Nondemolition observation of a free quantum particle

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Abstract

A stochastic model of a continuous nondemolition observation of a free quantum Brownian motion is presented. The nonlinear stochastic wave equation describing the posterior dynamics of the observed quantum system is solved in a Gaussian case for a free particle of mass $m > 0$. It is shown that the dispersion of the wave packet does not increase to infinity like for the free unobserved particle but tends to the finite limit $\tau_\infty^2 = (\hbar/2\lambda m)^{1/2}$ where λ is the accuracy coefficient of an indirect nondemolition measurement of the particle position, and $\hbar$ is Planck constant.

1 Introduction

The Schrödinger equation describes the time–development of the wave function of a quantum system only for the time intervals between the succeeding instants of measurements. At the instant of a measurement of some observable with a discrete spectrum, Z, the quantum system makes an immediate transition (jump) from the state $\psi(t)$ to the eigenstate $\psi_z(t)$ corresponding to the obtained eigenvalue z of Z with the probability $|\langle\psi(t)|\psi_z(t)\rangle|^2$. Such a stochastic time–behaviour of the system at the instant of the measurement assures the repeatability of the results of measurements, if a second measurement were taken immediately after the first one then for discrete observable Z the measurement would again give z [1]. It is intuitively obvious that if one would perform measurements with a high frequency – in a limit continuously in time – the quantum system would show a stochastic irreversible behaviour for the whole period of observation. Therefore the time–development of a continuously observed quantum system cannot be governed by the deterministic Schrödinger equation describing the reversible motion. This statement remains true also in the case of the measurements of an observable with a continuous spectrum though for observables with continuous spectra the repeatability hypothesis is

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not assumed [1–4] as, in general, there are non–zero (a priori) probabilities of the results of such a measurement belonging to disjoint Borel sets.

The irreversible and stochastic behaviour of the continuously observed quantum system expressed by the so–called collapse or reduction of the wave function has no analogue in the classical deterministic mechanics. The Hamilton equations do not depend (for a nondemolition observation) on whether the dynamical object is observed during its motion along its trajectory. Ignoring that difference in the behaviour of classical and quantum observed objects leads to various quantum paradoxes of Zeno kind [5–11] which can be explained only in the way of a consistent investigation of the disturbed stochastic dynamics of the quantum system undergoing an observation.

It is quite natural to discuss this problem in the framework of stochastic quantum mechanics of open systems [12, 13] on the basis of the theory of nondemolition measurements developed recently [14–17]. The principle of a nondemolition continuous observation of a quantum system can be formulated as follows [17]:

(i) for any quantum measurement there exist observables $\hat{Q}(r), r \leq t$, which commute for any t with all Heisenberg operators $\hat{Z}(t)$ of the system represented in the Hilbert space corresponding to "the system–measuring apparatus",

(ii) according to the causality principle one does not impose any conditions on the future observables $\hat{Q}(s), s > t$, with respect to the past observables of the system $\hat{Z}(r), r \leq t$. A non–trivial nondemolition observation in the above–mentioned sense is provided by indirect measurements which can be only realized by considering the observed quantum system as an open one.

From the experimental point of view it is natural to consider indirect measurements because any measurement is taken with the help of some experimental device. The indirect measurements allow to describe the state changes resulting from the measurements of observables with continuous spectra [4] which are assumed to be nonideal. The necessity to use indirect measurements for the existence of the continual limit (with $\Delta t \to 0$) for successive instantaneous measurements taken at instants separated by Δt is proved in Ref. [18].

In this paper we shall illustrate the approach of the continuous quantum nondemolition measurement on the example of resolving the quantum Zeno paradox for a three–dimensional free particle undergoing an observation modeling the measurement of a trajectory of a quantum particle in a bubble chamber briefly reported in Ref. [17] and for the one–dimensional case in [19].

Sec. II has a preparatory character, we present here a stochastic model of a continuous nondemolition observation of a quantum system interacting with M–dimensional Bose field reservoir modeling the measuring device, proposed in [14–17].

In Sec. III we derive the filtering equation – the stochastic differential equation describing the time–development of the wave function of the quantum system observed by means of the vector "field coordinate" process. This equation was recently obtained with the help of quantum filtration method [20, 21]. The presented derivation – via stochastic instrument in the sense of Davies and Lewis [2, 3] – generalizes the result of Ref. [22] to the case of multidimensional

observation (the infinite dimensional and general cases see in [23, 24]).

In Sec. IV we solve the filtering equation for the three–dimensional free quantum particle undergoing the continuous nondemolition observation of its position. We prove that the dispersion of the Gaussian wave packet does not spread out in time but tends to the finite limit $\lim_{t\to\infty}\tau^2(t) = (\hbar/2\lambda m)^{1/2}$, where $m > 0$ is the mass of the observed particle and λ stands for the accuracy coefficient of the indirect nondemolition measurement of the particle position.

