QUANTUM STOCHASTIC CALCULUS AND QUANTUM NONLINEAR FILTERING

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Abstract. A $\star$–algebraic indefinite structure of quantum stochastic (QS) calculus is introduced and a continuity property of generalized nonadapted QS integrals is proved under the natural integrability conditions in an infinitely dimensional nuclear space. The class of nondemolition output QS processes in quantum open systems is characterized in terms of the QS calculus, and the problem of QS nonlinear filtering with respect to nondemolition continuous measurments is investigated. The stochastic calculus of a posteriori conditional expectations in quantum observed systems is developed and a general quantum filtering stochastic equation for a QS process is derived. An application to the description of the spontaneous collapse of the quantum spin under continuous observation is given.

Introduction

The problem of description of continuous observation and filtering in quantum dynamical systems can be effectively solved in the framework of quantum stochastic (QS) calculus of nondemolition input-output processes first developed for quantum unitary Markovian evolutions in [1].

In contrast to classical probability theory, the conditional expectations defining à posteriori states of a quantum system with respect to a subalgebra may not exist in general and the existence depends on the algebra of observables and on the à priori state.

During the preparation of the measurement of a Quantum System the change necessary to produce the à priori compatible state as a mixture of the à posteriori states is referred to in quantum physics as the *demolition* of the system. The latter involves a change in the initial state by the reduction of the algebra of the system during such a preparation. The nondemolition principle provides a sufficient condition for the algebra of the quantum system to be prepared for the measurement in an initial state.

The mathematical formulation of the nondemolition principle for the observability of a class of quantum processes was given in [2] and investigated in subsequent papers [3, 4]. This fundamental principle of quantum measurement theory means that if a QS process X_t is indirectly observable by the measurement of another process Y_t then X and Y must satisfy the one sided commutativity condition $[X_t, Y_s] \equiv X_tY_s - Y_sX_t = 0$ for all $t \geq s$ but not for $t < s$.

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In the physical language this means that the measurements of Y in real time do not demolish the quantum system X (which has been prepared for the observation) at the present time or in the future. The condition given above, however, shows that, though the past of X (priori to t) can never be observed, it is demolished by the observation of the Y process. Mathematically it can be expressed as the decomposability of the algebra $\mathcal{A}_t$ generated by $\{X(s) : s \geq t\}$, describing the present and future of the system with respect to the spectral resolution of any Hermitian operator of the algebra $\mathcal{B}^t$ generated by $\{Y(s) : s \leq t\}$.

In this paper we show using method of quantum filtering that the nondemolition condition given above is necessary and sufficient for the evaluation of a posteriori mean values of $X \in \mathcal{A}_t$ given are arbitrary initial state. In other words we prove that a quantum system is statistically predictable by a measurement procedure, iff the observable process satisfies the nondemolition condition.

In the Sections 1 and 2 of the paper we develop the general QS calculus of nondemolition input–output quantum processes in Fock space, tensored by an initial Hilbert space. We introduce the QS calculus of such processes using the $\star$–algebraic Minkowski metric structure of the basic quantum processes and the simple and convenient notation developed in [5]. The Fock representation of this structure is closely connected with the Lindsay–Maassen kernel calculus of [6] but is given in terms of the matrix elements of operators for general quantum noise in Fock space instead of their kernels. We define the QS integrals in the framework of the new noncommutative stochastic analysis in the Fock scale which is described in the first section.

In the Sections 3 and 4 we give complete proofs of the results, first formulated in [7], for the general (non Markovian) quantum filtering from the viewpoint of QS calculus. The advantage of the $\star$ – matrix notation enables us to prove the main filtering theorem for general output process as by using the indefinite metric for the corresponding $\star$ – algebra of generators of these nondemolition processes.

The Markovian nonlinear filtering problem in the framework of quantum operational (non-stochastic) approach was first investigated in [8], and the possibility of deriving the stochastic equations of quantum filtering within this framework was shown in [9]. The Markovian filtering for the quantum Gaussian case and the corresponding quantum Kalman linear filter, first obtained for the one-dimensional case in [2, 3], is considered using the QS calculus approach in [10].

The present paper is devoted essentially to the study of the nonlinear problem, extending the innovation martingale methods of the classical filtering theory [11, 12] to the noncommutative set up of our problem. An application of the quantum filtering theory to the solution of the problem of the continuous observation of quantum spin states is given in Section 5.

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1. QS calculus of input Bose processes in Fock space

Let us denote by $\mathcal{F} = \Gamma(\mathcal{E})$ the state space of the one–dimensional Bose–noise, that is the Fock space over the Hilbert space $\mathcal{E} = L^2(\mathbb{R}^+)$ of square–integrable complex functions $t \mapsto \varphi(t)$ on the real half–line $\mathbb{R}^+$. One should consider $\mathcal{F}$ as the Hilbert space $\Gamma(\mathcal{E}) = L^2(\Omega(\mathbb{R}^+))$ of the square–integrable functions $\tau \mapsto \varphi(\tau)$ of $\tau = (t_1, \ldots, t_n)$ with $t_i \in \mathbb{R}^+$, $t_1 < \cdots < t_n$, $n = 0, 1, 2, \ldots$ and scalar product $< \varphi|\chi >= \int \varphi(\tau)^* \chi(\tau)\mathrm{d}\tau$,

$$\int \varphi(\tau)^* \chi(\tau)\mathrm{d}\tau = \sum_{n=0}^{\infty} \int_{t_1 \leq \cdots \leq t_n} \int \overline{\varphi}\left(t_1, \ldots, t_n\right) \chi\left(t_1, \ldots, t_n\right) \mathrm{d}t_1 \ldots \mathrm{d}t_n,$$

where the integral is taken over the set $\Omega(\mathbb{R}^+)$ of all finite chains τ on $\mathbb{R}^+$ with respect to the natural Lebesgue measure $\mathrm{d}\tau = \mathrm{d}t_1 \ldots \mathrm{d}t_n$ for every $n = |\tau| = 0, 1, \ldots$. Following [5] we shall identify the chains $\tau = (t_1, \ldots, t_n)$ with the finite subsets $\{t_1, \ldots, t_n\} \subset \mathbb{R}^+$, so that the empty chain ($n = 0$) is identified with the empty subset $\tau = \emptyset$ having $\mathrm{d}\tau = 1$ and $\tau = t$ ($n = 1$) is identified with the one–point subset $\{t\}$ having $\mathrm{d}\tau = \mathrm{d}t$. We shall also denote the normalized vacuum function as the Kronecker δ-function: $\delta_\emptyset(\tau) = 1$, if $\tau = \emptyset$; $\delta_\emptyset(\tau) = 0$, if $\tau \neq \emptyset$, and consider the Fock spaces $\mathcal{F}_s = \Gamma(\mathcal{E}_s)$, $\mathcal{F}_t^s = \Gamma(\mathcal{E}_t^s)$ over orthogonal subspaces $\mathcal{E}_s = \{\varphi(t) = 0 : t \leq s\}, \mathcal{E}_t^s = \{\varphi(r) = 0 : r \notin [t, s]\}$ as the function Hilbert spaces $L^2(\Omega_s)$, $L^2(\Omega_t^s)$ on the subsets $\Omega_s = \{\tau \subset]s, \infty[\}$, $\Omega_t^s = \{\tau \subset]t, s]\}$ of the chains $\tau > s$, $s \geq \tau > t$ correspondingly.

Note that for any $t > s$ a chain $\tau \in \Omega$ can be represented as the triple $\tau = (\tau^t, \tau_t^s, \tau_s)$ of the subchains $\tau_s = \{t_i \in \tau : t_i > s\}$, $\tau_t^s = \{t_i \in \tau : s \geq t_i > t\}$, $\tau^t = \{t_i \in \tau : t_i \leq t\}$ so that the direct product representation $\Omega = \Omega^t \times \Omega_t^s \times \Omega_s$ holds and, hence, the tensor representation $\mathcal{F} = \mathcal{F}^t \otimes \mathcal{F}_t^s \otimes \mathcal{F}_s$ with $\mathcal{F}^t = L^2\left(\Omega^t\right)$, $\Omega^t = \{\tau \in \Omega : \tau \leq t\}$.

The basic processes for QS calculus in Fock space $\mathcal{F}$ are the annihilation A_-, creation A^+ and quantum number N processes, represented for all $t > 0$ by the unbounded operators

$$\begin{aligned} (A_-(t)\varphi)(\tau) &= \int_0^t \varphi(\tau \sqcup s)\mathrm{d}s, \\ (A^+(t)\varphi)(\tau) &= \sum_{s \in \tau} \chi^t(s)\varphi(\tau \backslash s), \end{aligned}$$

with the common dense domain $\mathcal{D}^t = \left\{\varphi \in \mathcal{F} : \int |\tau^t||\varphi(\tau)|^2 \mathrm{d}t < \infty\right\}$, and

$$(N(t)\varphi)(\tau) = |\tau^t|\varphi(\tau), \tag{1.1}$$

where $\chi^t(s) = 1$, if $s \leq t$, $\chi^t(s) = 0$, if $s > t$, $|\tau^t| = \sum_{s \in \tau} \chi^t(s)$, the chain $\tau \sqcup s$ is defined almost everywhere as $(t_1, \ldots t_i, s, t_{i+1}, \ldots, t_n)$, if $t_i < s < t_{i+1}$, and $\tau \backslash s = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots t_n)$, if $s = t$, $\tau \backslash s = \tau$, if $s \neq t_i$ for all i. Note that the processes A_-, A^+ and N are non-commuting, but commuting with increments:

$$\begin{aligned} \left[A_-(t), A^+(t')\right] &= t \wedge t' I, \quad \text{where } t \wedge t' = \min(t, t'), \\ [A_-(t), N(t')] &= A_-(t \wedge t'), [N(t), A^+(t)] = A^+(t \wedge t'), \end{aligned} \tag{1.2}$$

the processes A_- and A^+ are mutually adjoint: $A_-^*(t) = A_-(t)^* = A^+(t)$, and N is selfadjoint: $N^* = N$.

Let us introduce the notations [5]

$$A^+_-(t) = tI,\ A^\circ_-(t) = A_-(t),\ A^+_\circ = A^+(t), A^\circ_\circ(t) = N(t), \tag{1.3}$$

where I is the identity operator in $\mathcal{F}$, thus defining a 3×3 matrix–valued QS process $\mathbf{A} = (A^\mu_\nu)$, indexed by $\mu, \nu \in \{-, o, +\}$ with $A^\mu_\nu = 0$, if $\mu = +$ or $\nu = -$. We shall consider the process $\mathbf{A}$ defined as a linear operator-valued function $A(\mathbf{c}, t) = \mathrm{tr}\{\mathbf{c}\ \mathbf{A}(t)\}$ in terms of a 3×3 – matrix $\mathbf{c} = (c^\mu_\nu)$,

$$A(\mathbf{c}, t) = Ic^-_+ t + A_-(c^-_\circ, t) + A^+(c^\circ_+, t) + N(c^\circ_\circ, t), \tag{1.4}$$

where $A_-(c^-_\circ) = c^-_\circ A_-$, $A^+(c^\circ_+) = c^\circ_+ A^+$, $N(c^\circ_\circ) = c^\circ_\circ N$, writing the matrix trace as $\mathrm{tr}\{\mathbf{c}\mathbf{A}\} = c^\mu_\nu A^\nu_\mu$ by the tensor notation of the sum $\sum c^\mu_\nu A^\nu_\mu$. The matrices $\mathbf{c}$ with $c^\mu_\nu = 0$ for $\mu = +$ or $\nu = -$ form a complex Lie $\star$-algebra with respect to the matrix commutator and the involution

(1.5)

$$\mathbf{c} = \begin{pmatrix} 0 & c^-_\circ & c^-_+ \\ 0 & c^\circ_\circ & c^\circ_+ \\ 0 & 0 & 0 \end{pmatrix} \mapsto \mathbf{c}^\star = \begin{pmatrix} 0 & c^{\circ *}_+ & c^{-*}_+ \\ 0 & c^{\circ *}_\circ & c^{-*}_\circ \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{g}\mathbf{c}^\dagger\mathbf{g},\ \mathbf{g} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where $c^{\mu *}_\nu = c^{-\mu}_{\ \nu} = c^{*\nu}_\mu$ and $\mathbf{g}^\dagger = \mathbf{g} = \mathbf{g}^{-1}$ is the indefinite metric matrix, defining a pseudo-scalar product in $\mathbb{C}^3$:

$$(\mathbf{x}|\mathbf{z}) = \overline{x}^+ z^- + \overline{x}^\circ z^\circ + \overline{x}^- z^+ = \mathbf{x}^\star \mathbf{z}\ ,$$

$\mathbf{x}^\star = (\overline{x}_+, \overline{x}_\circ, \overline{x}_-) = \mathbf{x}^\dagger \mathbf{g}$ is the row, conjugate to the column $\mathbf{x} = (x^\mu) \in \mathbb{C}^3$.

Now we can consider a multi-dimensional Bose noise, when $\mathcal{E}$ is a Hilbert space $L^2(\mathbb{R}^+ \to \mathbb{C}^m)$ of vector-functions $\varphi(t) = (\varphi^j)(t) \equiv \varphi^\circ(t), j = 1, \dots, m$ with

$$< \varphi|\varphi > = \int \sum_{j=1}^m \overline{\varphi}^j \varphi^j \mathrm{d}t.$$

It is enough to regard $c^-_\circ$ as a m-row with components $c^-_j \in \mathbb{C}, c^\circ_+$ as a m-column with components $c^j_+ \in \mathbb{C}$, and $c^\circ_\circ$ as a $m \times m$-matrix with elements $c^i_k \in \mathbb{C}$. The following theorems are valid also for the general situation $\mathcal{F} = \Gamma(L^2(\mathbb{R}^+ \to \mathcal{K}))$, if the indices μ, ν take values in the set $-, J, +$, where the one-point index value $\mu, \nu = 0$ is split into $m = |J|$ points $j \in J$ of an index set J for a basis in a Hilbert space $\mathcal{K}$ with the infinite cardinality $|J| = \dim \mathcal{K}$.

Proposition 1. *The basic QS process $A(\mathbf{c})$, defined by (1.1), (1.2), gives for each t an operator representation of the complex Lie $\star$-algebra of matrices (1.5): $A(\mathbf{c}, t)^* = A(\mathbf{c}^\star, t)$,*

$$[A(\mathbf{c}^\star, t), A(\mathbf{c}, t')] = A([\mathbf{c}^\star, \mathbf{c}], t \wedge t')\ . \tag{1.6}$$

The multiplication table [1] for Ito differentials $dA_-, \mathrm{d}A^+, \mathrm{d}N$, and $I\mathrm{d}t$ can be written in terms of $A(\mathbf{c}, \mathrm{d}t) = c^\mu_\nu \mathrm{d}A^\nu_\mu(t)$ as

$$A(\mathbf{c}^\star, \mathrm{d}t) A(\mathbf{c}, \mathrm{d}t') = A(\mathbf{c}^\star \mathbf{c}, \mathrm{d}t \cap \mathrm{d}t'), \tag{1.7}$$

where $\mathrm{d}t \bigcap \mathrm{d}t' = \emptyset$ for $t \neq t'$, $A(\cdot, \emptyset) = 0$, and $\mathrm{d}t \bigcap \mathrm{d}t' = \mathrm{d}t$ for $t = t'$.

Proof. Taking into account, that $A^*_- = A^+$ and $N^* = N$, one obtains

$$A(\mathbf{c}, t)^* = Ic^{-*}_+ t + A_-(c^{\circ *}_+, t) + A^+(c^{-*}_\circ, t) + N(c^{\circ *}_\circ, t) \tag{1.8}$$

The comparing of (1.6) with (1.2) gives the $\star$-property $A(\mathbf{c})^* = A(\mathbf{c}^\star)$ of the map $\mathbf{c} \mapsto A(\mathbf{c})$.

The Lie representation property follows directly from the canonical commutation relations

$$[A_-(b_\circ^-,t),A^+(d_+^\circ,t)] = tb_\circ^- d_+^\circ,\quad [N(b_\circ^\circ),N(d_\circ^\circ)] = N([b_\circ^\circ,d_\circ^\circ])\ ,$$
$$[N(b_\circ^\circ),A^+(d_+^\circ)] = A^+(b_\circ^\circ d_+^\circ),\quad [A_-(b_\circ^-),N(d_\circ^\circ)] = A_-(b_\circ^- d_\circ^\circ)\ ,$$

which give $[A(\mathbf{b}),A(\mathbf{d})]=A([\mathbf{b},\mathbf{d}])$, where we take into account that

$$(\mathbf{b}\,\mathbf{d})_\circ^- = b_\circ^- d_+^\circ, (\mathbf{b}\,\mathbf{d})_\circ^- = b_\circ^- d_\circ^\circ, (\mathbf{b}\,\mathbf{d})_+^\circ = b_\circ^\circ d_+^\circ,\ (\mathbf{b}\,\mathbf{d})_\circ^\circ = b_\circ^\circ d_\circ^\circ$$

for matrices $\mathbf{b},\mathbf{d}$ of the form (1.5).

