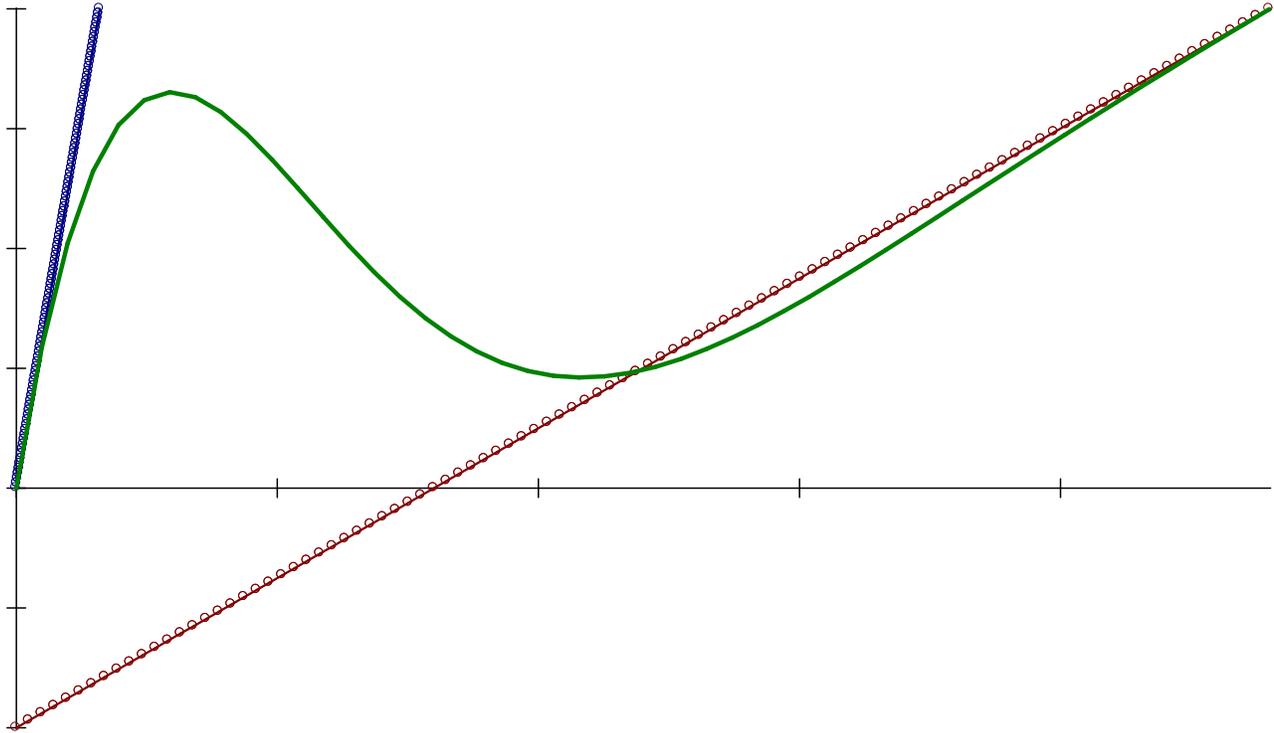
Acknowledgment:

I would like to acknowledge the help of Robin Hudson and some of my students attending the lecture course on Modern Quantum Theory who were the first who read and commented on these notes containing the answers on some of their questions. The best source on history and drama of quantum theory is in the biographies of the great inventors, Schrödinger, Bohr and Heisenberg [6, 49, 50], and on the conceptual development of this theory before the rise of quantum probability – in [48]. An excellent essay “The quantum age begins”, as well as short biographies with posters and famous quotations of all mathematicians and physicists mentioned here can be found on mathematics website at St Andrews University – <http://www-history.mcs.st-and.ac.uk/history/>, the use of which is acknowledged.

APPENDIX: The continuous collapse of a free quantum particle under the stationary nondemolition observation.

The posterior position expectation $q(t)$ in the absence of effective gravitation, $y''(t) = 0$, collapses to the registered linear trajectory $y(t) = ut - q$ with the rate $\kappa = (\lambda\hbar/2m)^{1/2}$, remaining not collapsed, $q_0(t) = v_0t$ in the framework where $q_0 = 0$, only in the classical limit $\hbar/m \rightarrow 0$ or absence of observation $\lambda = 0$:

$$q_0(t) = v_0t, \quad q(t) = ut + e^{-\kappa t} (q \cos \kappa t + (q + \kappa^{-1}(v_0 - u)) \sin \kappa t) - q.$$



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