2 Stochastic model of a continuous multidimensional diffusion observation of a quantum system

Let us assume that a quantum system $\mathcal{S}$ living in the Hilbert space $\mathcal{H}_0$ is coupled at instant $t = 0$ to the reservoir (measuring device) consisting of M independent Bose fields described by vector–operators $\mathbf{b}(t) = [b_j(t)]_1^M$, $\mathbf{b}^+(t) = [b_j^+(t)]_1^M$ acting in $\mathcal{F} = \mathcal{F}_{\mathrm{sym}}(\mathbb{C}^M \otimes L^2(\mathbb{R}_+))$, the symmetric Fock space over $\mathbb{C}^M \otimes L^2(\mathbb{R}_+)$. The Bose field operators satisfy the canonical commutation relations

$$[b_j(t),\ b_k^+(s)] = \delta_{jk}\delta(t-s)\ ,\ [b_j(t),\ b_k(s)] = 0 (j,\ k = 1,\ldots,\ M). \qquad (2.1)$$

The reservoir is assumed to be initially prepared in the vacuum state; $\langle b_k(t)\rangle_v = \langle b_k^+(t)\rangle_v = \langle b_k^+(t)b_k(s)\rangle_v = 0$, $\langle b_k(t)b_l^+(s)\rangle_v = \delta_{kl}\delta(t-s)$. The real and imaginary parts of $\mathbf{b}(t)$ defined as $\mathrm{Re}\,\mathbf{b}(t) = \frac{1}{2}(\mathbf{b}(t)+\mathbf{b}^+(t))$, $\mathrm{Im}\,\mathbf{b}(t) = \frac{1}{2\mathrm{i}}(\mathbf{b}(t)-\mathbf{b}^+(t)$ do not commute, but each of them has the statistical properties of the (classical) standard M–dimensional white noise. Similarly as in the classical case [25] the time–evolution of the system interacting with the reservoir can be described in a mathematically rigorous way in terms of a stochastic differential equation [12, 13]. A quantum stochastic calculus (QSC) of Ito type has been developed by Hudson and Parthasarathy [12]. Here we give the formal rules of QSC which will be needed in our paper.

Let us define annihilation and creation processes

$$B_j(t) = \int_0^t b_j(s)\mathrm{d}s\ ,\ B_j^+(t) = \int_0^t b_j^+(s)\mathrm{d}s\ , \qquad (2.2)$$

which satisfy the following commutation relations

$$[B_j(t)\ ,\ B_k^+(s)] = \delta_{jk}\min(t,s)\ ,\ [B_j(t)\ ,\ B_k(s)] = 0. \qquad (2.3)$$

The pair $\mathbf{B}(t) = [B_j(t)]_1^M$, $\mathbf{B}^+(t) = [B_j^+(t)]_1^M$ is the quantum analogue of standard complex M–dimensional Wiener diffusion process. The stochastic differentials of the processes in (2.2)

$$\mathrm{d}B_j(t) = B_j(t+\mathrm{d}t) - B(t)\ ,\ \mathrm{d}B_j^+(t) = B_j^+(t+\mathrm{d}t) - B_j^+(t)$$

satisfy the multiplication rules

$$\mathrm{d}B_j(t)\mathrm{d}B_k^+(t) = \delta_{jk}\mathrm{d}t \qquad (2.4)$$

and all other products involving $dB_j(t)$, $dB_j^+(t)$ and dt are equal to zero [12]. The Hudson–Parthasarathy differentiation formula [12] for the product $M(t)N(t)$ of the adapted processes (the operators on $\mathcal{H}_0 \otimes \mathcal{F}$ which depend on $\mathbf{B}(s)$ and $\mathbf{B}^+(s)$ only for times $s \leq t$) reads

$$d(M(t) \cdot N(t)) = dM(t) \cdot N(t) + M(t) \cdot dN(t) + dM(t) \cdot dN(t). \qquad (2.5)$$

We shall assume a unitary time–evolution of the compound quantum system in $\mathcal{H}_0 \otimes \mathcal{F}$. The unitary evolution operator $U(t)$ for the system $\mathcal{S}$ coupled to the Bose reservoir is assumed to satisfy the Ito quantum stochastic differential equation (QSDE) in the form [12, 13]

$$dU(t) = \left[\sum_j (L_j dB_j^+(t) - L_j^+ dB_j(t)) - K dt \right] U(t) \, , \; U(0) = I, \qquad (2.6)$$

where

$$K = \frac{i}{\hbar} H + \frac{1}{2} \sum_j L_j^+ L_j. \qquad (2.7)$$

In these formulas H stands for the Hamiltonian of $\mathcal{S}$, $i\hbar \sum (L_j dB_j^+ - L_j^+ dB_j)$ describes the interaction between $\mathcal{S}$ and the fields, $-\frac{1}{2} \sum L_j^+ L_j$ is the Ito correction term. (If one applied instead of (2.6) a QSDE based on the quantum Stratonovich integral [13] this term would disappear). With the help of (2.6) the Heisenberg equation of motion for any observable of $\mathcal{S}$ can be easily obtained. By applying to the product

$$\hat{Z}(t) = U^+(t) Z U(t) \qquad (2.8)$$

the quantum Ito formula (2.5), Eq. (2.6) and its adjoint equation, one can check with the help of (2.4) and (2.7) that the Heisenberg observable $\hat{Z}(t)$ satisfies the following QSDE

$$d\hat{Z} + \left(\hat{K}^+ \hat{Z} + \hat{Z}\hat{K} - \sum_j \hat{L}_j^+ \hat{Z} \hat{L}_j \right) dt = \sum_j \left(\left[\hat{Z}, \hat{L}_j \right] dB_j^+ + \left[\hat{L}_j^+, \hat{Z} \right] dB_j \right),$$
$$(2.9)$$

where we have employed the simplified notation: $\hat{Z}$ for $\hat{Z}(t)$ etc.

