QUANTUM STOCHASTIC POSITIVE EVOLUTIONS: CHARACTERIZATION, CONSTRUCTION, DILATION

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Abstract. A characterization of the unbounded stochastic generators of quantum completely positive flows is given. This suggests the general form of quantum stochastic adapted evolutions with respect to the Wiener (diffusion), Poisson (jumps), or general Quantum Noise. The corresponding irreversible Heisenberg evolution in terms of stochastic completely positive (CP) maps is constructed. The general form and the dilation of the stochastic completely dissipative (CD) equation over the algebra $\mathcal{L}(\mathcal{H})$ is discovered, as well as the unitary quantum stochastic dilation of the subfiltering and contractive flows with unbounded generators. A unitary quantum stochastic cocycle, dilating the subfiltering CP flows over $\mathcal{L}(\mathcal{H})$, is reconstructed.

Introduction

In quantum theory of open systems there is a well known Lindblad's form [1] of the quantum Markovian master equation, satisfied by the one-parameter semigroup of completely positive (CP) maps. This nonstochastical equation is obtained by averaging the stochastic Langevin equation for quantum diffusion [2] over the driving quantum noises. On the other hand the quantum Langevin equation is satisfied by a quantum stochastic process of dynamical representations, which are obviously completely positive due to *-multiplicativity of the homomorphisms, describing these representations. The homomorphisms give the examples of pure, i.e. extreme point CP maps, but among the extreme points of the convex cone of all CP maps there are not only the homomorphisms. This means a possibility to construct the dynamical semigroups by averaging of pure, i.e. non-mixing irreversible quantum stochastic CP dynamics, which is not driven by a Langevin equation.

The examples of such dynamics having recently been found in many physical applications, will be considered in the first section. The rest of the paper will be devoted to the mathematical derivation of the general structure for the quantum stochastic CP evolutions and the corresponding equations. The results of the paper not only generalize the Evans-Hudson (EH) flows [2] from homomorphism-valued maps to the general CP maps, but also prove the existence of the homomorphic dilations for the subfiltering and contractive CP flows. Here in the introduction we would like to outline this structure on the formal level.

The initial purpose of this paper was to extend the Evans–Lewis differential analog [3] of the Stinespring dilation [4] for the CP semigroups to the stochastic

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differentials, generating an Itô $*$–algebra

(0.1)
$$\mathrm{d}\Lambda(a)^{*}\,\mathrm{d}\Lambda(a)=\mathrm{d}\Lambda(a^{\star}a),\quad \sum\lambda_{i}\mathrm{d}\Lambda(a_{i})=\mathrm{d}\Lambda\left(\sum\lambda_{i}a_{i}\right),\quad \mathrm{d}\Lambda(a)^{*}=\mathrm{d}\Lambda(a^{\star})$$

with given mean values $\langle\mathrm{d}\Lambda(t,a)\rangle=l(a)\,\mathrm{d}t$, $a\in\mathfrak{a}$. Here $\mathfrak{a}$ is in general a non-commutative $\star$-algebra with a self-adjoint annihilator (death) $d=d^{\star}$, $\mathfrak{a}d=0$, corresponding to $\mathrm{d}t=\mathrm{d}\Lambda(t,d)$, and $l:\mathfrak{a}\to\mathbb{C}$ is a positive $l(a^{\star}a)\geq 0$ linear functional, normalized as $l(d)=1$, corresponding to the determinism $\langle\mathrm{d}t\rangle=\mathrm{d}t$. The functional l defines the GNS representation $a\mapsto\boldsymbol{a}=(a_{\nu}^{\mu})_{\nu=+,\bullet}^{\mu=-,\bullet}$ of $\mathfrak{a}$ in terms of the quadruples

$$a_{\bullet}^{\bullet}=\jmath(a),\quad a_{+}^{\bullet}=k(a),\quad a_{\bullet}^{-}=k^{*}(a),\quad a_{+}^{-}=l(a),\tag{0.2}$$

where $\jmath(a^{\star}a)=\jmath(a)^{*}\jmath(a)$ is the operator representation $\jmath(a)^{*}k(a)=k(a^{\star}a)$ on the pre-Hilbert space $\mathcal{E}=k(\mathfrak{a})$ of the Kolmogorov decomposition $l(a^{\star}a)=k(a)^{*}k(a)$, and $k^{*}(a)=k(a^{\star})^{*}$.

