

# CONTRAVARIANT DENSITIES, COMPLETE DISTANCES AND RELATIVE FIDELITIES FOR QUANTUM CHANNELS

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*In celebration of the 100th anniversary of the birth of John von Neumann*

ABSTRACT. Introducing contravariant trace-densities for quantum states, we restore one-to-one correspondence between quantum operations described by normal CP maps and their trace densities as Hermitian-positive operator-valued contravariant kernels. The CB-norm distance between two quantum operations is explicitly expressed in terms of these densities as the supremum over the input states. A larger C-distance is given as the natural norm-distance for the channel densities, and another, Helinger type complete distance (CH-distance), related to the minimax mean square fidelity optimization problem by purification of quantum channels, is also introduced and evaluated in terms of their contravariant trace-densities. It is proved that the CH distance between two channels is equivalent to the CB distance. An operational meaning for these distances and relative complete fidelity for quantum channels is given in terms of quantum encodings producing optimal entanglements of quantum states for an opposite and output systems.

## 1. INTRODUCTION

Quantum channels, which are usually described by trace preserving operations  $T : \mathcal{A}_* \rightarrow \mathcal{B}_*$ , preadjoint to normal completely positive (CP) maps  $\Psi = T^*$  of an output operator algebra  $\mathcal{B}$  into input  $\mathcal{A}$  with respect to Hilbert-Schmidt pairing

$$\langle T(\varrho) | B \rangle := \text{tr} \varrho^\dagger \Psi(B) \equiv \langle \varrho | \Psi(B) \rangle,$$

can be completely characterized in terms of their positive densities  $\Psi_\top$  with respect to the standard trace  $\text{tr}$ , see for example [1]. The channel density  $\Psi_\top$  can be seen as operator-valued generalization of quantum state density  $\rho_\top$  as non-commutative Radon-Nikodym derivative of normal unital maps  $\rho : \mathcal{A} \rightarrow \mathbb{C}$  with respect to a reference trace  $\tau = \text{tr}$  on  $\mathcal{A}$ . Such densities, called contravariant in finite dimensional case, are actually different from the usually used covariant densities, but in the case of states they are simply complex conjugated,  $\rho_\top = \bar{\varrho}$ , or equivalently transposed,  $\bar{\varrho} = \tilde{\varrho}$ , to the usual (covariant) densities  $\varrho \in \mathcal{A}_*$  of the positive unital maps  $\rho : A \mapsto \text{tr} \varrho A$ . The contravariant state densities  $\rho_\top \in \overline{\mathcal{A}}$ , defining the states as  $\rho(A) = \langle A, \rho_\top \rangle$  with respect to the bilinear pairing  $\langle A, \rho_\top \rangle = \langle \bar{\rho} | A \rangle$ , are more suitable for operational generalizations than the usual, covariant densities  $\rho_* = \varrho$ .

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The latter retain only partial (transpose) positivity when they are replaced by covariant channel densities  $\Psi_* \in \mathcal{A} \otimes \mathcal{B}_*$  defining the maps  $\Psi$  by the partial trace  $\Psi(B) = \text{tr}_{\mathcal{B}}(I \otimes B) \Psi_*$ . These positive, contravariant densities  $\Psi_{\top}$  describe CP maps  $\Psi$  by partial tracing corresponding to  $\langle B, \sigma_{\top} \rangle = \text{tr} \tilde{B} \Psi_{\top}$ , and they transform the contravariant input densities  $\rho_{\top}$  into contravariant output densities  $\sigma_{\top} = \zeta$  of  $\varsigma = T(\varrho)$  by partial tracing

$$\langle \Psi_{\top}, \rho_{\top} \rangle = \text{tr}_{\mathcal{A}} \Psi_{\top} (\hat{\rho}_{\top} \otimes I_{\mathcal{B}}).$$

In the infinite dimensional Hilbert space the channel densities  $\Psi_{\top}$ , unlike the state densities, might be unbounded, and in general should be understood in a distribution sense. Such densities were defined in [2] as Hermitian-positive kernels, characterized as unbounded self-adjoint operators in Stinespring representation [3] of another, dominating CP map  $\Phi$ . In the case of tracial  $\Phi(B) = I_{\mathcal{A}} \text{tr} B$  the channel densities  $\Psi_{\top}$  are defined as Hermitian-positive kernels affiliated to the tensor product  $\mathcal{A} \otimes \bar{\mathcal{B}}$  of the input channel algebra  $\mathcal{A}$  with the opposite (transposed) algebra  $\bar{\mathcal{B}}$  to the output  $\mathcal{B}$ . As in the case of state density operators, Schmidt decompositions of the positive densities  $\Psi_{\top}$  are in one-to-one correspondence with the Kraus decompositions of the channels  $\Psi$ , and the spectral decomposition of the positive self-adjoint operators  $\Psi_{\top}$ , corresponding to the orthogonal Kraus decompositions of  $\Psi$ , completely describe the properties of quantum channels in terms of their spectral measures on  $\mathbb{R}_+$ .

Here we develop a consistent metric space theory of contravariant quantum channel kernels describing quantum operations in terms of trace-pairings of the observables and the contravariant densities respectively to generalized traces, introduced in [1]. We discuss several distances for comparing two quantum channels  $\Phi$  and  $\Psi$ , among them the complete boundedness (CB) distance and the complete Helinger (CH), or operational Bures distance, and define the complete relative fidelity of these channels. All these distances, conditioned upon the input state, are explicitly evaluated in terms of the contravariant densities  $\Phi_{\top}, \Psi_{\top}$  of the channels. Thus, the CB distance is expressed as the maximum  $D_{\text{cb}}(\Phi, \Psi) = \sup_{\rho} D_{\text{cb}}^{\rho}(\Phi, \Psi)$  of the standard entanglement trace distance

$$D_{\text{cb}}^{\rho}(\Phi, \Psi) = \text{Tr} \left| \left( \rho_*^{1/2} \otimes I_{\mathcal{B}} \right) (\Phi_{\top} - \Psi_{\top}) \left( \rho_*^{1/2} \otimes I_{\mathcal{B}} \right) \right|,$$

over the input state densities  $\rho_* \in \mathcal{A}_*$  as contravariant densities of normal states on  $\bar{\mathcal{A}}$ . The CH distance  $d_c(\Phi, \Psi)$  between two quantum channels  $\Phi$  and  $\Psi$  is found as  $d_c(\Phi, \Psi)^2 / 2 = 1 - f_c(\Phi, \Psi)$  in terms of the minimum  $f_c(\Phi, \Psi) = \inf_{\rho} f_c^{\rho}(\Phi, \Psi)$  of the standard entanglement relative fidelity

$$f_c^{\rho}(\Phi, \Psi) = \text{Tr} \left| \Phi_{\top}^{1/2} (\rho_* \otimes I_{\mathcal{B}}) \Psi_{\top}^{1/2} \right|$$

for a given  $\rho$ . Here and below  $|A|$  means the "modulus" of an operator  $A$ , for which under the trace can be taken any of the expressions  $\sqrt{A^{\dagger}A}$  and  $\sqrt{AA^{\dagger}}$  (which are different in the case of not normal  $A$ ).

CB distance has recently found an extensive use quantum information theory, see for example the review paper [[4]]. While the CH distance, as will be shown here, is topologically equivalent to the CB distance between two channels, from the operational and computational point of view this new (Helinger) complete distance have certain advantages over the CB distance.

Note that in the finite dimensional case one can use for discriminating of the channels the standard entanglement relative fidelity with respect to the input tracial state  $\rho$  corresponding to  $\rho_* = d^{-1}I$ . In fact such fidelity has been recently suggested by Raginsky [5] who used to define his fidelity relative to the maximal entangled state. However the corresponding fidelity distance is not equivalent to the CB distance, and there is no such measure in the infinite dimensional case. The normalized tracial states do not exist on type one algebras if  $d = \infty$ , and there is no maximal entangled state. We prove that our complete fidelity distance is equivalent to the CB distance, and give an operational interpretation of this fidelity in terms of a minimax problem for quantum encodings and decompositions, purifying the channels, in complete parallel to Uhlman's theorem [6].

