

A New Wave Equation For a Continuous Nondemolition Measurement

V. P. Belavkin*

Institute of Physics, Copernicus University, Toruń (Poland)

December 1988

Published in: Physics Letters A, **140** (1989), No 7,8, pp 355–358

Abstract

A stochastic model for nondemolition continuous measurement in a quantum system is given. It is shown that the posterior dynamics, including a continuous collapse of the wave function, is described by a nonlinear stochastic wave equation. For a particle in an electromagnetic field it reduces the Schrödinger equation with extra imaginary stochastic potentials.

According to the statistical interpretation of the wave function $\psi(t, \mathbf{x})$, a quantum measurement of a simple observable \hat{X} of a particle reduces it to an eigenfunction of \hat{X} by a random jump (collapse, or reduction of the wave packet). Such a jump cannot be described by the Schrödinger equation because the latter corresponds to the unobserved particle. To ignore this fact gives rise to the various quantum paradoxes of the Zeno kind. The aim of this paper is to derive a dissipative wave equation with an extra stochastic nonlinear term describing the quantum particle under continuous nondemolition observation. This derivation can be obtained in the framework of quantum stochastic theory of continuous measurements developed in refs. [4, 2, 5], and the quantum nonlinear filtering method recently announced in [6]. The derivation can be done in general terms but for simplicity here we consider a quantum spinless particle in an electromagnetic field. The unperturbed dynamics of such a particle with mass m is given by the Schrödinger equation

$$i\hbar\partial\psi/\partial t = [(\mathbf{U} + i\hbar\nabla)^2/2m + V]\psi, \quad (1)$$

where $\mathbf{U} = e\mathbf{A}/c$ and $V = e\Phi$, $\mathbf{A}(t, \mathbf{x})$ and $\Phi(t, \mathbf{x})$ are the scalar and vector field potentials.

*Permanent address: M.I.E.M., B. Vusovsky 3/12, Moscow 109028, USSR.

A continuous process $\dot{\mathbf{Y}}$ is taken as the sum

$$\dot{\mathbf{Y}}(t) = (2\lambda)^{1/2} \hat{\mathbf{X}}(t) + \mathbf{e}(t) \quad (2)$$

of the standard white noise $\mathbf{e} = (e_1, e_2, e_3)$ (error),

$$\langle e_i(r) e_k(s) \rangle = \delta_{ik} \delta(r - s),$$

and the particle coordinate operator process $\mathbf{X} = (X_1, X_2, X_3)$ in the Heisenberg picture $\hat{\mathbf{X}}(t) = U(t)^* \mathbf{X} U(t)$ with amplification $\sqrt{2\lambda}$ ($\lambda > 0$ is the accuracy coefficient).

Indirect measurement of the position of the particle described by (2) perturbs its dynamics (1) in such a way that the vector process $\dot{\mathbf{Y}} = (\dot{Y}_1, \dot{Y}_2, \dot{Y}_3)$ is a commutative one (self-nondemolition),

$$[\dot{Y}_i(r), \dot{Y}_k(s)] = 0 \quad (3)$$

and satisfies the nondemolition principle [5, 6]

$$[\dot{Y}_i(t), \hat{Z}(s)] = 0, \quad t \leq s, \quad (4)$$

with respect to all future Heisenberg operators $\hat{Z}(t) = U(t)^* \mathbf{Z} U(t)$ of the particle. This means that the unitary evolution $U(t)$ can no longer be the resolving operator for eq. (2) but must be defined for an extended quantum system involving an apparatus with field coordinate described by the white noise \mathbf{e} . A very nice model of such an extended unitary evolution is based on the quantum stochastic Schrödinger equation

$$dU + KU dt = (d\mathbf{B}^+ \mathbf{L} - \mathbf{L}^+ d\mathbf{B}) U, \quad (5)$$

$K = \mathbf{L}^+ \mathbf{L} / 2 + iH/\hbar$, introduced by Hudson and Parthasarathy in ref. [8]. Here H is the Hamiltonian which in the spinless case is

$$H = [e\mathbf{A}(t, \mathbf{x})/c + i\hbar\nabla]^2 / 2m + e\Phi(t, \mathbf{x}), \quad (6)$$