Applying it to $\mathbf{b}=\mathbf{c}^\star,\mathbf{d}=\mathbf{c}$ and taking into account the commutativity of $A(\mathbf{c}^\star,t)$ with increment $A(\mathbf{c},t')-A(\mathbf{c},t)$, one obtains (1.6). In the same way one obtains (1.7) from the Hudson – Parthasarathy multiplication table

$$\begin{aligned} &\mathrm{d}A_-(b_\circ^-)\mathrm{d}A^+(d_+^\circ) = I\mathrm{d}t(b_\circ^- d_+^\circ),\quad \mathrm{d}A_-(b_\circ^-)\,\mathrm{d}N(d_\circ^\circ) = \mathrm{d}A_-(b_\circ^- d_\circ^\circ)\\ &\mathrm{d}N(b_\circ^\circ)\mathrm{d}A^+(d_+^\circ) = \mathrm{d}A^+(b_\circ^\circ d_+^\circ),\quad \mathrm{d}N(b_\circ^\circ)\mathrm{d}N(d_\circ^\circ) = \mathrm{d}N(b_\circ^\circ d_\circ^\circ), \end{aligned} \tag{1.9}$$

for $\mathrm{d}A_\nu^\mu(t) = A_\nu^\mu(t+\mathrm{d}t)-A_\nu^\mu(t),\mathbf{b}=\mathbf{c}^\star,\mathbf{d}=\mathbf{c}$. Due to complex linearity of the map $\mathbf{c}\mapsto A(\mathbf{c})$ the formulas (1.6), (1.7) can be always extended to arbitrary $\mathbf{b},\mathbf{d}$ by polarization formula

$$\begin{aligned} A(\mathbf{b}\,\mathbf{d}) &= \sum_{n=0}^{3} A\left((\mathbf{b}^\star+\mathrm{i}^n\mathbf{d})^\star(\mathbf{b}^\star+\mathrm{i}^n\mathbf{d})\right)/4\mathrm{i}^n \quad ,\quad \mathrm{i}=\sqrt{-1},\\ A(\mathbf{b})A(\mathbf{d}) &= \sum_{n=0}^{3} A(\mathbf{b}^\star+\mathrm{i}^n\mathbf{d})^\star A(\mathbf{b}^\star+\mathrm{i}^n\mathbf{d})/4\mathrm{i}^n. \end{aligned}$$

Hence, (1.6) is equivalent to (1.8) and (1.7) to (1.9). ∎

Let us now define a QS integral with respect to the basic process A for a matrix quantum process $\mathbf{C}(t)=(C_\nu^\mu)(t),\ \mu,\nu\in\{-,J,+\}$ in $\mathcal{F}$. Assuming that the operator–valued functions $t\mapsto C_\nu^\mu(t)$ are weakly measurable and adapted: $C(t)=C^t\otimes I_t$, where C^t are the operators in $\mathcal{F}^t$ for all $\mu\in\{-,J\}$ and $\nu\in\{J,+\}$, one can define in the case of finite J the QS–integral

$$\int_0^t A(\mathbf{C},\mathrm{d}s) := \int_0^t \sum_{\mu,\nu} C_\nu^\mu \mathrm{d}A_\mu^\nu \equiv \int_0^t C_\nu^\mu \mathrm{d}A_\mu^\nu$$

as the sum of the Lebesgue operator–valued integral $\int C_+^-(s)\mathrm{d}s$ and the Itô integrals $\int C_j^-\mathrm{d}A_-^j$, $\int C_+^j\mathrm{d}A_j^+$, $\int C_k^i\mathrm{d}N_i^k$ in the sense [13, 14].

In the general case $\mathcal{E}=L^2(\mathbb{R}^+\to\mathcal{K})$ we shall regard the QS–integral $\int C_\nu^\mu\mathrm{d}A_\mu^\nu$ as a continuous operator $\mathcal{F}^+\to\mathcal{F}_-$ on the projective limit $\mathcal{F}^+=\bigcap_{\eta>1}\mathcal{F}(\eta)$ into $\mathcal{F}_-=\bigcap_{\eta<1}\mathcal{F}(\eta)$ of Hilbert spaces $\mathcal{G}(\zeta)\subset\mathcal{F}\subset\mathcal{G}(\xi)$, $\zeta>1>\xi$, with respect to the scalar products

$$\|\varphi\|^2(\eta) = \int_\eta^{|\tau|} \eta^{|\tau|} \begin{matrix}\langle\varphi(\tau)|\varphi(\tau)\rangle_+\\ \langle\varphi(\tau)|\varphi(\tau)\rangle_-\end{matrix}\ \mathrm{d}\tau\ ,\qquad \begin{matrix}\varphi(\tau)\in\mathcal{E}(\tau)\ ,\eta>1\\ \varphi(\tau)\in\mathcal{E}'(\tau)\ ,\eta<1\ .\end{matrix}$$

Here $\langle\varphi|\varphi\rangle_+(\tau)\ge\|\varphi\|^2\ge\langle\varphi|\varphi\rangle_-(\tau)$ are the square-norms in the Hilbert tensor products $\mathcal{E}(\tau)=\otimes_{t\in T}\mathcal{E}(t)$, $\mathcal{K}^{\otimes|\tau|}$, $\mathcal{E}'(\tau)=\otimes_{t\in T}\mathcal{E}'(t)$ of Hilbert spaces $\mathcal{E}(t)\subseteq\mathcal{K}\subseteq\mathcal{E}'(t)$, forming a Gelfand triple for each $t\in\mathbb{R}^+$ with respect to the scalar product $\|\varphi\|^2=\langle\varphi|\varphi\rangle$ in a Hilbert space $\mathcal{K}$ (or simply $\mathcal{E}(t)=\mathcal{K}=\mathcal{E}'(t)$, if $\mathcal{K}=\mathbb{C}^m$).

We shall say that a weakly measurable function $t \mapsto \mathbf{C}(t)$ is locally QS–integrable if its components $C^{\mu}_{\nu}, \mu \in \{-, o\},\ \nu \in \{o, +\}$ are locally L^p–integrable as operator–valued functions

$$\begin{array}{ll} C^-_+(t) : \mathcal{G}^+ \to \mathcal{G}_-, & \|C^-_+(\cdot)\|^{\xi,1}_{\zeta,t} < \infty \quad (p = 1) \\ C^\circ_+(t) : \mathcal{G}^+ \to \mathcal{G}_- \otimes \mathcal{E}'(t), & \|C^o_+(\cdot)\|^{\xi,2}_{\zeta,t} < \infty \quad (p = 2) \\ C^-_\circ(t) : \mathcal{G}^+ \otimes \mathcal{E}(t) \to \mathcal{G}_-, & \|C^o_\circ(\cdot)\|^{\xi,2}_{\zeta,t} < \infty \quad (p = 2) \\ C^\circ_\circ(t) : \mathcal{G}^+ \otimes \mathcal{E}(t) \to \mathcal{G}_- \otimes \mathcal{E}'(t), & \|C^\circ_\circ(\cdot)\|^{\xi,\infty}_{\zeta,t} < \infty \quad (p = \infty) \end{array}$$

Here the norms are defined for any $t > 0$, $\xi \in]0, 1[$ and a sufficiently large $\zeta > 1$ by

$$\begin{aligned} \|C^-_+\|^{\xi,1}_{\zeta,t} &= \int_0^t \|C^-_+(s)\|^\xi_\zeta \mathrm{d}s, \quad \|C^\circ_\circ\|^{\xi,\infty}_{\zeta,t} = \mathrm{ess}_{s\le t} \sup \|C^\circ_\circ(s)\|^\xi_\zeta, \\ \|C^-_+\|^\xi_\zeta &= \sup_\varphi \{\|C^-_+\varphi\|(\xi)/\|\varphi\|(\zeta)\},\ \|C^\circ_\circ\|^\xi_\zeta = \sup_{\varphi^\circ} \{\|C^\circ_\circ\varphi^\circ\|(\xi)/\|\varphi^\circ\|(\zeta)\}\ , \end{aligned}$$

where

$$\begin{aligned} \varphi \in \mathcal{G}(\zeta),\ \|\varphi\|^2(\zeta) &= \ < \varphi|\varphi > (\zeta), \\ \varphi^\circ \in \mathcal{G}(\zeta) \otimes \mathcal{E}(s),\ \|\varphi^\circ\|(\zeta) &= \ < \varphi^\circ|\varphi^\circ > (\zeta) \end{aligned}$$

and $\|C^\circ_+\|^{\xi,2}_{\zeta,t} = \|C^{\circ t}_+\|^\xi_\zeta,\ \|C^-_\circ\|^{\xi,2}_{\zeta,t} = \|C^-_{\circ t}\|^\xi_\zeta$ are the norms

$$\|C\|^{\xi,2}_{\xi,t} = (\int_0^t (\|C(s)\|^\xi_\zeta)^2 \mathrm{d}s)^{1/2}$$

of the operators

$$\begin{array}{ll} C^{\circ t}_+ : \mathcal{G}(\zeta) \to \mathcal{G}(\xi) \otimes \mathcal{E}'^t, & \left(C^{\circ t}_+\varphi\right)(s) = C^\circ_+(s)\varphi,\ s \le t \\ C^-_{\circ t} : \mathcal{G}(\zeta) \otimes \mathcal{E}^t \to \mathcal{G}(\xi), & C^-_{\circ t}\varphi^\circ = \displaystyle\int_0^t C^-_\circ(s)\varphi^\circ(s)\mathrm{d}s \end{array}$$

in the Hilbert spaces $\mathcal{E}^t = \oplus \int_0^t \mathcal{E}(s)\mathrm{d}s$, $\mathcal{E}'^t = \oplus \int_0^t \mathcal{E}'(s)\mathrm{d}s$.

The following theorem shows the continuity of the QS–integral of an integrable $\mathbf{C}$, defined on $\mathcal{G}^+$ even for nonadapted $C^\mu_\nu(t)$ by the formula

$$\begin{aligned} \left(\int_0^t A(\mathbf{C}, \mathrm{d}s)\varphi\right)(\tau) &= \int_0^t \left(C^-_+(s)\varphi + C^-_\circ(s)\varphi^\circ_s\right)(\tau)\mathrm{d}s \\ &+ \sum_{s\in\tau}^{s\le t} \left(C^\circ_+(s)\varphi + C^\circ_\circ(s)\varphi^\circ_s\right)(\tau/s)\ , \end{aligned} \tag{1.10}$$

where $\varphi^\circ_t \in \mathcal{G}^+ \otimes \mathcal{E}(t)$ is defined almost everywhere as the tensor–function $\varphi^\circ_t(\tau) = \varphi(\tau \sqcup t)$.

Theorem 1. *Suppose that $\mathbf{C}(t)$ is a locally QS–integrable function i.e. for any $\xi < 1$, $t > 0$ there exists $\zeta > 1$, such that*

$$\|C^-_+\|^{\xi,1}_{\zeta,t} < \infty,\ \|C^\circ_+\|^{\xi,2}_{\zeta,t} < \infty,\ \|C^-_\circ\|^{\xi,2}_{\zeta,t} < \infty,\ \|C^\circ_\circ\|^{\xi,\infty}_{\zeta,t} < \infty.$$

Then the QS–integral (1.10) is defined as a continuous operator $\imath^t_0(\mathbf{C}) : \mathcal{G}^+ \to \mathcal{G}_-$ with the estimate

$$\left\|\int_0^t A(\mathbf{C}(s), \mathrm{d}s)\right\|^{\eta_-}_{\eta^+} \le \|C^-_+\|^{\xi,1}_{\zeta,t} + \frac{1}{\sqrt{\varepsilon}}\left(\|C^-_\circ\|^{\xi,2}_{\zeta,t} + \|C^\circ_+\|^{\xi,2}_{\zeta,t}\right) + \frac{1}{\varepsilon}\|C^\circ_\circ\|^{\xi,\infty}_{\zeta,t} \tag{1.11}$$

for the norms $\|\imath_0^t(\mathbf{C})\|_{\eta^+}^{\eta_-} = \sup_\varphi\{\|\imath_0^t(C)\varphi\|(\eta_-)/\|\varphi\|(\eta^+)\}$, *where* $\eta_- \leq \xi - \varepsilon$, $\eta^+ \geq \zeta + \varepsilon$ *and* $0 < \varepsilon < \xi$. *Moreover, the adjoint integral*

$$< \int_0^t A(\mathbf{C}, \mathrm{d}s)^* \varphi | \chi > = < \varphi | \int_0^t A(\mathbf{C}, \mathrm{d}s)\chi >, \ \varphi, \chi \in \mathcal{G}^+$$

is also densely defined on $\mathcal{G}^+ \subset \mathcal{G}_-$ *as the QS–integral* $\int_0^t A(\mathbf{C}^\star, \mathrm{d}s)$, *and the function* $\mathbf{C}^\star(t) = \mathbf{g}\mathbf{C}(t)^*\mathbf{g}$,

$$\mathbf{C}^\star(t)_+^- = C_+^-(t)^*, \ \mathbf{C}^\star(t)_+^\circ = C_\circ^-(t)^*, \ \mathbf{C}^\star(t)_\circ^- = C_+^\circ(t)^*, \ \mathbf{C}^\star(t)_\circ^\circ = C_\circ^\circ(t)^*$$

is locally QS–integrable with $\|\mathbf{C}^\star(t)_\nu^\mu\|_{1/\xi}^{1/\zeta} = \|C_\nu^\mu(t)^*\|_\zeta^\xi < \infty$ *for almost all* t.

Proof. . In order to show the continuity of the integral (1.10) in the projective topology of $\bigcap_{\zeta>1}\mathcal{G}(\zeta)$, one should prove that

$$\| \int_0^t A(\mathbf{C}(s), \mathrm{d}s)\varphi\|(\eta_-) \leq c\|\varphi\|(\eta^+), \|\varphi\|(\eta) = (< \varphi|\varphi > (\eta))^{1/2}$$

for any $\varphi \in \mathcal{G}(\eta^+)$, $\eta_- < \xi$ and a $\eta^+ > \zeta$, $c > 0$. Due to the definition

$$\begin{aligned} \left\| \int_0^t A(\mathbf{C}, \mathrm{d}s)\varphi \right\| &\leq \| \int_0^t C_\circ^- \mathrm{d}A_-^\circ \varphi\| + \| \int_0^t C_+^\circ \mathrm{d}A_\circ^+ \varphi\| \\ &+ \| \int_0^t C_\circ^\circ \mathrm{d}N_\circ^\circ \varphi\| + \| \int_0^t C_+^- \mathrm{d}s \varphi\|, \end{aligned}$$

where

$$\int_0^t C_\circ^- \mathrm{d}A_-^\circ \varphi = \int_0^t C_\circ^-(s)\varphi_s^\circ \mathrm{d}s, \ \left(\int_0^t C_+^\circ \mathrm{d}A_\circ^+ \varphi\right)(\tau) = \sum_{s\in\tau}^{s\leq t} (C_+^\circ(s)\varphi)(\tau\backslash s),$$

and

$$\left(\int_0^t C_\circ^\circ \mathrm{d}N_\circ^\circ \varphi\right)(\tau) = \sum_{s\in\tau}^{s\leq t} \left(C_\circ^\circ(s)\varphi_s^\circ\right)(\tau\backslash s).$$

The first two integrals in (1.10) can be easily estimated as

$$\begin{aligned} \left\| \int_0^t C_+^- \varphi \mathrm{d}s \right\|(\eta_-) &\leq \int_0^t \|C_+^-(s)\varphi\|(\xi)\mathrm{d}s \leq \int_0^t \|C_+^-(s)\|_\zeta^\xi \mathrm{d}s \|\varphi\|(\zeta) = \\ & \|C_+^-\|_{\zeta,t}^{\xi,1}\|\varphi\|(\zeta), \\ \left\| \int_0^t C_\circ^- \mathrm{d}A_-^\circ \varphi \right\|(\eta_-) &= \|C_{\circ t}^- \varphi^\circ\|(\xi) \leq \|C_{\circ t}^-\|_\zeta^\xi \|\varphi^\circ\|(\zeta) = \\ & \|C_\circ^-\|_{\zeta,t}^{\xi,2} \left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\|\varphi\|^2(\zeta)\right)^{1/2}, \end{aligned}$$

where we took into account that

$$\|\varphi^\circ\|^2(\zeta) = \int\int \zeta^{|\tau|}\|\varphi(\tau \sqcup t)\|^2 \mathrm{d}\tau\mathrm{d}t = \int |\tau|\zeta^{|\tau|-1}\|\varphi(\tau)\|^2\mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}\zeta}\|\varphi\|^2(\zeta).$$