Eq. (2.6) or Eqs. (2.9) describe the distorted dynamics of the initially closed quantum system $\mathcal{S}$ under the stochastic interaction with the Bose fields. The fields, however, do not only disturb the system. They also give some possibility of a continuous (in time) observation of $\mathcal{S}$. Let us first pay attention to their time–development. In the Heisenberg picture, the processes

$$\hat{B}_j(t) = U^+(t) B_j(t) U(t) \qquad (2.10)$$

exhibit a useful property [26]: they remain unchanged for all times $s \geq t$, i.e.

$$\hat{B}_j(t) = U^+(s) B_j(t) U(s) \, , \; s \geq t. \qquad (2.11)$$

Obviously, the same holds for the creation process $B^+(t)$. The property (2.11) results essentially from two facts: Eq. (2.6) is written in the interaction picture with respect to a free dynamics of the fields and the coupling between $\mathcal{S}$ and fields is singular. The vector annihilation and creation processes $\mathbf{B}(t)$ and $\mathbf{B}^+(t)$ are called input (annihilation, creation) processes while $\hat{\mathbf{B}}$, $\hat{\mathbf{B}}^+(t)$ are called output processes [13]. The input processes describe Bose fields before their interaction with $\mathcal{S}$, the output ones – after the interaction. Note that due to (2.11) the output processes satisfy the nondemolition conditions [17]

$$[\hat{\mathbf{B}}(s), \hat{Z}(t)] = U^+(t)[\mathbf{B}(s), Z]U(t) = 0 \quad \forall s \leq t \tag{2.12}$$

0Let us consider the continuous measurement of the output vector "field coordinate" ("diffusion") process

$$\hat{\mathbf{Q}}(t) = \hat{\mathbf{B}}(t) + \hat{\mathbf{B}}^+(t) = U^+(t)\mathbf{Q}(t)U(t),$$

where $\mathbf{Q}(t) = \mathbf{B}(t) + \mathbf{B}^+(t)$ is the input vector Wiener process. From (2.3) it follows that

$$[\hat{\mathbf{Q}}(t), \hat{\mathbf{Q}}(t')] = 0 \quad Vt, t' \geq 0 \tag{2.13}$$

i.e. the output Hermitian process $\hat{\mathbf{Q}}$ is selfnondemolition. Therefore the output process $\hat{\mathbf{Q}}(t)$ can be observed as a classical process. Due to (2.12), (2.13) the measurement of $\hat{\mathbf{Q}}$ is nondemolition [16, 17] with respect to the time–evolution of the system, for any Z

$$[\hat{\mathbf{Q}}(s), \hat{Z}(t)] = 0 \quad Vs \leq t. \tag{2.14}$$

This means that the measurement of $\mathbf{Q}$ disturbs neither the present nor the future state of the system $\mathcal{S}$. The stochastic differential equation for $\hat{\mathbf{Q}}(t)$, which can be easily obtained in the same way as Eq. (2.9), has the form

$$d\hat{\mathbf{Q}}(t) = (\hat{\mathbf{L}}(t) + \hat{\mathbf{L}}^+(t))dt + d\mathbf{Q}(t). \tag{2.15}$$

Therefore, the process $\hat{\mathbf{Q}}(t)$ contains some information about $\mathcal{S}$.

Eq. (2.6) does not include any observation, it describes the perturbed dynamics of the unobserved system $\mathcal{S}$ (represented in $\mathcal{H}_0 \otimes \mathcal{F}$). Following Refs. 16, 17 we shall call it the prior dynamics. Similarly, for any initial systematic observable Z, (2.9) is the equation for the unobserved process $\hat{Z}(t)$. But for each Z we have the possibility of considering Eq. (2.9) together with Eq. (2.15), consequently $\hat{Z}(t)$ for any initial Z becomes partially observed. As it is proved in Ref. [20] the condition (2.14) gives the possibility to define the posterior (observed) mean values of $\hat{Z}(t)$ under the condition of observation of any nonanticipating function of $\hat{\mathbf{Q}}$ up to the moment t. It turns out [16, 17] that if the Bose reservoir is initially prepared to be in the vacuum state and the initial state of $\mathcal{S}$ is pure, then the posterior state of $\mathcal{S}$ is a pure one.

3 Quantum filtering equation

In this section we shall derive the quantum filtering equation – the QSDE which describes the time–development of the posterior state of the quantum system $\mathcal{S}$ undergoing the M–dimensional diffusion observation (2.15). It shall be done with the help of the method of solving the differential equation for the generating map of the corresponding instrument [2, 3]. For $M = 1, \infty$ this approach was applied by one of us (V.P.B.) in Ref. [22–24].

Let us denote by $\nu = \otimes_{j=1}^{M} \nu_j$ the standard product Wiener probability measure on the space Ω of continuous trajectories $\mathbf{q} = \{\mathbf{q}(t)|t > 0\}$ of the observed process $\hat{\mathbf{Q}}$ restricted to the space $\Omega^t = \{\mathbf{q}(t)|\mathbf{z} \in \Omega\}$ of the trajectories stopped at $t : \mathbf{q}^t = \{\mathbf{q}(r)|r \leq t\}$. Consider the instrument $\mathcal{I}^t$ on the algebra of operators Z of the observed quantum system $\mathcal{S}$ as a function of the observed event $\mathrm{d}\mathbf{q}$ up to the instant t. $\mathcal{I}^t$, by its definition, defines the time–evolution $\rho \mapsto \rho^t(\mathrm{d}q)$ of an initial state functional $\rho : Z \mapsto \rho[Z]$ of $\mathcal{S}$ to the state $\rho^t(\mathrm{d}\mathbf{q}) = \rho \circ \mathcal{I}^t(\mathrm{d}\mathbf{q})$ normalized to the probability $\mu^t(\mathrm{d}\mathbf{q}) = \rho\left[\mathcal{I}^t(\mathrm{d}\mathbf{q})[I]\right]$.