$\mathbf{L}(t)$ and $\mathbf{B}(t)$ are the vector-operators (columns) of the particle and of a Bose field respectively in the Schrödinger picture, $\mathbf{L}^+ = (L_j^*)$ and $\mathbf{B}^+ = (B_j^*)$ are rows of conjugate operators, $\mathbf{L}^+ \mathbf{L} = \sum L_j^* L_j$, $\mathbf{B}^+ \mathbf{L} = \sum B_j^* L_j$, $\mathbf{L}^+ \mathbf{B} = \sum L_j^* B_j$. The Bose field operators $B_j(t)$ are defined in Fock space as annihilations by the canonical commutations relations

$$\begin{aligned} [B_i(r), B_k(s)] &= 0, \\ [\dot{B}_i(r), \dot{B}_k^*(s)] &= \delta_{ik} \delta(r - s) \end{aligned}$$

for the (generalized) derivations $\dot{B}_j(t) = dB_j/dt$, and $d\mathbf{B}$ in (5) are the forward increments $d\mathbf{B}(t) = \mathbf{B}(t + dt) - \mathbf{B}(t)$.

Let us respect the error in (2) as the operator-valued vector-process

$$\mathbf{e}(t) = 2\Re\dot{\mathbf{B}}(t) = \dot{\mathbf{B}}(t) + \dot{\mathbf{B}}(t)^*,$$

having the correlations of the standard white noise,

$$\langle e_i(r)e_k(s) \rangle = \langle \dot{B}_i(r)\dot{B}_k^*(s) \rangle = \delta_{ik}\delta(r-s),$$

with respect to the vacuum state of the Bose field. One can easily prove that the nondemolition principle (3), (4) is fulfilled if $\mathbf{L}(t) = (2\lambda)^{1/2}\mathbf{X}$ where \mathbf{X} is the coordinate vector-operator of the particle given in the Schrödinger representation as multiplication by $\mathbf{x} = (x_1, x_2, x_3)$. Indeed, in this case $\mathbf{Y}(t) = \int_0^t \dot{\mathbf{Y}}(r)dr$ is an output process $\mathbf{Y} = \hat{\mathbf{Q}}$ with respect to the evolution U in the sense of refs. [6, 3, 7, 1]:

$$\hat{\mathbf{Q}}_j(t) = U(s)^*Q - j(t)U(s), \quad s \leq t, \quad j = 1, 2, 3,$$

where

$$\mathbf{Q}(t) = \int_0^t \mathbf{e}(r)dr = 2\Re\mathbf{B}(t)$$

is the standard Wiener process in Fock space. Hence $\dot{Y}_j(t)$ commutes with $\dot{Y}_k(s)$ and $\hat{Z}(s)$ for all $s \geq t$ due to the commutativity of $e_j(t)$ with $e_k(s)$ and Z . The corresponding quantum Langevin equation

$$d\hat{Z} + (\hat{Z}\hat{K} + \hat{K}^*\hat{Z} - \hat{\mathbf{L}}^+ Z\hat{\mathbf{L}})dt = d\mathbf{B}^+[\hat{Z}, \hat{\mathbf{L}}] + [\hat{\mathbf{B}}^+, \hat{Z}]d\mathbf{B}, \quad (7)$$

where $\hat{K}(t) = U(t)^*KU(t)$, $\hat{L}_j(t) = U(t)^*L_jU(t)$, gives the quantum stochastic Lorentz equation for $\hat{Z} = \hat{X}_j$, $j = 1, 2, 3$,

$$\begin{aligned} m\hat{\ddot{\mathbf{X}}} &= e(\hat{\mathbf{E}} + \hat{\mathbf{X}} \times \hat{\mathbf{H}}/c) + \hbar(2\lambda)^{1/2}\Im\dot{\mathbf{B}}^*, \\ \hat{\mathbf{E}} &= -\nabla\Phi(\hat{\mathbf{X}}), \quad \hat{\mathbf{H}} = \nabla \times \mathbf{A}(\hat{\mathbf{X}}), \\ (\hat{\mathbf{X}} \times \hat{\mathbf{H}})_i &= \epsilon_{ijk}(\hat{X}_j\hat{H}_k + \hat{H}_k\hat{X}_j)/2. \end{aligned}$$