In order to estimate the integrals of C_{+}° and $C_{\circ}^{\circ}$ let us find

$$\left\|\int_0^t C_{+}^{\circ}\mathrm{d}A_{\circ}^{+}\varphi\right\|^2(\eta_-) = \int_\Omega \left\|\sum_{s\in\tau}^{s\leq t}(C_{+}^{\circ}(s)\varphi)(\tau/s)\right\|^2 \eta_-^{|\tau|}\mathrm{d}\tau =$$

$$\eta_-^2\int_0^t\int_0^t\int_\Omega < [C_{+}^{\circ}(s_1)\varphi_{s_2}^{\circ}](\tau)|[C_{+}^{\circ}(s_2)\varphi_{s_1}^{\circ}](\tau) > \eta_-^{|\tau|}\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2 +$$

$$\eta_-\int_0^t\int_\Omega \|[C_{+}^{\circ}(s)\varphi](\tau)\|^2\eta_-^{|\tau|}\mathrm{d}\tau\mathrm{d}s \leq \eta_-\left(1+\eta_-\frac{\mathrm{d}}{\mathrm{d}\eta_-}\right)\|C_{+}^{\circ}\varphi\|_t^2(\eta_-)$$

by Schwarz inequality. In the same way we get

$$\left\|\int_0^t C_{\circ}^{\circ}\mathrm{d}N_{\circ}^{\circ}\varphi\right\|^2(\eta_-) = \int_\Omega \left\|\sum_{s\in\tau}^{s\leq t}(C_{\circ}^{\circ}(s)\varphi^{\circ}(s))(\tau/s)\right\|^2 \eta_-^{|\tau|}\mathrm{d}\tau =$$

$$= \eta_-^2\int_0^t\int_0^t\int_\Omega < [C_{\circ}^{\circ}(s_1)\varphi_{s_1\,s_2}^{\circ\,\circ}](\tau)|[C_{\circ}^{\circ}(s_2)\varphi_{s_1\,s_2}^{\circ\,\circ}](\tau) > \eta_-^{|\tau|}\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2 +$$

$$\eta_-\int_0^t\int_\Omega \|[C_{\circ}^{\circ}(s)\varphi_s^{\circ}](\tau)\|^2\eta_-^{|\tau|}\mathrm{d}\tau\mathrm{d}s \leq \eta_-\left(1+\eta_-\frac{\mathrm{d}}{\mathrm{d}\eta_-}\right)\|C_{\circ}^{\circ}\varphi^{\circ}\|_t^2(\eta_-)\ ,$$

where $\varphi_{s_1,s_2}^{\circ\circ}(\tau) = \varphi(\tau\sqcup s_1\sqcup s_2)$.

Taking into account that for any $\varepsilon>0$, $\xi=\eta+\varepsilon$

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\|\varphi\|^2(\eta) \leq \frac{1}{\varepsilon}\left(\|\varphi\|^2(\eta+\varepsilon)-\|\varphi\|^2(\eta)\right) \leq \frac{1}{\varepsilon}\|\varphi\|^2(\xi)\ ,$$

one can find that $\left(1+\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\right)\|\varphi\|_t^2(\eta) \leq \frac{\xi}{\varepsilon}\|\varphi\|_t^2(\xi)$,

$$\begin{aligned}\left(1+\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\right)\|C_{+}^{\circ}\varphi\|_t^2(\eta) &\leq \frac{\xi}{\varepsilon}\left(\|C_{+}^{\circ}\|_{\zeta,t}^{\xi,2}\right)^2\|\varphi\|^2(\zeta)\ ,\\ \left(1+\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\right)\|C_{\circ}^{\circ}\varphi^{\circ}\|_t^2(\eta) &\leq \frac{\xi}{\varepsilon}\left(\|C_{\circ}^{\circ}\|_{\zeta,t}^{\xi,\infty}\right)^2\frac{\mathrm{d}}{\mathrm{d}\zeta}\|\varphi\|^2(\zeta)\ ,\end{aligned}$$

if $\varepsilon\leq\xi$. Hence, due to $\|\varphi(\eta^+)\geq\|\varphi\|(\zeta+\varepsilon)\geq\|\varphi\|(\zeta)$ for $\eta^+\geq\zeta+\varepsilon$, we obtain for $\eta_-\leq\eta=\xi-\varepsilon$, $\xi\leq 1$

$$\left\|\int_0^t A(\mathbf{C},\mathrm{d}s)\varphi\right\|(\eta^-) \leq \|C_{+}^{-}\|_{\zeta,t}^{\xi,1} + \frac{1}{\sqrt{\varepsilon}}\left(\|C_{\circ}^{-}\|_{\zeta,t}^{\xi,2}+\|C_{+}^{\circ}\|_{\zeta,t}^{\xi,2}\right) + \frac{1}{\varepsilon}\|C_{\circ}^{\circ}\|_{\zeta,t}^{\xi,\infty},$$

if $\|\varphi\|(\eta^+)\leq 1,\ 0<\varepsilon\leq\xi$, what is equivalent to (1.11).

Due to the duality $\mathcal{G}(\zeta)^* = \mathcal{G}(\zeta^{-1})$ of $\mathcal{F}(\zeta)$ and $\mathcal{G}(1/\zeta)$ the QS–matrix process $\mathbf{C}^\star(t)$ is also locally QS–integrable, and there exists the adjoint integral $\int_0^t A(\mathbf{C},\mathrm{d}s)^*$, defined as in (1.10) by $\mathbf{C}^\star$:

$$\begin{aligned}&<\varphi|\int_0^t A(C,\mathrm{d}s)\chi> = \int_0^t <\varphi|C_{+}^{-}(s)\chi + C_{\circ}^{-}(s)\chi^{\circ}(s)>\mathrm{d}s\\ &+ \int_{\circ}^t <\varphi_s^{\circ}|C_{+}^{\circ}(s)\chi + C_{\circ}^{\circ}(s)\chi_s^{\circ}>\mathrm{d}s = \int_0^t <C_{+}^{-}(s)^*\varphi + C_{+}^{\circ}(s)^*\varphi_s^{\circ}|\chi>\mathrm{d}s +\\ &+ \int_0^t <C_{\circ}^{-}(s)^*\varphi + C_{\circ}^{\circ}(s)^*\varphi_s^{\circ}|\chi^{\circ}>\mathrm{d}s = <\int_0^t A(\mathbf{C}^\star,\mathrm{d}s)\varphi|\chi>\ .\end{aligned}$$

Obviously, $\|\int_0^+ A(\mathbf{C}^\star,\mathrm{d}s)\|_{1/\eta_-}^{1/\eta_+} = \|\int_0^+ A(\mathbf{C},\mathrm{d}s)\|_{\eta_+}^{\eta_-}$ ∎

Corollary 1. *If $\mathbf{C}(t)$ are the simple measurable adapted functions, then the definition (1.10) coincides with the QS–integral, given by integral Ito's sums with respect to the processes (1.1). Moreover, the QS–integral (1.10) is a limit of such integral sums in the inductive operator topology, defined by the norms (1.11), if locally QS–integrable matrix–process $\mathbf{C}$ can be uniformly approximated by a sequence of simple operator–valued processes with respect to the defined L^p – norms on $]0,t]$.*

2. QS calculus of output nondemolition processes

Let us consider an initial Hilbert space $\mathcal{H}^0 = \mathfrak{h}$ with identity operator $\widehat{1}$, $\mathcal{H} = \mathfrak{h} \otimes \mathcal{G}$, and denote by $\mathcal{H}^t = \mathfrak{h} \otimes \mathcal{G}^t$ and by $\widehat{I}^t = \widehat{1} \otimes I^t$ the corresponding multipliers of the Hilbert space $\mathcal{H} = \mathcal{H}^t \otimes \mathcal{G}_t$ and identity operator $\widehat{I} = \widehat{I}^t \otimes \widehat{1}_t$. Let us identify the basic QS process $\mathbf{A} = (A^\mu_\nu)$ with the process $\widehat{\mathbf{A}} = \widehat{1} \otimes \mathbf{A}$, in $\mathcal{H}$: $\widehat{A}^-_+(t) = t\widehat{I}$, $\widehat{A}^j_- = \widehat{1} \otimes A^j_-$, $\widehat{A}^+_j = \widehat{1} \otimes A^+_j$, $\widehat{A}^i_k = \widehat{1} \otimes N^i_k$. A QS matrix process $\widehat{\mathbf{C}} = \left(\widehat{C}^\mu_\nu\right)$ with an $\widehat{C}^\mu_\nu(t)$ acting in $\mathcal{H}$ is called adapted, if $\widehat{\mathbf{C}}(t) = \widehat{\mathbf{C}}^t \otimes \widehat{I}_t$, for any t, where $\widehat{\mathbf{C}}^t$ is a matrix of operators in $\mathcal{H}^t$. We define the QS integral of an adapted QS matrix process $\widehat{\mathbf{C}}$ as in (1.10) by the sum of integrals

$$\int_0^t \widehat{A}\left(\widehat{\mathbf{C}}, \mathrm{d}s\right) = \int_0^t \left(\widehat{C}^-_+ \mathrm{d}s + \widehat{C}^-_\circ \mathrm{d}\widehat{A}^\circ_- + \widehat{C}^\circ_- \mathrm{d}\widehat{A}^+_\circ + \widehat{C}^\circ_\circ \mathrm{d}\widehat{N}^\circ_\circ\right) , \tag{2.1}$$

which exists as an adapted process with the QS differential $\widehat{A}(\widehat{\mathbf{C}}, \mathrm{d}t) = \widehat{C}^\mu_\nu(t)\mathrm{d}\widehat{A}^\nu_\mu(t)$ for weakly measurable, locally integrable functions $t \mapsto \widehat{C}^\mu_\nu(t)$, called below QS integrable processes.

Now let us consider an adapted process $\widehat{X}(t)$, defined by the QS differential equation

$$\mathrm{d}\widehat{X}(t) = \left(\widehat{F}^\mu_\nu(t) - \widehat{X}(t)\delta^\mu_\nu\right)\mathrm{d}\widehat{A}^\nu_\mu(t) , \qquad \widehat{X}(0) = \widehat{x} \otimes I,$$

having the solution $\widehat{X}(t) = \widehat{X} \otimes I + \int_0^t \widehat{C}^\mu_\nu \mathrm{d}\widehat{A}^\nu_\mu$, iff $\widehat{\mathbf{C}} = \widehat{\mathbf{F}} - \widehat{X} \otimes \boldsymbol{\delta}$ satisfies the conditions for the existence of the integral (2.1), where $\widehat{X} \otimes \boldsymbol{\delta} = \left(\widehat{X}\delta^\mu_\nu\right)$. We shall define the elements $\widehat{F}^\mu_\nu$ of matrix–operators $\widehat{\mathbf{F}}(t)$ also for $\mu = - = \nu$ and for $\mu = + = \nu$ by $\widehat{F}^-_- = \widehat{X} = \widehat{F}^+_+$, and assume that $\widehat{F}^\mu_\nu = 0$, if $\mu > \nu$ under the order $- < o < +$.

Proposition 2. *If the QS process $\widehat{X}$ satisfies the QS differential equation (**??**), then the process $(\widehat{X}^*\widehat{X})(t) = \widehat{X}(t)^*\widehat{X}(t)$ satisfies the equation*

$$\mathrm{d}(\widehat{X}^*\widehat{X}) = \left(\widehat{\mathbf{F}}^\star\widehat{\mathbf{F}} - \widehat{X}^*\widehat{X} \otimes \boldsymbol{\delta}\right)^\mu_\nu \mathrm{d}A^\nu_\mu, \qquad (\widehat{X}^*\widehat{X})(0) = \widehat{x}^*\widehat{x} \otimes I. \tag{2.2}$$

This QS Ito formula establishes an $*$-algebra isomorphism from the QS differentiable processes $\widehat{X}$ into the algebra of matrices of operator processes $\widehat{F}^\mu_\nu$ defined above. In particular, $\widehat{X}$ is formally normal (selfadjoint, unitary) iff $[\widehat{\mathbf{F}}, \widehat{\mathbf{F}}^\star] = 0, \left(\widehat{\mathbf{F}}^\star = \widehat{\mathbf{F}}, \widehat{\mathbf{F}}^\star = \widehat{\mathbf{F}}^{-1}\right)$ with repsect to the $\star$-operation $\mathbf{F}^\star(t) = \mathbf{g}\mathbf{F}(t)^*\mathbf{g}$, and X is partially isometric (isometric, orthoprojection), iff $\widehat{\mathbf{F}}\widehat{\mathbf{F}}^\star\widehat{\mathbf{F}} = \widehat{\mathbf{F}}$ ($\widehat{\mathbf{F}}^\star\widehat{\mathbf{F}} = \widehat{I} \otimes \boldsymbol{\delta}, \widehat{\mathbf{F}}^\star\widehat{\mathbf{F}} = \widehat{\mathbf{F}}$).

PROOF. Taking into account that

$$\begin{aligned} \mathrm{d}\widehat{X} &= \widehat{C}^{-}_{+}\mathrm{d}t + \widehat{C}^{-}_{\circ}\mathrm{d}\widehat{A}^{\circ}_{-} + \widehat{C}^{\circ}_{+}\mathrm{d}\widehat{A}^{+}_{\circ} + \widehat{C}^{\circ}_{\circ}\mathrm{d}\widehat{N}^{\circ}_{\circ}, \\ \mathrm{d}\widehat{X}^{*} &= \widehat{C}^{-*}_{+}\mathrm{d}t + \widehat{C}^{\circ *}_{+}\mathrm{d}\widehat{A}^{\circ}_{-} + \widehat{C}^{-*}_{\circ}\mathrm{d}\widehat{A}^{+}_{\circ} + \widehat{C}^{\circ *}_{\circ}\mathrm{d}\widehat{N}^{\circ}_{\circ}, \end{aligned}$$

and using the QS Itô formula [1], defining the product $(\widehat{X}^{*}\widehat{X})(t) = (\widehat{x}^{*}\widehat{x}) \otimes I + \int_0^t \mathrm{d}(\widehat{X}^{*}\widehat{X})$ by the QS differential

$$\mathrm{d}(\widehat{X}^{*}\widehat{X}) = \mathrm{d}\widehat{X}^{*}\widehat{X} + \widehat{X}^{*}\mathrm{d}\widehat{X} + \mathrm{d}\widehat{X}^{*}\mathrm{d}\widehat{X} =$$

$$= \left(C^{\star\mu}_{\nu}\widehat{X} + \widehat{X}^{*}\widehat{C}^{\mu}_{\nu} + C^{\star\mu}_{\mu}C^{\mu}_{\nu}\right)\mathrm{d}\widehat{A}^{\nu}_{\mu} =$$

$$\left((\widehat{\mathbf{C}} + \widehat{X}\otimes\boldsymbol{\delta})^{\star}(\widehat{\mathbf{C}} + \widehat{X}\otimes\boldsymbol{\delta}) - \widehat{X}^{*}\widehat{X}\otimes\boldsymbol{\delta}\right)^{\mu}_{\nu}\mathrm{d}\widehat{A}^{\nu}_{\mu},$$

we obtain the equation (2.2) with $\widehat{\mathbf{F}} = \widehat{X}\otimes\boldsymbol{\delta} + \widehat{\mathbf{C}}$. Due to the linearity of (2.2) with respect to the pairs $(\widehat{\mathbf{F}}, \widehat{X})$ and $(\widehat{\mathbf{F}}^{\star}, \widehat{X}^{*})$, it can be extended to

$$\mathrm{d}(\widehat{X}^{*}\widehat{X}') = \left(\widehat{\mathbf{F}}^{\star}\widehat{\mathbf{F}}' - \widehat{X}^{*}\widehat{X}'\otimes\boldsymbol{\delta}\right)^{\mu}_{\nu}\mathrm{d}\widehat{A}^{\nu}_{\mu} \tag{2.3}$$

by the polarization formula

$$\widehat{X}^{*}\widehat{X}' = \sum_{n=0}^{3}\left(\widehat{X} + \mathrm{i}^{n}\widehat{X}'\right)^{*}\left(\widehat{X} + \mathrm{i}^{n}\widehat{X}'\right)/4\mathrm{i}^{n}\ , \qquad \mathrm{i} = \sqrt{-1}.$$

Hence, the formula (2.2) is equivalent to QS Hudson – Parthasarathy Itô formula [1] and $\star$ – property

$$\mathrm{d}\widehat{X}^{*}(t) = \left(\widehat{\mathbf{F}}^{\star}(t) - \widehat{X}^{*}(t)\otimes\boldsymbol{\delta}\right)^{\mu}_{\nu}\mathrm{d}\widehat{A}^{\nu}_{\mu}, \qquad \widehat{X}^{*}(0) = \widehat{x}^{*}\otimes I,$$

which follows from it for $\widehat{\mathbf{F}}' = I\otimes\boldsymbol{\delta}$, corresponding to $\widehat{X}' = \widehat{I}$. So the map $\widehat{X} \mapsto \widehat{\mathbf{F}}$ is a homomorphism with respect to the associative operator algebra structure of $\widehat{X}$ and $\widehat{\mathbf{F}}$ with the appropriate involutions. Furthermore it is an injection, because if $\widehat{\mathbf{F}} = 0$, then $\widehat{X} = 0$, as $\widehat{F}^{-}_{-} = \widehat{X} = \widehat{F}^{+}_{+}$.