As was proved in [5], a quantum stochastic stationary processes $t\in\mathbb{R}_{+}\mapsto\Lambda(t,a),a\in\mathfrak{a}$ with $\Lambda(0,a)=0$ and independent increments $\mathrm{d}\Lambda(t,a)=\Lambda(t+\mathrm{d}t,a)-\Lambda(t,a)$, forming an Itô $\star$-algebra, can be represented in the Fock space $\mathfrak{F}$ over the space of $\mathcal{E}$ -valued square-integrable functions on $\mathbb{R}_{+}$ as $\Lambda_{\mu}^{\nu}(t,a_{\nu}^{\mu})=a_{\nu}^{\mu}\Lambda_{\mu}^{\nu}(t)$. Here

$$a_{\nu}^{\mu}\Lambda_{\mu}^{\nu}(t)=a_{\bullet}^{\bullet}\Lambda_{\bullet}^{\bullet}(t)+a_{+}^{\bullet}\Lambda_{\bullet}^{+}(t)+a_{\bullet}^{-}\Lambda_{-}^{\bullet}(t)+a_{+}^{-}\Lambda_{-}^{+}(t),\tag{0.3}$$

is the canonical decomposition of Λ into the exchange $\Lambda_{\bullet}^{\bullet}$, creation $\Lambda_{\bullet}^{+}$, annihilation $\Lambda_{-}^{\bullet}$ and preservation (time) $\Lambda_{-}^{+}=t\mathrm{I}$ processes of quantum stochastic calculus [6], [7] having the mean values $\left\langle\Lambda_{\mu}^{\nu}(t)\right\rangle=t\delta_{+}^{\nu}\delta_{\mu}^{-}$ with respect to the vacuum state in $\mathfrak{F}$. Thus the parametrizing algebra $\mathfrak{a}$ can be always identified with a $\star$-subalgebra of the algebra $\mathcal{Q}(\mathcal{E})$ of all quadruples $\boldsymbol{a}=(a_{\nu}^{\mu})_{\nu=+,\bullet}^{\mu=-,\bullet}$, where $a_{\nu}^{\mu}:\mathcal{E}_{\nu}\to\mathcal{E}_{\mu}$ are the linear operators on $\mathcal{E}_{\bullet}=\mathcal{E},\mathcal{E}_{+}=\mathbb{C}=\mathcal{E}_{-}$, having the adjoints $a_{\nu}^{\mu*}\mathcal{E}_{\mu}\subseteq\mathcal{E}_{\nu}$, with the Hudson–Parthasarathy (HP) multiplication table [8]

$$\boldsymbol{a}\bullet\boldsymbol{b}=(a_{\bullet}^{\mu}b_{\nu}^{\bullet})_{\nu=+,\bullet}^{\mu=-,\bullet},\tag{0.4}$$

the unique death $\boldsymbol{d}=\left(\delta_{-}^{\mu}\delta_{\nu}^{+}\right)_{\nu=+,\bullet}^{\mu=-,\bullet}$, and the involution $a_{-\nu}^{\star\mu}=a_{-\mu}^{\nu*}$, where $-(-)=+$, $-\bullet=\bullet$, $-(+)=-$.

The stochastic differential of a CP flow $\phi=(\phi_{t})_{t>0}$ over an operator algebra $\mathcal{B}$ is written in terms of the quantum canonical differentials as $\mathrm{d}\phi=\phi\circ\lambda_{\nu}^{\mu}\mathrm{d}\Lambda_{\mu}^{\nu}$ with $\phi_{0}=\imath$ at $t=0$, where $\imath(B)=B$ is the identical representation of $\mathcal{B}$. The main result of this paper is the construction of CP flows and their filtering dilation to the HP flows, based on the linear quantum stochastic evolution equation of the form

$$\mathrm{d}\phi_{t}(B)+\phi_{t}(K^{*}B+BK-L^{*}\jmath(B)L)\,\mathrm{d}t=\phi_{t}(L^{\bullet}\jmath(B)L_{\bullet}-B\otimes\delta_{\bullet}^{\bullet})\,\mathrm{d}\Lambda_{\bullet}^{\bullet}$$
$$+\phi_{t}(L^{\bullet}\jmath(B)L-K^{\bullet}B)\,\mathrm{d}\Lambda_{\bullet}^{+}+\phi_{t}(L^{*}\jmath(B)L_{\bullet}-BK_{\bullet})\,\mathrm{d}\Lambda_{-}^{\bullet},\tag{0.5}$$

where $\jmath$ is an operator representation of $\mathcal{B}$, $\delta_{\bullet}^{\bullet}$ is the identity operator in $\mathcal{E}$, and the operator K satisfies the conservativity condition $K+K^{*}=L^{*}L$ for the deterministic generator $\lambda=\lambda_{+}^{-}$. This form of the CP evolution equation was discovered in [9] as a result of the general CP differential structure

$$\boldsymbol{\lambda}(B)=\boldsymbol{L}^{*}\jmath(B)\boldsymbol{L}-\boldsymbol{K}^{*}B-B\boldsymbol{K}$$

