Measurements continuous in time and a posteriori states in quantum mechanics

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Abstract

Measurements continuous in time were consistently introduced in quantum mechanics and applications worked out, mainly in quantum optics. In this context a quantum filtering theory has been developed giving the reduced state after the measurement when a certain trajectory of the measured observables is registered (the a posteriori states). In this paper a new derivation of filtering equations is presented, in the cases of counting processes and of measurement processes of diffusive type. It is also shown that the equation for the a posteriori dynamics in the diffusive case can be obtained, by a suitable limit, from that one in the counting case. Moreover, the paper is intended to clarify the meaning of the various concepts involved and to discuss the connections among them. As an illustration of the theory, simple models are worked out.

Introduction

Usually in quantum mechanics only instantaneous measurements are considered, but by using the notion of *instrument* [1]–[3] also measurements

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continuous in time were consistently introduced [2, 4]–[15] and applications worked out [2, 7, 9, 12, 16]–[19]; see also [20, 23].

Now a natural question is: if during a continuous measurement a certain trajectory of the measured observable is registered, what is the state of the system soon after, conditioned upon this information (the "a posteriori" state)? By using ideas from the classical filtering theory for stochastic processes and the formulation of continuous measurements in terms of quantum stochastic differential equations [13, 14, 17, 22, 24], a stochastic equation for the a posteriori states has been obtained [25]–[28]. The main purpose of this paper is indeed that of clarifying the meaning of that equation by presenting a natural derivation of it in terms of instruments, independently from any notion related to quantum stochastic calculus, and by discussing some models. Moreover, we shall discuss the connections among various things appeared in the literature about what can be called a quantum version of the theory of stochastic processes (continuous measurements) and filtering theory (a posteriori states).

Let us start by recalling the important notions of instrument and of a posteriori states. The notion of instrument has been introduced in the operational approach to quantum mechanics [1]. Let a quantum system be described in a separable Hilbert space $\mathcal{H}$ and denote by $\mathcal{B}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ the Banach spaces of the bounded operators on $\mathcal{H}$ and the trace–class operators, respectively. Let (Ω, Σ) be a measurable space (Ω a set and Σ a σ-algebra of subsets of Ω). An *instrument* [1]–[3] $\mathcal{I}$ is a map from Σ into the space of the *linear* bounded operators on $\mathcal{T}(\mathcal{H})$ such that (i) $\mathcal{I}(B)$ is completely positive [29] for any $B \in \Sigma$, (ii) $\sum_j \mathcal{I}(B_j)\varrho = \mathcal{I}\big(\bigcup_j B_j\big)\varrho$ for any sequence of pairwise disjoint elements of Σ and any ϱ in $\mathcal{T}(\mathcal{H})$ (convergence in trace norm), (iii) $\mathrm{Tr}\{\mathcal{I}(\Omega)\varrho\} = \mathrm{Tr}\{\varrho\}, \quad \forall \varrho \in \mathcal{T}(\mathcal{H})$.

The instrument $\mathcal{I}$ is an operator–valued measure: (i) is the positivity condition, (ii) is σ-additivity, (iii) is normalization. The instruments represent measurement procedures and their interpretation is as follows. Ω is the set of all possible outcomes of the measurement $\big((\Omega, \Sigma)$ is called the value space$\big)$ and the probability of obtaining the result $\omega \in B$ ($B \in \Sigma$), when before the measurement the system is in a state ϱ $\big(\varrho \in \mathcal{T}(\mathcal{H}),\ \varrho \geq 0,\ \mathrm{Tr}\{\varrho\} = 1\big)$, is given by $P(B|\varrho) := \mathrm{Tr}\{\mathcal{I}(B)\varrho\}$. Moreover, let us consider a sequence of measurements represented by the instruments $\mathcal{I}_1$, $\mathcal{I}_2$, ..., $\mathcal{I}_n$ and performed in the natural order ($\mathcal{I}_2$ after $\mathcal{I}_1$ and so on). We assume any time specification to be included in the definition of the instruments (Heisenberg picture). Then, the joint probability of the sequence of results $\omega_1 \in B_1$, $\omega_2 \in B_2$, ...,

$\omega_n \in B_n$, when the *premeasurement state* is ϱ, is given by

$$P(B_1, B_2, \ldots, B_n|\varrho) = \mathrm{Tr}\left\{\mathcal{I}_n(B_n) \circ \mathcal{I}_{n-1}(B_{n-1}) \circ \cdots \circ \mathcal{I}_1(B_1)\varrho\right\} . \qquad (0.1)$$

If we consider the conditional probability of the results $\omega_2 \in B_2$, ..., $\omega_n \in B_n$ given the first result $\omega_1 \in B_1$, we can write

$$\begin{aligned} P(B_2, \ldots, B_n|B_1; \varrho) &\equiv \frac{P(B_1, B_2, \ldots, B_n|\varrho)}{P(B_1|\varrho)} = \\ = P\big(B_2, \ldots, B_n|\varrho(B_1)\big) &\equiv \mathrm{Tr}\left\{\mathcal{I}_n(B_n) \circ \cdots \circ \mathcal{I}_2(B_2)\varrho(B_1)\right\} , \qquad (0.2) \end{aligned}$$

where we have introduced the statistical operator $\varrho(B_1)$ representing the state after the first measurement, conditioned upon the result $\omega_1 \in B_1$. For a generic instrument $\mathcal{I}$ and set B, the conditioned state $\varrho(B)$ is defined by

$$\varrho(B) = \frac{\mathcal{I}(B)\varrho}{\mathrm{Tr}\{\mathcal{I}(B)\varrho\}} \equiv \frac{\mathcal{I}(B)\varrho}{P(B|\varrho)} . \qquad (0.3)$$

Let us note that joint probabilities (0.1) preserve mixtures, by the linearity of the instruments: for ϱ and σ statistical operators and $0 \leq \lambda \leq 1$, we have

$$\lambda P(B_1, \ldots, B_n|\varrho) + (1-\lambda)P(B_1, \ldots, B_n|\sigma) = P\big(B_1, \ldots, B_n\big|\lambda\varrho + (1-\lambda)\sigma\big) . \qquad (0.4)$$

However, this property is not shared by conditional probabilities (0.2), by the definition itself of conditioning, and, therefore, the expression (0.3) for the conditioned state is not linear in the premeasurement state ϱ, unless $B = \Omega$.

Formula (0.3) can be interpreted by saying that we perform some measurement on a statistical ensemble of systems and select those systems for which the result $\omega \in B$ has been found. Then, (0.3) is the state after the measurement of the systems selected in this way and depends not only on the result $\omega \in B$, but also on the perturbations due to the concrete measuring procedure and to the dynamics. If we perform the measurement, but no selection, we obtain by (iii) $\varrho(\Omega) = \mathcal{I}(\Omega)\varrho$. By the definition of instrument, this quantity is linear in ϱ and it is a statistical operator if ϱ is a state. We can call $\varrho(\Omega)$ the *a priori state*: if we know the premeasurement state ϱ and the measurement $\mathcal{I}$, $\varrho(\Omega)$ is the state we can "a priori" attribute to our systems, before knowing the result of the measurement.

Let us consider now the case of the most fine selection when in (0.3) the set B shrinks to an "infinitesimally small" set $\mathrm{d}\omega$ around the value ω.

According to the discussion above, the quantity

$$\varrho(\omega) = \frac{\mathcal{I}(d\omega)\varrho}{\mathrm{Tr}\{\mathcal{I}(d\omega)\varrho\}} \tag{0.5}$$

represents the state conditioned upon the result $\omega \in \mathrm{d}\omega$. The quantity $\varrho(\omega)$ is the state one can attribute to those systems for which the result ω has actually been found in the measurement and for this reason we call it a posteriori state [23].

More precisely, a family of statistical operators $\{\varrho(\omega), \omega \in \Omega\}$ is said to be a family of *a posteriori states* [30], for an initial state ϱ and an instrument $\mathcal{I}$ with value space (Ω, Σ), if (a) the function $\omega \to \varrho(\omega)$ is strongly measurable with respect to the probability measure

$$\mu_\varrho(B) := \mathrm{Tr}\{\mathcal{I}(B)\varrho\} \equiv P(B|\varrho) \tag{0.6}$$

for the observable associated with the instrument $\mathcal{I}$ and (b) $\forall Y \in \mathcal{B}(\mathcal{H})$, $\forall B \in \Sigma$,

$$\int_B \mathrm{Tr}\{Y\varrho(\omega)\}\mu_\varrho(\mathrm{d}\omega) = \mathrm{Tr}\{Y\,\mathcal{I}(B)\varrho\}\,. \tag{0.7}$$

Let us note that by definition the link between a priori and a posteriori states is given by

$$\varrho(\Omega) \equiv \mathcal{I}(\Omega)\varrho = \int_\Omega \varrho(\omega)\,\mu_\varrho(\mathrm{d}\omega)\,. \tag{0.8}$$

Let us stress that (0.7) defines the a posteriori states once the instrument $\mathcal{I}$ and the *premeasurement state* ϱ are given. On the contrary, if $\varrho(\omega)$ and μ_ϱ are given for any ϱ, (0.7) allows to reconstruct the instrument $\mathcal{I}$. We shall make use of this in the following sections.

Finally, let us note that there is no reason for $\varrho(\omega)$ to be a pure state if ϱ is pure: it depends on the concrete measuring procedure. Roughly speaking $\varrho(\omega)$ is pure if one has some property of minimal disturbance, some ability of the measurement to give a maximum of information, ...; we shall see various examples ($\varrho(\omega)$ pure and not pure) in the case of continuous measurements.

1 Counting processes

The first class of continuous measurements which has been introduced in quantum mechanics is that of counting processes [2, 4]–[9, 13, 17, 18, 28, 31]–

[34]. One or more counters act continuously on the system and register the times of arrival of photons or other kinds of particles.

Let us consider the case of d counters. They differ by their localization and/or by the type of particles to which they are sensible and/or by their operating way... We can describe this counting process by giving the so called *exclusive probability densities* (EPDs) [7, 9]. The quantity $P_{t_0}^t(0|\varrho)$ is the probability of having no count in the time interval $(t_0, t]$, when the system is prepared in the state ϱ at time t_0. The quantity $p_{t_0}^t(j_1, t_1; j_2, t_2; \ldots ; j_m, t_m|\varrho)$, $j_k = 1, \ldots, d$, $t_0 < t_1 < t_2 < \cdots < t_m \leq t$, is the multi–time probability density of having a count of type j_1 at time t_1, a count of type j_2 at time t_2, ..., and no other count in the rest of the interval $(t_0, t]$. Davies [4] (see also [5]–[9]) has shown that these EPDs can be consistently described in quantum mechanics in the following way.