Conversely, if $\widehat{X} = 0$, then $\int_0^t \widehat{F}^{\mu}_{\nu}\mathrm{d}\widehat{A}^{\nu}_{\mu} = 0$ for all t, but it implies $\widehat{F}^{\mu}_{\nu} = 0$ due to the independence of stochastic integrators [15]. ∎

Now let us consider an adapted selfadjoint QS process Y, satisfying a QS equation

$$\mathrm{d}Y(t) = (\mathbf{Z}^{\star}\mathbf{G}\mathbf{Z} - Y\otimes\boldsymbol{\delta})^{\mu}_{\nu}(t)\mathrm{d}\widehat{A}^{\nu}_{\mu}(t)\ , \qquad Y(0) = \widehat{y}\otimes I\ , \tag{2.4}$$

where $\mathbf{G}^{\star} = \mathbf{G}$ is a $\star$ – selfadjoint matrix adapted QS process with $G^{-}_{-} = Y = G^{+}_{+}$, $G^{\mu}_{\nu} = 0$ for $\mu > \nu$, and $\mathbf{Z} = (Z^{\mu}_{\nu})$ is a $\star$ – isometric or $\star$ – unitary matrix adapted process: $\mathbf{Z}^{\star}\mathbf{Z} = \widehat{I}\otimes\boldsymbol{\delta}$ $\left(\mathbf{Z}^{\star} = \mathbf{Z}^{-1}\right)$ such, that $\mathbf{G}\mathbf{Z}\mathbf{Z}^{\star} = \mathbf{G} = \mathbf{Z}\mathbf{Z}^{\star}\mathbf{G}$ (otherwise $\mathbf{G}$ should be replaced by $\mathbf{Z}\mathbf{Z}^{\star}\mathbf{G}\mathbf{Z}\mathbf{Z}^{\star}$).

We shall demand that $Z^{-}_{-} = \widehat{I} = Z^{+}_{+}$, $Z^{\mu}_{\nu} = 0$, if $\mu > \nu$, and $Z^{\mu}_{\nu}, \mu \neq +$ or $\nu \neq -$ satisfy the conditions for the existance of QS isometric (unitary) evolution $U(t) : \mathcal{H} \to \mathcal{H}$, defined by the QS equation (**??**) with $\widehat{X} = U, F^{\mu}_{\nu} = UZ^{\mu}_{\nu}, \widehat{x} = \widehat{1}$:

$$\mathrm{d}U(t) = U(t)\left(Z^{\mu}_{\nu}(t) - I\delta^{\mu}_{\nu}\right)\mathrm{d}\widehat{A}^{\nu}_{\mu}\ , \qquad U(0) = \widehat{I}. \tag{2.5}$$

Sufficient conditions for this are the conditions of local integrability of the weakly measurable processes Z^{μ}_{ν} in the sense of the L^{p}-norms [16]:

$$\|Z^{-}_{+}\|^{(1)}_{t} < \infty\ ,\ \|Z^{0}_{+}\|^{(2)}_{t} < \infty\ ,\ \|Z^{-}_{0}\|^{(2)}_{t} < \infty\ ,\ \|Z^{0}_{0}\|^{(\infty)}_{t} < \infty\ .$$

Let us call the process Y an output process, if Y is nondemolition with respect to the QS process $\mathbf{Z}$, generating the evolution (2.5). By this we mean the commutativity condition

$$[Y(t), X(s)] = 0, \qquad \forall t \le s \tag{2.6}$$

with respect to the all QS processes $X(t) = Z^\mu_\nu(t), \mu, \nu \in \{-, J, +\}$ (the conditions are nontrivial for $\mu \neq +$ and $\nu \neq -$).

Theorem 2. *The process Y is defined by (2.4) as an adapted selfadjoint QS process iff*

$$UY = \widehat{Y}U, \quad \widehat{Y}(t) = \widehat{y} \otimes I + \int_0^t \widehat{D}^\mu_\nu \mathrm{d}\widehat{A}^\nu_\mu, \tag{2.7}$$

where $\widehat{\mathbf{D}}$ is an adapted QS integrable matrix process satisfying the conditions $\widehat{D}^\mu_\nu U = UD^\mu_\nu, D^\mu_\nu = G^\mu_\nu - Y\delta^\mu_\nu$. The process Y is an output QS process, iff

$$U(s)Y(t) = \widehat{Y}(t)U(s) \quad , \quad \forall t \le s, \tag{2.8}$$

which is equivalent to the condition $[\widehat{Y}(t), \widehat{Z}^\mu_\nu(s)]U(s) = 0$ for $s \le t, \widehat{Z}^\mu_\nu U = UZ^\mu_\nu$. The output process Y satisfies the nondemolition condition (2.6) with respect to an adapted QS process X, defined by

$$UX = \widehat{X}U, \quad \widehat{X}(t) = \widehat{x} \otimes I + \int_0^t \widehat{C}^\mu_\nu \mathrm{d}\widehat{A}^\nu_\mu, \tag{2.9}$$

iff $[\widehat{Y}(t), \widehat{X}(s)]U(s) = 0$ for all $t \le s$. The last is equivalent to the commutativity conditions $[\widehat{Y}(t), \widehat{F}^\mu_\nu(s)]U(s) = 0, \forall t \le s$,

$$[\widehat{y}, \widehat{x}] = 0, \quad [\widehat{\mathbf{D}}, \widehat{\mathbf{F}}]U = 0, \tag{2.10}$$

where $\widehat{\mathbf{F}} = \widehat{\mathbf{C}} + \widehat{X} \otimes \boldsymbol{\delta}$, (the conditions are nontrivial for $\mu \neq +$ and $\nu \neq -$).

Proof. We obtain (2.4) with $G^\mu_\nu = U^*\widehat{G}^\mu_\nu U$ from (2.7) for U, satisfying (2.5) simply by applying to $Y = U^*\widehat{Y}U$ the QS Itô formula (2.4):

$$\mathrm{d}(U^*\widehat{Y}U) = \left(\mathbf{Z}^\star(U^* \otimes \boldsymbol{\delta})\widehat{\mathbf{G}}(U \otimes \boldsymbol{\delta})\mathbf{Z} - U^*\widehat{Y}U \otimes \boldsymbol{\delta}\right)^\mu_\nu \mathrm{d}\widehat{A}^\nu_\mu.$$

Conversely, we obtain

$$\mathrm{d}(UYU^*) = ((U \otimes \boldsymbol{\delta})\mathbf{Z}\mathbf{Z}^\star\mathbf{G}\mathbf{Z}\mathbf{Z}^\star(U^* \otimes \boldsymbol{\delta}) - UYU^* \otimes \boldsymbol{\delta})^\mu_\nu \, \mathrm{d}\widehat{A}^\nu_\mu,$$

so the QS–process $\widehat{Y} = UYU^*$ obviously satisfying the condition $\widehat{Y}U = UY$, is defined as QS integral in (2.7) with $G^\mu_\nu U^*$ due to the assumption $\mathbf{Z}\mathbf{Z}^\star\mathbf{G}\mathbf{Z}\mathbf{Z}^\star = \mathbf{G}$ for a weakly measurable, locally L^p-integrable $\mathbf{G}$.

If the processes Y and $X = Z^\mu_\nu$ satisfy the commutativity conditions (2.6), then the isometry

$$U(t,s) = \widehat{I} + \int_t^s U(t,r)\left(Z^\mu_\nu(r) - \widehat{I}\delta^\mu_\nu\right)\mathrm{d}\widehat{A}^\nu_\mu(r) \tag{2.11}$$

commutes with $Y(t)$, as it can be easily proved by induction with respect to $n = 1, 2, \ldots$ for the corresponding QS Itô integral sums

$$U_n(t,s) = \widehat{I} + \sum_{i=0}^{n-1} U_i(t,t_i)\left(Z^\mu_\nu(t_i) - \widehat{I}\delta^\mu_\nu\right)\left(\widehat{A}^\nu_\mu(t_{i+1}) - \widehat{A}^\nu_\mu(t_i)\right), \tag{2.12}$$

where $t_i = t + i(s-t)/n, U_0(t,t) = \widehat{I}$. Hence, taking into account that $U(s) = U(t)U(t,s)$, we obtain (2.4):

$$U(s)Y(t) = U(t)Y(t)U(t,s) = \widehat{Y}(t)U(s).$$

Conversely, multiplying (2.8) from the left side hand by $U(t)^*$, we obtain the commutativity condition for $Y(t)$ and $U(t,s)$, which is equivalent (2.6) for $X = Z^\mu_\nu$ due to the approximation (2.12) of (2.11) and adaptedness of Y. The condition (2.6) in the terms of $\widehat{Z}^\mu_\nu U = UZ^\mu_\nu$ can be written as:

$$U(s)\left[Y(t), Z^\mu_\nu(s)\right] = \left[\widehat{Y}(t), \widehat{Z}^\mu_\nu(s)\right] U(s) = 0, \forall t \leq s.$$

In the same way the non-demolition condition for an output process Y with respect to a QS process X can be written as $[\widehat{Y}(t), \widehat{X}(s)]U(s) = 0$ in terms of $\widehat{X}U = UX$.

Representing X in the case (2.9) in the form of (2.4) as the solution of the QS – equation

$$\mathrm{d}X = (\mathbf{Z}^\star\mathbf{F}\mathbf{Z} - X\otimes\boldsymbol{\delta})^\mu_\nu\,\mathrm{d}\widehat{A}^\nu_\mu, \quad X(0) = \widehat{x}\otimes I, \tag{2.13}$$

and taking into account that due to (2.3)

$$\mathrm{d}(YX) = (\mathbf{Z}^\star\mathbf{G}\mathbf{F}\mathbf{Z} - YX\otimes\boldsymbol{\delta})^\mu_\nu\mathrm{d}\widehat{A}^\nu_\mu, \tag{2.14}$$

one can easily obtain, that $[Y, X] = 0$, iff $[\widehat{y}, \widehat{x}] = 0$ and $[\mathbf{G}, \mathbf{F}] = 0$ due to $\mathbf{Z}\mathbf{Z}^\star\mathbf{F} = \mathbf{F} = \mathbf{F}\mathbf{Z}\mathbf{Z}^\star$. In order to satisfy the condition $[Y(t), X(s)] = 0$ for all $t \leq s$, it should be completed by $[Y(t), F^\mu_\nu(s)] = 0$ at least for $\mu \neq +$ or $\nu \neq -$ and all $t \leq s$ due to the QS integral representation

$$X(s) = X(t) + \int_t^s (\mathbf{Z}^\star\mathbf{F}\mathbf{Z} - X\otimes\boldsymbol{\delta})^\mu_\nu\mathrm{d}\widehat{A}^\nu_\mu \text{ for } s > t$$

and commutativity of $Y(t)$ with $\mathbf{Z}(s)$ at $s \geq t$. So $[\mathbf{D}, \mathbf{F}] = [\mathbf{G}, \mathbf{F}] - [Y\otimes\boldsymbol{\delta}, \mathbf{F}] = 0$, what gives the necessary and sufficient nondemolition conditions, which can be written in the terms of $\widehat{Y}, \widehat{\mathbf{D}}, \widehat{\mathbf{F}}$ as (2.10) by multiplication on the right by the corresponding U. ∎

Corollary 2. *The process X is an evolute transformation $X = U^*(\widehat{x}\otimes I)U$ of an initial operator $\widehat{x} \in \mathcal{B}(\mathfrak{h})$ with respect to a QS unitary process U, described by the QS equation (2.5) iff it satisfies the QS equation (2.13) with $\mathbf{F} = X\otimes\boldsymbol{\delta}$. The process Y is an output process with respect to the QS Markovian evolution defined on the von Neumann algebra $\mathcal{A} = \mathcal{B}(\mathfrak{h})$ by the transformation $Z^\mu_\nu = U^*\left(\widehat{z}^\mu_\nu \otimes I\right)U$ of the initial QS generators $\widehat{z}^\mu_\nu$, acting in $\mathfrak{h}$, iff $\left[\widehat{Y}(t), \widehat{z}^\mu_\nu\otimes I\right] = 0$ for all t and μ,ν. The output process $\widehat{Y}$ is nondemolition with respect to $X = U^*(\widehat{x}\otimes I)U$ for arbitrary $\widehat{x} \in \mathcal{B}(\mathfrak{h})$, iff $\widehat{Y} = \widehat{1}\otimes B$, where B is an adapted process in Fock space $\mathcal{G}$.*

Indeed, if $\widehat{X}(t) = \widehat{x}\otimes I$ is a time independent adapted process, then it satisfies the QS equation $\mathrm{d}\widehat{X} = \widehat{C}^\mu_\nu\mathrm{d}\widehat{A}^\nu_\mu$ corresponding to $\widehat{C}^\mu_\nu = 0 = \widehat{F}^\mu_\nu - \widehat{z}\otimes I\delta^\mu_\nu$. Hence, the process $\mathbf{Z} = U^*\widehat{\mathbf{Z}}U$ satisfies the equation (2.13) with $F^\mu_\nu = U^*(\widehat{x}\otimes I)U\delta^\mu_\nu = X\delta^\mu_\nu$.

The output condition $\left[\widehat{Y}(t), \widehat{Z}^\mu_\nu(s)\right]U(s) = 0$, for $\widehat{\mathbf{Z}}(s) = \widehat{\mathbf{z}}\otimes I$ and unitary U, means $\left[\widehat{Y}(t), \widehat{z}^\mu_\nu\otimes I\right] = 0$ for all t, μ, ν; moreover the nondemolition condition $[\widehat{Y}(t), \widehat{X}(s)]U(s) = 0$ for $\widehat{X}(s) = \widehat{z}\otimes I$ with arbitrary $\widehat{x} \in \mathcal{B}(\mathfrak{h})$ is possible only if $Y = \widehat{1}\otimes B$.

Note that an output QS process $Y = U^*\widehat{Y}U$ is defined as the sum of $\widehat{y} \otimes I$ and a QS – integral

$$\int_0^t A(\mathbf{D}, \mathrm{d}s) = U(t)^* \int_0^t \widehat{A}(\widehat{\mathbf{D}}, \mathrm{d}s)U(t)$$

with the QS differential $A(\mathbf{D}, \mathrm{d}t) = (\mathbf{Z}^\star\mathbf{D}\mathbf{Z})^\mu_\nu(t)\mathrm{d}\widehat{A}^\nu_\mu(t)$. In the case of commuting matrix elements $D^\kappa_\mu = U^*\widehat{D}^\kappa_\mu U$ and $Z^\iota_\nu = U^*\widehat{Z}^\iota_\nu U$, as it happens for $\widehat{Z}^\iota_\nu(t) = \widehat{z}^\iota_\nu \otimes I$, $\widehat{D}^\kappa_\mu = \widehat{1} \otimes \widehat{D}^\kappa_\mu$, this integral can be defined as the QS Itô integral

$$\int_0^t A(\mathbf{D}, \mathrm{d}s) = \int_0^t \left(D^-_+ \mathrm{d}s + D^-_\circ \mathrm{d}A^\circ_- + D^\circ_+ \mathrm{d}A^+_\circ + D^\circ_\circ \mathrm{d}N^\circ_\circ\right) = \int_0^t D^\mu_\nu \mathrm{d}A^\nu_\mu$$

with respect to output annihilation A°_-, creation $A^+_\circ$ and quantum number $N^\circ_\circ$ processes

$$A^\circ_-(t) = \int_0^t \left(Z^\circ_k \mathrm{d}\widehat{A}^k_- + Z^\circ_+ \mathrm{d}s\right) = A^+_\circ(t)^* \ , \qquad N^\circ_\circ(t) =$$

$$\int_0^t \left((Z^{\circ^*}_\circ Z^\circ_\circ)^i_k \mathrm{d}\widehat{N}^k_i + (Z^{\circ^*}_\circ Z^\circ_+)^i \mathrm{d}\widehat{A}^+_i + (Z^{\circ^*}_+ Z^\circ_\circ)_k \mathrm{d}\widehat{A}^k_- + (Z^{\circ *}_+ Z^\circ_+ \mathrm{d}s)\right)$$

as the unitary transformation $A^\mu_\nu = U^*\widehat{A}^\mu_\nu U$ of the input canonical processes $\widehat{A}^\mu_\nu$.