## 2. SOME FACTS AND NOTATION

*Quantum state*  $\sigma$ , identified as usual with a list of all expectations  $\sigma(A)$ , is defined as a linear normalized functional  $A \mapsto \sigma(A)$  on the algebra  $\mathcal{B}(\mathcal{H})$  of all bounded operators  $A$  in a separable Hilbert space  $\mathcal{H}$ . The functional also satisfies the positivity condition  $\sigma(A^\dagger A) \geq 0$  for any  $A \in \mathcal{B}(\mathcal{H})$  such that  $\sigma(I) = 1$  for the identity operator  $I$  on  $\mathcal{H}$ . As usual we shall consider only normal states  $\sigma(A) = \langle A, \sigma \rangle$ , identifying them with *density operators*  $\sigma = \sum |i\rangle\sigma^{ik}\langle k|$  defining the expectations  $\sigma(A)$  as sums, or the series

$$(2.1) \quad \langle A, \sigma \rangle = \sum_{ik} a_{ik}\sigma^{ik} \equiv a_{ik}\sigma^{ik}, \quad a_{ik} = \langle i|A|k\rangle.$$

Here  $\sigma$  is given in an orthogonal basis of real units  $\langle i| \in \mathcal{H}$  and  $|i\rangle = \langle i|^\dagger$  as a Hermit-positive  $\sigma \geq 0$  matrix of  $\sigma^{ik} = \sigma(|i\rangle\langle k|)$  with the unit trace

$$\text{Tr } \sigma = \sum_i \sigma^{ii} \equiv \langle I, \sigma \rangle = 1.$$

For the reason which will be explained later we prefer to use the tensor form (2.1) for the bilinear pairing  $A$  and  $\sigma$  rather than sesquilinear form  $\text{Tr } \bar{\sigma}A = \sum \sigma_{ik}a_{ki}$  which pairs  $A$  with the covariant form  $\bar{\sigma} = [\sigma_{ik}]$  of the density matrix  $\sigma = [\sigma^{ik}]$  in terms of the complex conjugated elements  $\bar{\sigma}^{ik} \equiv \sigma_{ik} = \sigma^{ki}$ , coinciding with the transposed ones due to  $\sigma^\dagger = \sigma$ . Equivalently (2.1) can be written as  $\langle A, \sigma \rangle = \text{Tr } \bar{\sigma}A$ , where  $\bar{\sigma}$  denotes the transposed matrix  $[\sigma^{ki}]$ . The matrix  $[\sigma^{ik}]$  is called the *contravariant density* of the state  $\sigma$  with respect to the standard trace  $\tau = \text{Tr}$ . As we shall see, contravariant density gives more adequate representation for quantum channels than the matrix  $\bar{\sigma} = [\langle i|\bar{\sigma}|k\rangle] = \bar{\sigma}$ , and its name can be explained in terms of the *contravariant transformations*. If  $\{v_i\}$  is an orthogonal basis of  $\mathcal{H}$  in terms of rows  $v_i$  such that their Hermitian adjoints  $v_i^\dagger$  form a unitary matrix  $U = (v_i^\dagger)$ , then the quantum expectations  $\sigma(A)$  do not change if the corresponding densities  $\sigma \in \mathcal{B}_\tau(\mathcal{H})$  are contravariantly transformed as  $U\sigma U^\dagger = v_i^\dagger v_k \sigma^{ik}$  with respect to the observables transformation  $V^\dagger AV = a_{ik}v^{i\dagger}v^k$ , where  $V = [v^k]$  is given as a column of the complex conjugate rows  $v^i = \bar{v}_i$  such that  $V_\tau := U = \tilde{V}$ . (They are inversely transformed in the case  $\bar{U} = U^\dagger$  of the *symmetric transformations*  $\tilde{V} = V$ , in which case  $U = V$ .)

Recall that the linear span of the convex set  $\mathcal{S}_\tau = \mathcal{S}(\mathcal{H})$  of all trace-densities  $\sigma$  for normal states on  $\mathcal{B}(\mathcal{H})$  is a Banach space  $\mathcal{B}_\tau(\mathcal{H})$  with respect to the trace norm  $\|A\|_\tau := \text{Tr } \sqrt{A^\dagger A}$ , consisting of all trace class operators in  $\mathcal{H}$ , and its dual

$\mathcal{B}_\tau(\mathcal{H})^\top$ , the space of all bounded functionals, coincides with  $\mathcal{B}(\mathcal{H})$  with respect to the pairing (2.1). Given a  $\sigma \in \mathcal{B}_\tau(\mathcal{H})$ , one can define the pair-transposed element  $A^\top$  as the functional

$$A^\top(\sigma) = \langle A, \sigma \rangle.$$

on the *predual* space  $\mathcal{B}_\tau(\mathcal{H})$  given by an operator  $A \in \mathcal{B}(\mathcal{H})$ . Note that  $A^\top$  is not the transposed operator in  $\mathcal{H}$  but is described in terms of the transposed operator  $\tilde{A} = \overline{A}^\dagger$  as  $A^\top(\sigma) = \text{Tr} \tilde{A}\sigma$ . The transposition  $A \mapsto \tilde{A}$  is related not to the operator but to the Hilbert space pairing  $\langle \tilde{\varphi}, \psi \rangle = \langle \varphi | \psi \rangle$  defining the *vector transpose*  $\psi \mapsto \tilde{\psi}$  as the reverse to  $|\psi\rangle \mapsto \langle \tilde{\psi}|$  by identifying the complex conjugate elements  $\tilde{\psi} \in \mathcal{H}$  with bra-vectors  $\langle \tilde{\psi}|$ .

All of that can be easily generalized [1] to an arbitrary operator subalgebra  $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$  with the standard, or a nonstandard normal faithful semifinite trace [7]  $\nu$  when the standard one,  $\tau = \text{Tr}$ , is not semifinite, e.g. is trivial on  $\mathcal{B}$  in the sense that  $\text{Tr} A^\dagger A = \infty$  for all operators  $A \neq 0$  from the algebra  $\mathcal{B}$ . Such a trace  $\nu$  is defined as a nonnormalized state on  $\mathcal{B}$ , finite on a weakly dense part of  $\mathcal{B}$ , with the property  $\nu(A^\dagger A) = \nu(AA^\dagger)$  and separating  $\mathcal{B}$  in the sense that  $\nu(A^\dagger A) = 0 \Rightarrow A = 0$ . A quantum system will be called *semifinite* if it is described by the semifinite algebra, i.e. admits a semifinite, not necessarily standard, trace  $\nu$ . (There exist also quantum infinite systems which are not semifinite.) The only difference is that the predual space  $\mathcal{B}_\tau$  of contravariant densities  $\sigma$  with respect to the trace  $\nu$  on  $\mathcal{B}$  may not be apart of the algebra  $\mathcal{B}$  but a part of an *opposite* algebra  $\overline{\mathcal{B}}$ , or, if unbounded, affiliated to  $\overline{\mathcal{B}}$ . The opposite algebra can be defined on the same Hilbert space equipped with a complex conjugation  $\chi \mapsto \bar{\chi}$ , i.e. isometric involution in  $\mathcal{H}$ , as the subalgebra  $\overline{\mathcal{B}} = \{A \in \mathcal{B}(\mathcal{H}) : \bar{A} \in \mathcal{B}\}$  of complex conjugated operators  $\bar{A} = \tilde{A}^\dagger$ . This  $\overline{\mathcal{B}}$ , equipped with the *reference* trace  $\bar{\nu}(\bar{A}) := \overline{\nu(A)}$ , does not necessarily coincide with  $(\mathcal{B}, \nu)$  as in the simple case  $\mathcal{B} = \mathcal{B}(\mathcal{H})$ . The "opposite" trace  $\mu = \bar{\nu}$ , coinciding with  $\tilde{\nu}(A) = \nu(\tilde{A})$  on the opposite algebra  $\overline{\mathcal{B}}$ , defines the  $\mu$ -pairing

$$(2.2) \quad \langle A, \sigma \rangle_\mu = \mu(\tilde{A}\sigma), \quad A \in \mathcal{B}, \sigma \in \mathcal{B}_\tau,$$

generalizing (2.1). Note that in the symmetric case  $\mathcal{B} = \overline{\mathcal{B}}$  the trace  $\mu$  is not distinguished from  $\nu$ :  $\bar{\nu}(A) = \nu(\tilde{A}) = \nu(A)$ .

Below  $\mathcal{S}_\tau \subseteq \mathcal{A}_*$  denotes the convex set of all positive normalized contravariant densities  $\varrho \geq 0$ ,  $\tau(\varrho) = 1$  for normal states on the von Neumann algebra  $\overline{\mathcal{A}}$  with respect to a normal faithful semifinite trace  $\lambda = \bar{\tau}$ . They can also be regarded as covariant densities,  $\rho_* = \varrho \in \mathcal{A}_*$  of the normal states  $\rho(A) = \tau(\varrho A)$  on the opposite algebra  $\mathcal{A}$  with the reference trace  $\tau$ .

### 3. QUANTUM BURES AND TRACE DISTANCES

The difference between two states  $\rho, \sigma \in \mathcal{S}(\mathcal{H})$  is usually measured by *trace-norm distance*

$$(3.1) \quad D(\rho, \sigma) := \text{Tr} |\rho - \sigma| \equiv \|\rho - \sigma\|_\tau,$$

given in terms of the modulus-difference of their trace densities  $\rho = \rho_\tau$  and  $\sigma = \sigma_\tau$ . It is also defined for non positive  $\rho$  and  $\sigma$  and thus doesn't have much information-operational meaning. Another measure of this difference is relative entropy [8]

which has clearly more information-operational meaning. However the quantum relative entropy is not symmetric and not unique, and is not a distance on  $\mathcal{S}_\tau$ .