Let $\mathcal{L}_0(t)$ be a Liouvillian $\big($the generator of a completely positive dynamics [29] on $\mathcal{T}(\mathcal{H})\big)$ and $\mathcal{J}_j(t)$, $j = 1, \ldots, d$, be completely positive maps on $\mathcal{T}(\mathcal{H})$. Let us introduce the positive operators $R_j(t)$ on $\mathcal{H}$ by

$$R_j(t) := \mathcal{J}_j(t)'\mathbf{1}\,. \tag{1.1}$$

For any operation $\mathcal{A}$ on $\mathcal{T}(\mathcal{H})$ its adjoint $\mathcal{A}'$ on $\mathcal{B}(\mathcal{H})$ is defined by

$$\mathrm{Tr}\{X\,\mathcal{A}\varrho\} = \mathrm{Tr}\{\varrho\,\mathcal{A}'X\}\,, \qquad \forall\varrho \in \mathcal{T}(\mathcal{H})\,, \quad \forall X \in \mathcal{B}(\mathcal{H})\,. \tag{1.2}$$

Finally, let $\mathcal{S}(t, t_0)$, $t \geq t_0$, be the family of completely positive maps on $\mathcal{T}(\mathcal{H})$ defined by the equations

$$\frac{\partial}{\partial t}\mathcal{S}(t, t_0) = \mathcal{A}(t)\mathcal{S}(t, t_0)\,, \qquad \mathcal{S}(t_0, t_0) = \mathrm{Id}\,, \tag{1.3}$$

$$\mathcal{A}(t)\varrho = \mathcal{L}_0(t)\varrho - \frac{1}{2}\sum_{j=1}^{d}\{R_j(t), \varrho\}\,. \tag{1.4}$$

Here $\{a, b\} = ab + ba$ and Id is the identity map on $\mathcal{T}(\mathcal{H})$. Then, the quantities

$$P_{t_0}^t(0|\varrho) = \mathrm{Tr}\{\mathcal{S}(t, t_0)\varrho\}\,, \tag{1.5}$$

$$\begin{aligned} p_{t_0}^t(j_1, t_1; j_2, t_2; \ldots ; j_m, t_m|\varrho) = \mathrm{Tr}\Big\{&\mathcal{S}(t, t_m)\mathcal{J}_{j_m}(t_m) \\ &\times\mathcal{S}(t_m, t_{m-1})\mathcal{J}_{j_{m-1}}(t_{m-1}) \cdots \mathcal{S}(t_2, t_1)\mathcal{J}_{j_1}(t_1)\mathcal{S}(t_1, t_0)\varrho\Big\} \end{aligned} \tag{1.6}$$

(where $t_0 < t_1 < t_2 < \cdots < t_m \leq t$, $j_k = 1, \ldots, d$) are a consistent family of EPDs.

The whole statistics of the counts can be reconstructed from the EPDs. For instance, the probability of m counts of type j in the time interval $(t_0, t_1]$, n counts of type i in the interval $(t_1, t_2]$ is given by

$$
\begin{aligned}
&P\big(m, j, (t_0, t_1]; n, i, (t_1, t_2] | \varrho\big) = \\
&\quad = \int_{t_1}^{t_2} \mathrm{d}r_n \int_{t_1}^{r_n} \mathrm{d}r_{n-1} \cdots \int_{t_1}^{r_2} \mathrm{d}r_1 \int_{t_0}^{t_1} \underset{.}{\mathrm{s}}_m \int_{t_0}^{s_m} \mathrm{d}s_{m-1} \cdots \\
&\quad \cdots \int_{t_0}^{s_2} \mathrm{d}s_1 \; p_{t_0}^{t_2}\big(j, s_1; j, s_2; \ldots; j, s_m; i, r_1; i, r_2; \ldots; i, r_n | \varrho\big) . \qquad (1.7)
\end{aligned}
$$

In a similar way all more complicated joint probabilities can be constructed.

One of the most significant problems treated by this theory is that of the electron shelving effect or quantum jumps. An atom with a peculiar level configuration and suitably stimulated by laser light emits a pulsed fluorescence light with random bright and dark periods. It is possible to use $\mathcal{L}_0(t)$ for describing the free atom and the driving term due to the laser and to use the operators $\mathcal{J}_j(t)$ for describing the emission process. Then, the full statistics of the fluorescence light can be computed and, in particular, the mean duration of the dark periods [17]–[19]. Other applications to quantum optics of the counting theory described here are given in [35]–[37].

Let us now consider the problem of the a posteriori states. Our counting process can be considered as a stochastic process whose associated probability measure (uniquely determined by (1.5) and (1.6)) is concentrated on step functions. Let us consider $t_0 = 0$ as initial time. A typical trajectory ω_t up to time t is specified by giving the sequence $(j_1, t_1; j_2, t_2; \ldots; j_n, t_n)$ of types of counts and instants of counts $t_1 < t_2 < \cdots < t_n$ up to time t. Let ω_t be the trajectory we have registered up to time t. Then, the conditional probability $P\big(0, (t, t+\bar{t}] | \omega_t; \varrho\big)$ of no count in the interval $(t, t+\bar{t}]$, given the state ϱ at time zero and the trajectory ω_t, is given by

$$
P\big(0, (t, t+\bar{t}] | \omega_t; \varrho\big) = \frac{p_0^{t+\bar{t}}(j_1, t_1; \ldots; j_n, t_n | \varrho)}{p_0^t(j_1, t_1; \ldots; j_n, t_n | \varrho)} \qquad (1.8)
$$

(cf. (0.2)). By (1.3), (1.5), (1.6), we obtain immediately that the probability (1.8) can be rewritten as

$$
P\big(0, (t, t+\bar{t}] | \omega_t; \varrho\big) = \mathrm{Tr}\{\mathcal{S}(t+\bar{t}, t)\varrho(t)\} = P_t^{t+\bar{t}}\big(0 | \varrho(t)\big) , \qquad (1.9)
$$

$$\varrho(t) = \frac{1}{C(t)}\,\mathcal{S}(t,t_n)\mathcal{J}_{j_n}(t_n)\mathcal{S}(t_n,t_{n-1})\cdots\mathcal{J}_{j_1}(t_1)\mathcal{S}(t_1,0)\varrho\,, \tag{1.10}$$

where $C(t)$ is the normalization factor determined by $\mathrm{Tr}\{\varrho(t)\} = 1$ (cf. (0.2), (0.3) and (0.5)). Similar results hold for the other EPDs conditioned upon some trajectory up to time t. Therefore, all conditional probabilities can be computed by (1.5) and (1.6) if one uses as initial state the expression (1.10). Equation (1.10) gives the state of the system at time t conditioned upon the trajectory ω_t up to time t (the a posteriori state).

The interpretation of (1.10) is that, when no count is registered, the system evolution is given by $\mathcal{S}(t,t_0)$ and that the action of the counter on the system, at the time t in which a count of type j is registered, is described by the map $\mathcal{J}_j(t)$. However, $\mathcal{S}(t,t_0)$ and $\mathcal{J}_j(t)$ do not preserve normalization and the normalization factor $C(t)$ is needed. This is due to the fact that they are the probabilities (1.5) and (1.6) which have to be correctly normalized and this is guaranteed by equations (1.1) and (1.4), connecting $\mathcal{J}_j(t)$ with $\mathcal{S}(t,t_0)$. According to this interpretation of (1.10), the state of the system in between two counts is

$$\varrho(t) = \frac{\mathcal{S}(t,t_r)\varrho(t_r)}{\mathrm{Tr}\{\mathcal{S}(t,t_r)\varrho(t_r)\}}\,, \tag{1.11}$$

where t_r is the time of the last count and $\varrho(t_r)$ the state just after this count. If we denote $\mathrm{Tr}\{R_j(t)\varrho(t)\}$ by $\langle R_j(t)\rangle_t$ and differentiate (1.11), we obtain

$$\frac{\mathrm{d}\varrho(t)}{\mathrm{d}t} = \mathcal{L}_0(t)\varrho(t) - \frac{1}{2}\sum_{j=1}^{d}\left\{R_j(t) - \langle R_j(t)\rangle_t, \varrho(t)\right\}. \tag{1.12}$$

Moreover, if at time t_r we have a count of type j, the state of the system soon after is

$$\varrho(t_r{+}\mathrm{d}t) = \frac{\mathcal{J}_j(t_r)\varrho(t_r)}{\mathrm{Tr}\{\mathcal{J}_j(t_r)\varrho(t_r)\}} = \frac{\mathcal{J}_j(t_r)\varrho(t_r)}{\langle R_j(t_r)\rangle_{t_r}}\,. \tag{1.13}$$

Now the typical trajectory $N_j(t)$ (number of counts of type j up to time t), $j = 1,\ldots,d$, of our stochastic process is a step function such that $N_j(t)$ increases by 1 if soon after time t there is a count of type j, otherwise $N_j(t)$ is constant. Therefore, the Itô differential

$$\mathrm{d}N_j(t) = N_j(t{+}\mathrm{d}t) - N_j(t) \tag{1.14}$$

is equal to one if at time t there is a count of type j and to zero otherwise. This gives $\big(\mathrm{d}N_j(t)\big)^2 = \mathrm{d}N_j(t)$. Moreover, the probability of more than one count in an interval $\mathrm{d}t$ vanishes more rapidly than $\mathrm{d}t$, i.e. between $\mathrm{d}N_j(t)$ and $\mathrm{d}N_i(t)$, $i \neq j$, at least one of the two must be zero. Moreover, $\mathrm{d}N_j(t)\,\mathrm{d}t$ is of higher order than $\mathrm{d}t$ and has to be taken vanishing. Summarizing, we have the Itô table

$$\mathrm{d}N_j(t)\,\mathrm{d}N_i(t) = \delta_{ij}\,\mathrm{d}N_j(t)\,, \qquad \mathrm{d}N_j(t)\,\mathrm{d}t = 0\,. \tag{1.15}$$

By using these results we can rewrite (1.12) and (1.13) in the form of a single stochastic differential equation in Itô sense $\big(\,\mathrm{d}\varrho(t) = \varrho(t{+}\mathrm{d}t) - \varrho(t)\,\big)$:

$$\mathrm{d}\varrho(t) = \mathcal{L}(t)\varrho(t)\,\mathrm{d}t + \sum_{j=1}^{d}\left(\frac{\mathcal{J}_j(t)\varrho(t)}{\langle R_j(t)\rangle_t} - \varrho(t)\right)\Big(\mathrm{d}N_j(t) - \langle R_j(t)\rangle_t\,\mathrm{d}t\Big)\,, \tag{1.16}$$

$$\mathcal{L}(t)\varrho = \mathcal{L}_0(t)\varrho + \sum_{j=1}^{d}\Big(\mathcal{J}_j(t)\varrho - \frac{1}{2}\left\{R_j(t),\varrho\right\}\Big)\,, \tag{1.17}$$

$$\langle R_j(t)\rangle_t = \mathrm{Tr}\big\{R_j(t)\varrho(t)\big\} = \mathrm{Tr}\big\{\mathcal{J}_j(t)\varrho(t)\big\}\,. \tag{1.18}$$

Indeed, when all $\mathrm{d}N_j(t)$ vanish, (1.16) reduces to (1.12); when one of the $\mathrm{d}N_j(t)$ is equal to one all the other terms in the r.h.s. of (1.16) are negligible and we obtain (1.13). Equation (1.16) was firstly obtained by quantum stochastic calculus methods in [25, 28, 38, 39].