of the bounded quantum stochastic generators $\boldsymbol{\lambda}=(\lambda_{\nu}^{\mu})_{\nu=+,\bullet}^{\mu=-,\bullet}$ over a von Neumann algebra $\mathcal{B}$ even in the nonlinear case. The dilation of the stochastic differentials for CP processes over arbitrary $*$-algebras, giving this structure for the bounded generators as a consequence of the Christensen-Evans theorem [10], was constructed in [11]. Here we shall prove that such a quantum stochastic extension of Lindblad's structure $\lambda(B) = L^* \jmath(B) L - K^* B - BK$, can be always used for the construction and the dilation of the CP flows also in the case of the unbounded maps $\lambda_\nu^\mu : \mathcal{B} \to \mathcal{B}$ over the algebra $\mathcal{B} = \mathcal{L}(\mathcal{H})$ of all operators in a Hilbert space $\mathcal{H}$. We shall prove that this structure is necessary at least in the case of the w*-continuous generators, which are extendable to the covariant ones over the algebra of all bounded operators $\mathcal{L}(\mathcal{H})$. The existence of a minimal CP solution which is constructed under certain continuity conditions proves that this structure is also sufficient for the CP property of any solution to this stochastic equation. The construction of the differential dilations and the CP solutions of such quantum stochastic differential equations with the bounded generators over the simple finite-dimensional Itô algebra $\mathfrak{a} = \mathcal{Q}(\mathcal{E})$ and the arbitrary $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ was recently discussed in [12, 13] (the latter paper contains also a characterization of the bounded generators for the contractive CP flows.)

The Evans–Lewis case $\Lambda(t,a) = \alpha t \mathrm{I}$ is described by the simplest one-dimensional Itô algebra $\mathfrak{a} = \mathbb{C}d$ with $l(a) = \alpha \in \mathbb{C}$, $\alpha^\star = \bar{\alpha}$, and the nilpotent multiplication $\alpha^\star \alpha = 0$ corresponding to the non-stochastic (Newton) calculus $(\mathrm{d}t)^2 = 0$ in $\mathcal{E} = 0$. The standard Wiener process $\mathrm{Q} = \Lambda_-^\bullet + \Lambda_\bullet^+$ in Fock space is described by the second order nilpotent algebra $\mathfrak{a}$ of pairs $a = (\alpha, \xi)$ with $d = (1,0)$, $\xi \in \mathbb{C}$, represented by the quadruples $a_+^- = \alpha$, $a_\bullet^- = \xi = a_+^\bullet$, $a_\bullet^\bullet = 0$ in $\mathcal{E} = \mathbb{C}$, corresponding to $\Lambda(t,a) = \alpha t \mathrm{I} + \xi \mathrm{Q}(t)$. The unital $\star$-algebra $\mathbb{C}$ with the usual multiplication $\zeta^\star \zeta = |\zeta|^2$ can be embedded into the two-dimensional Itô algebra $\mathfrak{a}$ of $a = (\alpha, \zeta)$, $\alpha = l(a)$, $\zeta \in \mathbb{C}$ as $a_\bullet^\bullet = \zeta$, $a_+^\bullet = +i\zeta$, $a_\bullet^- = -i\zeta$, $a_+^- = \zeta$. It corresponds to $\Lambda(t,a) = \alpha t \mathrm{I} + \zeta \mathrm{P}(t)$, where $\mathrm{P} = \Lambda_\bullet^\bullet + i\left(\Lambda_\bullet^+ - \Lambda_-^\bullet\right)$ is the representation of the standard Poisson process, compensated by its mean value t. Thus our results are applicable also to the classical stochastic differentials of completely positive processes, corresponding to the commutative Itô algebras, which are decomposable into the Wiener, Poisson and Newton orthogonal components.

1. Quantum filtering dynamics

The quantum filtering theory, which was outlined in [14, 15] and developed then since [16], provides the derivations for new types of irreversible stochastic equations for quantum states, giving the dynamical solution for the well-known quantum measurement problem. Some particular types of such equations have been considered recently in the phenomenological theories of quantum permanent reduction [17, 18], continuous measurement collapse [19, 20], spontaneous jumps [27, 21], diffusions and localizations [22, 23]. The main feature of such dynamics is that the reduced irreversible evolution can be described in terms of a linear dissipative stochastic wave equation, the solution to which is normalized only in the mean square sense.