Define the generating map of $\mathcal{I}^t$ in the following way (cf. also Refs. [27, 28])

$$\Gamma(\mathbf{l}, t)[Z] = \int_{\Omega^t} \exp\left\{\int_0^t \mathbf{l}(r)\mathrm{d}\mathbf{q}(r)\right\} \mathcal{I}^t(\mathrm{d}\mathbf{q})[Z], \qquad (3.1)$$

where $\mathbf{l}(t) = [l_j(t)]_1^M$ with components l_j being integrable c–valued functions. The generating map can also be defined by the condition

$$\langle \psi | \Gamma(\mathbf{l}, t)[Z]\psi \rangle = \langle \hat{Y}(\mathbf{l}, t)\hat{Z}(t) \rangle, \qquad (3.2)$$

where

$$\hat{Y}(\mathbf{l}, t) = \exp\left\{\sum_{j=1}^{M} \int_0^t l_j(r)\mathrm{d}\hat{Q}_j(r)\right\} = \prod_{j=1}^{M} \exp\left\{\int_0^t l_j(r)\mathrm{d}\hat{Q}_j(r)\right\}. \qquad (3.3)$$

The mean value on the right hand side of (3.2) is taken with respect to $\psi \otimes \delta_\phi$ with $\psi \in \mathcal{H}_0$ being an (arbitrary) initial pure state of $\mathcal{S}$ and $\delta_\phi \in \mathcal{F}$ the vacuum state vector for the fields. Note that the M–exponential output process $\hat{Y}(\mathbf{l}, t)$ given by (3.3) is nondemolition and selfnondemolition.

Let us now find the differential equation for the generating map $\Gamma(\mathbf{l}, t)$ of the instrument $\mathcal{I}^t$. According to (3.2) it can be done by finding the differential equation for the mean value $\langle \hat{Y}(\mathbf{l}, t)\hat{Z}(t) \rangle$. First we obtain the stochastic differential equation for $\hat{G}(t) = \hat{Y}(t)\hat{Z}(t)$. Let us write $\hat{G}(t)$ in the form $\hat{G}(t) = U^+(t)G(t)U(t) = U^+(t)Y(t)ZU(t)$, where $Y(t)$ is the input process corresponding to (3.3):

$$Y(\mathbf{l}, t) = \exp\left\{\sum_{j=1}^{M} \int_0^t l_j(r)\mathrm{d}Q_j(r)\right\}. \qquad (3.4)$$

Then from Ito's formula (2.5) applied to the product $\hat{G} = U^+ G U$ we get

$$
\begin{aligned}
\mathrm{d}\hat{G} &= U^+ \left[\sum_j \left(\frac{1}{2} l_j^2 G + L_j^+ G l_j + l_j G L_j + L_j^+ G L_j \right) - K^+ G - G K \right] U \mathrm{d}t \\
&+ U^+ \left[\sum_j \left(L_j^+ G + G(l_j - L_j^+) \right) \mathrm{d}B_j + (G L_j + (l_j - L_j)G)\mathrm{d}B_j^+ \right] U,
\end{aligned}
$$

where we have used (2.6), multiplication rules (2.4) and the stochastic differential of G, $\mathrm{d}G = \mathrm{d}Y \cdot Z$ with

$$
\mathrm{d}Y(\mathbf{l}, t) = \sum_j \left(l_j(t)\mathrm{d}Q_j(t) + \frac{1}{2} l_j^2(t)\mathrm{d}t \right) Y(\mathbf{l}, t) \tag{3.6}
$$

which can be obtained from (3.4) by classical Ito's formula [25].

Eq. (3.5) yields the following differential equation for the mean value of $\hat{G}(t) = \mathring{Y}(\mathbf{l}, t)\mathring{Z}(t)$

$$
\langle \mathrm{d}\hat{G} \rangle = \langle \hat{\eta}(t)| \sum_j [\frac{1}{2} l_j^2 G + l_j(L_j^+ G + G L_j) + L_j^+ G L_j] - (K^+ G + G K)|\hat{\eta}(t)\rangle \mathrm{d}t \tag{3.7}
$$

with $\hat{\eta}(t) = U(t)\eta$, $\eta = \psi \otimes \delta_\phi$. Note that the mean values of terms containing $\mathrm{d}B_j$ and $\mathrm{d}B_j^+$ in (3.5) do not appear in (3.7), they are equal to zero, because for each j

$$
\mathrm{d}B_j(t)U(t)\eta = U(t)\mathrm{d}B_j(t)\eta = 0. \tag{3.8}
$$

From (3.2) and (3.7) one can easily get the forward differential equation for the generating map Γ:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Gamma[Z] = \Gamma \left[\sum_j \left(\frac{1}{2} l_j^2 Z + l_j(L_j^+ Z + Z L_j) + L_j^+ Z L_j \right) - K^+ Z - Z K \right] \tag{3.9}
$$

with the initial condition $\Gamma(\mathbf{l}, 0)[Z] = Z$.