Formula (1.16) is the equation for the a posteriori states in the case of a counting measurement: it determines the state *at time t* depending on the (stochastic) trajectory *up to time t*. Let us stress that we know the solution of this equation: it is the state (1.10). In any case, it is very useful to have the differential stochastic equation (1.16) as we shall see in the rest of this section and in section 3.

Let $\langle \mathrm{d}N_j(t)\rangle(\omega_t)$ be the mean number of counts of type j in the interval $(t, t{+}\mathrm{d}t]$ conditioned upon the trajectory ω_t up to time t. Because probabilities of more that one count in a small interval are negligible, we have

$$\langle \mathrm{d}N_j(t)\rangle(\omega_t) \simeq p_t^{t+\mathrm{d}t}\big(j,t|\varrho(t)\big)\,\mathrm{d}t \simeq \mathrm{Tr}\big\{\mathcal{J}_j(t)\varrho(t)\big\}\,\mathrm{d}t = \langle R_j(t)\rangle_t\,\mathrm{d}t\,. \tag{1.19}$$

In other words the quantities $\langle R_j(t)\rangle_t\,\mathrm{d}t$ appearing in (1.16) are the *a posteriori mean values* of $\mathrm{d}N_j(t)$. Moreover, the differentials

$$\mathrm{d}M_j(t) = \mathrm{d}N_j(t) - \langle R_j(t)\rangle_t\,\mathrm{d}t\,, \tag{1.20}$$

appearing in (1.16), together with the initial condition $M_j(0)=0$, define the a posteriori centered processes $M_j(t)$, called *innovating martingales*.

Equation (1.16) is non–linear, but it is mathematically equivalent to a linear one. Let us introduce an arbitrary stochastic real factor $c(t)$ and define the trace–class operator $\varphi(t) := c(t)\varrho(t)$. If we know $\varphi(t)$ we can reobtain the state $\varrho(t)$ simply by normalization. The factor $c(t)$ can be chosen in such a way that $\varphi(t)$ obeys a linear stochastic differential equation; moreover, this choice is not unique. We shall do this in a very convenient way: a new linear stochastic equation is obtained giving both the a posteriori state $\varrho(t)$ and the EPDs (1.5) and (1.6) (cf. [39]). Let $\varphi(t)$ be a trace–class operator depending on the trajectory ω_t and defined by $\varphi(t)=\mathcal{S}(t,t_r)\varphi(t_r)$ if t_r is the time of the last count and by $\varphi(t_r{+}\mathrm{d}t)=\tau\mathcal{J}_j(t_r)\varphi(t_r)$ if at time t_r there is a count of type j; τ is an arbitrary parameter with dimensions of a time, which disappears from the physical quantities. For initial condition we take $\varphi(0)=\varrho$.

By the definition of $\varphi(t)$, the quantity

$$c(t)=\mathrm{Tr}\{\varphi(t)\} \tag{1.21}$$

gives the EPDs (1.5) and (1.6): in the case of a trajectory ω_t containing no jump we have

$$c(t)=P_0^t(0|\varrho) \tag{1.22}$$

and in the case of a trajectory with a jump of type j_1 at time t_1, ..., of type j_m at time t_m we have

$$c(t)=\tau^m\,p_0^t(j_1,t_1;\ldots;j_m,t_m|\varrho)\,. \tag{1.23}$$

Moreover, by the definition of $\varphi(t)$, $c(t)$ and $\varrho(t)$ the a posteriori state is

$$\varrho(t)=\varphi(t)/c(t)\,. \tag{1.24}$$

In the same way as for $\varrho(t)$, we can obtain the stochastic differential equation for $\varphi(t)$, which turns out to be $\big($cf. [28], equation (**??**20)$\big)$

$$\mathrm{d}\varphi(t)=\Big[\mathcal{L}_0(t)\varphi(t)-\frac{1}{2}\sum_{j=1}^{d}\{R_j(t),\varphi(t)\}\Big]\mathrm{d}t+\sum_{j=1}^{d}\big[\tau\mathcal{J}_j(t)\varphi(t)-\varphi(t)\big]\mathrm{d}N_j(t)\,. \tag{1.25}$$

By using Itô's calculus for counting processes it is possible to verify that indeed (1.21), (1.24) and (1.25) are equivalent to (1.16). Equation (1.25) determines all the probabilities via (1.21)–(1.23) and the a posteriori states via (1.21) and (1.24). Equation (1.25) is linear, once a realization $N_j(t)$, $j = 1, \cdots, d$, of the process is given. However, let us note that the statistics of $N_j(t)$ depends in its turn on the premeasurement state ϱ, as shown by (1.5), (1.6), (1.19). The possibility of finding a linear equation mathematically equivalent to (1.16) means that $\varrho(t)$ is linear in ϱ up to a normalization factor, as suggested by (0.5).

Let us stress that in general equation (1.16) does not transform pure states into pure states. This simply means that in the course of time we loose information due to some dissipation mechanism, for instance the system interacts also with some external bath or similar things. In any case, the situation in which pure states are preserved is particularly interesting. This is the case [20, 21, 25, 26] when

$$\mathcal{L}_0(t)\varrho = -\mathrm{i}[H(t), \varrho]\,, \qquad \mathcal{J}_j(t) = Z_j(t)\varrho Z_j(t)^\dagger\,, \tag{1.26}$$

where $Z_j(t)$ and $H(t)$ are operators on $\mathcal{H}$, $H(t)^\dagger = H(t)$. Then, we have $R_j(t) = Z_j(t)^\dagger Z_j(t)$ and

$$\mathcal{L}(t)\varrho = -\mathrm{i}[H(t), \varrho] + \sum_{j=1}^d \left(Z_j(t)\varrho Z_j(t)^\dagger - \frac{1}{2}\left\{ Z_j(t)^\dagger Z_j(t), \varrho \right\} \right). \tag{1.27}$$

Then, (1.16) becomes

$$\begin{aligned} \mathrm{d}\varrho(t) \;&=\; -\mathrm{i}[H(t), \varrho(t)]\mathrm{d}t - \frac{1}{2}\sum_{j=1}^d \left\{ Z_j(t)^\dagger Z_j(t) - \langle Z_j(t)^\dagger Z_j(t)\rangle_t, \varrho(t) \right\} \mathrm{d}t + \\ &+\; \sum_{j=1}^d \left(\frac{Z_j(t)\varrho(t) Z_j(t)^\dagger}{\langle Z_j(t)^\dagger Z_j(t)\rangle_t} - \varrho(t) \right) \mathrm{d}N_j(t)\,. \end{aligned} \tag{1.28}$$

By using Itô formula (1.15), one cane prove that $\varrho(t+\mathrm{d}t)^2 = \varrho(t+\mathrm{d}t)$, if $\varrho(t)^2 = \varrho(t)$; therefore, (1.28) transforms pure states into pure states and it is equivalent to a stochastic differential equation for a wave function. Indeed, let $\psi(t) \in \mathcal{H}$ satisfy the "a posteriori Schrödinger equation" [28, 39]

$$\begin{aligned} \mathrm{d}\psi(t) \;&=\; \left[-\mathrm{i}H(t) - \frac{1}{2}\sum_{j=1}^d \left(Z_j(t)^\dagger Z_j(t) - \langle Z_j(t)^\dagger Z_j(t)\rangle_t \right) \right] \psi(t)\mathrm{d}t + \\ &+\; \sum_{j=1}^d \left(\frac{Z_j(t)}{\sqrt{\langle Z_j(t)^\dagger Z_j(t)\rangle_t}} - \mathbf{1} \right) \psi(t)\mathrm{d}N_j(t)\,. \end{aligned} \tag{1.29}$$

with $\langle Z_j(t)^\dagger Z_j(t)\rangle_t = \langle\psi(t)|Z_j(t)^\dagger Z_j(t)\,\psi(t)\rangle$; then, by (1.15) one obtains that $\varrho(t) = |\psi(t)\rangle\langle\psi(t)|$ satisfies (1.28).

It is interesting to note that in between two counts (when $\mathrm{d}N_j(t) = 0$) (1.29) becomes a nonlinear Schrödinger equation of the type studied, for instance, in [40, 41]. However, this equation has a quite different interpretation in the quoted references, where the problem is to find evolution equations compatible with the Hilbert space structure and preserving "properties" in the sense of quantum logic.

Now we have the a posteriori states defined by (1.16) and a probability measure on the trajectory space, which is implicitly defined by the EPDs (1.5) and (1.6). Therefore, we can reconstruct the instruments associated to our measurement by means of (0.7). As in [28, 39], we shall do this by using the notion of characteristic operator, a concept introduced in [10]–[13], and Itô formula for counting processes.