The simplest dynamics of this kind is described by the continuous filtering wave propagators $V_t(\omega)$, defined on the space Ω of all Brownian trajectories as an adapted operator-valued stochastic process in the system Hilbert space $\mathcal{H}$, satisfying the stochastic diffusion equation

$$\mathrm{d}V_t + KV_t \mathrm{d}t = LV_t \mathrm{dQ}, \quad V_0 = I \tag{1.1}$$

in the Itô sense. Here $\mathrm{Q}(t,\omega)$ is the standard Wiener process, which is described by the independent increments $\mathrm{dQ}(t)=\mathrm{Q}(t+\mathrm{d}t)-\mathrm{Q}(t)$, having the zero mean values $\langle \mathrm{dQ}\rangle=0$ and the multiplication property $(\mathrm{dQ})^2=\mathrm{d}t$, K is an accretive operator, $K+K^{\dagger}\geq L^{*}L$, defined on a dense domain $\mathcal{D}\subseteq\mathcal{H}$, with $K^{\dagger}=K^{*}|\mathcal{D}$, and L is a linear operator $\mathcal{D}\to\mathcal{H}$. This stochastic wave equation was first derived [25] from a unitary cocycle evolution by a quantum filtering procedure. A sufficient analyticity condition, under which it has the unique solution in the form of a stochastic multiple integral even in the case of unbounded K and L is given in the Appendix. Using the Itô formula

$$\mathrm{d}(V_t^{*}V_t)=\mathrm{d}V_t^{*}V_t+V_t^{*}\mathrm{d}V_t+\mathrm{d}V_t^{*}\mathrm{d}V_t, \tag{1.2}$$

and averaging $\langle\cdot\rangle$ over the trajectories of Q, one obtains $\langle V_t^{*}V_t\rangle\leq I$ as a consequence of $\mathrm{d}\langle V_t^{*}V_t\rangle\leq 0$. Note that the process V_t is not necessarily unitary if the filtering condition $K^{\dagger}+K=L^{*}L$ holds, and even if $L^{\dagger}=-L$, it might be only isometric, $V_t^{*}V_t=I$, in the unbounded case.

Another type of the filtering wave propagator $V_t(\omega):\psi_0\in\mathcal{H}\mapsto\psi_t(\omega)$ in $\mathcal{H}$ is given by the stochastic jump equation

$$\mathrm{d}V_t+KV_t\mathrm{d}t=LV_t\mathrm{dP},\quad V_0=I \tag{1.3}$$

at the random time instants $\omega=\{t_1,t_2,...\}$. Here $L=J-I$ is the jump operator, corresponding to the stationary discontinuous evolutions $\psi_{t+}=J\psi$ at $t\in\omega$, and $\mathrm{P}(t,\omega)$ is the standard Poisson process, counting the number $|\omega\cap[0,t)|$ compensated by its mean value t. It is described as the process with independent increments $\mathrm{dP}(t)=\mathrm{P}(t+\mathrm{d}t)-\mathrm{P}(t)$, having the values $\{0,1\}$ at $\mathrm{d}t\to 0$, with zero mean $\langle\mathrm{dP}\rangle=0$, and the multiplication property $(\mathrm{dP})^2=\mathrm{dP}+\mathrm{d}t$. This stochastic wave equation was first derived in [24] by the conditioning with respect to the spontaneous reductions $J:\psi_t\mapsto\psi_{t+}$. An analyticity condition under which it has the unique solution in the form of the multiple stochastic integral even in the case of unbounded K and L is also given in the Appendix. Using the Itô formula (1.2) with $\mathrm{d}V_t^{*}\mathrm{d}V_t=V_t^{*}L^{*}LV_t(\mathrm{dP}+\mathrm{d}t)$, one can obtain

$$\mathrm{d}(V_t^{*}V_t)=V_t^{*}\left(L^{*}L-K-K^{\dagger}\right)V_t\mathrm{d}t+V_t^{*}\left(L^{\dagger}+L+L^{*}L\right)V_t\mathrm{dP}.$$

Averaging $\langle\cdot\rangle$ over the trajectories of P, one can easily find that $\mathrm{d}\langle V_t^{*}V_t\rangle\leq 0$ under the sub-filtering condition $L^{*}L\leq K+K^{\dagger}$. Such evolution is not needed to be unitary even if $L^{*}L=K+K^{\dagger}$, but it might be isometric, $V_t^{*}V_t=I$ if the jumps are isometric, $J^{*}J=I$.

This proves in both cases that the stochastic wave function $\psi_t(\omega)=V_t(\omega)\psi_0$ is not normalized for each ω, but it is normalized in the mean square sense to the survival probability $\langle||\psi_t||^2\rangle\leq||\psi_0||^2=1$ for the quantum system not to be demolished during its observation up to the time t. If $\left\langle||\psi_t||^2\right\rangle=1$, then the positive stochastic function $||\psi_t(\omega)||^2$ is the probability density of a diffusive $\widehat{\mathrm{Q}}$ or counting $\widehat{\mathrm{P}}$ output process up to the given t with respect to the standard Wiener Q or Poisson P input processes.