3. QS nonlinear nondemolition filtering

Let us consider a selfadjoint family $Y = (Y_i)$ of commuting output processes $Y_i, i = 1, \ldots, n$, defined by

$$Y_i(t) = \widehat{y}_i \otimes I + \int_0^t (\mathbf{Z}^\star \mathbf{D}_i \mathbf{Z})^\mu_\nu \mathrm{d}\widehat{A}^\nu_\mu, \tag{3.1}$$

which are nondemolition with respect a QS process X:

$$[Y_i(t), Z^\mu_\nu(s)] = 0,\ [Y_i(t), X(s)] = 0 \quad \forall t \leq s \tag{3.2}$$

As follows from (2.10) for $\widehat{x} = \widehat{y}_k$, $\widehat{\mathbf{F}} = \widehat{\mathbf{D}}_k + \widehat{Y}_k \otimes \boldsymbol{\delta}$, the family Y satisfies the selfnondemolition condition

$$[Y_i(t), Y_k(s)] = 0, \quad \forall s, t; i, k, \tag{3.3}$$

iff $\left[\widehat{Y}_i(t), \widehat{D}^\mu_\nu(s)_k\right] U(s) = 0$ for all $t \leq s$, and

$$[\widehat{y}_i, \widehat{y}_k] = 0, \quad \left[\widehat{\mathbf{D}}_i, \widehat{\mathbf{D}}_k\right] U = 0, \quad \forall i, k. \tag{3.4}$$

Let us denote by $\mathcal{A}_t = \{Y^t_i : i = 1, \ldots, n\}'$ the reduced algebra of bounded operators in $\mathcal{H}$, corresponding to the measurements of the process $Y^t = \{Y(s) : s \leq t\}$ up to a time t, defined as the commutant of all $Y^t_i = U^*\widehat{Y}^t_i U$, and by $\mathcal{O} = \{y_1, \ldots y_n\}'$ the initial algebra, defining $\mathcal{A}_0 = \mathcal{O} \otimes \mathcal{B}(\mathcal{F})$. The nonincreasing family $(\mathcal{A}_t)$ is the family of maximal von Neumann subalgebras $\mathcal{A}_s \subseteq \mathcal{A}_t \subseteq \mathcal{A}_0,\ s \geq t \geq 0$ of the initial reduced algebra $\mathcal{A}_0$, with respect to which Y is a nondemolition commutative vector process in the sense of the definition $Y_i(t) \in \mathcal{A}'_t \quad \forall t$ (or $Y_i(t)$ is affiliated to $\mathcal{A}'_t$) of a nondemolition QS process given in [4]. The Abelian algebra $\mathcal{B}^t = \mathcal{A}'_t$ generated by Y^t, with $\mathcal{B}^0 = \mathcal{O}' \otimes I$, generated by $Y^0 = y \otimes I$, forms the center $\mathcal{B}^t = \mathcal{A}_t \bigcap \mathcal{A}'_t$ of $\mathcal{A}_t$, hence $\mathcal{A}_t$ is a decomposable algebra, having the conditional expectations with respect to $\mathcal{B}^t = \mathcal{A}_t$ for any normal initial state on $\mathcal{A}_0 \supseteq \mathcal{A}_t$.

As follows from the next theorem, the nondemolition principle is not only sufficient, but also necessary for the existence of compatible conditional expectation on $\mathcal{A}_t$ with respect to $\mathcal{B}_t$ for an arbitrary initial state vector ξ.

We shall explicitly construct the conditional expectation not only for bounded $X \in \mathcal{A}_t$, but also for X affiliated to $\mathcal{A}_t$. An operator B is said to be defined almost everywhere with respect to the pair $(\mathcal{B}^t, \xi)$, if it is densely defined in the support subspace $\mathcal{K}^t = P^t\mathcal{H}$, where $P^t = \inf\{P = P^*P \in \mathcal{B}^t : P\xi = \xi\}$, $\|\xi\| = 1$.

Theorem 3. *Let $\mathcal{B}^t \subset \mathcal{A}_t$ be a von Neumann subalgebra on a Hilbert space $\mathcal{H}$. Then a conditional expectation ϵ_t as a positive projection onto $\mathcal{B}^t$, satisfying the compatibility condition $<\xi|\epsilon_t(X)\xi> = <\xi|X\xi>$, for all $X \in \mathcal{A}$ exists on $\mathcal{A}_t$ for an arbitrary $\xi \in \mathcal{H}$, iff $\mathcal{B}^t$ commutes with $\mathcal{A}_t : \mathcal{B}^t \subseteq \mathcal{A}_t'$. In this case the algebra $\mathcal{A}_t$ can be extended to the commutant of $\mathcal{B}^t$, such that for any operator X, commuting on the domain $\mathcal{A}_t\xi$ with $\mathcal{B}^t = \mathcal{A}_t'$, the expectation $\epsilon_t(X)$ is given on $\mathcal{A}_t\xi$ by*

$$\epsilon_t(X)A\xi = AE_tX\xi, \quad \forall A \in \mathcal{A}_t, \tag{3.5}$$

where $E_t \in \mathcal{A}_t$ is the orthoprojector on $\overline{\mathcal{B}^t\xi}$. The formula (3.5) uniquely defines $\epsilon_t(X)$ as an operator $\epsilon_t(X)P^t$ affiliated to $\mathcal{B}^t$ on $\overline{\mathcal{A}_t\xi}$ even for unbounded X.

Proof. Let us suppose that $[X, B] \neq 0$ for an $X \in \mathcal{A}_t$ and $B \in \mathcal{B}^t$, and that $\epsilon_t : \mathcal{A}_t \to \mathcal{B}^t$ is defined as a positive projection, compatible with $\xi \in \mathcal{H}$, for which $<\xi|[X, B]\xi> \neq 0$. Then, due to the modularity property

$$\epsilon_t(XB) = \epsilon_t(X)B \ , \ \epsilon_t(BX) = B\epsilon_t(X) \ ,$$

where $X \in \mathcal{A}_t$, $B \in \mathcal{B}^t$, we would have $<\xi|[\epsilon_t(x), B]\xi> = <\xi|[X, B]\xi> \neq 0$, what would be possible only if $\mathcal{B}^t$ would be non-Abelian. But for non-Abelian $\mathcal{B}^t$ the conditional expectation does not exist for all vectors $\xi \in \mathcal{H}$, as can be easily shown for a factor $\mathcal{B}^t \neq \mathbb{C}\widehat{I}$. Indeed, in this case such a vector ξ has to be of the form $\xi_0 \otimes \xi_1$, and $\epsilon(A \otimes B) = <\xi_0|A\xi_0> I_0 \otimes B$, where $\xi_0 \in \mathcal{H}_0 \neq \mathbb{C}$, if $\mathcal{A}_t \neq \mathcal{B}^t$, $\xi_1 \in \mathcal{H}_1 = \overline{\mathcal{B}^t\xi}$, corresponding to the decomposition $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1$. So, it is necessary that $\mathcal{B}^t \subseteq \mathcal{A}_t'$.

Let us define ϵ_t for such an Abelian algebra $\mathcal{B}^t$ by (3.5) with $\mathcal{A}_t' = \mathcal{B}^t$ and a fixed $\xi \in \mathcal{H}$. The orthoprojector E_t commutes with $\mathcal{A}_t'$ due to the invariance of $\mathcal{E}_t = \overline{\mathcal{A}_t'\xi}$ with respect to the action of the algebra $\mathcal{A}_t'$. Hence the operator E_tXE_t commutes with $\mathcal{A}_t'E_t$:

$$E_tXE_tBE_t = E_tXBE_t = E_tBXE_t = BE_tXE_t = E_tBE_tXE_t \ ,$$

if the operator X commutes with the all $B \in \mathcal{A}_t'$. But this means that E_tXE_t is affiliated with the reduced von Neumann algebra $E_t\mathcal{A}_tE_t$ on $\mathcal{E}_t$, coinciding with its commutant $\mathcal{A}_t'E_t$ on $\mathcal{E}_t$ because the induced Abelian algebra $\mathcal{A}_t'E_t$ has the cyclic vector ξ in $\mathcal{E}_t$. The commutativity of $E_t\mathcal{A}_tE_t = \mathcal{A}_t'E_t$ helps to establish the correctness of the definition (3.5) of the linear operator $\epsilon_t(X)$ on $\mathcal{A}_t\xi$ $A\xi = 0 \Rightarrow \epsilon_t(X)A\xi = 0$. Indeed, $\|\epsilon_t(x)A\xi\| =$

$$\|AE_tX\xi\| = \|(E_tA^*AE_t)^{1/2}E_tXE_t\xi\| = \|E_tXE_t(E_tA^*AE_t)^{1/2}\xi\|$$

because $E_t\xi = \xi$ and $(E_tA^*AE_t)^{1/2}\ \xi = 0$, if $A\xi = 0$.

The operator $\epsilon_t(X) : A\xi \to AE_tX\xi$ having the range $\mathcal{A}_tE_tX\xi \subseteq \mathcal{K}^t = \overline{\mathcal{A}_t\xi}$, commutes with arbitrary $A \in \mathcal{A}_t$ due to the definition (3.5), so $P^t\epsilon_t(X)$ is affiliated to $P^t\mathcal{A}_t'P^t$, coinciding with $\mathcal{A}_t'P^t$ because $P^t \in \mathcal{A}_t' \bigcap \mathcal{A}_t = \mathcal{A}_t'$, if P^t is the orthoprojector on $\mathcal{K}^t$.

The map $X \mapsto \epsilon_t(X)$ satisfies the unital property $\epsilon^t(\widehat{I})A\xi = AE_t\xi = A\xi$ due to $\xi \in \mathcal{E}_t$, and the modularity property

$$\epsilon_t(XB)A\xi = AE_tBX\xi = ABE_tX\xi = BAE_tX\xi = B\epsilon_t(X)A\xi$$

for all $A \in \mathcal{A}_t$ and $B \in \mathcal{A}'_t$, and, hence, maps the algebra $\mathcal{A}_t$ on the subalgebra $\mathcal{A}'_t \subset \mathcal{A}_t$, represented on $\mathcal{K}^t$.

Now let us prove the uniqueness of the representation (3.5) of conditional expectation ϵ_t as a map onto factor subalgebra $\mathcal{A}'_t/\mathcal{A}'_tP^t_1 = \mathcal{A}'_tP^t$, where $P^t = \widehat{I} - P^t_1 \in \mathcal{A}'_t$ is the support of ξ which is the orthoprojector on $\overline{\mathcal{A}_t\xi} = \mathcal{K}^t$. Due to the commutativity of $\epsilon(X)$ with $\mathcal{A}_t$ we have $\epsilon_t(X)A\xi = A\epsilon_t(X)\xi$ for $A \in \mathcal{A}_t$. So we have to prove, that $\epsilon_t(X)\xi = E_tX\xi$. But $\epsilon_t(X)\xi \in \mathcal{A}'_t\xi$, because $\epsilon_t(X) \in \mathcal{A}'_t$ for $X \in \mathcal{A}_t$; hence we should prove, that $< B\xi|\epsilon_t(X)\xi >=< B\xi|E_tX\xi >$ for all $B \in \mathcal{A}'_t$, which is a consequence of modularity and compatibility conditions:

$$< B\xi|\epsilon_t(X)\xi >=< \xi|\epsilon_t(B^*X)\xi >=< \xi|B^*X\xi >=< B\xi|X\xi >=< B\xi|E_tX\xi > \ .$$

Remark. Note that one should identify the factor-algebra $\mathcal{B}^tP^t$ with the space $L^\infty(\mathcal{V}^t)$ of essentially bounded measurable complex functions on the probability space $\mathcal{V}^t$ of all observed values $v^t = \{v(s) : s \le t\}$, $v(t) = (v_i)(t)$ of the commutative vector process Y^t, stopped at t. The probability measure $\mu(\mathrm{d}v^t) =< \xi|I(\mathrm{d}v^t)\xi >$ is induced on the Borel σ-algebra of $\mathcal{V}^t$ by the spectral resolution $Y^t = \int v^t I(\mathrm{d}v^t)$. If $P^t = \int^\otimes_{\mathcal{V}^t} P_{v^t}\mu(\mathrm{d}v^t)$ is the corresponding decomposition of $P^t \in \mathcal{A}'_t$, then

$$P^t\epsilon_t(X) = \int^\otimes_{\mathcal{V}^t} < X >_{v^t} P_{v^t}\mu(\mathrm{d}v^t) \ , \tag{3.6}$$

where $< X >_{v^t}=< \xi_{v^t}|X\xi_{v^t} >$, and the vectors $\xi_{v^t} = P_{v^t}\xi/\|P_{v^t}\xi\|$ define the resolution $E_t = \int^\otimes_{\mathcal{V}^t} |\xi_{v^t} >< \xi_{v^t}|\mu(\mathrm{d}v^t)$. Hence one should consider $\epsilon_t(X)$ for an $X \in \mathcal{A}_t$ as a function $\epsilon_t(X) : \mathcal{V}^t \to \mathbb{C}P_{v^t}$ giving for almost all trajectories $v^t \in \mathcal{V}^t$, observed up to a time t, the posterior mean values $< X >_{v^t}$ of a QS nondemolished process $X(t)$. The initial conditional expectation ϵ_0 with respect to $\mathcal{B}^0 = \mathcal{O}' \otimes I$ and $\xi = \psi \otimes \varphi$ is given for $X = \widehat{x} \otimes I$ as $\epsilon(\widehat{X}) \otimes I$ by

$$\widehat{p}\epsilon(\widehat{x}) = \int^\oplus_{\mathcal{V}_0} < \psi_v|\widehat{x}\psi_v > \widehat{p}_v\mu(\mathrm{d}v) \ , \quad \mu(\mathrm{d}v) = \|\widehat{1}(\mathrm{d}v)\psi\|^2 \ . \tag{3.7}$$

Here the vectors $\psi_v = \widehat{p}_v\psi/\|\widehat{\psi}_v\psi\|$, $v \in \mathcal{V}_0$ and the decomposition $\widehat{e} = \int^\otimes_{\mathcal{V}} |\psi_v >< \psi_v|\mu(\mathrm{d}v)$ for $E_0 = \widehat{e} \otimes I$, $\{p_v\}$ define the decomposition $\widehat{p} = \int^\otimes \widehat{p}_v\mu(\mathrm{d}v)$ for $P^0 = \widehat{p} \otimes I$, corresponding to the orthogonal resolution $\widehat{y} = \int v\widehat{1}(\mathrm{d}v)$ on the spectrum $\mathcal{V}_0$ of the commutative family $\widehat{y} = (\widehat{y}_i)$ of the initial operators $Y(0) = \widehat{y} \otimes I$.

4. QS calculus of a posteriori expectations

Now let us suppose that $\xi = \psi \otimes \delta_\emptyset$, the output commuting processes (3.1) are nondemolition with respect to

$$X(t) = \widehat{x} \otimes I + \int_0^t (\mathbf{Z}^\star\mathbf{F}\mathbf{Z} - X \otimes \boldsymbol{\delta})^\mu_\nu \mathrm{d}\widehat{A}^\nu_\mu \tag{4.1}$$

with $\widehat{x} \in \mathcal{O}$, and $\mathbf{F}(t) \in \mathcal{F}_t$ where $\mathcal{F}_t$ is the $\star$-algebra of matrix-operators $\mathbf{F} = (F^\mu_\nu)$, commuting with $Y(s)$, $s \le t$ and $\mathbf{D}(t) : \mathcal{F}_t = \{F^\mu_\nu \in \mathcal{A}_t : [\mathbf{D}_i, \mathbf{F}] = 0, \, i = 1, \dots, n\}$. In the following we shall also demand that the process $Y = (Y_i)$ is continuous from the right in the sense $\mathcal{A}_t' = \bigcap_{s>t} \mathcal{A}_s'$, what is equivalent to $D_i(t)^\mu_\nu \in \mathcal{A}_t'$ for all i and t.