Let f be any function of the trajectories of our stochastic process and let us denote by $\langle f\rangle_{\mathrm{st}}$ the mean value of f with respect to the measure associated to the EPDs (1.5) and (1.6). The quantity

$$\Phi_t[\vec{k}] = \left\langle \exp\left\{ \mathrm{i}\sum_{j=1}^{d}\int_0^t k_j(s)\,\mathrm{d}N_j(s)\right\}\right\rangle_{\mathrm{st}} \tag{1.30}$$

is called the *characteristic functional* of the process. Here $\vec{k}(s)$ is a test function, i.e. $k_j(s)$ is a real compact support C^∞-function on $(0,+\infty)$. $\Phi_t[\vec{k}]$ determines uniquely the whole counting process up to time t: roughly speaking $\Phi_t[\vec{k}]$ is the Fourier transform of the probability measure of the process. More explicitly [17], we have

$$\Phi_t[\vec{k}] = P_0^t(0|\varrho) + \sum_{m=1}^{\infty}\sum_{\{j_k\}=1}^{\infty}\int_0^t \mathrm{d}t_m \int_0^{t_m}\mathrm{d}t_{m-1}\cdots$$
$$\cdots\int_0^{t_2}\mathrm{d}t_1\,\exp\left\{\mathrm{i}\sum_{l=1}^{m}k_{j_l}(t_l)\right\} p_0^t\big(j_1,t_1;j_2,t_2;\ldots;j_m,t_m|\varrho\big)\,. \tag{1.31}$$

Let us set now

$$V_t[\vec{k}] = \exp\left\{\mathrm{i}\sum_{j=1}^{d}\int_0^t k_j(s)\,\mathrm{d}N_j(s)\right\}. \tag{1.32}$$

According to (0.7), we can write

$$\langle V_t[\vec{k}]\varrho(t)\rangle_{\rm st} = \mathcal{G}_t[\vec{k}]\varrho\,, \tag{1.33}$$

where $\varrho(t)$ is the a posteriori state at time t and $\mathcal{G}_t[\vec{k}]$ is an operator on $\mathcal{T}(\mathcal{H})$ which represents the "functional Fourier transform" of the instrument $\mathcal{I}_t$ associated to our measurement up to time t. The quantity $\mathcal{G}_t[\vec{k}]$ can be called *characteristic operator* and it is the operator analogue of the characteristic functional of a stochastic process [11]–[13]. By the normalization of $\varrho(t)$ and (1.30) and (1.33), we obtain

$$\Phi_t[\vec{k}] = \mathrm{Tr}\big\{\mathcal{G}_t[\vec{k}]\varrho\big\}\,. \tag{1.34}$$

An equation for $\mathcal{G}_t[\vec{k}]$ can be found by differentiating (1.33). The differential of $\varrho(t)$ is given by (1.16), while the differential of $V_t[\vec{k}]$ is

$$\mathrm{d}V_t[\vec{k}] = V_t[\vec{k}]\left[\sum_{j=1}^{d}\left(\mathrm{e}^{\mathrm{i}k_j(t)}-1\right)\,\mathrm{d}N_j(t)\right]. \tag{1.35}$$

This formula can be easily obtained from (1.32) by expanding the exponential and using (1.15). By using the formula

$$\mathrm{d}\Big(V_t[\vec{k}]\varrho(t)\Big) = \Big(\mathrm{d}V_t[\vec{k}]\Big)\,\varrho(t) + V_t[\vec{k}]\big(\mathrm{d}\varrho(t)\big) + \Big(\mathrm{d}V_t[\vec{k}]\Big)\,\big(\mathrm{d}\varrho(t)\big)\,, \tag{1.36}$$

where the *Itô correction* $\Big(\mathrm{d}V_t[\vec{k}]\Big)\,(\mathrm{d}\varrho(t))$ has to be computed by means of the Itô table (1.15), we obtain

$$\begin{aligned}\mathrm{d}\Big(V_t[\vec{k}]\varrho(t)\Big) \;=\;& V_t[\vec{k}]\Bigg\{\mathcal{L}(t)\varrho(t)\,\mathrm{d}t + \sum_{j=1}^{d}\left(\mathrm{e}^{\mathrm{i}k_j(t)}-1\right)\mathcal{J}_j(t)\varrho(t)\,\mathrm{d}t\,+\\ &+\sum_{j=1}^{d}\left[\mathrm{e}^{\mathrm{i}k_j(t)}\,\frac{\mathcal{J}_j(t)\varrho(t)}{\langle R_j(t)\rangle_t}-\varrho(t)\right]\Big(\mathrm{d}N_j(t)-\langle R_j(t)\rangle_t\,\mathrm{d}t\Big)\Bigg\}\end{aligned} \tag{1.37}$$

Now let us take the stochastic mean of (1.37). We compute this mean in the following way. First we take the mean with respect to the probability measure on the future (with respect to t) conditioned upon the given trajectory. All the quantities in the r.h.s. of (1.37) depend only on the past (they are *adapted*), but the quantity $\mathrm{d}N_j(t)$, whose a posteriori mean value is just $\langle R_j(t)\rangle_t\,\mathrm{d}t$ (equation (1.19)). Therefore, the last term in (1.37) vanishes. Then, we take the mean value also on the past and, by (1.33), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_t[\vec{k}] = \mathcal{K}_t\big(\vec{k}(t)\big)\mathcal{G}_t[\vec{k}]\,, \tag{1.38}$$

$$\mathcal{K}_t\big(\vec{k}(t)\big) = \mathcal{L}(t) + \sum_{j=1}^{d}\left(\mathrm{e}^{\mathrm{i}k_j(t)} - 1\right)\mathcal{J}_j(t)\,. \tag{1.39}$$

Together with the initial condition

$$\mathcal{G}_0[\vec{k}] = \mathrm{Id} \tag{1.40}$$

(which follows from the definition (1.33)), equation (1.38) determines uniquely $\mathcal{G}_t[\vec{k}]$ and implicitly the instruments on the trajectory space. This kind of equations has been obtained for the first time in [13].

If no selection is made according to the results of the measurement (let us say: the results are not read), the state of the system at time t will be (cf. (0.8))

$$\sigma(t) = \langle\varrho(t)\rangle_{\mathrm{st}}\,; \tag{1.41}$$

$\sigma(t)$ is the *a priori state* for the case of the continuous measurement described in this section. According to (1.32), (1.33), (1.38), (1.39), we have that the a priori states satisfy the *quantum master equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma(t) = \mathcal{L}(t)\sigma(t)\,, \tag{1.42}$$

with the new Liouvillian (1.17): the unperturbed Liouvillian $\mathcal{L}_0(t)$ corrected by the measurement effect term $\sum_{j=1}^{d}\left(\mathcal{J}_j(t)\varrho - \frac{1}{2}\left\{R_j(t),\varrho\right\}\right)$. The fact that we have obtained a linear equation for the a priori states is due to linearity and normalization of the instruments (cf. (0.8)).

2 An example of counting process: a two–level atom

Let us consider an example of counting measurement on the simplest quantum system: a two–state system, described in the Hilbert space $\mathcal{H} = \mathbf{C}^2$.

We can think of a two–level atom, an unstable particle, a spin... While the general case could be handled, for concreteness we treat a two–level atom with pumping and damping. This section has to be considered just as an illustration of the theory developed before.

The (time independent) unperturbed Liouvillian is given by

$$\mathcal{L}_0\varrho = -\frac{\mathrm{i}}{2}\,\omega[\sigma_3,\varrho] + \mathcal{J}_0\varrho - \frac{1}{2}\,\{R_0,\varrho\}\,, \tag{2.1}$$

$$\mathcal{J}_0\varrho = \lambda_+\,\sigma_+\varrho\sigma_- + \lambda_-\,\sigma_-\varrho\sigma_+\,, \tag{2.2}$$

$$R_0 = \mathcal{J}_0'\sigma_0 = \frac{1}{2}\,\lambda_+(\sigma_0-\sigma_3) + \frac{1}{2}\,\lambda_-(\sigma_0+\sigma_3)\,. \tag{2.3}$$

Here $\omega > 0$, $\lambda_\pm \geq 0$, σ_i, $i = 1,2,3$, are the Pauli matrices, σ_0 is the 2×2 identity matrix and $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)$.

We consider a single counter ($d = 1$) and take

$$\mathcal{J}_1\varrho = \lambda_1\,\sigma_-\varrho\sigma_+\,, \qquad \lambda_1 > 0\,; \tag{2.4}$$

the map $\mathcal{J}_1$ describes the emission of photons (or other types of particles), which are then counted by some electronic device. In the present case, the rate operator (1.1) is

$$R_1 = \mathcal{J}_1'\sigma_0 = \lambda_1\,\sigma_+\sigma_- \equiv \frac{1}{2}\,\lambda_1(\sigma_0+\sigma_3) \tag{2.5}$$

and the generator $\mathcal{L}$ (1.17) of the full dynamics is

$$\mathcal{L}\varrho = -\frac{\mathrm{i}}{2}\,\omega[\sigma_3,\varrho] + \sum_{j=0}^{1}\left(\mathcal{J}_j\varrho - \frac{1}{2}\,\{R_j,\varrho\}\right)\,. \tag{2.6}$$

We can interpret the terms with λ_+ as pumping, the terms with λ_- as incoherent damping and the terms with λ_1 as electromagnetic decay; $\Gamma = \lambda_1$ is the electromagnetic transition rate. If $\lambda_+ = 0$, we can interpret the system as a Wigner atom (or another unstable particle). In this case the electromagnetic transition rate is $\Gamma = \lambda_- + \lambda_1$; $\lambda_- \neq 0$ means that not all the photons are collected by the photocounter; $\varepsilon = \lambda_1/(\lambda_- + \lambda_1)$ is the efficiency of the counter [17].

In order to perform computations, it is convenient to represent selfadjoint trace–class operators φ as

$$\varphi = \frac{1}{2}\left(c\sigma_0 + \zeta\sigma_+ + \zeta^*\sigma_- + \xi\sigma_3\right)\,, \qquad c,\xi\in\mathbb{R}\,, \quad \zeta\in\mathbf{C}\,. \tag{2.7}$$

The operator φ is positive if $c \geq \left(\xi^2 + |\zeta|^2\right)^{1/2}$ and it is a density matrix if also $c = 1$.

Let us consider (1.25) and represent $\varphi(t)$ in the form (2.7) with $c \to c(t)$, $\zeta \to \zeta(t)$, $\xi \to \xi(t)$. The stochastic equation (1.25), choosing $\tau = \lambda_1^{-1}$, becomes

$$\mathrm{d}c(t) + \frac{1}{2}\,\lambda_1[c(t) + \xi(t)]\mathrm{d}t = \frac{1}{2}\,[\xi(t) - c(t)]\mathrm{d}N(t)\,, \tag{2.8}$$

$$\mathrm{d}\xi(t) + \left[\left(2\kappa - \frac{1}{2}\,\lambda_1\right)\xi(t) + \left(\alpha + \frac{1}{2}\,\lambda_1\right)c(t)\right]\mathrm{d}t = -\frac{1}{2}\,[c(t) + 3\xi(t)]\mathrm{d}N(t)\,, \tag{2.9}$$

$$\mathrm{d}\zeta(t) + (\mathrm{i}\omega + \kappa)\zeta(t)\,\mathrm{d}t = -\zeta(t)\,\mathrm{d}N(t)\,, \tag{2.10}$$

where $\kappa = \frac{1}{2}(\lambda_+ + \lambda_- + \lambda_1)$, $\alpha = \lambda_- - \lambda_+$. It is convenient to rewrite (2.8) and (2.9) in terms of the stochastic parameters

$$\pi_0(t) = \frac{1}{2}\left(c(t) - \xi(t)\right), \qquad \pi_1(t) = \frac{1}{2}\left(c(t) + \xi(t)\right); \tag{2.11}$$

this gives

$$\begin{aligned} \mathrm{d}\pi_0(t) + [\mu_\uparrow\pi_0(t) - \kappa_\downarrow\pi_1(t)]\mathrm{d}t &= [\pi_1(t) - \pi_0(t)]\mathrm{d}N(t)\,, \\ \mathrm{d}\pi_1(t) + [\mu_\downarrow\pi_1(t) - \kappa_\uparrow\pi_0(t)]\mathrm{d}t &= -\pi_1(t)\mathrm{d}N(t)\,, \end{aligned} \tag{2.12}$$

where $\mu_\uparrow = \kappa_\uparrow = \lambda_+$, $\kappa_\downarrow = \lambda_-$, $\mu_\downarrow = \lambda_1 + \lambda_-$.