Note that the extra stochastic force

$$\mathbf{f}(t) = \hbar(2\lambda)^{1/2}\Im\dot{\mathbf{B}}^* = \frac{\hbar}{i}(\lambda/2)^{1/2}(\dot{\mathbf{B}}^* - \dot{\mathbf{B}})$$

perturbs the Hamiltonian dynamics of the particle and to the observation (2) is another white noise of intensity $\hbar^2\lambda/2$, which does not commute with the error \mathbf{e} :

$$[e_j(t), f_k(s)] = \frac{\hbar}{i}(2\lambda)^{1/2}\delta_{jk}\delta(t-s). \quad (8)$$

Due to the openness of the observed particle as a quantum system in the Bose reservoir, its prior dynamics is a mixing described by the irreversible master equation

$$\frac{d}{dt} \langle Z \rangle_t + \langle ZK + K^*Z - L^+ZL \rangle_t = 0, \quad (9)$$

obtained by averaging (7) with respect to the product $\varphi_0 = \psi \otimes |0\rangle$ of an initial wave function ψ of the particle and the vacuum state $|0\rangle$ of the Bose field. This means that the prior expectations $\langle Z \rangle_t = \langle \varphi_t | Z \varphi_t \rangle$ of the particle observables Z for $\varphi_t = U(t)\varphi_0$ cannot be described in terms of a wave function involving only the particle, in spite of the fact that the initial state of the particle is a pure one.

The posterior dynamics of the particle is described by posterior mean values $\hat{z}(t) = \langle Z \rangle^t$, which are defined as conditional expectations

$$\langle Z \rangle^t = \epsilon^t(U(t)^* Z U(t)) = \epsilon^t(\hat{Z}(t)) \quad (10)$$

of $\hat{Z} = U^* Z U$ with respect to the observables $\hat{\mathbf{Q}}^t = \{\mathbf{Q}(r) | r \leq t\}$ up to the current time instant t and the initial state vector φ_0 . As it was proved in ref. [3], the nondemolition principle (3), (4) is necessary and sufficient for the existence of the conditional expectation $\epsilon^t : \hat{Z} \mapsto \epsilon^t(\hat{Z})$ with respect to any initial φ_0 . For a fixed \hat{Z} , it is a non-anticipating c -valued function $\hat{z} : \mathbf{q} \mapsto z(t, \mathbf{q}) = \langle Z \rangle(\mathbf{q}^t)$ on the observed trajectories $\mathbf{q} = \{\mathbf{q}(t)\}$ of the output process $\mathbf{Y} = \hat{\mathbf{Q}}$; the conditional expectation ϵ^t must satisfy the positive projection conditions

$$\epsilon^t(\hat{Z}^* \hat{Z}) \geq 0, \quad \epsilon^t(z(t, \hat{\mathbf{Q}}))(\mathbf{q}) = z(t, \mathbf{q}).$$

Hence for any particle operator Z the posterior process $\hat{z}(t) = \langle Z \rangle^t$ is a classical (commutative) stochastic one; its averaging over all observable trajectories \mathbf{q} coincides with the prior mean value: $\langle \hat{z}(t) \rangle = \langle Z \rangle_t$. As a linear map $Z \mapsto \langle Z \rangle^t$ it is described by the quantum filtering equation

$$d\langle Z \rangle^t + \langle ZK + K^*Z - L^+ZL \rangle^t dt = \langle \tilde{Z}L + L^+\tilde{Z} \rangle^t d\tilde{\mathbf{Q}}, \quad (11)$$

obtained in ref. [6] for the case considered

$$\mathbf{Y}(t) = \int_0^t 2\Re[L(r)dr + d\mathbf{B}] = \hat{\mathbf{Q}}(t)$$

and ref. [3] for general output nondemolition processes with respect to the initial vacuum Bose state. Here $\tilde{Z}(t) = Z - \hat{z}(t)I$ is the deviation of Z in the Schrödinger picture from the posterior mean value $\hat{z}(t) = \langle Z \rangle^t$ and

$$\tilde{\mathbf{Q}}(t) = \tilde{\mathbf{Q}}(t) - \int_0^t [\hat{l}(r) + \hat{l}^*(r)] dr \quad (12)$$

is the observed innovating Wiener process, $\hat{\mathbf{l}}(t)(\mathbf{q}) = \langle \mathbf{L} \rangle(\mathbf{q}^t)$.