We shall prove that the solution of (3.9) has the form

$$
\Gamma(\mathbf{l}, t)[Z] = \int_{\Omega^t} Y(\mathbf{l}, \mathbf{q}^t)V^+(\mathbf{q}^t)ZV(\mathbf{q}^t)\mathrm{d}\nu(\mathbf{q}^t) \tag{3.10}
$$

with the stochastic propagator $V(t)$ being the solution of a QSDE in the form

$$
\mathrm{d}V(t) = -KV(t)\mathrm{d}t + \sum_j L_j \mathrm{d}Q_j(t) , \ V(0) = I. \tag{3.11}
$$

Let us define the stochastic map $\Phi(t)$ from the algebra of observables of $\mathcal{S}$ into itself

$$
\Phi(t)[Z] = V^+(t)ZV(t). \tag{3.12}
$$

Then from Ito's formula (2.5) applied to the product appearing in (3.12) we get

$$d(\Phi(t)[Z]) = dV^+(t)ZV(t) + V^+(t)ZdV(t) + dV^+(t)ZdV(t).$$

By making use of (3.11) we obtain the recursive filtering equation for the stochastic map $\Phi(t)$

$$d(\Phi(t)[Z]) = \Phi(t)\left[\sum_j L_j^+ ZL_j - K^+ Z - ZK\right] + \sum_j \Phi(t)[L_j^+ Z + ZL_j]dQ_j(t)$$
(3.13)

with $\Phi(0)[Z] = Z$. The stochastic map (3.12) defines for any trajectory $\mathbf{q}$ the selective instrument, $\Phi(t)(\mathbf{q})[Z] = \Phi(\mathbf{q}^t)[Z] = V^+(\mathbf{q}^t)ZV(\mathbf{q}^t)$. Taking into account that

$$d(Y(\mathbf{l},t)\Phi(t)[Z]) = dY(\mathbf{l},t)\Phi(t)[Z] + Y(\mathbf{l},t)d\Phi(t)[Z] + dY(\mathbf{l},t)d\Phi(t)[Z]$$

$$+ Y(\mathbf{l},t)\Phi(t)\Big[\sum_j (\tfrac{1}{2}l_j^2(t)Z + l_j(t)(L_j^+ Z + ZL_j) + L_j^+ ZL_j) - K^+ Z - ZK\Big]dt$$

and averaging it with respect to the standard product Wiener measure one obtains (3.9) for the mean value (3.10) of the product $Y(\mathbf{l},\mathbf{q}^t)\Phi(\mathbf{q}^t)[Z]$.

So, the wave function $\hat{\chi}(t) = V(t)\psi$ of the system $\mathcal{S}$ under the continuous nondemolition diffusion observation of $\hat{\mathbf{Q}}$, satisfies the stochastic dissipative differential equation

$$d\hat{\chi}(t) + \left(\frac{\mathrm{i}}{\hbar}H + \frac{1}{2}\sum_j L_j^+ L_j\right)\hat{\chi}(t)dt = \sum_j L_j \hat{\chi}(t)dQ_j(t), \ \hat{\chi}(0) = \psi. \quad (3.14)$$

Eq. (3.14) plays an analogous role to the Schrödinger equation for the unobserved quantum system. (In (3.14) $d\mathbf{q}$ can be replaced with $d\hat{\mathbf{Q}}$ because in the Schrödinger picture $\mathbf{Q}$ and $\hat{\mathbf{Q}}$ coincide.) The posterior wave function $\hat{\chi}(t)$ is normalized to the probability density

$$p(\mathbf{q}^t) = \langle V(\mathbf{q}^t)\psi | V(\mathbf{q}^t)\psi \rangle \equiv \hat{p}(t)(\mathbf{q})$$

of the observed process $\hat{\mathbf{Q}}$ with respect to the standard product Wiener measure of the input process $\mathbf{Q}$. It follows from the integral representation of (3.2)

$$\langle \hat{Y}(\mathbf{l},t)\hat{Z}(t)\rangle = \int_{\Omega^t} Y(\mathbf{l},q^t)\langle V(\mathbf{q}^t)\psi | ZV(\mathbf{q}^t)\rangle d\nu(\mathbf{q}^t) \quad (3.15)$$

giving for $Z = I$ the mean value of the output process (3.3) as the generating function of the output probability measure

$$d\mu(\mathbf{q}^t) = p(\mathbf{q}^t)d\nu(\mathbf{q}^t).$$

The formula (3.15) defines the posterior mean value $\langle Z\rangle(q^t)$ as

$$\langle Z\rangle(\mathbf{q}^t) = \langle \psi(\mathbf{q}^t) | Z\psi(\mathbf{q}^t)\rangle \equiv \hat{z}(t)(\mathbf{q})$$

in terms of the normalized posterior wave function $\hat{\psi}(t)\mathbf{q}) = \psi(\mathbf{q}^t)\psi(\mathbf{q}^t) = \chi(\mathbf{q}^t)/q(t)^{1/2}$.

The normalized posterior wave function $\hat{\psi}(t)$ satisfies the nonlinear stochastic wave equation [16, 17]

$$\mathrm{d}\hat{\psi}(t) + \left(\frac{1}{2}\sum_j \tilde{L}_j^+(t)\tilde{L}_j(t) + \frac{\mathrm{i}}{\hbar}\tilde{H}(t)\right)\hat{\psi}(t)\mathrm{d}t = \sum_j \tilde{L}_j(t)\mathrm{d}\tilde{Q}_j(t)\hat{\psi}(t), \quad (3.16)$$

where
$$\tilde{L}_j(t) = L_j - \mathrm{Re}\,\hat{l}_j(t)\ ,\ \tilde{H}(t) = H - \hbar\sum_j \mathrm{Re}\,\hat{l}_j(t)\,\mathrm{Im}\,L_j,$$

and $\mathrm{d}\tilde{Q}_j(t) = \mathrm{d}Q_j(t) - 2\,\mathrm{Re}\,\hat{l}_j(t)\mathrm{d}t$ is the so–called Wiener innovating process.