The solution of (2.10) is very simple:

$$\zeta(t) = \begin{cases} \mathrm{e}^{-(\mathrm{i}\omega+\kappa)t}\zeta(0), & \text{if } t \leq t_1, \\ 0, & \text{if } t > t_1, \end{cases} \tag{2.13}$$

where t_1 is the instant of the first jump of $N(t)$. About (2.12), let us denote by $\pi_j(t|a,b)$ the solution of (2.12) with $\mathrm{d}N(t) = 0$ and initial conditions $\pi_0(0) = a$, $\pi_1(0) = b$. Then, the solution of the stochastic system (2.12) is

$$\pi_j(t) = \begin{cases} \pi_j\big(t|\pi_0(0), \pi_1(0)\big), & \text{if } t \leq t_1, \\ \pi_j\big(t - t_r|\pi_1(t_r), 0\big), & \text{if } t_r < t \leq t_{r+1},\ r \geq 1, \end{cases} \tag{2.14}$$

where t_r are the instants of the jumps of $N(t)$.

By (1.21), (1.24), (2.7), the matrix elements of the a posteriori state $\varrho(t)$ are given by

$$\begin{aligned}
\langle 1|\varrho(t)|1\rangle &\equiv \operatorname{Tr}\Big\{\frac{1}{2}\,(\sigma_0+\sigma_3)\varrho(t)\Big\} = \pi_1(t)/c(t)\,, \\
\langle 0|\varrho(t)|0\rangle &\equiv \operatorname{Tr}\Big\{\frac{1}{2}\,(\sigma_0-\sigma_3)\varrho(t)\Big\} = \pi_0(t)/c(t)\,, \\
\langle 1|\varrho(t)|0\rangle &\equiv \operatorname{Tr}\big\{\sigma_-\varrho(t)\big\} = \zeta(t)/[2c(t)]\,, \\
\langle 0|\varrho(t)|1\rangle &\equiv \operatorname{Tr}\big\{\sigma_+\varrho(t)\big\} = \zeta(t)^*/[2c(t)]\,,
\end{aligned} \tag{2.15}$$

with $c(t) = \pi_0(t) + \pi_1(t)$. Equations (2.13)–(2.15) shows that at a jump of $N(t)$ the system surely goes into the ground state, because $\zeta = 0$ and $\pi_1 = 0$, and that for $t > t_1$ the system is surely in a mixture of ground and excited states, because $\zeta = 0$. The EPDs are implicitly given by $c(t) = \pi_0(t)+\pi_1(t)$, $\tau = \lambda_1^{-1}$, (1.22), (1.23), (2.14).

Just as an example let us discuss the case of the Wigner atom ($\lambda_+ = 0$). Equations (2.14) become

$$\pi_1(t) = \begin{cases} \pi_0(0) + \frac{\lambda_-}{2\kappa}\,(1-\mathrm{e}^{-2\kappa t})\,\pi_1(0), & \text{if } t \le t_1, \\ \exp[-2\kappa t_1]\,\pi_1(0), & \text{if } t_1 < t \le t_2, \\ 0, & \text{if } t > t_2, \end{cases} \tag{2.16}$$

with $\kappa = \frac{1}{2}(\lambda_- + \lambda_1)$, $\pi_0(0) + \pi_1(0) = 1$. Equations (2.15) give $\varrho(t) = |0\rangle\langle 0|$ for $t > t_1$: after the first registered emission the atom is in the ground state. Finally the EPDs are

$$P_0^t(0|\varrho) = \pi_0(0) + \frac{1}{2\kappa}\left(\lambda_- + \lambda_1 \mathrm{e}^{-2\kappa t}\right)\pi_1(0)\,, \tag{2.17}$$

$$p_0^t(j_1, t_1|\varrho) = \lambda_1\,\exp[-2\kappa t_1]\,\pi_1(0)\,, \tag{2.18}$$

$$p_0^t\big(j_1, t_1; \ldots; j_m, t_m|\varrho\big) = 0\,, \qquad m \ge 2\,. \tag{2.19}$$

These equations say that there is at most a count, as it must be because there is no pumping.

3 Diffusion processes

In the classical case Gaussian diffusion processes can be obtained from Poissonian counting ones by centering and scaling. Similarly, in the quantum case we can obtain some kind of "quantum diffusion measuring processes" from the quantum counting processes of section 1.

Let us take the maps $\mathcal{J}_j(t)$, describing the action of the counters, of the following form:

$$\mathcal{J}_j(t)\varrho = \left[Z_j(t) + \frac{1}{\varepsilon}\, f_j(t) \right] \varrho \left[Z_j(t)^\dagger + \frac{1}{\varepsilon}\, f_j(t)^* \right] , \tag{3.1}$$

where the $Z_j(t)$ are operators on $\mathcal{H}$, the f_j are complex functions and $\varepsilon > 0$ is a parameter which we want to make vanishing at the end. Moreover, instead of $\mathcal{L}_0(t)$ we take as unperturbed Liouvillian the expression

$$\mathcal{L}_0^\varepsilon(t)\varrho = \mathcal{L}_0(t)\varrho + \frac{\mathrm{i}}{2\varepsilon} \sum_{j=1}^{d} \left[\mathrm{i} f_j(t)^* Z_j(t) - \mathrm{i} f_j(t) Z_j(t)^\dagger ,\, \varrho \right] . \tag{3.2}$$

Then, the generator $\mathcal{L}(t)$ of the a priori dynamics $\big($cf. (1.17) and (1.42)$\big)$ becomes

$$\mathcal{L}(t)\sigma = \mathcal{L}_0(t)\sigma + \sum_{j=1}^{d} \Big(Z_j(t)\sigma Z_j(t)^\dagger - \frac{1}{2} \left\{ Z_j(t)^\dagger Z_j(t) \,,\, \sigma \right\} \Big) . \tag{3.3}$$

The expression (3.2) has been assumed in order to have $\mathcal{L}(t)$ independent of the parameter ε. Physically, the structure (3.1)–(3.3) is related to heterodyne detection [42].

Moreover, we make a linear transformation on the outputs: we call $Y_j^\varepsilon(t)$ the new observed processes, related to the old processes $N_j(t)$ by

$$\mathrm{d}Y_j^\varepsilon(t) := \varepsilon \, \mathrm{d}N_j(t) - \frac{1}{\varepsilon}\, |f_j(t)|^2 \, \mathrm{d}t \,; \tag{3.4}$$

this means that we rescale the outputs and subtract a known deterministic signal. Then, by (3.4) and (1.15) we obtain

$$\mathrm{d}Y_j^\varepsilon(t)\, \mathrm{d}Y_i^\varepsilon(t) = \varepsilon^2 \delta_{ij}\, \mathrm{d}N_j(t) = \varepsilon\, \delta_{ij}\, \mathrm{d}Y_j^\varepsilon(t) + \delta_{ij}\, |f_j(t)|^2 \, \mathrm{d}t \,. \tag{3.5}$$

In order to have the characteristic operator associated to this new processes, we have to rescale the test function $\vec{k}(s)$, appearing in (1.30), (1.34), (1.38)–(1.40), by changing $k_j(t)$ into $\varepsilon\, k_j(t)$ and we have to shift the mean values of $\varepsilon N_j(t)$ as in (3.4) by adding to $\mathcal{K}_t\big(\vec{k}(t)\big)$ the term $-\frac{i}{\varepsilon} \sum_j k_j(t)\, |f_j(t)|^2$. The final result is that the generator $\mathcal{K}_t\big(\vec{k}(t)\big)$ of the characteristic operator $\mathcal{G}_t[\vec{k}]$ becomes

$$\begin{aligned}\mathcal{K}_t\big(\vec{k}(t)\big)\varrho &= \mathcal{L}(t)\varrho + \sum_{j=1}^{d}\Big\{-\frac{1}{2}\,k_j(t)^2\,|f_j(t)|^2\,\varrho + \mathrm{i}k_j(t)\big[f_j(t)^* Z_j(t)\varrho \\ &+ f_j(t)\varrho Z_j(t)^\dagger\big] + \big[\mathrm{e}^{\mathrm{i}\varepsilon k_j(t)}-1\big]\,Z_j(t)\varrho Z_j(t)^\dagger \\ &+ \frac{1}{\varepsilon}\big[\mathrm{e}^{\mathrm{i}\varepsilon k_j(t)}-1-\mathrm{i}\varepsilon k_j(t)\big]\,\big[f_j(t)^* Z_j(t)\varrho + f_j(t)\varrho Z_j(t)^\dagger\big] \\ &+ \frac{1}{\varepsilon^2}\,|f_j(t)|^2\left[\mathrm{e}^{\mathrm{i}\varepsilon k_j(t)}-1-\mathrm{i}\varepsilon k_j(t)+\frac{1}{2}\,\varepsilon^2 k_j(t)^2\right]\varrho\Big\}\,. \end{aligned} \tag{3.6}$$

Also equation (1.16) for the a posteriori states can be expressed in terms of the new processes $Y_j^\varepsilon(t)$. By (1.1), (1.18), (3.1), (3.3) and (3.4), we obtain

$$\begin{aligned}\mathrm{d}\varrho(t) &= \mathcal{L}(t)\varrho(t)\,\mathrm{d}t + \sum_{j=1}^{d}\Big\{\varepsilon\,Z_j(t)\varrho Z_j(t)^\dagger - \varepsilon\,\langle Z_j(t)^\dagger Z_j(t)\rangle_t\varrho(t) + \\ &+ f_j(t)^*\big[Z_j(t)-\langle Z_j(t)\rangle_t\big]\varrho(t) + f_j(t)\varrho(t)\big[Z_j(t)^\dagger - \langle Z_j(t)^\dagger\rangle_t\big]\Big\} \\ &\times \Big[\varepsilon^2\langle Z_j(t)^\dagger Z_j(t)\rangle_t + \varepsilon f_j(t)^*\langle Z_j(t)\rangle_t + \varepsilon f_j(t)\langle Z_j(t)^\dagger\rangle_t + |f_j(t)|^2\Big]^{-1} \\ &\times \Big[\mathrm{d}Y_j^\varepsilon(t) - \varepsilon\langle Z_j(t)^\dagger Z_j(t)\rangle_t\,\mathrm{d}t - f_j(t)^*\langle Z_j(t)\rangle_t\,\mathrm{d}t \\ &-f_j(t)\langle Z_j(t)^\dagger\rangle_t\,\mathrm{d}t\Big]\,, \end{aligned} \tag{3.7}$$

where, for any operator X on $\mathcal{H}$, $\langle X\rangle_t$ is defined by

$$\langle X\rangle_t = \mathrm{Tr}\{X\,\varrho(t)\}\,. \tag{3.8}$$

Moreover, from (3.4), (1.19) and (3.1), we have that the a posteriori mean values of $\mathrm{d}Y_j^\varepsilon(t)$ are given by

$$\langle\mathrm{d}Y_j^\varepsilon(t)\rangle(\omega_t) = \big[f_j(t)\langle Z_j(t)^\dagger\rangle_t + f_j(t)^*\langle Z_j(t)\rangle_t\big]\,\mathrm{d}t + \varepsilon\langle Z_j(t)^\dagger Z_j(t)\rangle_t\,\mathrm{d}t\,. \tag{3.9}$$