Let us prove that for any initial wave function $\psi(\mathbf{x})$ of the open particle the posterior state (9) is pure and is given by

$$\hat{z}(t) = \int \hat{\varphi}(t, \mathbf{x})^* Z \hat{\varphi}(t, \mathbf{x}) d\mathbf{x} = \hat{\varphi}(t)^+ Z \hat{\varphi}(t), \quad (13)$$

where the posterior wave function $\hat{\varphi}(t, \mathbf{x})(\mathbf{q}) = \varphi(\mathbf{q}^t, \mathbf{x})$ satisfies the stochastic wave equation

$$d\hat{\varphi} + \tilde{K} \hat{\varphi} dt = \tilde{L} \hat{\varphi} d\tilde{\mathbf{Q}}, \quad \hat{\varphi}(0, \mathbf{x}) = \psi(\mathbf{x}). \quad (14)$$

Indeed, if $\hat{\varphi}$ satisfies eq. (14) in the Ito form, then $\langle Z \rangle = \hat{\varphi}^+ Z \hat{\varphi}$ satisfies the following equation,

$$d\langle Z \rangle = \langle Z \tilde{K} + \tilde{K}^* \tilde{Z} - \tilde{L}^+ Z \tilde{L} \rangle dt = \langle Z \tilde{L} + \tilde{L}^+ Z \rangle d\tilde{\mathbf{Q}}, \quad (15)$$

obtained by using Ito's formula

$$d(\hat{\varphi}^+ Z \hat{\varphi}) = d\hat{\varphi}^+ Z \hat{\varphi} + \hat{\varphi}^+ Z d\hat{\varphi} + d\hat{\varphi}^+ Z d\hat{\varphi} \quad (16)$$

and the Ito multiplication table $d\tilde{Q}_k d\tilde{Q}_l = \delta_{kl} dt$. Comparing (11) and (15) and taking into account the relation $\langle (Z - \hat{z}) \mathbf{L} \rangle = \langle Z(\mathbf{L} - \hat{\mathbf{l}}) \rangle$, one obtains $\tilde{K} = \tilde{L}^+ \tilde{L} / 2 + i\tilde{H} / \hbar$ with

$$\tilde{L} = \mathbf{L} - \Re \hat{\mathbf{l}} + i\hat{\mathbf{r}} / \hbar, \quad (17)$$

where $\hat{\mathbf{r}}(t) = (\hat{r}_1, \hat{r}_2, \hat{r}_3)(t)$ and $\hat{s}(t)$ are arbitrary (inessential) real functions of \mathbf{q}^t and

$$\tilde{H} = H - \hbar \Re \hat{\mathbf{l}} \Im \mathbf{L} - \hat{\mathbf{r}} \Re \mathbf{L} - \hat{s} \quad (18)$$

is the Hamiltonian of the particle. Putting $\hat{\mathbf{r}} = 0$, $\hat{s} = 0$, we obtain the following stochastic dissipative equation,

$$d\hat{\varphi} + (\tilde{L}^+ \tilde{L} / 2 + i\tilde{H} / \hbar) \hat{\varphi} dt = \tilde{L} \hat{\varphi} d\tilde{\mathbf{Q}}, \quad (19)$$

which is nonlinear because \tilde{L} and \tilde{H} depend on $\hat{\varphi}$ (17). Multiplying the posterior normalized wave function $\hat{\varphi}(t, \mathbf{x})$ by the stochastic amplitude $\hat{c}(t)$ which satisfies the Ito equation

$$d\hat{c} + (\Re \hat{\mathbf{l}})^2 \hat{c} dt / 2 = (\Re \hat{\mathbf{l}}) \hat{c} d\tilde{\mathbf{Q}}, \quad \hat{c}(0) = 1, \quad (20)$$