Let us denote by $\mathcal{C}$ the linear span of the initial operators $\{y_i\}$ with the operators $y_0 \in \mathcal{C}_0$ from the commutative ideal $\mathcal{C}_0 = \{b \in \mathcal{O}' :< b\psi | b\psi >= 0\}$. We also denote by $\mathcal{D}^t$ the $\mathcal{A}_t'$-span of the operator-matrices $\{\mathbf{D}_i\}(t)$ with the ideal

$$\mathcal{D}_0^t = \{\mathbf{D} \in \mathcal{F}_t' : D_-^- = 0 = D_+^+, (\mathbf{D}Z_+ | \mathbf{D}Z_+) = 0\}$$

of the commutative $\star$-algebra

$$\mathcal{F}_t' = \{D^\mu_\nu \in \mathcal{A}_t' | [\mathbf{D}, \mathbf{F}] = 0, \mathbf{F} \in \mathcal{F}\},$$

corresponding to the kernel of the pseudoscalar product

$$(\mathbf{Z}_+ | \mathbf{Z}_+) =< \xi | (\mathbf{Z}^\star \mathbf{Z})_+^- \xi >=< \xi | \mathbf{Z}_+^* \mathbf{g} \mathbf{Z}_+ \xi > \ ,$$

where $\mathbf{Z}_+^* = \left(Z_+^{-*}, Z_+^{\circ *}, \widehat{I} \right)$ is the conjugate row to the column-operator $\mathbf{Z}_+$. Now we can formulate the main theorem.

Theorem 4. *Suppose that the output observed process (3.1) is nondemolition with respect to a QS process (4.1), and the spans $\mathcal{C} \subseteq \mathcal{O}'$ and $\mathcal{D}^t \subseteq \mathcal{F}_t'$ are * – and $\star$ – algebras correspondingly. Then the posterior mean value $\epsilon_t(X(t))$ for an initial state vector $\xi = \psi \otimes \delta_\emptyset, \psi \in \mathfrak{h}$, is defined by an adapted commutative vector–process $\kappa_t = \left(\kappa_t^i\right), \kappa_t^i \in \mathcal{A}_t', i = 1, \ldots, n$, almost everywhere as an $\mathcal{A}_t'$ linear nonaticipating transformation of the output process Y by the stochastic Itô equation*

$$\mathrm{d}\epsilon_t(X(t)) = \epsilon_t(\mathbf{Z}^\star \mathbf{F} \mathbf{Z})_+^-(t)\mathrm{d}t + \kappa_t^i(X(t))\mathrm{d}\widetilde{Y}_i(t) \tag{4.2}$$

Here $\epsilon_t(\mathbf{F})_+^- = \epsilon_t(F_+^-), \epsilon_0(\widehat{x} \otimes I) = \epsilon(\widehat{x}) \otimes I$, $\kappa^i \mathrm{d}Y_i \equiv \sum_{i=1}^n \kappa^i \mathrm{d}Y_i$, and

$$\mathrm{d}\widetilde{Y}_i(t) = \mathrm{d}Y_i(t) - \epsilon_t(\mathbf{Z}^\star \mathbf{D}_i \mathbf{Z})_+^-(t)\mathrm{d}t,\ \widetilde{Y}_i(0) = \widetilde{y}_i \otimes I \tag{4.3}$$

are the observed martingales with respect to the filtration (ϵ_t), and state vector ξ, called the innovating process for $(\mathcal{A}_t')$. The process κ_t is defined uniquely up to the kernel of the correlation matrix–process

$$\varrho_{ik}(t) = \epsilon_t \left(\mathbf{Z}^\star \mathbf{D}_i^\star \mathbf{D}_k \mathbf{Z}\right)_+^- (t) = \epsilon_t[(D_\circ^\circ Z_\circ^\circ + D_+^\circ)_i^* (D_\circ^\circ Z_+^\circ + D_+^\circ)_k](t) \tag{4.4}$$

by the linear algebraic equation

$$\varrho_{ik}(t)\kappa_t^k = \epsilon_t \left(\mathbf{Z}^\star \mathbf{D}_i^\star \mathbf{F} \mathbf{Z}\right)_+^- (t) - \epsilon_t(X(t))\epsilon_t \left(\mathbf{Z}^\star \mathbf{D}_i^\star \mathbf{Z}\right)_+^- (t), \tag{4.5}$$

having in the case $\mathbf{F} = X \otimes \boldsymbol{\delta}$, corresponding to $\widehat{X}(t) = \widehat{x} \otimes I$, the form

$$\varrho_{ik}(t)\kappa_t^k = \epsilon_t \left(Z_+^* D_{\circ i}^{\circ *} \tilde{X} Z_+^\circ\right)(t) + \epsilon_t \left(\widetilde{X} D_{+i}^{\circ *} Z_+^\circ + Z_+^{\circ *} D_{\circ i}^{-*} \widetilde{X}\right)(t), \tag{4.6}$$

where $\widetilde{X}(t) = X(t) - \epsilon_t(X(t))$. The initial a posteriori mean value $\epsilon(\widehat{x})$ is the linear combination $\epsilon(\widehat{x}) =< \psi | \widehat{x}\psi > + \kappa^i(\widehat{x})\widetilde{y}_i$, of $\widetilde{y}_i = \widehat{y}_i - < \psi | \widehat{y}_i \psi > \widehat{1}$, where $\kappa = (\kappa^i)$ is defined by the equation

$$\varrho_{ik}\kappa^k =< \psi | \widehat{y}_i^* \widehat{x} \psi >, \quad \varrho_{ik} =< \psi | \widetilde{y}_i^* \widetilde{y}_k \psi >, \tag{4.7}$$

with $\widetilde{x} = \widetilde{x} - < \psi | \widehat{x}\psi > \widehat{1}$, uniquely up to the kernel of the initial correlation matrix $\varrho = (\varrho_{ik})$.

In order to prove this fundamental filtering theorem we need the following lemmas.

Lemma 1. *If the process X satisfies the equation (4.1), then there exists such a martingale M_t with respect to (ϵ_t, ξ), affiliated with $\mathcal{A}_t'$ on $\mathcal{A}_t\xi$, such that almost everywhere*

$$\epsilon_t(X(t)) = \epsilon(\widehat{x}) \otimes I + \int_0^t \epsilon_s(\mathbf{Z}^\star\mathbf{F}\mathbf{Z})_+^-(s)\mathrm{d}s + M_t. \tag{4.8}$$

Proof. Let us define M_t on ξ by

$$M_t\xi = (E_t - E_0)\, X(0)\xi + \int_0^t (E_t - E_s)\,(\mathbf{Z}^\star\mathbf{F}\mathbf{Z})_\nu^\mu \mathrm{d}A_\mu^\nu\xi\ .$$

Obviously, that $M_t\xi$ satisfies the (ϵ_t, ξ) – martingale condition $E_sM_t\xi = M_s\xi$ for all $s \leq t$, and

$$E_tX(t)\xi = (\widehat{e}\widehat{x} \otimes I)\,\xi + \int_0^t E_s(\mathbf{Z}^\star\mathbf{F}\mathbf{Z})_+^-(s)\xi\mathrm{d}s + M_t\xi$$

due to $E_s(\mathbf{Z}^\star\mathbf{F}\mathbf{Z})_\nu^\mu\mathrm{d}A_\mu^\nu\xi = E_s(\mathbf{Z}^\star\mathbf{F}\mathbf{Z})_+^-\xi\mathrm{d}t$ for $\xi = \psi \otimes \delta_\emptyset$. The operator M_t, affiliated with $\mathcal{A}_t'$ can be correctly defined almost everywhere by

$$M_tA\xi = AE_tM_t\xi = AM_t\xi, \quad \forall A \in \mathcal{A}_t,$$

as in the case of (3.5) for $X = M_t,\ \epsilon_t(M_t) = M_t$.

So, for any $A \in \mathcal{A}_t$ we have

$$\epsilon_t(X(t))A\xi = A\left(\widehat{e}\widehat{x}\psi \otimes \delta_\emptyset + \int_0^t E_s(\mathbf{Z}^\star\mathbf{F}\mathbf{Z})_+^-(s)\xi\mathrm{d}s + M_t\xi\right) =$$

$$= (\epsilon(\widehat{x}) \otimes I)\, A\xi + \int_0^t \epsilon_s(\mathbf{Z}^\star\mathbf{F}\mathbf{Z})_+^-(s)A\xi\mathrm{d}s + M_tA\xi,$$

and, hence, (4.8) holds on the dense linear manifold $\mathcal{A}_t\xi$ of the support $\mathcal{K}^t$ of the state ξ on $\mathcal{A}_t'$.

Lemma 2. *A process $M_t = \int_0^t(\mathbf{Z}^\star\mathbf{D}\mathbf{Z})_\nu^\mu\mathrm{d}\widehat{A}_\mu^\nu$ with $\mathbf{D}(t) \in \mathcal{D}^t$ is a martingale with respect to (ϵ_t, ξ), iff $\epsilon_t(\mathbf{Z}^\star\mathbf{D}\mathbf{Z})_+^-(t) = 0$ for all t, that is almost everywhere*

$$D_+^-(t) + D_\circ^-(t)\epsilon_t(Z_+^\circ) + \epsilon_t(Z_+^\circ)^*D_+^\circ(t) + \epsilon_t(Z_+^{\circ *}D_\circ^\circ Z_+^\circ)(t) = 0 \tag{4.9}$$

and is the zero martingale (almost everywhere), iff $\mathbf{D}(t) \in \mathcal{D}_0^t$, which is equivalent to $\epsilon_t(\mathbf{Z}^\star\mathbf{D}^\star\mathbf{D}\mathbf{Z})_+^-(t) = 0$ for all t almost everywhere, that is

$$\epsilon_t[(D_+^\circ + D_\circ^\circ Z_+^\circ)^*(D_+^\circ + D_\circ^\circ Z_+^\circ)](t) = 0 \tag{4.10}$$

and, hence, $D_+^-(t) = \epsilon_t(Z_+^{\circ *}D_\circ^\circ Z_+^\circ)(t)$.

Proof. Due to commutativity of $M(t)$ with $\mathcal{A}_t$, we have to prove only that $E_tM_r\xi = M_t\xi$ for all $r \geq t$, iff $E_t(\mathbf{Z}^\star\mathbf{D}\mathbf{Z})_+^-(t) = 0$ for all t. Indeed, $E_t(M_r - M_t)\xi = \int_t^r E_t(\mathbf{Z}^\star\mathbf{D}\mathbf{Z})_+^-(s)\xi\mathrm{d}s = 0$ for all $r > t$ iff $E_t(\mathbf{Z}^\star\mathbf{D}\mathbf{Z})_+^-(s)\xi = 0$ for all $t \leq s$, which is equivalent to $E_t(\mathbf{Z}^\star\mathbf{D}\mathbf{Z})_+^-(t)\xi = 0$ for all t due to $E_tE_s = E_t$ for $t \leq s$, written in the form (4.10) for

$$(\mathbf{Z}^\star\mathbf{D}\mathbf{Z})_+^- = D_+^- + D_\circ^-Z_+^\circ + Z_+^{\circ *}D_+^\circ + Z_+^{\circ *}D_\circ^\circ Z_+^\circ\ .$$

If M_t is a martingale, then

$$\epsilon_t[(M_r - M_t)^*(M_r - M_t)] = \epsilon_t(M_r^*M_r) - M_t^*M_t =$$

$$\int_t^r \epsilon_t(\mathbf{Z}^\star\mathbf{D}^\star\mathbf{D}\mathbf{Z})_+^-(s)\mathrm{d}s \geq 0 \ .$$

Hence, if M_t is a zero martingale, $|M_r - M_t|^2 = 0$, and $\epsilon_t(\mathbf{Z}^\star\mathbf{D}^\star\mathbf{D}\mathbf{Z})_+^-(s) = 0$ for all $t \geq s$, what is equivalent to $\epsilon_t(\mathbf{Z}^\star\mathbf{D}^\star\mathbf{D}\mathbf{Z})_+^-(t) = 0$ for all t, or to (4.10) in view of

$$(\mathbf{Z}^\star\mathbf{D}^\star\mathbf{D}\mathbf{Z})_+^- = (D_+^\circ + D_\circ^\circ Z_+^\circ)^*(D_+^\circ + D_\circ^\circ Z_+^\circ) \ .$$

But this means, that $(D_+^\circ + D_\circ^\circ Z_+^\circ)\xi = 0$, i.e. $\mathbf{D}(t) \in \mathcal{D}_0^t$. Conversely if $\mathbf{D}(t) \in \mathcal{D}_0^t$, i.e. if $\langle\xi|(\mathbf{Z}^\star\mathbf{D}^\star\mathbf{D}\mathbf{Z})_+^-\xi\rangle = 0$, then $E_t(\mathbf{Z}^\star\mathbf{D}^\star\mathbf{D}\mathbf{Z})_+^-(t)\xi = 0$ because

$$\langle B\xi|(\mathbf{Z}^\star\mathbf{D}^\star\mathbf{D}\mathbf{Z})_+^-\xi\rangle = \langle\xi|(\mathbf{Z}^\star\mathbf{D}^\star(B\otimes\boldsymbol{\delta})\mathbf{D}\mathbf{Z})_+^-\xi\rangle = 0$$

for any $B \in \mathcal{A}_t'$.

Lemma 3. *Let the linear complex span of $\{\widehat{y}_i\}$ and the span of $\{\mathbf{D}_i\}(t)$ with the coefficients in $\mathcal{A}_t'$, be commutative * – and $\star$ – algebras $\mathcal{C}$ and $\mathcal{D}^t$ up to the ideals $\mathcal{C}_\circ \subseteq \mathcal{C}$ and $\mathcal{D}_0^t \subset \mathcal{D}^t$ correspondingly. Then the locally bounded process*

$$B(t) = (\widehat{y}_0 + \lambda_0^i\tilde{y}_i)\otimes I + \int_0^t (\mathbf{Z}^\star(\mathbf{D}_0 + \lambda_s^i\tilde{\mathbf{D}}_i)\mathbf{Z})_\nu^\mu(s)\mathrm{d}A_\mu^\nu(s) \ , \tag{4.11}$$

where $y_0 \in \mathcal{C}$, $\tilde{y}_i = \widehat{y}_i - <\psi|\widehat{y}_i\psi> \widehat{1}$, $D_0 \in \mathcal{D}_0^t$, $\tilde{D}_\nu^\mu = D_\nu^\mu$, if $(\mu,\nu) \neq (-,+)$, and $\tilde{D}_+^-(t) = D_+^-(t) - \epsilon_t(\mathbf{Z}^\star\mathbf{D}\mathbf{Z})_+^-(t)$, defined by weakly measurable locally bounded functions $t \mapsto \lambda_t^i \in \mathcal{A}_t'$, $\lambda_0^i \in \mathbb{C}$, compose a weakly dense *-algebra $\mathcal{C}^t$ in $\mathcal{A}_t'$.

Proof. Using the QS Itô formula (2.2), one obtains for $\mathrm{d}B = (\mathbf{Z}^\star\mathbf{D}\mathbf{Z})_\nu^\mu\mathrm{d}A_\mu^\nu$ with $\mathbf{D}(t) = \mathbf{D}_0(t) + \lambda_t^i\hat{\mathbf{D}}_i(t) \in \mathcal{D}^t$

$$\mathrm{d}(B^*B) = (\mathbf{Z}^\star(\mathbf{D}^\star(B\otimes\boldsymbol{\delta}) + (B\otimes\boldsymbol{\delta})^*\mathbf{D} + \mathbf{D}^\star\mathbf{D})\mathbf{Z})_\nu^\mu\mathrm{d}\widehat{A}_\mu^\nu$$

due to the commutativity of $B(t)$ with $Z_\nu^\mu(t)$. But $\mathbf{D}^\star(t)\mathbf{D}(t) \in \mathcal{D}^t$ and, hence $(\mathbf{D}^\star(B\otimes\boldsymbol{\delta}) + (B\otimes\boldsymbol{\delta})^*\mathbf{D} + \mathbf{D}^\star\mathbf{D})(t) \in D^t$ is an $\mathcal{A}_t'$ linear combination of $\{\mathbf{D}_i(t)\}$ and a $\mathbf{G} \in D_0^t$, as well as $b^*b \in \mathcal{C}$ for $b = y_0 + y_i\lambda^i \in \mathcal{C}$ is a linear combination of $\widehat{y}^i$ and a $\widehat{b} \in \mathcal{C}_0$. Hence, B^*B is a process of the same form as $B(t)$, what means that the operators $B(t)$ compose a *-subalgebra $\mathcal{C}^t$ of $\mathcal{A}_t'$. The algebra b is a weakly dense in $\mathcal{A}_t'$ because it has the same commutant $\mathcal{A}_t$, as the family $\{Y(s) : s \geq t\}$, and hence generates the same von Neumann algebra $\mathcal{A}_t'$.