We assume $|f_j(t)|\neq 0$, $\forall t$. From (3.5)–(3.7) and (3.9), it is apparent that the limit $\varepsilon\downarrow 0$ exists. In this limit we obtain that the characteristic operator is given by (1.38) and (1.40) with generator

$$\begin{aligned}\mathcal{K}_t\big(\vec{k}(t)\big)\varrho &= \mathcal{L}(t)\varrho \\ &+\sum_{j=1}^{d}\Big\{-\frac{1}{2}\,k_j(t)^2\,|f_j(t)|^2\,\varrho + \mathrm{i}k_j(t)\big[f_j(t)^* Z_j(t)\varrho + f_j(t)\,\varrho Z_j(t)^\dagger\big]\Big\} \end{aligned} \tag{3.10}$$

By setting $Y_j(t) = \lim_{\varepsilon\downarrow 0} Y_j^\varepsilon(t)$, the equation for the a posteriori states becomes

$$\begin{aligned} \mathrm{d}\varrho(t) &= \mathcal{L}(t)\varrho(t)\,\mathrm{d}t \\ &+ \sum_{j=1}^{d}\Big\{ f_j(t)^*\big[Z_j(t) - \langle Z_j(t)\rangle_t\big]\varrho(t) + f_j(t)\varrho(t)\big[Z_j(t)^\dagger - \langle Z_j(t)^\dagger\rangle_t\big]\Big\} \\ &\times \frac{1}{|f_j(t)|^2}\left[\mathrm{d}Y_j(t) - f_j(t)^*\langle Z_j(t)\rangle_t\,\mathrm{d}t - f_j(t)\langle Z_j(t)^\dagger\rangle_t\,\mathrm{d}t\right]. \end{aligned} \tag{3.11}$$

Moreover, the a posteriori mean value of $\mathrm{d}Y_j(t)$ becomes

$$\langle \mathrm{d}Y_j(t)\rangle(\omega_t) = 2\mathrm{Re}\big[f_j(t)^*\langle Z_j(t)\rangle_t\big]\,\mathrm{d}t \tag{3.12}$$

and the processes $M_j(t)$, defined by

$$\mathrm{d}M_j(t) = \mathrm{d}Y_j(t) - 2\mathrm{Re}\big[f_j(t)^*\langle Z_j(t)\rangle_t\big]\,\mathrm{d}t\,, \qquad M_j(0) = 0\,, \tag{3.13}$$

are again innovating martingales. Finally, the multiplication rule for the differentials $\mathrm{d}Y_j(t)$ is the limit of (3.5) under $\varepsilon \downarrow 0$; by taking into account also the second of equations (1.15), we have the Itô table

$$\mathrm{d}Y_j(t)\,\mathrm{d}Y_i(t) = \delta_{ji}\,|f_j(t)|^2\,\mathrm{d}t\,, \qquad \mathrm{d}Y_j(t)\,\mathrm{d}t = 0\,. \tag{3.14}$$

By the procedure we have followed, it turns out that also the connection between a posteriori states $\varrho(t)$ and characteristic operator $\mathcal{G}_t[\vec{k}]$ given by (1.33) continues to hold, but now $\varrho(t)$ satisfies (3.11), $\mathcal{G}_t[\vec{k}]$ satisfies (1.38), (1.40) with generator given by (3.10) and $V_t[\vec{k}]$ is given by

$$V_t[\vec{k}] = \exp\left[\mathrm{i}\sum_{j=1}^{d}\int_0^t k_j(s)\,\mathrm{d}Y_j(s)\right]. \tag{3.15}$$

Alternatively, equation (1.33) can be proved by taking the stochastic differential of both its sides, as done in the case of counting processes.

By taking the mean value of (3.12) on the past, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Y_j(t)\rangle_{\mathrm{st}} = \mathrm{Tr}\Big\{\big[f_j(t)^*\,Z_j(t) + f_j(t)\,Z_j(t)^\dagger\big]\sigma(t)\Big\}\,, \tag{3.16}$$

where $\sigma(t)$ are the a priori states satisfying equation (1.42) with Liouvillian (3.3). The same result can be obtained by functional differentiation of the characteristic functional $\mathrm{Tr}\left\{\mathcal{G}_T[\vec{k}]\,\varrho\right\}$, $T > t$, with respect to $k_j(t)$ [11].

Equations (3.15) and (3.16) show us two things. First, our continuous measurement gives the statistics of the generalized derivatives [43] $y_j(t) = \dot{Y}_j(t)$ $\big($or of the increments $\mathrm{d}Y_j(t)\big)$ more than the statistics of the $Y_j(t)$ themselves. The same was true in the case of counting processes, but in that case this difference was irrelevant, because we had the natural initial condition $N_j(0) = 0$. Second, (3.16) can be interpreted by saying that $y_j(t)$ is the output of a continuous measurement of the quantum observables (selfadjoint operators) $f_j(t)^* Z_j(t) + f_j(t) Z_j(t)^\dagger$, which are in general noncommuting [10]–[12, 22]–[25].

Measuring processes defined by a characteristic operator with generator of the type (3.10) were introduced in [10]–[12] and equation (3.11) was obtained by quantum stochastic calculus methods in [25]–[27, 38, 44]. By linear transformations on the outputs, the most general diffusive case can be reached; moreover, by taking prescription (3.1) only for a subset of the $\mathcal{J}_j$, mixtures of diffusive and Poissonian contributions can be obtained [38]. Itô equations for the a posteriori states in the [45, 46, 47] purely diffusive case have been considered also by Diósi [45]–[47].

As in the case of counting processes there exists a (not unique) linear stochastic equation mathematically equivalent to (3.11). For instance, let $\varphi(t)$ be a trace class operator satisfying the equation [38]

$$\mathrm{d}\varphi(t) = \mathcal{L}(t)\varphi(t)\mathrm{d}t + \sum_{j=1}^{d} \left[\frac{1}{f_j(t)} Z_j(t)\varphi(t) + \frac{1}{f_j(t)^*}\varphi(t) Z_j(t)^\dagger \right] \mathrm{d}Y_j(t) \tag{3.17}$$

and set $c(t) := \mathrm{Tr}\{\varphi(t)\}$. Then, by Itô's calculus one shows that $\varrho(t) = \varphi(t)/c(t)$ satisfies (3.11). To the linear equation (3.17) the same comments apply as to (1.25).

In the case of an unperturbed Liouvillian of a purely Hamiltonian form,

$$\mathcal{L}_0(t)\,\varrho = -\mathrm{i}[H(t), \varrho]\,, \tag{3.18}$$

equation (3.11) transforms pure states into pure ones; for proving this it is sufficient to show that $\varrho(t+\mathrm{d}t)^2 = \varrho(t+\mathrm{d}t)$ if $\varrho(t)^2 = \varrho(t)$. In this case, which we can call of complete measurement, (3.11) is equivalent to a stochastic differential equation for a wave function, as in the case of counting measurements. Indeed, let $\psi(t) \in \mathcal{H}$ satisfy the "a posteriori Schrödinger equation" [26, 44]

$$\begin{aligned} \mathrm{d}\psi(t) &= -\Big\{\mathrm{i}H(t)+\frac{1}{2}\sum_{j=1}^{d}\big[Z_j(t)^\dagger Z_j(t)-2\langle Z_j(t)^\dagger\rangle_t Z_j(t)+|\langle Z_j(t)\rangle_t|^2\big]\Big\}\psi(t)\mathrm{d}t \\ &+ \sum_{j=1}^{d}\frac{1}{f_j(t)}\big[Z_j(t)-\langle Z_j(t)\rangle_t\big]\psi(t)\big[\mathrm{d}Y_j(t)-f_j(t)^*\langle Z_j(t)\rangle_t\mathrm{d}t \\ &\qquad -f_j(t)\langle Z_j(t)^\dagger\rangle_t\mathrm{d}t\big]\,, \end{aligned} \tag{3.19}$$

with $\langle Z_j(t)\rangle_t = \langle\psi(t)|Z_j(t)\,\psi(t)\rangle$; then, by Itô's calculus, one obtains that $\varrho(t)\equiv|\psi(t)\rangle\langle\psi(t)|$ satisfies (3.11).

It is interesting to note that stochastic equations of the type of (3.11) and (3.19), with $f_j(t)=1$, have been appeared in the literature also in connection with *dynamical theories of wave–function reduction* [48]–[52]. The idea is that the wave–function reduction associated to a measurement is some kind of stochastic process and an equation of the type of (3.19) is postulated. Apart from the different interpretations, another important difference is that in the dynamical reduction theories the noise comes from outside, while for us it is determined by the system itself.