Eq. (3.16) can be obtained from Eq. (3.14) in the following way. Writing $\hat{\psi}(t)$ in the form $\hat{\psi}(t) = \hat{\chi}(t)(\hat{\chi}^+(t)\hat{\chi}(t))^{-1/2}$ we get

$$\mathrm{d}\hat{\psi} = \mathrm{d}\hat{\chi}\cdot(\hat{\chi}^+\hat{\chi})^{-1/2} + \hat{\chi}\cdot\mathrm{d}[(\hat{\chi}^+\hat{\chi})^{-1/2}] + \mathrm{d}\hat{\chi}\cdot\mathrm{d}[(\hat{\chi}^+\hat{\chi})^{-1/2}]\ . \quad (3.17)$$

For $\hat{\chi}$ satisfying Eq. (3.14) one finds easily

$$\mathrm{d}(\hat{\chi}^+\hat{\chi}) = 2\sum_j \hat{\chi}^+(\mathrm{Re}\,L_j)\hat{\chi}\mathrm{d}Q_j$$

and by the classical Ito formula

$$\mathrm{d}[(\hat{\chi}^+\hat{\chi})^{-1/2}] = (\hat{\chi}^+\hat{\chi})^{-1/2}\left\{-\sum_j \mathrm{Re}\,\hat{l}_j(t)\mathrm{d}Q_j + \frac{3}{2}\sum_j(\mathrm{Re}\,\hat{l}_j(t))^2\mathrm{d}t\right\} \quad (3.18)$$

Finally combining (3.17), (3.18) and (3.14) yields Eq. (3.16).

4   Watchdog effect

The Schrödinger equation for a free particle

$$\dot{\psi} - \frac{\mathrm{i}\hbar}{2m}\Delta\psi = 0 \tag{4.1}$$

describes the effect of spreading out of the wave packet. The probability of detection of the quantum particle in any finite coordinate region tends to zero as time increases.

On the other hand experimental data on observed quantum particles show well–localized paths of quantum particles (for instance in bubble chamber experiments). This phenomenon being an example of the watchdog effect is also known as quantum Zeno paradox [5] because it is in contradiction with predictions of Eq. (4.17). The above paradox of the orthodox quantum mechanics

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can be resolved in the framework of posterior quantum dynamics for observed quantum systems by using nondemolition filtering methods.

The typical observations in quantum systems are indirect (in the bubble chamber the path of an ionizing particle is made visible by a string of vapor bubbles), moreover one has to consider the interaction with the measuring device, hence the observed quantum object should be considered as an open quantum system.

The aim of this section is to demonstrate the watchdog effect that occurs for a free quantum particle coupled to the three–dimensional Bose field in the vacuum state (measuring device), the position of which is continuously observed. We shall consider an indirect measurement of the particle position $\mathbf{X} = [X_1, X_2, X_3]$ therefore we choose the coupling operator $\mathbf{L}$ (cf. (2.6) and (2.15)) to be proportional to $\mathbf{X}$,

$$\mathbf{L} = (\lambda/2)^{1/2}\mathbf{X}. \tag{4.2}$$

With such a choice of $\mathbf{L}$ we get the QSDEs describing the perturbed dynamics of the particle in the Heisenberg picture by putting for Z in Eq. (2.9) the position and momentum components

$$d\hat{\mathbf{X}}(t) = \frac{1}{m}\hat{\mathbf{P}}(t)dt \ , \ d\hat{\mathbf{P}}(t) = (2\lambda)^{1/2}\hbar \ d(\mathrm{Im}\,\mathbf{B}^+(t)). \tag{4.3}$$

Eqs. (4.3) describe the motion of the particle upon the stochastic (Langevin) force $f(t) = (2\lambda)^{1/2}\hbar \ \mathrm{Im}\,\mathbf{b}^+(t) = -(2\lambda)^{1/2}\hbar \ \mathrm{Im}\,\mathbf{b}(t)(cf.(2.2))$ from the Bose reservoir.

The observed nondemolition field coordinate process $\hat{\mathbf{Q}}(t)$ ((2.13)) satisfies due to (2.15) and (4.2) the QSDE in the form

$$d\hat{\mathbf{Q}}(t) = (2\lambda)^{1/2}\hat{\mathbf{X}}(t)dt + d\mathbf{Q}(t). \tag{4.4}$$

Eq. (4.4) describes the indirect (and imperfect) measurement of the particle position. Note that in terms of generalized derivatives of the processes $\hat{\mathbf{Q}}$ and $\mathbf{Q}$ Eq. (4.4) can be written as $\dot{\hat{\mathbf{Q}}}(t) = (2\lambda)^{1/2}\hat{\mathbf{X}}(t) + 2\,\mathrm{Re}\,\dot{\mathbf{B}}(t) = (2\lambda)^{1/2}\hat{\mathbf{X}}(t) + 2\,\mathrm{Re}\,\mathbf{b}(t)$, therefore the (generalized) stochastic process $\dot{\hat{\mathbf{Q}}}(t)$ describes the measurement of $\hat{\mathbf{X}}(t)$ together with a random error given by the standard vector white noise $2\,\mathrm{Re}\,\mathbf{b}(t)$. From the last formula one can see that the positive constant λ can be interpreted as the measurement accuracy coefficient.