Proof of the Theorem 4. We shall look for the martingale M, defining the decomposition (4.8) in the Lemma 1. Let us suppose, that it is a stochastic integral nonanticipating span

$$M_t = \int_0^t (\mathbf{Z}^\star\hat{\mathbf{D}}_i\mathbf{Z})_\nu^\mu(s)\kappa_s^i\mathrm{d}A_\mu^\nu(s) \ , \quad \kappa_t^i \in \mathcal{A}_t^1$$

of the observable martingales

$$\tilde{Y}_i(t) = Y_i(t) - \int_0^t \epsilon_s(\mathbf{Z}^\star\mathbf{D}_i\mathbf{Z})_+^-(s)\mathrm{d}s = \int \mathbf{Z}^\star\hat{\mathbf{D}}_i\mathbf{Z}\mathrm{d}\widehat{A} \ ,$$

where $\tilde{Y}_i(t)$ should not be taken into account, if $D_i(t) \in \mathcal{D}_0^t$, as it is a zero almost everywhere martingale according to the Lemma 2. Due to the weak density of $\mathcal{C}^t$ in $\mathcal{A}_t'$, proved in Lemma 3, it is sufficient to find the coefficient κ_t^i from the condition

$$<\xi|B(t)^*X(t)\xi> = <\xi|B(t)^*\epsilon_t(X(t))\xi>$$

for all $B(t)$ in the form (4.11). Using the QS Itô formula (2.3) for $\mathrm{d}B = (\mathbf{Z}^\star\mathbf{D}\mathbf{Z})^\mu_\nu \mathrm{d}A^\nu_\mu$, where $\mathbf{D} = \mathbf{D}_0 + \hat{\mathbf{D}}_i\lambda^i$, one can obtain

$$\begin{aligned} \mathrm{d} &< \xi|B^* X\xi> = <\xi|(\mathbf{Z}^\star(B\otimes\boldsymbol{\delta} + \mathbf{D})^\star\mathbf{F}\mathbf{Z})^-_+\xi> \mathrm{d}t = \\ &= <\xi|B^*\epsilon_t(\mathbf{Z}^\star\mathbf{F}\mathbf{Z})^-_+ + (\mathbf{Z}^\star\mathbf{D}^\star\mathbf{F}\mathbf{Z})^-_+|\xi> \mathrm{d}t\ . \end{aligned}$$

On the other hand, taking into account that $\mathrm{d}\epsilon_t(X(t)) = \epsilon_t(\mathbf{Z}^\star\mathbf{F}\mathbf{Z})^-_+(t)\mathrm{d}t + \mathrm{d}M_t$, $\mathrm{d}M = (\mathbf{Z}^\star\hat{\mathbf{D}}_i\mathbf{Z})^\mu_\nu\kappa^i\mathrm{d}\widehat{A}^\nu_\mu$, one can obtain

$$\begin{aligned} \mathrm{d} &< \xi|B^*\epsilon_t(X)\xi> = <\xi|B^*\epsilon_t(\mathbf{Z}^\star\mathbf{F}\mathbf{Z})^-_+\xi> \mathrm{d}t + \\ + &< \xi|(\mathbf{Z}^\star\mathbf{D}^\star\mathbf{Z})^-_+\epsilon_t(X) + (\mathbf{Z}^\star\mathbf{D}^\star\widetilde{\mathbf{D}}_i\mathbf{Z})^-_+\kappa^i_t|\xi> \mathrm{d}t\ . \end{aligned}$$

Hence, $<\xi|\mathbf{Z}^\star\mathbf{D}^\star\widetilde{\mathbf{D}}_i\mathbf{Z}|\kappa^i_t\xi>^-_+ = <\xi|\left\{(\mathbf{Z}^\star\mathbf{D}^\star\mathbf{F}\mathbf{Z})^-_+ - (\mathbf{Z}^\star\mathbf{D}^\star\mathbf{Z})^-_+\epsilon_t(X)\right\}\xi>$, what is equivalent for $\mathbf{D} = \mathbf{D}_0 + \widetilde{\mathbf{D}}_i\lambda^i$ to (4.5) and

$$<\xi|(\mathbf{Z}^\star\mathbf{D}_0^\star\mathbf{D}_i\mathbf{Z})^-_+\kappa^i_t\xi> = <\xi|\{(\mathbf{Z}^\star\mathbf{D}_0^\star\mathbf{F}\mathbf{Z})^-_+ - (\mathbf{Z}^\star\mathbf{D}_0^\star\mathbf{Z})^-_+\epsilon_t(X)\}\xi> \tag{4.12}$$

due to $\mathbf{D}^\star\widetilde{\mathbf{D}}_i = \mathbf{D}^\star\mathbf{D}_i$ and arbitrariness of $\lambda^i_t \in \mathcal{A}'_t$. But the left hand side of the last equation (4.12) due to Schwarz inequality is zero:

$$<\xi|(\mathbf{Z}^\star\mathbf{D}_0^\star\mathbf{D}_0\mathbf{Z})^-_+\xi> = <(D^\circ_\circ Z^\circ_+ + D^\circ_+)_0\xi|(D^\circ_\circ Z^\circ_+ + D^\circ_+)_i\xi> = 0$$

as $<\xi|(\mathbf{Z}^\star\mathbf{D}_0^\star\mathbf{D}_0\mathbf{Z})^-_+\xi> = \|(D^\circ_\circ Z^\circ_+ + D^\circ_+)_0\xi\|^2 = 0$ for $D_0 \in \mathcal{D}^t_0$. On the other hand, taking into account, that

$$<\xi|(\mathbf{Z}^\star\mathbf{D}_0^\star\mathbf{F}\mathbf{Z})^-_+\xi> = <\xi|(\mathbf{Z}^\star\mathbf{D}_0(X\otimes\boldsymbol{\delta})\mathbf{Z})^-_+\xi>,$$

as $<\xi|(\mathbf{Z}^\star\mathbf{D}_0^\star\mathbf{C}\mathbf{Z})^-_+\xi> = <(D^\circ_\circ Z^\circ_+ + D^\circ_+)_0\xi|(C^\circ_\circ Z^\circ_+ + C^\circ_+)\xi> = 0$ for $\mathbf{C} = \mathbf{F} - X\otimes\boldsymbol{\delta}$, and due to $\star$ − normality

$$D^\circ_\circ D^{\circ *}_\circ = D^{\circ *}_\circ D^\circ_\circ,\ D^\circ_\circ D^{-*}_\circ = D^{\circ *}_\circ D^\circ_+,\ D^-_\circ D^{-*}_\circ = D^{\circ *}_+ D^\circ_+$$

of $\mathbf{D}_0 \in \mathcal{D}^t_0$ as for a matrix–operator of the commutative matrix $\star$–algebra $\mathcal{F}'_t$, one can obtain

$$\begin{aligned} &<\xi|(Z^\star\mathbf{D}_0^\star FZ)^-_+\xi> = <\xi|(D^-_+ + D^-_\circ Z^\circ_+)^*_0 X\xi> + \\ &+ < (D^\circ_\circ Z^\circ_+ + D^\circ_+)_0\xi|XZ^\circ_+\xi> = <\xi|(D^{-*}_+ X + Z^{\circ *}_+ X D^{\circ *}_\circ Z^\circ_+)_0\xi> = \\ &= <\xi|(D^{-*}_+ + D^{\circ *}_+ D^{\circ +}_\circ D^\circ_+)_0 X\xi> = <\xi|(D^-_+ + D^{\circ *}_+ D^{\circ +}_\circ D^\circ_+)^*_0\epsilon_t(X)\xi>\ . \end{aligned}$$

Here $D^{\circ^+}_\circ$ is quasi-inverse conjugate matrix–operator for normal $D^\circ_\circ = D^{\circ^*}_\circ D^{\circ^+}_\circ D^\circ_\circ$, and we used

$$(D^\circ_\circ Z^\circ_+ + D^\circ_+)_0\xi = 0, \quad (D^{\circ *}_\circ Z^\circ_+ + D^{-*}_\circ)_0\xi = 0$$

as for $\star$ − normal $\mathbf{D}_0 \in \mathcal{D}^t_0$. Hence, the right side of equation (4.12) is also zero:

$$<\xi|(\mathbf{Z}^\star\mathbf{D}_0^\star\mathbf{Z})^-_+\xi> = <\xi|(\mathbf{Z}^\star\mathbf{D}_0^\star\mathbf{Z})^-_+\epsilon_t(X)\xi>,$$

because in the same way one can obtain

$$\begin{aligned} <\xi|(\mathbf{Z}^\star\mathbf{D}_0^\star\mathbf{Z})^-_+\epsilon_t(X)\xi> &= <\xi|(D^-_+ + D^-_\circ Z^\circ_+)^*_0\epsilon_t(X)\xi> + \\ + <(D^\circ_\circ Z^\circ_+ + D^\circ_+)_0\xi|Z^\circ_+\epsilon_t(X)\xi> &= \xi|(D^-_+ + D^\circ_+ D^{\circ +}_\circ D^\circ_+)^*_0\epsilon_t(X)\xi>\ . \end{aligned}$$

This proves also the uniqueness of the solution of the equation (4.5) up to the kernel of the correlation matrix $\varrho(t) = (\varrho_{ik})(t)$ because if $\varrho_{ik}\lambda^k_0 = 0$ for an $\mathcal{A}'_t$ − adapted vector process $\lambda_0 = (\lambda_0^i)$, then $\mathbf{D}_0 = \lambda_0^i \mathbf{D}_i \in \mathcal{D}_0$, and, hence,

$$\begin{aligned} < \lambda^i \xi | \left\{ \epsilon(\mathbf{Z}^\star \mathbf{D}_i^\star \mathbf{F} \mathbf{Z})_+^- - \epsilon(\mathbf{Z}^\star \mathbf{D}_i^\star \mathbf{Z})_+^- \epsilon(X) \right\} \xi > &= \\ + < \xi | \left\{ (\mathbf{Z}^\star \mathbf{D}_0^\star \mathbf{F} \mathbf{Z})_+^- (\mathbf{Z}^\star \mathbf{D}_0^\star \mathbf{Z})_+^- \epsilon(X) \right\} \xi > &= 0. \end{aligned}$$

In the case $F_\nu^\mu = X \delta_\nu^\mu$ taking into account that

$$\epsilon(\mathbf{Z}^\star \mathbf{D}(X \otimes \boldsymbol{\delta}) \mathbf{Z})_+^- = D_+^- \epsilon(X) + D_\circ^- \epsilon(X Z_+^\circ) + \epsilon(Z_+^{\circ *} X) D_+^\circ + \epsilon(Z_+^{\circ *} D_\circ^\circ X Z_+^\circ)$$

and $\epsilon(\mathbf{Z}^\star \mathbf{D} \mathbf{Z})_+^- \epsilon(Z) =$

$$D_+^- \epsilon(X) + D_\circ^- \epsilon(X) \epsilon(Z_+^\circ) + \epsilon(Z_+^\circ)^* \epsilon(X) D_+^\circ + \epsilon(Z_+^{\circ *} D_\circ^\circ Z_+^\circ) \epsilon(X),$$

one can easily obtain the equation (4.6) from (4.5).

One should look for the initial condition $\epsilon_0(\widehat{x} \otimes I) = \epsilon(\widehat{x}) \otimes I$ for the equation (4.2) in the linear form $\epsilon(\widehat{x}) =< \psi | \widehat{x} \psi > + \tilde{y}_i \kappa^i$, where κ^i should be found from $< \widehat{b} \psi | \widehat{z} \psi >=< \widehat{b} \psi | \epsilon(\widehat{z}) \psi >$ for all $\widehat{b} = \widehat{y}_0 + \Sigma \tilde{y}_i \lambda^i$, where $\widehat{y}_0 \in \mathcal{C}$ and $\lambda^i \in \mathbb{C}$. This gives the initial equation (4.7). ∎

Corollary 3. *If $\{y_i\}$ are commuting orthogonal projectors in $\mathfrak{h}$, and also $\mathbf{D}_i^\star = \mathbf{D}_i = \mathbf{D}_i^2$ are commuting $\star$-matrix projectors, then the conditions of the Theorem 4 are fullfilled, and they are fullfilled also in the case $D_{\circ i}^\circ = 0$. In particular, for the case $D_\circ^\circ = I \otimes \delta_\circ^\circ$, $D_\circ^- = 0 = D_+^\circ$, $D_+^- = 0$, corresponding to the counting output process $Y = N$, the equation (4.6) gives*

$$\kappa_t = \epsilon_t(Z_+^{\circ *} X Z_+^\circ)(t) / \epsilon_t(Z_+^{\circ *} Z_+^\circ)(t) - \epsilon_t(X)(t) \ ,$$

if $\epsilon_t(Z_+^{\circ *} Z_+^\circ)(t) \neq 0$. In the other case $D_\circ^- = 1 = D_+^\circ$, $D_\circ^\circ = 0$, $D_+^- = 0$, corresponding to the output coordinate observation $Y = Q$, one obtains

$$\kappa_t = \epsilon_t(X Z_+^\circ + Z_+^{\circ *} X) - \epsilon_t(X) \epsilon_t(Z_+^\circ + Z_+^{\circ *}) \ . \tag{4.13}$$

Indeed, the linear, span of commuting orthoprojectors $\{y_i\}$, and also $\mathcal{A}_t'$-span of $\star$-projectors $\{D_i\}$ is a $*$-and $\star$-algebra $\mathcal{C}$ and $\mathcal{D}^t$ correspondingly. In the case $D_{i\circ}^\circ = 0$ the product $\mathbf{D}_i^\star \cdot \mathbf{D}_k$ is in $\mathcal{D}_0^t$ as matrix with $(\mathbf{D}_i^\star \mathbf{D}_k)_\circ^\circ = 0$, $(\mathbf{D}_i^\star \mathbf{D}_k)_+^\circ = 0$, $(\mathbf{D}_i^\star \mathbf{D}_k)_\circ^- = 0$. Such commutative matrices also form a commutative $\star$-algebra $\mathcal{D}^t$ up to the ideal $\mathcal{D}_0^t$, because $\mathbf{G}^\star \mathbf{G} = 0$, and, hence, $(\mathbf{G} \mathbf{Z}_+ | \mathbf{G} \mathbf{Z}_+) = 0$ for a matrix-operator $\mathbf{G}$ with $G_\nu^\mu = 0$ for $(\mu, \nu) \neq (-, +)$.

5. An application of the QS filtering

The applications of the filtering equation (4.2) to the derivation of a posteriori Schrödinger equation for the coordinate observation are given in [17, 18], and for the counting observation are given in [19, 20].

In contrast to the usual Schrödinger equation, describing a closed quantum system without observation, these new stochastic wave equations give the dynamics of an open quantum system undergoing the nondemolition measurements which are continuous in time. Thus the continual wave packed reduction problem is solved by the quantum filtering method for the typical QS models of observation such as a quantum particle is a bubble chamber [18] (diffusive observation) and an atom radiating the photons [20] (counting observation). Here we consider another example of the quantum nonlinear filtering – the QS spin localization, describing the continuous collapse of the vector polarization $\vec{p} = (p_1, p_2, p_3)$ for the spin $\frac{1}{2}$ of an electron under a continuous nondemolition measurement in a magnetic field. The polarization $\vec{p}(t)$ at the time instant $t > 0$ is given by the conditional expectations (3.5)

$$p_j(t) = \epsilon_t(X_j(t)) \ , \ X_j(t) = U^*(t)(\hat{x}_j \otimes I)U(t) \ , \tag{5.1}$$

where $\hat{x}_j = \hat{\sigma}_j$ are the Pauli matrices

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $U(t)$ is a QS unitary evolution in the Hilbert space $\mathcal{H}^0 \otimes \mathcal{F}$. Here $\mathcal{H}^0$ is the Hilbert space $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$ of the spinors $\psi(\vec{r}) = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}(\vec{r})$, where $\psi_\pm(\vec{r})$, $\vec{r} \in \mathbb{R}^3$ are the wave functions of the nonrelativistic electron with the definite z-projections $\pm\frac{1}{2}$ of its spin $\hat{\vec{s}} = \frac{1}{2}(\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ and probabilistic normalization $\|\psi\|^2 \int \psi(\vec{r})^+\psi(\vec{r})\mathrm{d}r = 1$, where $\psi^+\psi = |_-|^2 + |\psi_+|^2$, and $\mathcal{F} = \Gamma(\mathcal{E})$ is the Fock space over the Hilbert space $\mathcal{E} = L^2(\mathbb{R}^+) \otimes \mathbb{C}^n$.
The initial polarization $\vec{p}(0) = (p_1^0, p_2^0, p_3^0) = \vec{p}_0$,

$$p_j^0 = \int \psi(\vec{r})^\dagger \hat{\sigma}_j \psi(\vec{r})\mathrm{d}\vec{r} \ , \quad j = 1, 2, 3 \ ,$$

has the values in the unite ball $\mathcal{B} = \{\vec{p} \in \mathbb{R}^3 : \ |\vec{p}| \leq 1\}$, where $|\vec{p}|^2 = (\vec{p}, \vec{p}) \equiv p^2$, i.e. can be mixed $\sum_{j=1}^3 (p_j^0)^2 < 1$ even in the pure (vector) state $\psi \in \mathcal{H}^0$, $\|\psi\|^2 = \int \psi(\vec{r})^\dagger \psi(\vec{r})\mathrm{d}\vec{r} = 1$.