Sometimes it is useful to have at disposal a complexified version of diffusion processes. Let us consider the case of an even d. By redefining d and the index j, the sum appearing in (3.10) and (3.11) can be reorganized as a double sum over λ, $\lambda=1,2$, and j, $j=1,\ldots,d$. Then, we take $f_{1j}(t)=1$, $f_{2j}(t)=\mathrm{i}$, $Z_{1j}(t)=Z_{2j}(t)\equiv Z_j(t)$ and set $\kappa_j(t)=k_{1j}(t)+\mathrm{i}k_{2j}(t)$, $W_j(t)=\frac{1}{2}\left(Y_{1j}(t)+\mathrm{i}Y_{2j}(t)\right)$. Then, (3.3), (3.10)–(3.12), (3.15) become

$$\mathcal{L}(t)\sigma=\mathcal{L}_0(t)\sigma+2\sum_{j=1}^{d}\Big(Z_j(t)\sigma Z_j(t)^\dagger-\frac{1}{2}\,\big\{Z_j(t)^\dagger Z_j(t)\,,\,\sigma\big\}\Big)\,, \tag{3.20}$$

$$\mathcal{K}_t\big(\vec{\kappa}(t)\big)\varrho=\mathcal{L}(t)\varrho+\sum_{j=1}^{d}\Big\{-\frac{1}{2}\,|\kappa_j(t)|^2\,\varrho+\mathrm{i}\big[\kappa_j(t)^*Z_j(t)\varrho+\kappa_j(t)\,\varrho Z_j(t)^\dagger\big]\Big\}\,, \tag{3.21}$$

$$\begin{aligned} \mathrm{d}\varrho(t) &= \mathcal{L}(t)\varrho(t)\,\mathrm{d}t+2\sum_{j=1}^{d}\Big\{\big[Z_j(t)-\langle Z_j(t)\rangle_t\big]\varrho(t)\big[\mathrm{d}W_j(t)^*-\langle Z_j(t)^\dagger\rangle_t\mathrm{d}t\big] \\ &+ \big[\mathrm{d}W_j(t)-\langle Z_j(t)\rangle_t\mathrm{d}t\big]\varrho(t)\big[Z_j(t)^\dagger-\langle Z_j(t)^\dagger\rangle_t\big]\Big\}\,, \end{aligned} \tag{3.22}$$

$$\mathrm{d}W_j(t)\,\mathrm{d}W_i(t) = 0\,, \quad \mathrm{d}W_j(t)^*\,\mathrm{d}W_i(t) = \frac{1}{2}\,\delta_{ji}\,\mathrm{d}t\,, \quad \mathrm{d}W_j(t)\,\mathrm{d}t = 0\,, \tag{3.23}$$

$$\langle \mathrm{d}W_j(t)\rangle(\omega_t) = \langle Z_j(t)\rangle_t \mathrm{d}t\,. \tag{3.24}$$

By taking the mean value of (3.24) on the past, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle W_j(t)\rangle_{\mathrm{st}} = \mathrm{Tr}\big\{Z_j(t)\,\sigma(t)\big\}\,, \tag{3.25}$$

which allows to interpret the equations above as describing a continuous measurement of the noncommuting, nonselfadjoint operators $Z_j(t)$. Filtering equation (3.22) for linear systems (quantum oscillators) was introduced in [21, 22].

4 An example of diffusion process

Let us close the paper with a simple example of the theory developed in Section 3, in the complexified version (3.20)–(3.25). A real–valued Gaussian example for an observed particle in a quadratic potential can be found in [25, 53]. We consider a single–mode field in a cavity and with a source,

$$H(t) = \omega a^\dagger a + g(t)a^\dagger + g(t)^* a\,, \qquad \omega > 0\,, \tag{4.1}$$

interacting with a thermal bath,

$$\mathcal{L}_0(t)\varrho = -\mathrm{i}[H(t),\varrho] + \lambda_\downarrow\big([a\varrho, a^\dagger] + [a, \varrho a^\dagger]\big) + \lambda_\uparrow\big([a^\dagger\varrho, a] + [a^\dagger, \varrho a]\big)\,, \tag{4.2}$$

$\lambda_\downarrow, \lambda_\uparrow \geq 0$, and subjected to the measurement of a single complex observable ($d = 1$) proportional to the annihilation operator,

$$Z = \eta a\,, \qquad \eta \in \mathbf{C}\,. \tag{4.3}$$

The fact that Z is proportional to a means that we are considering a passive, purely absorbing detector.

By scaling the output in such a way that we have exactly a measurement of a $\big(\mathrm{d}W(t)/\eta \to \mathrm{d}W(t)$, $\eta^*\kappa(t) \to \kappa(t)\big)$, equations (3.20)–(3.25) become

$$\mathcal{L}(t)\sigma = \mathcal{L}_0(t)\sigma + |\eta|^2\left([a\sigma, a^\dagger] + [a, \sigma a^\dagger]\right)\,, \tag{4.4}$$

$$\mathcal{K}_t\big(\kappa(t)^*, \kappa(t)\big)\varrho = \mathcal{L}(t)\varrho - \frac{1}{2}\,|\kappa(t)/\eta|^2\,\varrho + \mathrm{i}\left[\kappa(t)^* a\varrho + \kappa(t)\varrho a^\dagger\right]\,, \tag{4.5}$$

$$\begin{aligned} \mathrm{d}\varrho(t) &= \mathcal{L}(t)\varrho(t)\mathrm{d}t + 2\,|\eta|^2 \Big\{ [a - \langle a\rangle_t]\, \varrho(t)\, \big[\mathrm{d}W(t)^* - \langle a^\dagger\rangle_t \mathrm{d}t\big] \\ &\qquad + [\mathrm{d}W(t) - \langle a\rangle_t \mathrm{d}t]\, \varrho(t)\, \big[a^\dagger - \langle a^\dagger\rangle_t\big] \Big\}\,, \end{aligned} \tag{4.6}$$

$$\big(\mathrm{d}W(t)\big)^2 = 0\,, \qquad |\mathrm{d}W(t)|^2 = \mathrm{d}t/\big(2\,|\eta|^2\big)\,, \qquad \mathrm{d}W(t)\,\mathrm{d}t = 0\,, \tag{4.7}$$

$$\mathrm{d}\langle W(t)\rangle(\omega_t) = \langle a\rangle_t \mathrm{d}t\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle W(t)\rangle_{\mathrm{st}} = \mathrm{Tr}\{a\,\sigma(t)\}\,. \tag{4.8}$$

Equations (1.38), with generator (4.5), and (4.6) can be solved by anti-normal ordering expansion of trace–class operators. Let us define on $\mathcal{T}(\mathcal{H})$ a "tilde" operation by $\varphi \in \mathcal{T}(\mathcal{H}) \longrightarrow \tilde{\varphi}(\xi^*, \xi)$,

$$\tilde{\varphi}(\xi^*, \xi) = \mathrm{Tr}\left\{ \mathrm{e}^{-\mathrm{i}\xi^* a}\, \varphi\, \mathrm{e}^{-\mathrm{i}\xi a^\dagger} \right\}\,, \tag{4.9}$$

which can be inverted by

$$\varphi = \frac{1}{\pi} \int \mathrm{d}_2\xi\, \mathrm{e}^{\mathrm{i}\xi^* a}\, \mathrm{e}^{\mathrm{i}\xi a^\dagger}\, \tilde{\varphi}(\xi^*, \xi)\,. \tag{4.10}$$

Let us set

$$\varphi(t) \equiv \varphi(\kappa^*, \kappa; t) = \mathcal{G}_t[\kappa^*, \kappa]\, \varrho\,; \tag{4.11}$$

then (1.38) and (4.5) give, by standard computations,

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{\varphi}(\xi^*, \xi; t) &= \Big[-\left(\mathrm{i}\omega + \frac{1}{2}\,\Gamma\right) \xi^* \partial^* + \left(\mathrm{i}\omega - \frac{1}{2}\,\Gamma\right) \xi \partial - \kappa(t)^* \partial^* - \kappa(t)\partial \\ &\quad - \; 2\lambda_\uparrow\, |\xi|^2 - g(t)\xi^* + g(t)^*\xi - \frac{1}{2}\, |\kappa(t)/\eta|^2 \Big] \tilde{\varphi}(\xi^*, \xi; t)\,, \end{aligned} \tag{4.12}$$

where $\partial = \partial/\partial\xi$, $\partial^* = \partial/\partial\xi^*$ and

$$\Gamma = 2\big(|\eta|^2 + \lambda_\downarrow - \lambda_\uparrow\big)\,. \tag{4.13}$$

We suppose Γ be strictly positive.

If the initial condition is "Gaussian",

$$\tilde{\varrho}(\xi^*, \xi) = \exp\left[-\mathrm{i}(\xi^* \alpha_0 + \xi \alpha_0^*) - \frac{1}{2}\, \left(\xi^{*2}\mu_0 + \xi^2 \mu_0^*\right) - |\xi|^2\, \nu_0 \right]\,, \tag{4.14}$$

then $\tilde{\varphi}$ maintains this structure at any time. Indeed, by inserting

$$\tilde{\varphi}(\xi^*, \xi; t) = \exp\Big\{ -\mathrm{i}\,[\xi^* b(t) + \xi c(t)^*] - \frac{1}{2}\, \left[\xi^{*2} d(t) + \xi^2 d(t)^*\right] - |\xi|^2\, f(t) - h(t) \Big\} \tag{4.15}$$

into (4.12), we obtain the differential equations for the coefficients (f is real):

$$\begin{aligned}
\dot{b}(t) &= -\left(\mathrm{i}\omega + \frac{1}{2}\Gamma\right) b(t) + \mathrm{i}\kappa(t)^* d(t) + \mathrm{i}\kappa(t) f(t) - \mathrm{i}g(t)\,, \\
\dot{c}(t) &= -\left(\mathrm{i}\omega + \frac{1}{2}\Gamma\right) c(t) - \mathrm{i}\kappa(t)^* d(t) - \mathrm{i}\kappa(t) f(t) - \mathrm{i}g(t)\,, \\
\dot{d}(t) &= -(2\mathrm{i}\omega + \Gamma) d(t)\,, \\
\dot{f}(t) &= -\Gamma f(t) + 2\lambda_\uparrow\,, \\
\dot{h}(t) &= -\mathrm{i}\kappa(t)^* b(t) - \mathrm{i}\kappa(t) c(t)^* + \frac{1}{2}\,|\kappa(t)/\eta|^2\,.
\end{aligned} \tag{4.16}$$

The solution of these equations can be easily written down.