using Ito's formula one can easily obtain

$$\begin{aligned} d(\hat{c} \hat{\varphi}) &= \{ \Re \hat{\mathbf{l}} d\tilde{\mathbf{Q}} - (\Re \hat{\mathbf{l}})^2 dt / 2 + \tilde{L} d\tilde{\mathbf{Q}} - \tilde{K} dt + \tilde{L} \Re \hat{\mathbf{l}} dt \} \hat{c} \hat{\varphi} \\ &= \{ (\Re \hat{\mathbf{l}} + \tilde{L}) d\tilde{\mathbf{Q}} - [(\Re \hat{\mathbf{l}})^2 / 2 + \tilde{K} + \tilde{L} \Re \hat{\mathbf{l}}] dt \} \hat{c} \hat{\varphi} = (\mathbf{L} d\tilde{\mathbf{Q}} - K dt) \hat{c} \hat{\varphi}, \end{aligned}$$

where the relation $d\hat{Q}_i d\hat{Q}_k = \delta_{ik} dt$ for $d\tilde{\mathbf{Q}} = d\hat{\mathbf{Q}} - 2\Re\hat{\mathbf{l}}dt$, $\tilde{\mathbf{L}} = \mathbf{L} - \Re\hat{\mathbf{l}}$, $\tilde{K} = K - \mathbf{L}\Re\hat{\mathbf{l}} + (\Re\hat{\mathbf{l}})^2/2$ was taken into account. Hence the nonnormalized posterior wave function $\hat{\chi}(t, \mathbf{x}) = \hat{c}(t)\hat{\varphi}(t, \mathbf{x})$ satisfies the linear stochastic equation

$$d\hat{\chi} + (\mathbf{L}^+ \mathbf{L}/2 + \imath H/\hbar)\hat{\chi}dt = \mathbf{L}\hat{\chi}d\hat{\mathbf{Q}}. \quad (21)$$

The last equation can be transformed to a nonstochastic linear equation for $\hat{\psi}(t) = \exp[-\mathbf{L}\hat{\mathbf{Q}}(t)]\hat{\chi}(t)$:

$$\imath\hbar\partial\hat{\psi}(t)/\partial t = H(\hat{\mathbf{Q}}(t))\hat{\psi}(t), \quad \hat{\psi}(0) = \psi, \quad (22)$$

where

$$\imath H(\mathbf{Q}(t)) = \hbar \exp[-\mathbf{L}\hat{\mathbf{Q}}(t)](K + \mathbf{L}^2/2) \exp[\mathbf{L}\mathbf{Q}(t)]$$

is a perturbed Hamiltonian $W(t)HW^*(t)$, $W(t) = \exp[-\mathbf{L}\hat{\mathbf{Q}}(t)]$ ($W^* = W^{-1}$, if $\mathbf{L}^* = -\mathbf{L}$). Indeed, with the help of Ito's formula, we obtain

$$\begin{aligned} d\hat{\psi} &= e^{-\mathbf{L}\hat{\mathbf{Q}}}d\hat{\chi} + de^{-\mathbf{L}\hat{\mathbf{Q}}}\hat{\chi} - \mathbf{L}^2e^{-\mathbf{L}\hat{\mathbf{Q}}}\hat{\chi}dt \\ &= e^{-\mathbf{L}\hat{\mathbf{Q}}}(\mathbf{L}d\hat{\mathbf{Q}} - Kdt)\hat{\chi} - (\mathbf{L}d\hat{\mathbf{Q}} - \mathbf{L}^2dt/2)e^{-\mathbf{L}\hat{\mathbf{Q}}}\hat{\chi} - \mathbf{L}^2\hat{\psi}dt \\ &= -e^{-\mathbf{L}\hat{\mathbf{Q}}}(K + \mathbf{L}^2/2)e^{\mathbf{L}\hat{\mathbf{Q}}}\hat{\psi}dt = \frac{1}{\imath\hbar}H(\hat{\mathbf{Q}}(t))\hat{\psi}dt. \end{aligned}$$