The natural operational distance between two quantum states is *quantum Bures distance* which can be defined as the square root of the minimal Euclidean squared distance

$$(3.2) \quad d(\rho, \sigma)^2 := \inf_{\chi^\dagger \chi = \rho, \psi^\dagger \psi = \sigma} \text{Tr} (\chi - \psi)^\dagger (\chi - \psi).$$

Here the infimum is taken over all all Schmidt decompositions of the trace-densities  $\rho$  and  $\sigma$ . Uhlman's theorem [6] states that this infimum is actually achieved at the value  $d(\rho, \sigma)^2 / 2 = 1 - f(\rho, \sigma)$ , where

$$(3.3) \quad f(\rho, \sigma) = \text{Tr} \sqrt{\rho^{1/2} \sigma \rho^{1/2}} =: \text{Tr} \left| \rho^{1/2} \sigma^{1/2} \right| := \text{Tr} \sqrt{\sigma^{1/2} \rho \sigma^{1/2}}$$

is the relative fidelity of the states  $\rho$  and  $\sigma$ . Note that this can obviously be written as the trace of the matrix

$$\rho^{1/2} \sqrt{\rho^{1/2} \sigma \rho^{1/2}} \rho^{-1/2} \equiv \sqrt{\rho \sigma} \equiv \sigma^{-1/2} \sqrt{\sigma^{1/2} \rho \sigma^{1/2}} \sigma^{1/2}.$$

giving a much simpler formula  $\text{Tr} \sqrt{\rho \sigma}$  for the fidelity  $f(\rho, \sigma)$ . It is also valid even if  $\rho$  and/or  $\sigma$  are not invertible as soon as the product  $\rho \sigma = T \Lambda T^{-1}$  remains similar to a positive diagonal matrix  $\Lambda$  to make sense of  $\sqrt{\rho \sigma} = T \Lambda^{1/2} T^{-1}$ .

Thus, quantum Bures distance is a noncommutative generalization of the statistical Helinger distance which in the case of the commutative  $\rho$  and  $\sigma$  has the familiar form  $\text{Tr} (\sqrt{\rho} - \sqrt{\sigma})^2$ . Since Schmidt decompositions  $\psi^\dagger \psi = \sum \psi^\dagger |j\rangle \langle j| \psi$  correspond to purifications of the states  $\sigma$ , the infimum (3.2) has clear information-operational meaning.

Although the squared fidelity distance  $d^2$  is smaller [6] than  $D$ , they achieve the same maximal value  $d^2 = 2 = D$  on  $\mathcal{S}(\mathcal{H})$  when  $f = 0$ , i.e. when the range of  $\rho$  is orthogonal to the range of  $\sigma$ . In fact, as it follows from the inequality  $D \leq 2\sqrt{1 - f^2}$ , see for example [9],  $D$  cannot be larger than  $2d$ . Thus both distances are topologically equivalent,

$$(3.4) \quad d(\rho, \sigma)^2 \leq D(\rho, \sigma) \leq 2d(\rho, \sigma),$$

and  $d = 0 = D$  iff  $f = 1$ , i.e.  $\rho = \sigma$ .

All of that can be easily generalized to a more general semifinite algebra  $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$  with respect to a trace  $\nu$ , inducing the (nonstandard) reference trace  $\mu = \bar{\nu}$  on the opposite algebra  $\bar{\mathcal{B}}$ , and the pairing of  $\mathcal{B}$  with the predual space  $\mathcal{B}_\tau$  affiliated to the opposite algebra  $\bar{\mathcal{B}}$  is understood in the sense of (2.2). The only difference is that the contravariant densities  $\rho = \rho_\mu, \sigma = \sigma_\mu$ , normalized with respect to trace  $\mu = \bar{\nu}$  on  $\bar{\mathcal{B}}$ , may not be bounded, but they are still uniquely described by the positive selfadjoint operators in  $\mathcal{H}$  affiliated to  $\bar{\mathcal{B}}$ . The trace and the Bures distance formulae remain the same with the reference trace  $\mu = \bar{\nu}$  replacing the standard trace, and the fidelity formula with respect to the arbitrary trace  $\mu$  is generalized to

$$(3.5) \quad f(\rho, \sigma) = \mu(\sqrt{\rho \sigma}) = \mu \left( \left| \rho^{1/2} \sigma^{1/2} \right| \right) = \mu(\sqrt{\sigma \rho}).$$

Here  $\sqrt{\rho \sigma}$  (and  $\sqrt{\sigma \rho}$ ) is still well-defined as the function  $T\sqrt{\Lambda}T^{-1}$  of the operator  $\rho \sigma = T \Lambda T^{-1}$  (and  $\sqrt{\sigma \rho} = T \Lambda T^{-1}$ ) as being similar to a positive diagonal operator  $\Lambda$  with respect to a similarity transformation  $T$ . If  $\rho$  and  $\sigma$  are invertible, one can take either  $\sigma^{-1/2} U$  or  $\rho^{1/2} V^\dagger$  as such  $T$  (respectively either  $\rho^{-1/2} V$  or  $\sigma^{1/2} U^\dagger$ ),

where  $U$  and  $V$  are unitary transformation diagonalizing respectively  $\sigma^{1/2}\rho\sigma^{1/2}$  and  $\rho^{1/2}\sigma\rho^{1/2}$ .

#### 4. OPERATIONAL DENSITIES AND QUANTUM CHANNELS

In order to compare *quantum channels* we need to generalize these results to linear completely positive (CP) trace preserving mappings  $\Phi_\top$  from the predual space  $\mathcal{A}_\top$  of the "Alice" algebra  $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{g})$  on an *input* Hilbert space  $\mathfrak{g}$  into the predual space  $\mathcal{B}_\top$  of the "Bob" algebra  $\mathcal{B} \subseteq \mathcal{B}(\mathfrak{h})$  on the same or different *output* Hilbert space  $\mathfrak{h}$ . The notation  $\mathcal{H}$  will be kept for the Hilbert product  $\mathfrak{g} \otimes \mathfrak{h}$  with the total trace  $\text{Tr} = \tau_\mathfrak{g} \otimes \tau_\mathfrak{h}$  inducing the trace  $\lambda \otimes \bar{\mu}$  on the entangled input-output system  $\bar{\mathcal{A}} \otimes \mathcal{B}$  as the product of the standard traces  $\lambda = \tau_\mathfrak{g}|_{\bar{\mathcal{A}}}$  and  $\bar{\mu} = \tau_\mathfrak{h}|_{\mathcal{B}}$  if  $\tau_\mathfrak{g} = \text{tr}_\mathfrak{g}$  and  $\tau = \text{tr}$  remain on these subalgebras semifinite. Otherwise we should take non-standard reference traces  $\tau$  on  $\mathcal{A}$  and  $\nu$  on  $\mathcal{B}$  and define  $\lambda \otimes \bar{\mu}$  as the opposite trace to  $\tau \otimes \mu$  on  $\mathcal{A} \otimes \mathcal{B}$ .

As in the case of the states it is more convenient to describe quantum channels by the normal unital CP operations defined as the maps  $\Phi : \mathcal{B} \rightarrow \mathcal{A}$  on the *output algebra*  $\mathcal{B}$  into the *input algebra*  $\mathcal{A}$ , like the expectations  $\sigma$  mapping  $\mathcal{B}$  into the trivial algebra  $\mathcal{A} = \mathbb{C}$ . These can always be defined as the dual  $\Phi_\top^\dagger$  to  $\Phi_\top : \mathcal{A}_\top \rightarrow \mathcal{B}_\top$  with respect to the  $\lambda$ -pairing of  $\mathcal{A}$  with  $\mathcal{A}_\top$   $\mu$ -pairing (2.2):

$$(4.1) \quad \langle \Phi(B), \rho \rangle_\lambda = \langle B, \Phi_\top(\rho) \rangle_\mu \quad \forall \rho \in \mathcal{A}_\top, B \in \mathcal{B}.$$

In particular, if  $\Phi_\top(\rho) = F_\top \rho F_\top^\dagger$ , then  $\Phi(B) = F^\dagger B F$ , where  $F$  is the transposed operator to  $F_\top = \tilde{F}$ .

Recall that a normal map  $\Phi : \mathcal{B} \rightarrow \mathcal{A}$  is CP iff the map  $I_0 \otimes \Phi_\top$  is positive on  $\mathcal{B}_\top(\ell^2) \otimes \mathcal{A}_\top$  into  $\mathcal{B}_\top(\ell^2) \otimes \mathcal{B}_\top$ , where  $I_0 = \text{id}$  is the identity map on  $\mathcal{B}_\top(\ell^2)$ . Moreover, at least in the case of the simple algebra  $\mathcal{B} = \mathcal{B}(\mathcal{H})$ ,  $\Phi$  is normal on  $\mathcal{B}$  iff it is weak continuation of the CP map  $\Phi|_{\mathcal{B}_0(\mathfrak{h})}$ . This means that  $\Phi$  like a normal state is completely determined on finite rank operators  $B = \sum_{ik} |i\rangle b_{ik} \langle k|$  by the operator entries  $\Phi_\mu^{ik} = \Phi(|i\rangle \langle k|) \in \mathcal{A}$  of the Hermitian-positive matrix  $\Phi_\mu = [\Phi_\mu^{ik}]$ , and thus has the form  $\Phi(B) = \langle B, \Phi_\mu \rangle_\mu$  in terms of the  $(\mathcal{B}, \mathcal{B}_\top)_\mu$  pairing

$$\langle B, \Phi_\mu \rangle_\mu = \sum_{ik} b_{ik} \Phi_\mu^{ik}, \quad b_{ik} = \langle i|B|k \rangle.$$

of this  $\mathcal{B}$  with  $\mathcal{B}_\top = \mathcal{B}_\top(\mathfrak{h})$  with respect to  $\mu = \text{tr}_\mathfrak{h} = \tau$ . Here  $\Phi_\mu = \sum |i\rangle \Phi_\mu^{ik} \langle k| \geq 0$  is a kernel on  $\mathfrak{g} \otimes \mathfrak{h}$  given in an orthogonal basis of real units  $\langle i| \in \mathfrak{h}$  and  $|i\rangle = \langle i|^\dagger$  by the Hermit-positive  $\mathcal{A}$ -valued matrix  $[\Phi_\mu^{ik}]$  having the unit partial trace