The characteristic functional of our generalized process [43] $\dot{W}(t)$ is given by (see (1.34) and (1.40))

$$\Phi_t[\kappa^*,\kappa] = \mathrm{Tr}\big\{\varphi(\kappa^*,\kappa;t)\big\} = \tilde{\varphi}(0,0;t) = \exp[h(t)] \tag{4.17}$$

with

$$\begin{aligned}
h(t) &= -\mathrm{i}\int_0^t \mathrm{d}s\,[\kappa(s)^*\alpha(s) + \kappa(s)\alpha(s)^*] + \int_0^t \mathrm{d}s\,\mathrm{d}s'\,\Big[\kappa(s)^*\kappa(s')\Delta_1(s,s') \\
&+ \frac{1}{2}\,\kappa(s)\kappa(s')\Delta_2(s,s')^* + \frac{1}{2}\,\kappa(s)^*\kappa(s')^*\Delta_2(s,s')\Big]\,,
\end{aligned} \tag{4.18}$$

$$\alpha(t) = \mathrm{e}^{-(\mathrm{i}\omega+\frac{1}{2}\Gamma)t}\alpha_0 - \mathrm{i}\int_0^t \mathrm{d}s\,g(s)\,\mathrm{e}^{-(\mathrm{i}\omega+\Gamma/2)(t-s)}\,, \tag{4.19}$$

$$\begin{aligned}
\Delta_1(s,s') &= \frac{1}{2\,|\eta|^2}\delta(s-s') + \vartheta(s-s')\,\mathrm{e}^{-(\mathrm{i}\omega+\Gamma/2)(s-s')}\,C(s') \\
&\quad + \vartheta(s'-s)\,\mathrm{e}^{(\mathrm{i}\omega-\Gamma/2)(s'-s)}\,C(s)\,,
\end{aligned} \tag{4.20}$$

$$C(s) = \frac{2\lambda_\uparrow}{\Gamma} + \left(\nu_0 - \frac{2\lambda_\uparrow}{\Gamma}\right)\mathrm{e}^{-\Gamma s}\,, \tag{4.21}$$

$$\Delta_2(s,s') = \mathrm{e}^{-(\mathrm{i}\omega+\Gamma/2)(s+s')}\,\mu_0\,, \tag{4.22}$$

where ϑ is the usual step function. $\Phi_T[\kappa^*,\kappa]$ is the characteristic functional of a Gaussian complex process with covariance (4.20), (4.22) and (a priori) mean values

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle W(t)\rangle_{\mathrm{st}} = \mathrm{i}\frac{\delta}{\delta\kappa(t)}\Phi_t[\kappa^*,\kappa]\Big|_{\kappa=\kappa^*=0} = \mathrm{Tr}\big\{a\sigma(t)\big\} = \alpha(t)\,. \tag{4.23}$$

The a priori states are given by $\sigma(t) = \mathcal{G}_t[0]\,\varrho$ or $\tilde\sigma(\xi^*,\xi;t) = \tilde\varphi(\xi^*,\xi;t)\Big|_{\kappa=\kappa^*=0}$. By (4.15) and (4.16) we obtain

$$\tilde\sigma(\xi^*,\xi;t) = \exp\left\{-\mathrm{i}\left[\xi^*\alpha(t) + c.c.\right] - \frac{1}{2}\,\left[\xi^{*2}\mathrm{e}^{-(2\mathrm{i}\omega+\Gamma)t}\,\mu_0 + c.c.\right] - |\xi|^2\,C(t)\right\}\,. \tag{4.24}$$

This gives

$$\mathrm{Tr}\left\{a^\dagger a\,\sigma(t)\right\} = C(t)\,, \quad \mathrm{Tr}\left\{a^2\,\sigma(t)\right\} = \exp\left[-2\left(\mathrm{i}\omega + \frac{1}{2}\,\Gamma\right)t\right]\mu_0\,. \tag{4.25}$$

Note the links between the covariance (4.25) of the a priori states $\sigma(t)$ and the covariance (4.20), (4.22) of the process $\dot W(t)$.

By the “tilde" transformation (4.9), we can solve also the equation for the a posteriori states (4.6). From (4.6), (4.9), (4.10) we obtain

$$\begin{aligned} \mathrm{d}\tilde\varrho(\xi^*,\xi;t) &= \Big[-\left(\mathrm{i}\omega + \frac{1}{2}\,\Gamma\right)\xi^*\partial^* + \left(\mathrm{i}\omega - \frac{1}{2}\,\Gamma\right)\xi\partial - 2\lambda_\uparrow\,|\xi|^2 - g(t)\xi^* \\ &+ g(t)^*\xi\Big]\,\tilde\varrho(\xi^*,\xi;t)\,\mathrm{d}t + 2\,|\eta|^2\,\Big\{\left[\mathrm{d}W(t)^* - \langle a^\dagger\rangle_t\mathrm{d}t\right]\left[\mathrm{i}\partial^* - \langle a\rangle_t\right] \\ &+ \left[\mathrm{d}W(t) - \langle a\rangle_t\mathrm{d}t\right]\left[\mathrm{i}\partial - \langle a^\dagger\rangle_t\right]\Big\}\,\tilde\varrho(\xi^*,\xi;t)\,. \end{aligned} \tag{4.26}$$

This equation can be rewritten in terms of the stochastic function

$$l(\xi^*,\xi;t) := -\ln\tilde\varrho(\xi^*,\xi;t)\,. \tag{4.27}$$

By using Itô's formula $\mathrm{d}\tilde\varrho/\tilde\varrho = -\mathrm{d}l + \frac{1}{2}(\mathrm{d}l)^2$, which in turn implies $(\mathrm{d}\tilde\varrho/\tilde\varrho)^2 = (\mathrm{d}l)^2$, and Itô's table (4.7), we obtain

$$\begin{aligned} \mathrm{d}l &= \Big[-2\,|\eta|^2\,\partial^* l\,\partial l - \left(\mathrm{i}\omega + \frac{1}{2}\,\Gamma\right)\xi^*\partial^* l + \left(\mathrm{i}\omega - \frac{1}{2}\,\Gamma\right)\xi\partial l \\ &+ 2\lambda_\uparrow\,|\xi|^2 + g(t)\xi^* - g(t)^*\xi - 2\,|\eta|^2\,|\langle a\rangle_t|^2\Big]\mathrm{d}t \\ &+ 2\,|\eta|^2\left[\left(\mathrm{i}\partial^* l + \langle a\rangle_t\right)\mathrm{d}W(t)^* + \left(\mathrm{i}\partial l + \langle a^\dagger\rangle_t\right)\mathrm{d}W(t)\right]\,. \end{aligned} \tag{4.28}$$

With the initial condition (4.14) the solution of (4.28) remains quadratic in ξ and ξ^*. Indeed, let us write

$$l(\xi^*,\xi;t) = \mathrm{i}\left[\xi^*\langle a\rangle_t + \xi\langle a^\dagger\rangle_t\right] + \frac{1}{2}\left[\xi^{*2}\mu(t) + \xi^2\mu(t)^*\right] + |\xi|^2\,\nu(t)\,, \tag{4.29}$$

where $\nu(t) \geq 0$; the term independent of ξ is lacking because of normalization of $\varrho(t)$ and the linear term must have just the form we have written because $\langle a\rangle_t$ is the a posteriori mean value of $\dot{W}(t)$. By inserting (4.29) into (4.28) and equating the coefficients of the same order in ξ and ξ^*, we obtain

$$\mathrm{d}\langle a\rangle_t + \left[\left(\mathrm{i}\omega + \frac{1}{2}\Gamma\right)\langle a\rangle_t + \mathrm{i}g(t)\right]\mathrm{d}t = \\ 2\left|\eta\right|^2\left\{\mu(t)\left[\mathrm{d}W(t)^* - \langle a^\dagger\rangle_t\mathrm{d}t\right] + \nu(t)\left[\mathrm{d}W(t) - \langle a\rangle_t\mathrm{d}t\right]\right\}, \quad (4.30)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu(t) + (2\mathrm{i}\omega + \Gamma)\,\mu(t) = -4\left|\eta\right|^2\mu(t)\nu(t)\,, \quad (4.31)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\nu(t) + \Gamma\nu(t) = -2\left|\eta\right|^2\left(\left|\mu(t)\right|^2 + \nu(t)^2\right) + 2\lambda_\uparrow\,, \quad (4.32)$$

with $\langle a^\dagger\rangle_t = \langle a\rangle_t^*$ and the initial conditions $\langle a\rangle_0 = \alpha_0$, $\mu(0) = \mu_0$, $\nu(0) = \nu_0$.

In the case $\mu_0 = 0$, we obtain $\mu(t) = 0$ (the stationary solution of (4.31)) and (4.32) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\nu(t) + \Gamma\nu(t) = -2\left|\eta\right|^2\nu(t)^2 + 2\lambda_\uparrow\,, \quad (4.33)$$

which is Riccati's equation and has the stationary positive solution ν_∞

$$\nu_\infty = \frac{\Gamma}{4\left|\eta\right|^2}\left[\left(1 + 16\left|\eta\right|^2\frac{\lambda_\uparrow}{\Gamma^2}\right)^{1/2} - 1\right]. \quad (4.34)$$

Equations (4.30) (for $\mu = 0$) and (4.33) were obtained for the first time in [21, 22, 24] as optimal filtering equations for linear systems.

After a transient any memory of the initial condition is lost. The characteristic functional is given by (4.17) and (4.18) with a priori mean values

$$\alpha(t) = -\mathrm{i}\int_0^t \mathrm{e}^{-(\mathrm{i}\omega+\Gamma/2)(t-s)}\,g(s)\,\mathrm{d}s \quad (4.35)$$

and covariance $\Delta_2(s,s') = 0$,

$$\Delta_1(s,s') = \frac{1}{2\left|\eta\right|^2}\delta(s-s') + \frac{2\lambda_\uparrow}{\Gamma}\,\mathrm{e}^{-(\Gamma/2)|s-s'|}\,\mathrm{e}^{-\mathrm{i}\omega(s-s')}\,. \quad (4.36)$$

The a priori states are given by

$$\tilde{\sigma}_\infty(\xi^*,\xi;t) = \exp\left\{-\mathrm{i}\left[\xi^*\alpha(t) + \xi\alpha(t)^*\right] - \frac{2\lambda_\uparrow}{\Gamma}\left|\xi\right|^2\right\}, \quad (4.37)$$

while the a posteriori states are

$$\tilde{\varrho}_\infty(\xi^*,\xi;t) = \exp\left\{-\mathrm{i}\left[\xi^*\langle a\rangle_t + \xi\langle a^\dagger\rangle_t\right] - \nu_\infty\,|\xi|^2\right\}\,, \tag{4.38}$$

with a posteriori mean values

$$\begin{aligned}\langle a\rangle_t \;&=\; \int_0^t \exp\left[-\left(\mathrm{i}\omega + \frac{1}{2}\,\Gamma + 2\,|\eta|^2\,\nu_\infty\right)(t-s)\right] \\ &\quad \left[-\mathrm{i}g(s)\,\mathrm{d}s + 2\,|\eta|^2\,\nu_\infty\,\mathrm{d}W(s)\right]. \end{aligned} \tag{4.39}$$

Note that $\nu_\infty > 2\lambda_\uparrow/\Gamma$ for $\lambda_\uparrow > 0$ and $\nu_\infty = 0$ for $\lambda_\uparrow = 0$. In this last case the asymptotic a priori and a posteriori mean values coincides $\big(\alpha(t) = \langle a\rangle_t\big)$ and the same holds for a priori and a posteriori states

$$\sigma_\infty(t) = \varrho_\infty(t) = |\alpha(t)\rangle\langle\alpha(t)| \tag{4.40}$$

where $|\alpha\rangle$ denotes the usual coherent states and $\alpha(t)$ is given by (4.35).

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