Eq. (22) for any observed trajectory $\mathbf{q}(t)$ can be viewed as the Schrödinger equation (1) with complex potentials \mathbf{U} and V . In the case of position observation, $\mathbf{L} = (\lambda/2)^{1/2}\mathbf{x}$, these potentials have the form

$$\begin{aligned} \mathbf{U}(t, \mathbf{x}) &= eA(t, \mathbf{x})/c + \imath\hbar(\lambda/2)^{1/2}\mathbf{q}(t), \\ V(t, \mathbf{x}) &= e\hat{\Phi}(t, \mathbf{x}) - \imath\hbar[(\lambda/2)^{1/2}\mathbf{x}]^2, \end{aligned} \quad (23)$$

and we get

$$\begin{aligned} H(\mathbf{q}) &= \exp[-(\lambda/2)^{1/2}\mathbf{x}\mathbf{q}](H - \frac{1}{2}\imath\hbar\lambda\mathbf{x}^2) \exp[\mathbf{x}\mathbf{q}(\lambda/2)^{1/2}] = \\ &= \frac{1}{2m}\{(e/c)A + [\nabla - (\lambda/2)^{1/2}\mathbf{q}]^2\} + e\Phi + \frac{\hbar\lambda}{2i}\mathbf{x}^2. \end{aligned}$$

The solution $\hat{\psi}(t, \mathbf{x})(\mathbf{q}) = \psi(\mathbf{q}^t, \mathbf{x})$ of the wave equation (22) for an observed trajectory $\mathbf{q} = \{\mathbf{q}(t)\}$ defines the posterior normalized wave function

$$\varphi(\mathbf{q}^t, \mathbf{x}) = \exp[(\lambda/2)^{1/2}\mathbf{x}\mathbf{q}(t) - \ln c(\mathbf{q}^t)]\psi(\mathbf{q}^t, \mathbf{x}), \quad (24)$$

where

$$\ln \hat{c}(t) = \int_0^t (\lambda/2)^{1/2}\hat{\mathbf{x}}d\hat{\mathbf{Q}} - \lambda\hat{\mathbf{x}}^2dt/2$$

can be obtained from the normalization condition

$$c(\mathbf{q}^t)^2 = \int \exp[(2\lambda)^{1/2}\mathbf{x}\mathbf{q}(t)]|\psi(\mathbf{q}^t, \mathbf{x})|^2d\mathbf{x}. \quad (25)$$

Note that the indirect nondemolition measurement considered is complete in the sense that the posterior state of the particle is pure if the initial state is pure. Hence the posterior dynamics of such indirectly completely observed particle is pure contrary to the prior dynamics which is always mixed (for $\lambda \neq 0$) even for vacuum quantum noise. In the case of noncomplete measurement, if for instance, an extra bath is added, or if the noise has a nonzero temperature, as is supposed in ref. [5], this fact is no more true.

The author is grateful to Professor R. S. Ingarden and Dr. P. Staszewski for their hospitality and useful discussions during his stay at the Institute of Physics, N. Copernicus University, Toruń, where this work was completed.

References

- [1] Barchielli A. Input and Output channels in quantum systems and quantum stochastic differential equations. In L. Accardi and von Waldenfels W., editors, *Quantum Probability and Applications III*. Berlin: Springer-Verlag, 1988. Volume 37.
- [2] Barchielli A. and Lupieri G. Quantum stochastic calculus, operation valued stochastic processes and continual measurement in quantum mechanics. *Math. Phys.*, 26:2222–2230, 1985.
- [3] Belavkin V. P. Quantum stochastic calculus and quantum nonlinear filtering. *Journal of Multivariate analysis*, to be published in 1992.
- [4] Belavkin V. P. Optimal filtering of markov signals with quantum white noise. *Radio Eng Electron Physics*, (25):1445–1453, 1980.
- [5] Belavkin V.P. Non-demolition measurement and control in quantum dynamical systems. In Blaquiere, editor, *Proc of C I S M Seminar on Information Complexity and Control in Quantum Physics, Udine 1985.*, pages 311–329, Springer-Verlag, Wien-New York, 1987.
- [6] Belavkin V.P. Non-demolition stochastic calculus in Fock space and nonlinear filtering and control in quantum systems. In *Proc of Fourteenth Winter School in Theor Phys, Karpacz 1988*, ‘Stochastic Methods in Mathematics and Physics’, pages 310–324, Singapore, 1989. World Scientific.
- [7] Belavkin V.P. and P Staszewski. A quantum particle undergoing continuous observation. *Phys Letters A*, (140):359–362, 1989.
- [8] Hudson R.L. and Parthasarathy K.R. Quantum Ito’s formula and stochastic evolution. *Comm. Math. Phys.*, 93:301–323, 1984.