$$\mu(\Phi_\mu) := \sum_i \Phi_\mu^{ii} \equiv \langle I_\mathfrak{h}, \Phi_\mu \rangle_\mu = I_\mathfrak{g}$$

if  $\Phi(I_\mathfrak{h}) = I_\mathfrak{g}$ . The kernel  $\Phi_\mu$ , called the *density* of a quantum operation  $\Phi : \mathcal{B} \rightarrow \mathcal{A}$  with respect to the trace  $\bar{\mu}(B) = \text{tr}_\mathfrak{h} B$  on  $\mathcal{B}$ , defines the density operators of output states  $\sigma = \rho \circ \Phi$  by the partial tracing

$$(4.2) \quad \sigma = \tau[(\bar{\rho} \otimes I_\mathfrak{h}) \Phi_\mu] \equiv \langle \Phi_\mu, \rho \rangle_\lambda.$$

in terms of the  $(\mathcal{A}, \mathcal{A}_\top)$  pairing with respect to the reference trace  $\lambda = \bar{\tau}$  on  $\bar{\mathcal{A}}$ . Note that the usual trace-form of the pairing defines the covariant form  $\bar{\sigma} = \tau(\Phi'_\mu(\bar{\rho} \otimes I_\mathfrak{h}))$  of the output matrix  $\sigma$  by the partial transposition  $\Phi'_\mu = [\Phi_\mu^{ki}]$

which may be not Hermitian-positive for the positive  $\Phi_\mu$ . This indicates that covariant trace-pairing is not natural for describing quantum channelling and explains our preference to use the tensor form of the pairings corresponding to the contravariant form of the trace-densities. Otherwise there would not be one-to-one correspondence between complete positivity of operations and Hermitian positivity of their densities as kernels.

In general the density  $\Phi_\mu$  is an unbounded Hermitian-positive operator, or even a generalized one, defined only as a kernel of a positive Hermitian form in  $\mathfrak{g} \otimes \mathfrak{h}$ . Nevertheless we will still use the notations of the partial tracings

$$\Phi_\top(\rho) = \langle \Phi_\mu, \rho \rangle_\lambda, \quad \Phi(B) = \langle B, \Phi_\mu \rangle_\mu$$

and write  $\mu(\Phi_\mu) = \langle I_\mathfrak{h}, \Phi_\mu \rangle_\mu$ , defining the channel densities by the Hermitian positivity and the normalization conditions

$$\Phi_\mu \geq 0, \quad \mu(\Phi_\mu) = I_\mathfrak{g}.$$

condition  $\Phi(I_\mathfrak{h}) = I_\mathfrak{g}$ . In particular, the density  $\Phi_\mu$  of the operation  $\Phi(B) = F^\dagger B F$  with respect to the standard trace  $\mu = \text{tr}_\mathfrak{h}$  is described as the kernel  $\Phi_\mu = |F\rangle\langle F|$ , where the generalized bra-vector  $\langle F|$  is well-defined on the finite linear combinations of the products  $\xi \otimes \eta \in \mathfrak{g} \otimes \mathfrak{h}$  by the partial transposition

$$(4.3) \quad \langle F|(|\xi\rangle \otimes |\eta\rangle) = \eta F|\xi\rangle \quad \forall \xi \in \mathfrak{g}, \eta \in \mathfrak{h}.$$

It cannot be defined as an element of the Hilbert space  $\mathcal{H} = \mathfrak{g} \otimes \mathfrak{h}$  for any isometry, a unitary operator  $F = U$  say, if the space  $\mathfrak{g}$  has infinite dimensionality  $\dim \mathfrak{g} = \infty$ .

Note that for the arbitrary output algebra  $\mathcal{B}$  the quantum channel density as a positive self-adjoint, possibly unbounded operator (kernel) affiliated to  $\mathcal{A} \otimes \overline{\mathcal{B}}$  was first introduced in [2] with respect to operator-valued weights  $\phi$  as CP maps into  $\mathcal{A}$  densely defined in  $\mathcal{B}$ , which correspond in our case to  $\phi(B) = \nu(B) I_\mathfrak{g}$ . However, in order to avoid technicalities one can consider here only the channels with the bounded densities  $\Phi_\mu$  with respect to the trace  $\mu = \bar{\nu}$ .

## 5. AN EXPLICIT FORMULA FOR OPERATIONAL CB DISTANCE

As a measure of difference between two quantum channels  $\Phi$  and  $\Psi : \mathcal{B} \rightarrow \mathcal{A}$  one can adopt the usual boundedness norm distance  $D_b(\Phi, \Psi) := \|\Phi - \Psi\|_b$  (or B-distance for short), defined in terms of the density operators  $\Phi_\mu$  and  $\Psi_\mu$  as

$$D_b(\Phi, \Psi) = \sup_{\rho \in \mathcal{S}_\lambda} \mu |\Phi_\top(\rho) - \Psi_\top(\rho)| = \sup_{\rho \in \mathcal{S}_\lambda} \mu (|\tau[\Delta_\mu(\rho)]|),$$

where  $\tau[\Delta_\mu(\rho)]$  is dual action  $\Delta_\top(\rho) = \langle \Delta_\mu, \rho \rangle_\lambda$  of  $\Delta = \Phi - \Psi$  which is equal to the partial tracing (4.2) of the operator

$$(5.1) \quad \Delta_\mu(\rho) := \left( \tilde{\rho}^{1/2} \otimes I_\mathfrak{h} \right) \Delta_\mu \left( \tilde{\rho}^{1/2} \otimes I_\mathfrak{h} \right) \in \mathcal{S}_{\tau \otimes \mu}(\Phi_\top(\rho)) - \mathcal{S}_{\tau \otimes \mu}(\Psi_\top(\rho)) \subset (\overline{\mathcal{A}} \otimes \mathcal{B})_\top$$

in terms of the  $\mu$ -density operator  $\Delta_\mu = \Phi_\mu - \Psi_\mu$ . Here  $\mathcal{S}_{\tau \otimes \mu}(\sigma)$  denotes the convex set of normal state density operators  $\omega \in (\overline{\mathcal{A}} \otimes \mathcal{B})_\top$  with respect to the product trace  $\tau \otimes \mu$  in  $\mathcal{A} \otimes \overline{\mathcal{B}}$ , having fixed partial trace  $\tau(\omega) = \sigma$ .

However it is more appropriate to use the larger *CB-distance*  $D_{cb}(\Phi, \Psi) := \|\Delta\|_{cb}$  since the difference  $\Delta = \Phi - \Psi$  of two CP maps is not just bounded, but it is also completely bounded,  $\|\Delta\|_{cb} < \infty$ , where  $\|\cdot\|_{cb} \geq \|\cdot\|_b$  is the so-called norm of complete boundedness [10] (or CB-norm for short). Instead of maximizing over

normal input states  $\rho \in \mathcal{S}_\lambda$  one should maximize over normal *input entanglements* with any probe quantum system  $\mathbb{A}$ , or at least with the standard one, described by the "matrix algebra"  $\mathbb{A} = \mathcal{B}(\mathfrak{k}) = \overline{\mathbb{A}}$  on the Hilbert space  $\mathfrak{k} = \ell^2$  of all square-summable sequences indexed by  $\mathbb{N}$ . The normal entanglements are usually given by the densities  $\hat{\rho} \in \mathcal{S}_{\hat{\tau}}$  of the normal states  $\hat{\rho}$  on the algebra  $\mathbb{A} \otimes \mathcal{A}$  with respect to the standard trace  $\hat{\tau} = \tau_{\mathfrak{k}} \otimes \tau$ .

This maximization can be written as the supremum

$$D_{\text{cb}}(\Phi, \Psi) = \sup_{\rho \in \mathcal{S}_\lambda} D_{\text{cb}}^\rho(\Phi, \Psi)$$

over all input states of the *conditional* CB semidistance as the maximal trace distance

$$(5.2) \quad D_{\text{cb}}^\rho(\Phi, \Psi) = \sup_{\hat{\rho} \in \mathcal{S}_{\hat{\tau}}} \{ \hat{\tau} | \check{\Phi}_\top(\hat{\rho}) - \check{\Psi}_\top(\hat{\rho}) | : \tau_{\mathfrak{k}}(\hat{\rho}) = \rho \},$$

of the entangled output states  $\hat{\varphi} = \hat{\rho} \circ \check{\Phi}$  and  $\hat{\sigma} = \hat{\rho} \circ \check{\Psi}$  on  $\mathbb{A} \otimes \mathcal{B}$  with respect to the input entangled states  $\hat{\rho}$  having fixed margin  $\rho$  on  $\mathcal{A}$ . Here  $\check{\Phi}_\top = \text{Id} \otimes \Phi_\top$  and  $\check{\Psi}_\top = \text{Id} \otimes \Psi_\top$  are left ampliations  $\mathbb{A}_\top \otimes \mathcal{A}_\top \rightarrow \mathbb{A}_\top \otimes \mathcal{B}_\top$  of  $\Phi_\top$  and  $\Psi_\top$  on the predual space  $\mathbb{A}_\top = \mathcal{B}_\top(\mathfrak{k})$  of trace class matrices  $[\lambda(\hat{\rho}^{ik})]$  in the natural basis of  $\ell^2$  and  $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{g})$ , with  $\hat{\tau} = \tau_{\mathfrak{k}} \otimes \mu$ . They are applied as  $\hat{\Delta}_\top(\hat{\rho}) = [\Delta_\top(\hat{\rho}^{ik})]$  on the convex set  $\mathcal{S}_{\hat{\tau}}(\rho)$  of the predual space  $(\mathbb{A} \otimes \mathcal{A})_\top = \mathbb{A}_\top \otimes \mathcal{A}_\top$  of all  $\mathcal{A}_\top$ -valued density matrices  $\hat{\rho} = [\hat{\rho}^{ik}]$ ,  $\hat{\rho}^{ik} \in \mathcal{A}_\top$  with respect to the trace  $\hat{\tau}$  on  $\overline{\mathbb{A}} \otimes \overline{\mathcal{A}} = \mathbb{A} \otimes \overline{\mathcal{A}}$ , having the partial trace  $\tau_{\mathfrak{k}}(\hat{\rho}) := \sum_i \hat{\rho}^{ii} = \rho$ .