Using the Itô formula for $\phi_t(B)=V_t^{*}BV_t$, one can obtain the stochastic equations

$$\mathrm{d}\phi_t(B)+\phi_t(K^{*}B+BK-L^{*}BL)\,\mathrm{d}t=\phi_t(L^{*}B+BL)\,\mathrm{dQ}, \tag{1.4}$$

$$\mathrm{d}\phi_t(B)+\phi_t(K^{*}B+BK-L^{*}BL)\,\mathrm{d}t=\phi_t(J^{*}BJ-B)\,\mathrm{dP}, \tag{1.5}$$

describing the stochastic evolution $Y_t = \phi_t(B)$ of a bounded system operator $B \in \mathcal{L}(\mathcal{H})$ as $Y_t(\omega) = V_t(\omega)^* BV_t(\omega)$. The maps $\phi_t : B \mapsto Y_t$ are Hermitian in the sense that $Y_t^* = Y_t$ if $B^* = B$, but in contrast to the usual Hamiltonian dynamics, are not multiplicative in general, $\phi_t(B^*C) \neq \phi_t(B)^* \phi_t(C)$, even if they are not averaged with respect to ω. Moreover, they are usually not normalized, $R_t(\omega) := \phi_t(\omega, I) \neq I$, although the stochastic positive operators $R_t = V_t^* V_t$ under the filtering condition are usually normalized in the mean, $\langle R_t \rangle = I$, and satisfy the martingale property $\epsilon_t[R_s] = R_t$ for all $s > t$, where ϵ_t is the conditional expectation with respect to the history of the processes P or Q up to time t.

Although the filtering equations (1.3), (1.1) look very different, they can be unified in the form of the quantum stochastic equation

$$\mathrm{d}V_t + KV_t\mathrm{d}t + K^- V_t \mathrm{d}\Lambda_- = (J - I) V_t \mathrm{d}\Lambda + L_+ V_t \mathrm{d}\Lambda^+, \tag{1.6}$$

where $\Lambda^+(t)$ is the creation process, corresponding to the annihilation $\Lambda_-(t)$ on the interval $[0, t)$, and $\Lambda(t)$ is the number of quantums on this interval. These canonical quantum stochastic processes, representing the quantum noise with respect to the vacuum state $|0\rangle$ of the Fock space $\mathcal{F}$ over the single-quantum Hilbert space $L^2(\mathbb{R}_+)$ of square-integrable functions of $t \in [0, \infty)$, are formally given in [26] by the integrals

$$\Lambda_-(t) = \int_0^t \Lambda_-^r \mathrm{d}r, \quad \Lambda^+(t) = \int_0^t \Lambda_r^+ \mathrm{d}r, \quad \Lambda(t) = \int_0^t \Lambda_r^+ \Lambda_-^r \mathrm{d}r,$$

where Λ_-^r, Λ_r^+ are the generalized quantum one-dimensional fields in $\mathcal{F}$, satisfying the canonical commutation relations

$$\left[\Lambda_-^r, \Lambda_s^+\right] = \delta(s - r) I, \quad \left[\Lambda_-^r, \Lambda_-^s\right] = 0 = \left[\Lambda_r^+, \Lambda_s^+\right].$$

They can be defined by the independent increments with

$$\langle 0|\mathrm{d}\Lambda_-|0\rangle = 0, \quad \langle 0|\mathrm{d}\Lambda^+|0\rangle = 0, \quad \langle 0|\mathrm{d}\Lambda|0\rangle = 0 \tag{1.7}$$

and the noncommutative multiplication table

$$\mathrm{d}\Lambda\mathrm{d}\Lambda = \mathrm{d}\Lambda, \quad \mathrm{d}\Lambda_-\mathrm{d}\Lambda = \mathrm{d}\Lambda_-, \quad \mathrm{d}\Lambda\mathrm{d}\Lambda^+ = \mathrm{d}\Lambda^+, \quad \mathrm{d}\Lambda_-\mathrm{d}\Lambda^+ = \mathrm{d}tI \tag{1.8}$$

with all other products being zero: $\mathrm{d}\Lambda\mathrm{d}\Lambda_- = \mathrm{d}\Lambda^+\mathrm{d}\Lambda = \mathrm{d}\Lambda^+\mathrm{d}\Lambda_- = 0$. The standard Poisson process P as well as the Wiener process Q can be represented in $\mathfrak{F}$ by the linear combinations [8]

$$\mathrm{P}(t) = \Lambda(t) + i\left(\Lambda^+(t) - \Lambda_-(t)\right), \quad \mathrm{Q}(t) = \Lambda^+(t) + \Lambda_-(t), \tag{1.9}$$