Let us suppose that the evolution $U(t)$ defines the system of Langevin equations (2.13) of the form

$$\mathrm{d}\vec{X} + (\mathrm{i}[\vec{X}, H] + \frac{1}{2}\sum_{i=1}^n [[\vec{X}, L_j], L_j])\mathrm{d}t = \mathrm{i}\sum_{j=1}^n [\vec{X}, L_i]\mathrm{d}V_i \ . \tag{5.2}$$

Here $\vec{X}(t) = (X_1, X_2, X_3)(t)$, $H(t) = \frac{1}{2}\sum_{j=1}^3 u^j(t)X_j(t)$ is the spin-Hamiltonian, corresponding to the magnetic tense $\vec{u}(t) = (u^1, u^2, u^3)(t) \in \mathbb{R}^3$,

$$L_i(t) = \frac{1}{2}\sum_{j=1}^n r_i^j(t)X_j(t) \equiv \frac{1}{2}R_i(t)$$

are spin-operators, defined by the real vectors $\vec{r}_i(t) = (r_i^1, r_i^2, r_i^3)(t) \in \mathbb{R}^3$, $i = 1, \ldots, n$, and $V_i = \hat{1} \otimes 2\Im A_i^+ \ i = 1, \ldots, n$ are the independent standard Wiener processes, represented by the input operators $\frac{1}{\mathrm{i}}(A_i^+(t) - A_-^i(t))$, $\mathrm{i} = \sqrt{-1}$ in the Fock space $\mathcal{F}$ with respect to the initial vacuum state $\delta_\emptyset \in \mathcal{F}$. The stochastic system of the operator equations (5.2) corresponds to the unitary Markovian evolution (2.5) in $\mathfrak{h} \otimes \mathcal{F}$, $\mathfrak{h} = \mathbb{C}^2$ with the generators

$$Z_\nu^\mu(t) = U(t)^*(\hat{Z}_\nu^\mu(t) \otimes I)U(t) \ \begin{matrix} \mu = -, i, \ldots, n \\ \nu = 1, \ldots, n, + \end{matrix}$$

defined by the spin-operators

$$\hat{z}_k^i = \delta_k^i \hat{1} \ , \quad \hat{z}_+^i = \frac{1}{2}\hat{r}_i = \hat{z}_i^- \ , \ \hat{z}_+^- = -\frac{1}{2}\left(\frac{1}{4}\hat{r}^2 + \mathrm{i}\hat{u}\right) \ , \quad \mathrm{i} = \sqrt{-1} \ ,$$

where $\hat{r}_i(t) = \sum_{i=1}^3 r_i^j(t)\hat{\sigma}_j$, $\hat{r}^2(t) = \sum_{j=1}^n r_i^2(t)\hat{1}$, $\hat{u}(t) = \sum\limits_{j=1}^{3} u^j(t)\hat{\sigma}_j$. Such the evolution realizes the output coordinate processes

$$Y_i(t) = U(t)^* W_i(t) U(t) = Q_i(t), \quad i = 1, \ldots, n,$$

satisfying the QS equations (2.4) in the form

$$\mathrm{d}Y_i = R_i \mathrm{d}t + \mathrm{d}W_i \ , \quad \mathrm{d}W_i = \hat{1} \otimes 2\Re A_i^+ \tag{5.3}$$

of the indirect non-demolition observation of the noncommuting spin-operators

$$R_i(t) = U(t)^* (\hat{r}_i(t) \otimes I) U(t) \ , \quad i = 1, \ldots, n \ .$$

The standard Wiener processes W_i, $i = 1, \ldots, n$, represented by the commuting operators $A_i^+(t) + A_-^i(t)$ in $\mathcal{F}$, describe the independent errors $\dot{Y}_i - R_i$ as the white noises $\dot{W}_i$. They do not commute with the white noises $\dot{V}_i$ of the perturbations in the quantum system (5.2):

$$[\dot{V}_i(s) \ , \ \dot{W}_k(t)] = 2\mathrm{i}\delta(s-t)\delta_{ik}\hat{I} \ , \tag{5.4}$$

due to

$$[V_i(s) \ , \ \dot{W}_k(t)] = 2\mathrm{i}[\hat{A}_-^i(s), \ \hat{A}_k^+(t)] = 2\mathrm{i}\min(s,t)\delta_{ik}\hat{I} \ .$$

Proposition 3. *Under the given assumptions the a posteriori spin polarizations (5.1) satisfy the following system of nonlinear stochastic equations*

$$\mathrm{d}\vec{p} + (\vec{p} \wedge \vec{u} + \frac{1}{2}\sum_{i=1}^{n}(r_i^2 \vec{p} - (\vec{p}, \vec{r}_i)\vec{r}_i))\mathrm{d}t = \sum_{i=1}^{n}(\vec{r}_i - (\vec{p}, \vec{r}_i)\vec{p})\mathrm{d}\tilde{Y}_i \ , \tag{5.5}$$

where $\mathrm{d}\tilde{Y}_i(t) = \mathrm{d}Y_i(t) - (\vec{p}(t), \ \vec{r}_i(t))\mathrm{d}t$.

PROOF. Let us consider the nonlinear filtering equation (4.2) for the spin-operators $X_j(t)$, $j = 1, 2, 3$, which are equivalent to the Pauli matrices $\hat{\sigma}_j$, $j = 1, 2, 3$. We can use (4.2) for the evaluation of the expectations (5.1) because the conditions of the Theorem 4 are fulfilled (see the Corollary of sec. 4). The innovating martingales $\mathrm{d}\tilde{Y}_i = \mathrm{d}Y_i - \epsilon_t(r_i)\mathrm{d}t$ in this case are given by the differences $\mathrm{d}\tilde{Y}_i = \mathrm{d}Y_i - (\vec{p}, \vec{r}_i)\mathrm{d}t$ because

$$\epsilon_t(R_i(t)) = \sum_{j=1}^{3} r_i^j(t)\epsilon_t(X_j(t)) = \sum_{j=1}^{3} r_i^j(t) p_j(t) \ .$$

Due to $\rho_{ik}(t)\mathrm{d}t = \epsilon_t(\mathrm{d}\tilde{Y}_i(t)\mathrm{d}\tilde{Y}_k(t)) = \delta_{ik}\mathrm{d}t$, the coefficients $\kappa_t^i(X_j(t))$ are given by

$$\begin{aligned} \kappa_t^i(\vec{x}) &= \frac{1}{2}\epsilon_t(\vec{X}(t)R_i(t) + R_i(t)\vec{X}(t)) - \epsilon_t(\vec{X}(t))\epsilon_t(R_i(t)) = \\ &= \vec{r}_i(t) - (\vec{p}(t), \ \vec{r}_i(t))\vec{p}(t) \ , \end{aligned}$$

because $\hat{\sigma}_j \hat{r}_j + \hat{r}_j \hat{\sigma}_j = 2r_i^j \hat{1}$ for $\hat{r}_i = \sum\limits_{j=1}^{3} r_i^j \hat{\sigma}_j$, and

$$X_j R_i + R_i X_j = U(t)^* (\hat{\sigma}_j \hat{r}_i + \hat{r}_i \hat{\sigma}_j) U(t) = 2r_i^j \hat{I} \ .$$

The vector-product $\vec{p}(t) \wedge \vec{u}(t)$ in (5.5) represents the expectations

$$\mathrm{i}\epsilon_t([\vec{X}(t) \ , \ H(t)]) = \mathrm{i}\epsilon_t \left(\frac{1}{2}\sum_{j=1}^{3}[\vec{X}(t) \ , \ X_j(t)]u_j(t) \right) \ ,$$

because $[\hat{\vec{\sigma}}, \hat{u}] = \Sigma_{i=1}^{3}[\hat{\vec{\sigma}}, \hat{\sigma}_i]u^i = \frac{2}{\mathrm{i}}\hat{\vec{\sigma}} \wedge \vec{u}$, and

$$[\vec{X}(t)\ ,\ H(t)] = \frac{1}{2}U(t)^*([\hat{\vec{\sigma}}, \hat{u}(t)] \otimes I)U(t).$$

In the same way one can obtain

$$r_i^2(t)\vec{p}(t) - (\vec{p}(t), \vec{r}_i(t))\vec{r}_i(t) = (\vec{p}(t) \wedge \vec{r}_i(t)) \wedge \vec{r}_i(t)$$

as for the vector representation of the double commutator $\frac{1}{2}[[\vec{X}(t)\ ,\ L_i(t)]\ ,\ L_i(t)]$, defining together with $\mathrm{i}[\vec{X}(t)\ ,\ H(t)]$ the products $(\mathbf{Z}^\star(t)\vec{X}(t)\mathbf{Z}(t))_+^-$ in (4.2). ∎

Now we can prove, that the continuous indirect nondemolition measurement (5.3) of the quantum spin reduces any initial state of the electron at the limit $t \to \infty$, to the completely polarized one. This gives a kind of the stochastic ergodicity property of the nonlinear system of quantum filtering equation (5.5).

Theorem 5. *Let $\vec{p}(0) = \vec{p}_0 \in \mathcal{B}$ be an arbitrary initial polarization for the nonlinear quantum filtering equation (5.5). Then this equation has a unique stochastic solution $\vec{p}(t) \in \mathcal{B}$, and $p^2(t) = (\vec{p}(t)\ ,\ \vec{p}(t)) \to 1$ at $t \to \infty$ almost surely, if $\lambda(t) = \int_0^t \sum_{i=1}^{n} |r_i(s)|^2 \mathrm{d}s \to \infty$.*

Proof. The vector stochastic equation (5.5) up to a renormalization $\vec{f}(t) = \rho(t)\vec{p}(t)$ is equivalent to the linear stochastic equation

$$\mathrm{d}\vec{f} + (\vec{f} \wedge \vec{u} + \frac{1}{2}\sum_{i=1}^{u}(r_i^2\vec{f} - (\vec{f}, \vec{r}_i)\vec{r}_i))\mathrm{d}t = \rho\sum_{i=1}^{n}\vec{r}_i\mathrm{d}Y_i\ . \tag{5.6}$$

Indeed, let $\rho(t)$ be the stochastic Itô's integral

$$\rho(t) = 1 + \int_0^t \sum_{i=1}^{u}(\vec{f}(s),\ \vec{r}_i(s))\mathrm{d}Y_i(s) \tag{5.7}$$

defined by the unique solution $\vec{f}(t)$ of this ordinary linear stochastic differential equation with the initial nonstochastic vector $\vec{f}(0) = \vec{p}_0$. Then $\mathrm{d}\rho = \sum_{i=1}^{u}(\vec{f}, \vec{r}_i)\mathrm{d}Y_i$, and by Itô's formula

$$\mathrm{d}(\rho\vec{p}) = \mathrm{d}\rho\vec{p} + \mathrm{d}\rho\mathrm{d}\vec{p} + \rho\mathrm{d}\vec{p}$$

we obtain the equation for $\vec{f} = \rho\vec{p}$ iff $\vec{p}(t)$ satisfies the equation (5.5):

$$\begin{aligned}
&\mathrm{d}\vec{f} + (\vec{f} \wedge \vec{u} + \frac{1}{2}\sum_{i=1}^{u} r_i^2\vec{f} - (\vec{f}, \vec{r}_i)\vec{r}_i))\mathrm{d}t = \mathrm{d}\rho\vec{p} + \mathrm{d}\rho\mathrm{d}\vec{p} + \\
&\quad + \quad \rho\sum_{i=1}^{u}(\vec{r}_i - (\vec{p}, \vec{r}_i)\vec{p})\mathrm{d}\tilde{Y}_i = \sum_{i=1}^{u}(\vec{f}, \vec{r}_i)\vec{p}\mathrm{d}Y_i + \sum_{i=1}^{n}(\vec{r}_i - (\vec{p}, \vec{r}_i)\vec{p})(\vec{f}, \vec{r}_i)\mathrm{d}t \\
&\quad + \quad \sum_{i=1}^{n}(\rho\vec{r}_i - (\vec{f}, \vec{r}_i)\vec{p})(\mathrm{d}Y_i - (\vec{p}, \vec{r}_i)\mathrm{d}t) = \rho\sum_{i=1}^{n}\vec{r}_i\mathrm{d}Y_i\ .
\end{aligned}$$

This the unique solution of the nonlinear filtering equation (5.5) with $\vec{p}(0) = \vec{p}_0$ can be written almost surely ($\rho(t) \neq 0$) as $\vec{p}(t) = \vec{f}(t)/\rho(t)$, where $\vec{f}(t)$ is the solution of the linear equation (5.6) with $\vec{f}(0) = \vec{p}_0$, and $\rho(t)$ is the integral (5.7).

In order to prove that almost surely $|\vec{p}(t)| \leq 1$, if $|\vec{p}_0| \leq 1$, it is sufficient to show, that

$$f^2(t) = (\vec{f}(t), \vec{f}(t)) \leq \rho(t)^2 \quad \text{if} \quad \vec{f}(0) = \vec{p}_0\ .$$

Using the Itô's formula we obtain

$$\begin{aligned} \mathrm{d}f^2 &= 2(\vec{f},\ \mathrm{d}\vec{f}) + (\mathrm{d}\vec{f}\ ,\ \mathrm{d}\vec{f}) = 2\rho\sum_{i=1}^{u}(\vec{f},\vec{r}_i)\mathrm{d}Y_i - \\ &- \sum_{i=1}^{u}(r_i^2 f^2 - (\vec{f},\vec{r}_i)^2 - \rho^2 r_i^2)\mathrm{d}t = \mathrm{d}\rho^2 + (\rho^2 - f^2)\sum_{i=1}^{u} r_i^2 \mathrm{d}t\ , \end{aligned}$$

where $\mathrm{d}\rho^2 = 2\rho\mathrm{d}\rho + (\mathrm{d}\rho)^2 = 2\rho\sum_{i=1}^{u}(\vec{f}\ ,\ \vec{r}_i)(\mathrm{d}Y_i + (\vec{f},\vec{r}_i))$. Hence

$$\mathrm{d}(f^2 - \rho^2) = \dot{\lambda}(\rho^2 - f^2)\mathrm{d}t,$$

where $\dot{\lambda} = \sum_{i=1}^{u} r_i^2 \geq 0$, and

$$\rho^2(t) - (\vec{f}(t),\ \vec{f}(t)) = e^{-\lambda(t)}(1 - (\vec{p}_0,\ \vec{p}_0)), \quad \forall t\ .$$

Thus $f^2(t) \leq \rho^2(t)$, if $|\rho^0| \leq 1$, and $f^2(t) \to \rho^2(t)$ exponentially at $t \to \infty$, if $\lambda(t) \to \infty$ ($f^2(t) = \rho^2(t)$, $\forall t$, if $|\vec{p}_0| = 1$). This proves that $\vec{p}(t) = \vec{f}(t)/\rho(t) \to 1$ almost surely ($\rho(t) \neq 0$) due to the positivity of $\rho(t)$. ∎

Remark. The model (5.2) of continual nondemolition measurements of noncommuting spin-operators $R_i(t)$, $i = 1, \ldots, n$ in the quantum stochastic system (5.2) is unique in the Fock space $\mathcal{F} = \Gamma(\mathcal{E})$ over the minimal Hilbert space $\mathcal{E} = L^2(\mathbb{R}_+)\otimes\mathbb{C}^n$. It can not be realized in the framework of classical probability theory due to the noncommutativity (5.4) of the quantum stochastic processes $V_i(t)$ and $W_i(t)$ though each of them can be described as the classical one separately due to the selfnondemolition (commutativity) property $[V_i(t)\ ,\ V_k(s)] = 0 = [W_i(t)\ ,\ W_k(s)]$.

The result obtained here in a rigorous mathematical way corresponds to a rather intuitive physical picture of the continual spontaneous collapse of the quantum spin under the non-demolition observation. This proves the appropriateness of the given quantum stochastic setup for the theory of continuous measurements and quantum filtering.

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