Taking into account that any  $\hat{\rho} \in \mathcal{S}_{\hat{\tau}}(\rho)$  can be written in blocks as  $[\rho^{1/2} \Pi^{ik} \rho^{1/2}]$  with  $\Pi^{ik} \in \overline{\mathcal{A}}$  defining a normal unital CP map  $\Pi(\overline{A}) = \overline{a}_{ik} \Pi^{ik}$  on the matrix algebra  $\mathbb{A}$  into  $\overline{\mathcal{A}}$ , we can write

$$\Delta_\top(\hat{\rho}^{ik}) = \left\langle \Delta_\mu, \rho^{1/2} \Pi^{ik} \rho^{1/2} \right\rangle_\lambda = \langle \Pi^{ik}, \Delta_\mu(\rho) \rangle_\tau \equiv \hat{\Pi}_\top^{ik}[\Delta_\mu(\rho)] \quad \forall \hat{\rho} \in \mathcal{S}_{\hat{\tau}}(\rho),$$

where  $\hat{\Pi} = \Pi \otimes \text{Id}$  is the right ampliation  $\mathbb{A} \otimes \mathcal{B} \rightarrow \overline{\mathcal{A}} \otimes \mathcal{B}$  of the normal unital CP map  $\Pi$  given by the  $\overline{\mathcal{A}}$ -valued density matrix  $[\Pi^{ik}]$  with respect to  $\tau_{\mathfrak{k}}$ . Thus the supremum in (5.2) can be expressed in the form of the supremum over all normal unital CP maps  $\Pi$  as

$$(5.3) \quad D_{\text{cb}}^\rho(\Phi, \Psi) = \sup_{\Pi} \hat{\tau} \left| \hat{\Pi}_\top[\Delta_\mu(\rho)] \right| \leq (\tau \otimes \mu) |\Delta_\mu(\rho)|,$$

where we used the obvious inequality  $\left\| \hat{\Pi}_\top(\omega) \right\|_\tau \leq \|\omega\|_\tau := (\tau \otimes \mu) |\omega|$  valid for any linear trace preserving CP map  $\Pi_\top$  on  $\overline{\mathcal{A}}_\top$  into  $\mathbb{A}_\top = \mathcal{B}_\top(\mathfrak{k})$  and  $\omega \in (\overline{\mathcal{A}} \otimes \mathcal{B})_\top$ .

The inequality in (5.3) is actually the equality

$$(5.4) \quad D_{\text{cb}}^\rho(\Phi, \Psi) = (\tau \otimes \mu) \left| \left( \hat{\rho}^{1/2} \otimes I_{\mathfrak{h}} \right) (\Phi_\mu - \Psi_\mu) \left( \hat{\rho}^{1/2} \otimes I_{\mathfrak{h}} \right) \right|,$$

in the case of separable  $\mathcal{A}$  as it can easily be seen for the simple algebras  $\mathcal{A} = \mathcal{B}(\mathfrak{g})$  when the supremum is obviously achieved at  $\Pi = \text{Id}$  on the opposite input algebra  $\mathbb{A} = \overline{\mathcal{A}}$  coinciding with  $\mathcal{B}(\mathfrak{k})$  in a representation  $\mathfrak{g} = \mathfrak{k}$ .

In order to obtain the same formula for an arbitrary semifinite algebra  $\mathcal{A}$ , the supremum in (5.2) over  $\hat{\rho} \in \mathcal{S}_{\hat{\tau}}(\rho)$  should also be extended to the nonstandard  $\mathbb{A}$ . This can be seen as optimization of the operational distance via all *quantum encodings* [1] as CP mappings

$$\pi(\overline{A}) = \rho^{1/2} \Pi(\overline{A}) \rho^{1/2} = \overline{a}_{ik} \hat{\rho}^{ik}, \quad \overline{A} = [\overline{a}_{ik}]$$

of not only standard algebra  $\mathbb{A} = \mathcal{B}(\mathfrak{k})$  into  $\mathcal{A}_\tau$  but also less simple algebras, including  $\mathbb{A} = \overline{\mathcal{A}}$ . Quantum encodings were defined in [1] by normal CP maps  $\pi : \mathbb{A} \rightarrow \overline{\mathcal{A}}$  on any semifinite "quantum alphabet" algebra  $\mathbb{A}$  with fixed normalization  $\pi(I_{\mathfrak{k}}) = \rho$ . They describe *normal couplings* of the corresponding state  $\rho$  on  $\mathcal{A}$  with the normal states on  $\mathbb{A}$ , having the densities  $P = \Pi_\tau(\tilde{\rho}) = \pi_\tau(I_{\mathfrak{g}})$  with respect to a trace  $\tau$  on  $\mathbb{A}$ , by

$$\hat{\rho} = \left( I_{\mathfrak{k}} \otimes \rho^{1/2} \right) \Pi_\tau \left( I_{\mathfrak{k}} \otimes \rho^{1/2} \right) \equiv \Pi_\tau(\rho),$$

where  $\Pi_\tau \in \mathbb{A}_\tau \otimes \overline{\mathcal{A}}$  is the density operator of the unital CP map  $\Pi$ . The maximal distance over all such encodings is obviously achieved on the *standard coupling*  $\pi^*(\bar{A}) = \rho^{1/2} \bar{A} \rho^{1/2}$  as in the case of the simple algebra  $\mathcal{A} = \mathcal{B}(\mathfrak{g})$ , but the optimal "encoding alphabet" system algebra  $\mathbb{A} = \overline{\mathcal{A}}$  does not coincide with  $\mathcal{A}$  if  $\mathcal{A} \neq \overline{\mathcal{A}}$  but be antiisomorphic (opposite) to  $\mathcal{A}$ . The standard entanglement defines the *optimal compound state*

$$\langle \bar{A} \otimes A, \omega \rangle_{\tau \otimes \bar{\tau}} = \bar{\tau} \left( v \bar{A} v \tilde{A} \right) = (\bar{\tau} \otimes \tau) (\bar{\omega} (\bar{A} \otimes A)), \quad A \in \mathcal{A}, \bar{A} \in \overline{\mathcal{A}},$$

where  $v = \rho^{1/2}$  and  $\omega := |v\rangle\langle v| \equiv \omega(\rho)$  is the optimal density operator  $\hat{\rho}^*$  of this standard entangled state with respect to  $\hat{\tau} = \tau \otimes \lambda$  and the partial trace  $\tau(\omega) = \rho$ .

Thus the formula (5.4) gives an expression for the *CB-distance* as the supremum

$$(5.5) \quad D_{\text{cb}}(\Phi, \Psi) = \sup_{\rho \in \mathcal{S}_\lambda} (\tau \otimes \mu) |(\check{\Phi}_\tau - \check{\Psi}_\tau)[\omega(\rho)]| = \sup_{\rho \in \mathcal{S}_\lambda} (\tau \otimes \mu) |\Delta_\mu(\rho)|$$

of the conditional CB semidistance in terms of the standard input entangled states  $\omega(\rho) \in (\overline{\mathcal{A}} \otimes \mathcal{A})_\tau$ . It is maximal trace-distance of the optimally entangled states on  $\overline{\mathcal{A}} \otimes \mathcal{B}$  described by the densities  $\hat{\phi} = \check{\Phi}_\tau(\omega)$  and  $\hat{\sigma} = \check{\Psi}_\tau(\omega)$  with the same partial trace  $P = \lambda(\omega) = \bar{\rho}$ .

One can also show that the CB-distance is majorized by the natural density-operator distance (*complete distance*)  $D_c(\Phi, \Psi) \geq D_{\text{cb}}(\Phi, \Psi)$  having the particularly simple form

$$(5.6) \quad D_c(\Phi, \Psi) = \sup_{\rho \in \mathcal{S}_\lambda} \langle \mu(|\Phi_\mu - \Psi_\mu|), \rho \rangle_\lambda.$$

(Obviously  $D_b(\Phi, \Psi) \leq D_c(\Phi, \Psi)$ , but it is not so obvious that  $D_{\text{cb}}(\Phi, \Psi) \leq D_c(\Phi, \Psi)$ .) However  $D_c$  is not equivalent to the CB-distance, that is closeness of  $\Psi_n$  to  $\Phi$  in the sense  $D_{\text{cb}}(\Phi, \Psi_n) \searrow 0$  does not guarantee the closeness with respect to  $D_c(\Phi, \Psi_n) \geq D_{\text{cb}}(\Phi, \Psi_n)$ , and it is difficult to give an operational meaning of the optimality criterion defined by this distance. This is why an operational fidelity distance is even more desirable for quantum channels than for states.