so Eq. (1.6) corresponds to the stochastic diffusion equation (1.1) if $J = I$, $L_+ = L = -K^-$, and it corresponds to the stochastic jump equation (1.3) if $J = I + L$, $L_+ = iL = K^-$. The quantum stochastic equation for $\phi_t(B) = V_t^* BV_t$ has the following general form

$$\mathrm{d}\phi_t(B) + \phi_t\left(K^*B + BK - L^-BL_+\right)\mathrm{d}t = \phi_t(J^*BJ - B)\mathrm{d}\Lambda$$

$$+\phi_t(J^*BL_+ - K_+B)\mathrm{d}\Lambda^+ + \phi_t\left(L^-BJ - BK^-\right)\mathrm{d}\Lambda_-, \tag{1.10}$$

where $L^- = L_+^*, K_+^* = K^-$, coinciding with either (1.4) or with (1.5) in the particular cases. Equation (1.10) is obtained from (1.6) by using the Itô formula (1.2) with the multiplication table (1.8). The sub-filtering condition $K + K^\dagger \geq L^- L_+$ for Eq. (1.6) defines in both cases the positive operator-valued process $R_t = \phi_t(I)$

as a sub-martingale with $R_0 = I$, or a martingale in the case $K + K^\dagger = L^- L_+$. In the particular case

$$J = S, \quad K^- = L^- S, \quad L_+ = SK_+, \quad S^* S = I,$$

corresponding to the Hudson–Evans flow [2] if $S^* = S^{-1}$, the evolution is isometric, and identity preserving, $\phi_t(I) = I$, at least in the case of bounded K and L.

In the next sections we define a multidimensional analog of the quantum stochastic equation (1.10) and will show that the suggested general structure of its generator indeed follows just from the property of complete positivity of the map ϕ_t for all $t > 0$ and the normalization condition $\phi_t(I) = R_t$ to a form-valued sub-martingale with respect to the natural filtration of the quantum noise in the Fock space $\mathfrak{F}$.

2. Quantum completely positive flows

Throughout the complex pre-Hilbert space $\mathcal{D} \subseteq \mathcal{H}$ is a reflexive Fréchet space, $\mathcal{E} \otimes \mathcal{D}$ denotes the projective tensor product (π-product) with another such space $\mathcal{E}$, $\mathcal{D}' \supseteq \mathcal{H}$ denotes the dual space of continuous antilinear functionals $\eta' : \eta \in \mathcal{D} \mapsto \langle \eta | \eta' \rangle$, with respect to the canonical pairing $\langle \eta | \eta' \rangle$ given by $\|\eta\|^2$ for $\eta' = \eta \in \mathcal{H}$, $\mathcal{B}(\mathcal{D})$ denotes the linear space of all continuous sesquilinear forms $\langle \eta | B\eta \rangle$ on $\mathcal{D}$, identified with the continuous linear operators $B : \mathcal{D} \to \mathcal{D}'$ (kernels), $B^\dagger \in \mathcal{B}(\mathcal{D})$ is the Hermit conjugated form (kernel) $\langle \eta | B^\dagger \eta \rangle = \langle \eta | B\eta \rangle^*$, and $\mathcal{L}(\mathcal{D}) \subseteq \mathcal{B}(\mathcal{D})$ denotes the algebra of all strongly continuous operators $B : \mathcal{D} \to \mathcal{D}$. Any such space $\mathcal{D}$ can be considered as a projective limit with respect to an increasing sequence of Hilbertian norms $\|\cdot\|_p > \|\cdot\|$ on $\mathcal{D}$; for the definitions and properties of this standard topological notions see for example [28]. The space $\mathcal{D}'$ will be equipped with weak topology induced by its predual (= dual) $\mathcal{D}$, and $\mathcal{B}(\mathcal{D})$ will be equipped with w*-topology (induced by the predual $\mathcal{B}_*(\mathcal{D}) = \mathcal{D} \otimes \mathcal{D}$), coinciding with the weak topology on each bounded subset with respect to a norm $\|\cdot\|_p$. Any operator $A \in \mathcal{L}(\mathcal{D})$ with $A^\dagger \in \mathcal{L}(\mathcal{D})$ can be uniquely extended to a weakly continuous operator onto $\mathcal{D}'$ as $A^{\dagger *}$, denoted again as A, where A^* is the dual operator $\mathcal{D}' \to \mathcal{D}'$, $\langle \eta | A^* \eta' \rangle = \langle A\eta | \eta' \rangle$, defining the involution $A \mapsto A^*$ for such continuations $A : \mathcal{D}' \to \mathcal{D}'$. We say that the operator A commutes with a sesquilinear form, $BA = AB$ if $\langle \eta | BA\eta \rangle = \left\langle A^\dagger \eta | B\eta \right\rangle$ for all $\eta \in \mathcal{D}$. The commutant $\mathcal{A}^c = \{B \in \mathcal{B}(\mathcal{D}) : [A, B] = 0, \forall A \in \mathcal{A}\}$ of an operator $*$-algebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{D})$ is weakly closed in $\mathcal{B}(\mathcal{D})$, so that the weak closure $\overline{\mathcal{B}} \subseteq \mathcal{B}(\mathcal{D})$ of any $\mathcal{B} \subseteq \mathcal{A}^c$ also commutes with $\mathcal{A}$.