Let us denote by $\hat{\mathbf{q}}(t) = [\hat{q}_j(t)]_{j=1}^3$ and $\hat{\mathbf{q}}(t) = [\hat{q}_j(t)]_{j=1}^3$ the posterior mean values of position and momentum of the observed particle. We have

$$\hat{\mathbf{q}}(t) = \int \hat{\psi}(t, \mathbf{x})^* \mathbf{x}\hat{\psi}(t, \mathbf{x})d\mathbf{x} \ , \ \hat{\mathbf{p}}(t) = \int \hat{\psi}(t, \mathbf{x})^* \frac{\hbar}{\mathrm{i}}\nabla\hat{\psi}(t, \mathbf{x})d\mathbf{x}. \tag{4.5}$$

According to (3.16) the posterior (normalized) wave function satisfies in the considered case the stochastic wave equation which in the coordinate representation has the form

$$d\hat{\psi} - \left(\frac{\mathrm{i}\hbar}{2m}\Delta\hat{\psi} - \frac{\lambda}{4}(\mathbf{x} - \hat{\mathbf{q}})^2\hat{\psi}\right)dt = \hat{\psi}\left(\frac{\lambda}{2}\right)^{1/2}(\mathbf{x} - \hat{\mathbf{q}})d\tilde{\mathbf{Q}} \ , \ \hat{\psi}(0) = \psi \quad (4.6)$$

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with

$$d\tilde{\mathbf{Q}}(t) = d\mathbf{Q}(t) - (2\lambda)^{1/2}\,\hat{\mathbf{q}}(t)dt.$$

Let us now discuss the time–development of the posterior wave function assuming that the initial state ψ has the form of the Gaussian wave packet,

$$\psi(\mathbf{x}) = (2\sigma_q^2\pi)^{-3/4}\exp\left\{-\frac{1}{4\sigma_q^2}(\mathbf{x}-\mathbf{q})^2 + \frac{i}{\hbar}\mathbf{q}\mathbf{x}\right\}\,, \qquad (4.7)$$

$\mathbf{p}$ and $\mathbf{q}$ denote the initial mean values of position and momentum of the particle and σ_q^2 stands for the initial dispersion of the wave packet. We shall prove that the solution of Eq. (4.6) corresponding to the initial condition (4.7) has the form of Gaussian packet

$$\hat{\psi}(t,\mathbf{x}) = \hat{c}(t)\exp\left\{-\frac{1}{2}\omega(t)(\mathbf{x}-\hat{\mathbf{q}}(t))^2 + \frac{i}{\hbar}\hat{\mathbf{p}}(t)\mathbf{x}\right\} \qquad (4.8)$$

with posterior mean values $\hat{\mathbf{q}}(t), \hat{\mathbf{p}}(t)$, cf. (4.5), fulfilling linear filtration equations and $\omega(t)$ satisfying the Riccatti differential equation. In Eq. (4.8) $\hat{c}(t) = (2\tau_q^2\pi)^{-3/4}$ up to unessential stochastic phase factor and $\tau_q^2 = \hat{\mathbf{q}^2} - \hat{\mathbf{q}}^2$ is the posterior position dispersion.

It is convenient to rewrite Eq. (4.6) in terms of the complex osmotic velocity. By introducing

$$T(t,\mathbf{x}) = R(t,\mathbf{x}) + iS(t,\mathbf{x}) = \hbar\,ln\hat{\psi}(t,\mathbf{x}),$$

next by Ito's rule

$$dT(\hat{\psi}) = T'(\hat{\psi})d\hat{\psi} + \frac{1}{2}T''(\hat{\psi})(d\hat{\psi})^2$$

applied to the function $T = \hbar\ln\psi$ and by taking into account that

$$(d\hat{\psi})^2 = \frac{\lambda}{2}(\mathbf{x}-\hat{\mathbf{q}})^2\hat{\psi}^2\,dt$$

we obtain Eq. (4.6) in terms of T. From this equation we get for the complex osmotic velocity

$$\mathbf{W}(t,\mathbf{x}) = \frac{1}{m}\nabla T(t,\mathbf{x}) = \mathbf{U}(t,\mathbf{x}) + i\mathbf{V}(t,\mathbf{x})$$

the following equation

$$d\mathbf{W} + \left[\frac{\hbar\lambda}{m}(\mathbf{x}-\hat{\mathbf{q}}) - \frac{i}{2}(\nabla\mathbf{W}^2 + \frac{\hbar}{m}\Delta\mathbf{W})\right]dt = \left(\frac{\lambda}{2}\right)^{1/2}\frac{\hbar}{m}d\tilde{\mathbf{Q}}. \qquad (4.9)$$

We shall look for the solution of Eq. (4.9) corresponding to the initial condition

$$\mathbf{W}(0,\mathbf{x}) = \frac{\hbar}{m}\nabla\ln\psi(\mathbf{x}) = \frac{\hbar}{2m\sigma_q^2}(\mathbf{q}-\mathbf{x}) + \frac{i}{m}\mathbf{p} \qquad (4.10)$$

in the linear form

$$\mathbf{W}(t,x) = \hat{\mathbf{W}}(t) - \frac{\hbar}{m}\omega(t)\mathbf{x} \tag{4.11}$$

where in accordance with (4.8)

$$\hat{\mathbf{w}}(t) = \frac{\hbar}{m}\omega(t)\hat{\mathbf{q}}(t) + \frac{\mathrm{i}}{m}\hat{\mathbf{p}}(t). \tag{4.12}$$