## 6. HELINGER DISTANCE AND RELATIVE CHANNEL FIDELITY

One can define a Helinger like operational square-distance

$$(6.1) \quad d(\Phi, \Psi)^2 := \sup_{\rho \in \mathcal{S}_\lambda} \mu \left( \Phi_\tau(\rho) + \Psi_\tau(\rho) - 2\sqrt{\Phi_\tau(\rho) \Psi_\tau(\rho)} \right)$$

between two quantum operations  $\Phi$  and  $\Psi$  as the Bures square-distance  $d(\varphi, \sigma)^2$  for the output states  $\varphi = \rho \circ \Phi$ ,  $\sigma = \rho \circ \Psi$  in terms of their output  $\mu$ -densities  $\sigma = \Psi_\tau(\rho)$ ,

$$\Psi_\tau(\rho) = \tau[(\tilde{\rho} \otimes I_{\mathfrak{h}}) \Psi_\mu] = \tau[\Psi_\mu(\rho)]$$

and similar for  $\phi = \Phi_{\top}(\rho)$  in the notation of the previous section, maximized over the input states  $\rho$ . In the case of the channels described by the normal unital CP maps  $\Phi$  and  $\Psi$  this can be expressed as  $d(\Phi, \Psi)^2/2 = 1 - f(\Phi, \Psi)$  in terms of the the minimal fidelity

$$f(\Phi, \Psi) = \inf_{\rho \in \mathcal{S}_{\lambda}} \mu \left( \sqrt{\tau [(\tilde{\rho} \otimes I_{\mathfrak{h}}) \Phi_{\mu}] \tau [(\tilde{\rho} \otimes I_{\mathfrak{h}}) \Psi_{\mu}]} \right)$$

of the output states over all inputs  $\rho \in \mathcal{S}_{\lambda}$  with respect to  $\lambda = \bar{\tau}$ . As for any two output states  $\phi, \sigma$  on  $\mathcal{B}$  the following equivalence inequality obviously holds for this Helinger (H-) distance:

$$d(\Phi, \Psi)^2 \leq D_{\mathfrak{b}}(\Phi, \Psi) \leq 2d(\Phi, \Psi).$$

However there is no such equivalence inequality between this fidelity distance and the CB distance (5.2), and it is difficult to give a definition of this fidelity without reference to the input states  $\rho$ .

Since the map  $\Phi$  and  $\Psi$  are not just positive but CP, it is more appropriate to define the *complete* fidelity distance of the operations  $\Phi$  and  $\Psi$  as the supremum

$$d_c(\Phi, \Psi) = \sup_{\rho \in \mathcal{S}_{\lambda}} d_c^{\rho}(\Phi, \Psi)$$

of the maximal Bures distance

$$d_c^{\rho}(\Phi, \Psi) = \sup_{\hat{\rho} \in \mathcal{S}_{\bar{\tau}}} \{d(\hat{\rho} \circ \check{\Phi}, \hat{\rho} \circ \check{\Psi}) : \tau_{\mathfrak{k}}(\hat{\rho}) = \rho\}$$

for the input-output entangled states  $\hat{\varphi} = \hat{\rho} \circ \check{\Phi}$  and  $\hat{\sigma} = \hat{\rho} \circ \check{\Psi}$  over the densities  $\hat{\rho} \in (\mathbb{A} \otimes \mathcal{A})_{\top}$  describing quantum encodings of a fixed input state  $\rho$ , similar to (5.2). Since

$$\sup_{\hat{\rho} \in \mathcal{S}_{\bar{\tau}}(\rho)} d(\hat{\rho} \circ \check{\Phi}, \hat{\rho} \circ \check{\Psi})^2 = \mu(\Phi_{\top}(\rho) + \Psi_{\top}(\rho)) - 2 \inf_{\hat{\rho} \in \mathcal{S}_{\bar{\tau}}(\rho)} \hat{\tau} \left( \sqrt{\check{\Phi}_{\top}(\hat{\rho}) \check{\Psi}_{\top}(\hat{\rho})} \right),$$

in the case of the unital  $\Phi, \Psi$  this can be written as  $d_c^{\rho}(\Phi, \Psi)^2/2 = 1 - f_c^{\rho}(\Phi, \Psi)$  in terms of the *complete relative quantum channel fidelity*

$$(6.2) \quad f_c^{\rho}(\Phi, \Psi) = \inf_{\hat{\rho} \in \mathcal{S}_{\bar{\tau}}} \left\{ \hat{\tau} \left( \sqrt{\check{\Phi}_{\top}(\hat{\rho}) \check{\Psi}_{\top}(\hat{\rho})} \right) : \tau_{\mathfrak{k}}(\hat{\rho}) = \rho \right\}$$

conditioned by an input state  $\rho$ . Here as in (5.2) the minimization is given over quantum encodings

$$\hat{\rho} = \left( I_{\mathfrak{k}} \otimes \rho^{1/2} \right) \Pi_{\tau} \left( I_{\mathfrak{k}} \otimes \rho^{1/2} \right) \in \mathcal{S}_{\bar{\tau}}(\rho)$$

described by normal unital CP maps  $\Pi : \mathbb{A} \rightarrow \bar{\mathcal{A}}$  in terms of their densities  $\Pi_{\tau}$  with respect to  $\tau = \tau_{\mathfrak{k}}$  for a fixed  $\rho \in \mathcal{S}_{\lambda}$ . Using the monotonicity [8, 9]

$$f \left( \hat{\Pi}_{\top}[\Phi_{\mu}(\rho)], \hat{\Pi}_{\top}[\Psi_{\mu}(\rho)] \right) \geq f(\Phi_{\mu}(\rho), \Psi_{\mu}(\rho))$$

of the quantum state relative fidelity with respect to such  $\Pi$ , we obtain the inequality

$$f_c^{\rho}(\Phi, \Psi) = \inf_{\Pi} \hat{\tau} \left( \sqrt{\hat{\Pi}_{\top}[\Phi_{\mu}(\rho)] \hat{\Pi}_{\top}[\Psi_{\mu}(\rho)]} \right) \geq (\tau \otimes \mu) \left( \sqrt{\Phi_{\mu}(\rho) \Psi_{\mu}(\rho)} \right).$$

This inequality is obviously equality achieved at  $\Pi = \text{Id}$  in the case the simple input algebra  $\mathcal{A} = \mathcal{B}(\mathfrak{g})$  on  $\mathfrak{g} = \ell^2$ , and this lower bound is also achieved for any separable  $\mathcal{A}$ . Thus we arrive to the following formula

$$(6.3) \quad f_c^\rho(\Phi, \Psi) = (\tau \otimes \mu) \left( \sqrt{\Phi_\mu(\rho)^{1/2} \Psi_\mu(\rho) \Phi_\mu(\rho)^{1/2}} \right)$$

for the complete relative fidelity  $f_c$  of quantum channels in terms of their densities  $\Phi_\mu$  and  $\Psi_\mu$  with respect to the trace  $\mu$  and a given input state  $\rho$ . This defines the complete Helinger operational half-square semidistance for each  $\rho$ :

$$(6.4) \quad d_c^\rho(\Phi, \Psi)^2 / 2 = 1 - (\tau \otimes \mu) \left( \left| \Phi_\mu^{1/2}(\tilde{\rho} \otimes I_{\mathfrak{g}}) \Psi_\mu^{1/2} \right| \right),$$

where we used an equivalent representation (7.8) for the fidelity (6.3) which will be derived in the next Section. Since this is simply relative fidelity for two optimal input-output entangled states described by the densities  $\hat{\phi} = \Phi_\mu(\rho)$  and  $\hat{\sigma} = \Psi_\mu(\rho)$ , the following equivalence inequality obviously holds for each  $\rho$ :

$$d_c^\rho(\Phi, \Psi)^2 \leq D_{\text{cb}}^\rho(\Phi, \Psi) \leq 2d_c^\rho(\Phi, \Psi).$$

By allowing the arbitrary semifinite quantum alphabet algebras  $\mathbb{A}$  we can extend this formula to the complete channel fidelity as the infimum

$$(6.5) \quad f_c(\Phi, \Psi) = \inf_{\rho \in \mathcal{S}_\lambda} (\tau \otimes \mu) \left( \left| \Phi_\mu^{1/2}(\tilde{\rho} \otimes I_{\mathfrak{g}}) \Psi_\mu^{1/2} \right| \right)$$

over all input states  $\rho$  on any algebra  $\mathcal{A}$ . It defines the complete operational Helinger (CH-) distance by  $d_c^2/2 = 1 - f_c$  as the supremum of (6.4) over all standard encodings  $\omega$  such that  $\check{\Phi}_\top(\omega) = \Phi_\mu(\rho)$  and  $\check{\Psi}(\omega) = \Psi_\mu(\rho)$ , corresponding to the optimal quantum alphabet algebra  $\mathbb{A} = \overline{\mathcal{A}}$ . Applying the equivalence inequality (3.4) to these states and taking then the supremum over all  $\rho \in \mathcal{S}_\lambda$  we obtain the equivalence inequality

$$(6.6) \quad d_c(\Phi, \Psi)^2 \leq D_{\text{cb}}(\Phi, \Psi) \leq 2d_c(\Phi, \Psi)$$

also for the complete fidelity distance and the CB-distance (5.5).