1. Let $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ be a unital $*$-algebra of bounded operators $B : \mathcal{H} \to \mathcal{H}$, $\|B\| < \infty$, and $(\Omega, \mathfrak{A}, P)$ be a probability space with a filtration $(\mathfrak{A}_t)_{t>0}$, $\mathfrak{A}_t \subseteq \mathfrak{A}$ of σ-algebras on Ω. One can assume that the filtration $\mathfrak{A}_t \subseteq \mathfrak{A}_s, \forall t < s$ is generated by $x_t = \{r \mapsto x(r) : r < t\}$ of a stochastic process $x(t, \omega)$ with independent increments $\mathrm{d}x(t) = x(t + \Delta) - x(t)$, and the probability measure P is invariant under the measurable representations $\omega \mapsto \omega_s \in \Omega$, $A_s^{-1} = \{\omega : \omega_s \in A\} \in \mathfrak{A}$, $\forall A \in \mathfrak{A}$ on $\Omega \ni \omega$ of the time shifts $t \mapsto t + s, s > 0$, corresponding to the shifts of the random increments

$$\mathrm{d}x(t, \omega_s) = \mathrm{d}x(t + s, \omega), \quad \forall \omega \in \Omega, t \in \mathbb{R}_+.$$

The *filtering dynamics* over $\mathcal{B}$ with respect to the process $x(t)$ is described by a cocycle flow $\phi = (\phi_t)_{t>0}$ of linear completely positive [4] w*-continuous stochastic adapted maps $\phi_t(\omega) : \mathcal{B} \to \overline{\mathcal{B}}$, $\omega \in \Omega$ such that the stochastic process $y_t(\omega) =$

$\langle\eta|\phi_t(\omega,B)\eta\rangle$ is causally measurable for each $\eta\in\mathcal{D}$, $B\in\mathcal{B}$ in the sense that $y_t^{-1}(B)\in\mathfrak{A}_t$, $\forall t>0$ and any Borel $B\subseteq\mathbb{C}$. The maps ϕ_t can be extended on the $\mathfrak{A}$-measurable functions $Y:\omega\mapsto Y(\omega)$ with values $Y(\omega)\in\overline{\mathcal{B}}$ as the normal maps $\phi_t[Y](\omega)=\overline{\phi}_t(\omega,Y(\omega_t))$, defined for each $\omega\in\Omega$ by the normal extension $\overline{\phi}_t(\omega)$ onto $\overline{\mathcal{B}}$, and the cocycle condition $\phi_r(\omega)\circ\phi_s(\omega_r)=\phi_{r+s}(\omega)$, $\forall r,s>0$ reads as the semigroup condition $\phi_r[\phi_s[Y]]=\phi_{r+s}[Y]$ of the extended maps. As it was noted in the previous section, the maps $\phi_t(\omega)$ are not considered to be normalized to the identity, and can be even unbounded, but they are supposed to be normalized, $\phi_t(\omega,I)=R_t(\omega)$, to an operator-valued martingale $R_t=\epsilon_t[R_s]\geq 0$ with $R_0(\omega)=I$, or to a positive submartingale, $R_t\geq\epsilon_t[R_s],\forall s>t$ in the subfiltering case, where ϵ_t is the conditional expectation over ω with respect to $\mathfrak{A}_t$.

2. Now we give a noncommutative generalization of the filtering (subfiltering) CP flows for an arbitrary Itô algebra, which was suggested in [32] for a Gaussian Itô algebra of finite dimensional quantum thermal noise, and in [9] for the simple quantum Itô algebra $\mathcal{Q}(\mathbb{C}^d)$ even in the nonlinear case.