By putting $\nabla \mathbf{W}^2 = -\frac{2\hbar\omega}{m}\mathbf{W}$, $\Delta\mathbf{W} = 0$ into (4.9) we obtain the following system of equations for coefficients $\hat{\mathbf{w}}(t)$ and $\omega(t)$

$$\mathrm{d}\hat{\mathbf{w}}(t) + \frac{\mathrm{i}\hbar}{m}\omega(t)\hat{\mathbf{w}}(t)\mathrm{d}t = \left(\frac{\lambda}{2}\right)^{1/2}\frac{\hbar}{m}\mathrm{d}\mathbf{Q}(t), \quad \hat{\mathbf{w}}(0) = \frac{\hbar}{2m\sigma_q^2}\mathbf{q} + \frac{\mathrm{i}}{m}\mathbf{p}, \tag{4.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega(t) + \frac{\mathrm{i}\hbar}{m}\omega(t)^2 = \lambda, \quad \omega(0) = \frac{1}{2\sigma_q^2}, \tag{4.14}$$

which define the solution of Eq. (4.9) in the form (4.11). From (4.12) we get $\hat{\mathbf{q}}(t) = m\,\mathrm{Re}\,\hat{\mathbf{w}}(t)/\hbar\,\mathrm{Re}\,\omega(t)$ which is the root of the equation $\nabla R(t,\mathbf{x}) = m\mathbf{U}(t,\mathbf{x}) = 0$ for which the maximum of the posterior density $|\hat{\psi}(t,\mathbf{x})|^2 = \exp\left\{\frac{2}{\hbar}R(t,\mathbf{x})\right\}$ is attained. The posterior mean value of momentum $\hat{p}(t)$ coincides with

$$m\mathbf{V}(t,\hat{\mathbf{q}}(t)) = \nabla S(t,\mathbf{x})_{|\mathbf{x}=\hat{\mathbf{q}}(t)} \text{ and by (4.12) } \hat{\mathbf{p}}(t) = \mathrm{Im}(m\hat{\mathbf{w}}(t) - \hbar\omega(t)\hat{\mathbf{q}}(t)).$$

Eq. (4.12) gives the time–development of posterior mean values of position and momentum, with the help of (4.13) and (4.14) we obtain the Hamilton–Langevin equations

$$\mathrm{d}\hat{\mathbf{q}}(t) - \frac{1}{m}\hat{\mathbf{p}}(t)\mathrm{d}t = \frac{(\lambda/2)^{1/2}}{\mathrm{Re}\,\omega(t)}\mathrm{d}\tilde{\mathbf{Q}}(t), \quad \hat{\mathbf{q}}(0) = \mathbf{q}, \tag{4.15}$$

$$\mathrm{d}\hat{\mathbf{p}}(t) = -\hbar\frac{(\lambda/2)^{1/2}\,\mathrm{Im}\,\omega(t)}{\mathrm{Re}\,\omega(t)}\mathrm{d}\tilde{\mathbf{Q}}(t), \quad \hat{\mathbf{p}}(0) = \mathbf{p}.$$

They are classical stochastic equations describing continuously and indirectly observed position and momentum of a free quantum particle disturbed by the measuring device (in the mean $\hat{\mathbf{p}}(t)$ and $\hat{\mathbf{q}}(t)$ coincide with $\mathbf{p}(t) = \mathbf{p}$, $\mathbf{q}(t) = \mathbf{p}t/m$).

One can check easily that for the posterior wave function in the form (4.8), posterior momentum and position dispersions are given by formulas

$$\tau_q^2(t) = 1/2\,\mathrm{Re}\,\omega(t), \quad \tau_p^2(t) = \hbar^2|\omega(t)|^2/2\,\mathrm{Re}\,\omega(t) \tag{4.16}$$

with $\omega(t)$ being the solution of Eq. (4.14). These formulas yield the Heisenberg inequality $\tau_q^2\tau_p^2 \geq \hbar^2/4$.

The general solution of Eq. (4.14) has the form

$$\omega(t) = \alpha\frac{\omega(0) + \alpha\tanh(\frac{\lambda}{\alpha}t)}{\omega(0)\tanh(\frac{\lambda}{\alpha}t) + \alpha}, \quad \alpha = \left(\frac{\lambda m}{2\hbar}\right)^{1/2}(1 - \mathrm{i}). \tag{4.17}$$

Obviously, $\lim_{t\to\infty} \omega(t) = \alpha$, i.e. α is the asymptotic stationary solution of Eq. (4.14). Consequently, the posterior dispersions of position and momentum tend to finite limits independent of its initial values

$$\tau_q^2(\infty) = (\hbar/2\lambda m)^{1/2}, \ \tau_p^2(\infty) = \hbar(\lambda m\hbar/2)^{1/2} \tag{4.18}$$

giving the localization of the observed quantum particle. As it follows from (4.18) the asymptotic localization of the particle in the coordinate representation is inversely proportional to its mass and the measurement accuracy λ. It means that the particle of mass zero cannot be localized by any measurement and heavy particles ($m \to \infty$) can be localized at a point. Note that according to the dimension of λ, $[\lambda] = (m^2 \sec)^{-1}$, the measurement accuracy coefficient can be interpreted as inversely proportional to the scattering cross–section and characteristic time of transition process in a bubble chamber.

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