Thus the complete relative fidelity (6.2) refines the CB-norm inequalities  $D_{\text{b}} \leq D_{\text{cb}} \leq D_c$  by providing the channel fidelity distance  $d_c$  which satisfies the equivalence inequalities (6.6) due to the inequality  $D_{\text{cb}} \leq 2\sqrt{1 - f_c^2}$ . The complete channel fidelity  $f_c$  has clear operational meaning as the minimal relative fidelity of the compound states achieved over all input-output entanglements via two quantum channels described by the densities  $\Phi_\mu, \Psi_\mu$  [1]. In particular, as we show in the next section

$$(6.7) \quad f_c(\Phi, \Psi) = \inf_{\rho \in \mathcal{S}_\lambda} \left( \sum_j \left| \tau \left( \tilde{\rho} F_j^\dagger V \right) \right|^2 \right)^{1/2}$$

if  $\Phi$  is given in Kraus form  $\Phi(B) = \sum F_j^\dagger B F_j$  and  $\Psi$  is pure  $\Psi(B) = V^\dagger B V$ , given by an isometry  $V$ .

## 7. OPERATIONAL CH-DISTANCE AS A MINIMAX PROBLEM

Following analogy with quantum Bures distance as a variational problem (3.2) we should define the operational *Bures distance* as the square root of the natural

quadratic distance between generalized Schmidt decompositions  $\Gamma, \Upsilon \in \mathcal{B}(\mathfrak{f} \otimes \mathfrak{g})$  of their density operators  $\Phi_\mu$  and  $\Psi_\mu$ :

$$(7.1) \quad d_c^*(\Phi, \Psi)^2 = \inf_{\Gamma, \Upsilon} \left\{ \left\| \mu \left[ (\Gamma - \Upsilon)^\dagger (\Gamma - \Upsilon) \right] \right\| : \Gamma^\dagger \Gamma = \Phi_\mu, \Upsilon^\dagger \Upsilon = \Psi_\mu \right\}.$$

without explicit reference to the input state  $\rho$ . In the case of the standard trace  $\mu$  induced by  $\text{tr}_{\mathfrak{h}}$  this can be written as the minimization of the Hilbert module square distance

$$d_c^*(\Phi, \Psi)^2 = \inf_{\mathbf{F}, \mathbf{V}} \left\{ \left\| (\mathbf{F} - \mathbf{V})^\dagger (\mathbf{F} - \mathbf{V}) \right\| : |\mathbf{F}\rangle\langle\mathbf{F}| = \Phi_\mu, |\mathbf{V}\rangle\langle\mathbf{V}| = \Psi_\mu \right\}$$

over equivalent Kraus [11] decompositions

$$(7.2) \quad \Phi(B) = \sum_j F_j^\dagger B F_j \equiv \mathbf{F}^\dagger B \mathbf{F}, \quad \Psi(B) = \sum_j V_j^\dagger B V_j \equiv \mathbf{V}^\dagger B \mathbf{V}$$

corresponding to the purified Schmidt decomposition

$$(7.3) \quad \Phi_\mu = \sum_j |F_j\rangle\langle F_j| \equiv \Gamma^\dagger \Gamma, \quad \Psi_\mu = \sum_j |V_j\rangle\langle V_j| \equiv \Upsilon^\dagger \Upsilon.$$

Here  $(\mathbf{F}|$  and  $(\mathbf{V}|$  are the columns of  $(F_j|, (V_j|$  which are the components of

$$\Gamma = \sum_j |j\rangle\langle F_j|, \quad \Upsilon = \sum_j |j\rangle\langle V_j|,$$

in an orthonormal basis  $\{|j\rangle\}$  of  $\mathcal{H}$ , say  $|j\rangle = |i\rangle \otimes |k\rangle \equiv |i, k\rangle$ , defining the bounded operators  $F_j, V_j : \mathfrak{g} \rightarrow \mathfrak{f}$  as acting on the right of the bra-vectors  $\langle k| \in \mathfrak{g}$ :

$$\langle k|F_j|i\rangle = (F_j|(|i\rangle \otimes |k\rangle) = \langle j|\Gamma|i, k\rangle, \quad \langle k|V_j|i\rangle = (V_j|(|i\rangle \otimes |k\rangle) = \langle j|\Upsilon|i, k\rangle.$$

over all input densities  $\rho \in \mathcal{S}_\lambda$  with respect to  $\lambda = \bar{\tau}$ .

Taking into account that  $\|A^\dagger A\| = \sup_{\rho \in \mathcal{S}_\lambda} \tau(\tilde{\rho} A^\dagger A)$ , and that the positive function

$$c(\Gamma - \Upsilon; \rho) = (\tau \otimes \mu) \left[ (\Gamma - \Upsilon) (\tilde{\rho} \otimes I_{\mathfrak{g}}) (\Gamma - \Upsilon)^\dagger \right]$$

given by the total trace  $\tau \otimes \mu$  is convex with respect to  $\Gamma - \Upsilon$  and concave with respect to  $\rho$ , we can rewrite the minimal distance (7.1) in the form

$$(7.4) \quad d_c^*(\Phi, \Psi)^2 = \inf_{\substack{\Gamma: \Gamma^\dagger \Gamma = \Phi_\mu \\ \Upsilon: \Upsilon^\dagger \Upsilon = \Psi_\mu}} \sup_{\rho \in \mathcal{S}_\lambda} c(\Gamma - \Upsilon; \rho) = \sup_{\rho \in \mathcal{S}_\lambda} \inf_{\substack{\Gamma: \Gamma^\dagger \Gamma = \Phi_\mu \\ \Upsilon: \Upsilon^\dagger \Upsilon = \Psi_\mu}} c(\Gamma - \Upsilon; \rho),$$

where we exchanged the extrema since since  $\mathcal{S}_\lambda$  is convex and the function  $c$  is actually a positive quadratic form with respect to  $\Gamma - \Upsilon$ , certainly achieving its infimum. Thus, (7.1) can be represented in the form

$$d_c^*(\Phi, \Psi)^2 = \sup_{\rho \in \mathcal{S}_\lambda} \{ (\tau \otimes \mu) [(\Phi_\mu + \Psi_\mu) (\tilde{\rho} \otimes I_{\mathfrak{g}})] - 2f_c^*(\Phi, \Psi; \rho) \},$$

where the first term achieves the value 2 for the unital  $\Phi$  and  $\Psi$ , and

$$f_c^*(\Phi, \Psi; \rho) = \sup_{\Gamma^\dagger \Gamma = \Phi_\mu, \Upsilon^\dagger \Upsilon = \Psi_\mu} \text{Re} (\tau \otimes \mu) [(\tilde{\rho} \otimes I_{\mathfrak{g}}) \Gamma^\dagger \Upsilon].$$

Now we will prove that this supremum is in fact the conditional relative fidelity (6.3), and the complete relative fidelity (6.5) for quantum channels coincides with minimax

$$(7.5) \quad f_c^*(\Phi, \Psi) = \inf_{\rho \in \mathcal{S}_\lambda} \sup_{\Gamma^\dagger \Gamma = \Phi_\mu, \Upsilon^\dagger \Upsilon = \Psi_\mu} \text{Re} (\tau \otimes \mu) [(\tilde{\rho} \otimes I_{\mathfrak{g}}) \Gamma^\dagger \Upsilon]$$

for the real part of the Hilbert-Schmidt scalar product  $(\tau \otimes \mu)(X^\dagger Y)$  of  $X = \Gamma(\tilde{\rho} \otimes I_{\mathfrak{g}})^{1/2}$  and  $Y = \Upsilon(\tilde{\rho} \otimes I_{\mathfrak{g}})^{1/2}$  with respect to the total trace  $\tau \otimes \mu$ . Since the operators  $X$  and  $Y$  define the Schmidt decompositions  $\Phi_\mu(\rho) = X^\dagger X$  and  $\Psi_\mu(\rho) = Y^\dagger Y$  of the density operators  $\hat{\Phi}_\tau[\omega(\rho)]$  and  $\hat{\Psi}_\tau[\omega(\rho)]$  for the optimal entangled states on the algebra  $\mathbb{A} \otimes \mathcal{B} = \overline{\mathcal{A}} \otimes \mathcal{B}$ , the complete fidelity (6.5) is simply the minimax relative fidelity for the input-output entangled states obtained by two different transformations  $\check{\Phi}_\tau$  and  $\check{\Psi}_\tau$  of the input standard entangled states on algebra  $\mathbb{A} \otimes \mathcal{A} = \overline{\mathcal{A}} \otimes \mathcal{A}$ . It is first maximized over all Schmidt decompositions corresponding to the Kraus decompositions (7.2) of  $\Phi$  and  $\Psi$ , and then minimized with respect to all density operators  $\rho$  of the input states.

**Lemma 1.** *Let  $R, S \in \mathcal{A} \otimes \overline{\mathcal{B}}$  be positive bounded operators such that they have finite trace  $\tau \otimes \mu$ . Then*

$$(7.6) \quad \sup_{X, Y} \{(\tau \otimes \mu)(X^\dagger Y + Y^\dagger X) : X^\dagger X = R, Y^\dagger Y = S\} = 2(\tau \otimes \mu)(T),$$

where  $T = \sqrt{X S X^\dagger}$ . This supremum is achieved at any  $X \in \mathcal{A} \otimes \overline{\mathcal{B}}$  satisfying the condition  $X^\dagger X = R$ ,  $X = R^{1/2} \equiv X_o$  say, and at  $Y = Y_o$  satisfying the equation  $Y_o X^\dagger = T = X Y_o^\dagger$

*Proof.* First we observe, by applying the Schwarz inequality

$$\begin{aligned} (\tau \otimes \mu)(X^\dagger Y + Y^\dagger X) &\leq 2\sqrt{(\tau \otimes \mu)(X^\dagger X)(\tau \otimes \mu)(Y^\dagger Y)} \\ &= 2\sqrt{(\tau \otimes \mu)(R)(\tau \otimes \mu)(S)} \leq \infty, \end{aligned}$$

that this supremum is finite. In order to find it one can use Lagrangian multiplier method. Fixing  $X$  satisfying  $X^\dagger X = R$ ,  $X_o = R^{1/2}$  say, we can write the Lagrangian function as

$$\ell = (\tau \otimes \mu)(X^\dagger Y + Y^\dagger X - Y^\dagger Y L),$$

where  $L = L^\dagger \in \mathcal{A} \otimes \overline{\mathcal{B}}$  is the Lagrangian multiplier corresponding to the Hermitian condition  $S = Y^\dagger Y = S^\dagger$ . At the stationary point

$$\delta \ell = (\tau \otimes \mu)[(X^\dagger - L Y^\dagger) \delta Y + (X - Y L) \delta Y^\dagger] = 0,$$

so  $Y = Y_o$  must satisfy the equation  $Y L = X$  (the other equation  $L Y^\dagger = X^\dagger$  is Hermitian adjoint, corresponding to  $Y^\dagger = Y_o^\dagger$ ). Thus  $Y_o = X L^{-1}$ , where  $L^{-1}$  should be determined from  $L^{-1} X^\dagger X L^{-1} = S$ . Multiplying this from the left by  $X$  and from the right by  $X^\dagger$  this gives  $(X L^{-1} X^\dagger)^2 = X S X^\dagger$ , or  $X L^{-1} X^\dagger = \sqrt{X S X^\dagger}$ . Thus, indeed  $Y_o X^\dagger = \sqrt{X S X^\dagger} = X Y_o^\dagger$ , and therefore

$$(7.7) \quad (\tau \otimes \mu)(Y_o X^\dagger + X Y_o^\dagger) = 2(\tau \otimes \mu)\left(\sqrt{X S X^\dagger}\right).$$

This extremal value is the maximal value because of convexity of the maximizing function in (7.6). Note that due to  $\sqrt{U^\dagger X S X^\dagger U} = U^\dagger \sqrt{X S X^\dagger} U$  for any unitary  $U$ , the supremum (7.6), coinciding with (7.7) does not depend on the choice of  $X$  satisfying  $X^\dagger X = R$ . Indeed, by virtue of polar decomposition  $X = U R^{1/2}$

$$2(\tau \otimes \mu)\left(\sqrt{X S X^\dagger}\right) = 2(\tau \otimes \mu)\left(U^\dagger \sqrt{X S X^\dagger} U\right) = 2(\tau \otimes \mu)\left(\sqrt{R^{1/2} S R^{1/2}}\right).$$

Rewriting the trace  $(\tau \otimes \mu)(T) = (\tau \otimes \mu)(Y_o X^\dagger)$  as  $(\tau \otimes \mu)(X^\dagger Y_o)$  with

$$X^\dagger Y_o = R^{1/2} \sqrt{R^{1/2} S R^{1/2}} R^{-1/2} \equiv \sqrt{R S}$$

corresponding to  $X = R^{1/2}$ , we obtain the optimal value in (7.6) in either form  $(\tau \otimes \mu)(T) = (\tau \otimes \mu)\left(\sqrt{RS}\right)$  or  $(\tau \otimes \mu)(T) = (\tau \otimes \mu)\left(\sqrt{R^{1/2}SR^{1/2}}\right)$ .  $\square$

Note that choosing  $X = R^{1/2}$  for  $R = \Phi_\mu(\rho)$  and  $Y = R^{-1/2}\sqrt{RS}$  for  $S = \Psi_\mu(\rho)$  gives the formula (6.3). This proves that the operational Bures distance  $d_c^*$  and the operational Helinger distance  $d_c$  are the same:  $d_c^* = d_c$ . Another useful representation gives the choice  $X = \Phi_\mu^{1/2}(\tilde{\rho} \otimes I_{\mathfrak{g}})^{1/2}$  corresponding to  $\Gamma = \Phi_\mu^{1/2}$  gives the equivalent formula

$$(7.8) \quad f_c^\rho(\Phi, \Psi) = (\tau \otimes \mu) \left( \left| \Phi_\mu^{1/2}(\tilde{\rho} \otimes I_{\mathfrak{g}}) \Psi_\mu^{1/2} \right| \right) :$$

$$(7.9) \quad = (\tau \otimes \mu) \left( \sqrt{\Phi_\mu^{1/2}(\tilde{\rho} \otimes I_{\mathfrak{g}}) \Psi_\mu(\tilde{\rho} \otimes I_{\mathfrak{g}}) \Phi_\mu^{1/2}} \right)$$

for the complete relative fidelity (6.3) conditioned by the input state.

### 8. EXAMPLE: RELATIVE FIDELITY OF A PURE QUANTUM CHANNEL

Let us consider the case of pure quantum channel  $\Psi(B) = V^\dagger B V$  given on  $\mathcal{B} = \mathcal{B}(\mathfrak{h})$  by an isometry  $V$ ,  $V^\dagger V = I_{\mathfrak{g}}$ . In this case the input states are mapped as  $\Psi_\top(\rho) = V_\top^\dagger \rho V_\top$ , where  $V_\top = \tilde{V}$ , the Hilbert space transposition of  $V$ , and the density operator  $\Psi_\mu$  with respect to the standard trace  $\mu(B) = \text{tr}_{\mathfrak{h}} B$  is  $\Psi_\mu = |V\rangle\langle V|$ , where the generalized bra-vector  $\langle V|$  is defined in (4.3), and  $|V\rangle = (V|^\dagger$  is its Hermitian adjoint. Since

$$(L| = (V|(\tilde{\rho} \otimes I_{\mathfrak{g}}) \Phi_\mu^{1/2}$$

is defined as well-defined as Hilbert space element,

$$(L|L) = (V|(\tilde{\rho} \otimes I_{\mathfrak{h}}) \Phi_\mu(\tilde{\rho} \otimes I_{\mathfrak{h}}) |V) \leq 1,$$

we can use the formula  $\sqrt{|L\rangle\langle L|} = |L\rangle(L|L)^{-\frac{1}{2}}(L|$  in (7.8). Thus we obtain in this simple case

$$(8.1) \quad f_c^\rho(\Phi, \Psi) = (V|(\tilde{\rho} \otimes I_{\mathfrak{h}}) \Phi_\tau(\tilde{\rho} \otimes I_{\mathfrak{h}}) |V)^{1/2},$$

as  $(\tau \otimes \mu)(|L\rangle\langle L|) = (L|L)$  is induced by standard trace  $\text{Tr} = \text{tr}_{\mathfrak{g} \otimes \mathfrak{h}}$ . Using the Kraus decomposition  $\Phi(B) = \sum F_j^\dagger B F_j$ , this can be written as

$$f_c^\rho(\Phi, \Psi) = \left( \sum_j |(V|(\tilde{\rho} \otimes I_{\mathfrak{h}}) |F_j)|^2 \right)^{\frac{1}{2}} = \left( \sum_j \left| \text{tr}_{\mathfrak{g}}(\tilde{\rho} F_j^\dagger V) \right|^2 \right)^{\frac{1}{2}}$$

where we used the identity  $(V|(\tilde{\rho} \otimes I_{\mathfrak{h}}) |F) = \text{tr}_{\mathfrak{g}}(\tilde{\rho} F^\dagger V)$ . This defines the expression used in (6.7), and if  $\Phi$  is also pure,  $\Phi(B) = F^\dagger B F$ , we obtain even simpler formula  $f_c^\rho(\Phi, \Psi) = \left| \text{tr}_{\mathfrak{g}}(\tilde{\rho} F^\dagger V) \right|$ . Therefore, the CF-distance of such  $\Phi$  and  $\Psi$  is evaluated in  $d_c^2/2 = 1 - f_c^\rho(\Phi, \Psi)$  by the infimum

$$f_c(\Phi, \Psi) = \inf_{\rho \in \mathcal{S}_\lambda} \left( \sum_j \left| \text{tr}_{\mathfrak{g}}(\tilde{\rho} F_j^\dagger V) \right|^2 \right)^{1/2}.$$

which is minimal magnitude  $\inf_\rho \left| \text{tr}_{\mathfrak{g}}(\tilde{\rho} F^\dagger V) \right|$  of correlation between  $F$  and  $V$  in the case of pure  $\Phi$ . Note that the CB distance cannot be so explicitly evaluated in

the case of one pure channel, and it does not have such simple interpretation even when both channels are pure.

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