The role of the classical process $x(t)$ will play the quantum stochastic process

$$X(t)=A\otimes I+I\otimes\Lambda(t,a),\quad A\in\mathcal{A},a\in\mathfrak{a}$$

indexed by an operator algebra $\mathcal{A}\subset\mathcal{L}(\mathcal{D})$ and a noncommutative Itô algebra $\mathfrak{a}$. Here $\Lambda(t,a)$ is the process with independent increment on a dense subspace $\mathfrak{F}\subset\Gamma(\mathfrak{E})$ of the Fock space $\Gamma(\mathfrak{E})$ over the space $\mathfrak{E}=L^2_{\mathcal{E}}(\mathbb{R}_+)$ of all square-norm integrable $\mathcal{E}$-valued functions on $\mathbb{R}_+$, where $\mathcal{E}$ is a pre-Hilbert space of the representation $a\in\mathfrak{a}\mapsto(a^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ for the Itô $\star$-algebra $\mathfrak{a}$. Assuming that $\mathcal{E}$ is a Fréchet space, given by an increasing sequence of Hilbertian norms $\|e^\bullet\|(\xi)>\|e^\bullet\|$, $\xi\in\mathbb{N}$, we define $\mathfrak{F}$ as the projective limit $\cap_\xi\Gamma(\mathfrak{E},\xi)$ of the Fock spaces $\Gamma(\mathfrak{E},\xi)\subseteq\Gamma(\mathfrak{E})$, generated by coherent vectors $f^\otimes$, with respect to the norms

(2.1)
$$\left\|f^\otimes\right\|^2(\xi)=\int_\Gamma\left\|f^\otimes(\tau)\right\|^2(\xi)\,\mathrm{d}\tau:=\sum_{n=0}^\infty\frac{1}{n!}\left(\int_0^\infty\|f^\bullet(t)\|^2(\xi)\,\mathrm{d}t\right)^n=e^{\|f^\bullet\|^2(\xi)}.$$

Here $f^\otimes(\tau)=\bigotimes_{t\in\tau}f^\bullet(t)$ for each $f^\bullet\in\mathfrak{E}$ is represented by tensor-functions on the space Γ of all finite subsets $\tau=\{t_1,...,t_n\}\subseteq\mathbb{R}_+$ (for a simple example of the Fock scale see the Appendix.) Moreover, we shall assume that the Itô algebra $\mathfrak{a}$ is realized as a $\star$-subalgebra of Hudson-Parthasarathy (HP) algebra $\mathcal{Q}(\mathcal{E})$ of all quadruples $\boldsymbol{a}=(a^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ with $a^\bullet_\bullet\in\mathcal{L}(\mathcal{E})$, strongly representing the $\star$-semigroup $1+\mathfrak{a}$ on the Fréchet space $\mathcal{E}$ by projective contractions $\delta^\bullet_\bullet+a^\bullet_\bullet\in\mathcal{L}(\mathcal{E})$ in the sense that for each $\zeta\in\mathbb{N}$ there exists ξ such that $\|e^\bullet+a^\bullet_\bullet e^\bullet\|(\zeta)\leq\|e^\bullet\|(\xi)$ for all $e^\bullet\in\mathcal{E}$. The following theorem proves that these are natural assumptions (which are not restrictive in the simple Fock scale for a finite dimensional $\mathfrak{a}$.)

Proposition 2.1. *The exponential operators $W(t,a)=:\exp[\Lambda(t,a)]:$ defined as the solutions to the quantum Itô equation*

(2.2) $$\mathrm{d}W_t(g)=W_t(g)\,\mathrm{d}\Lambda(t,g(t)),\quad W_0(g)=I,g(t)\in\mathfrak{a}$$

with $g(t)=a$, are strongly continuous, $W(t,a)\in\mathcal{L}(\mathfrak{F})$, if all $\widehat{a}^\bullet_\bullet=\delta^\bullet_\bullet+a^\bullet_\bullet$ are projective contractions on $\mathcal{E}$. They give an analytic representation

(2.3) $$W(t,a\star a)=W(t,a)^\dagger W(t,a),\quad W(t,0)=I,\quad W(t,d)=e^tI$$

of the unital $\star$-semigroup $1+\mathfrak{a}$ for the Itô $\star$-algebra $\mathfrak{a}$ with respect to the $\star$-product $a\star a=a+a^\star a+a^\star$.

Proof. The solutions $W(t,a)$ are uniquely defined on the coherent vectors as analytic functions
(2.4)
$$W(t,a)\, f^{\otimes}(\tau) = \otimes_{r\in\tau}^{r<t} \left(\widehat{a}_{\bullet}^{\bullet} f^{\bullet}(r) + a_{+}^{\bullet}\right) \exp\left[\int_0^t \left(a_{\bullet}^{-} f^{\bullet}(r) + a_{+}^{-}\right) \mathrm{d}r\right] \otimes_{r\in\tau}^{r\geq t} f^{\bullet}(r),$$

which obey the properties (2.3), see for example [5]. Thus the span of coherent vectors is invariant, and it is also invariant under $W(t,a)^{\dagger} = W(t,a^{\star})$. They can be extended on $\mathfrak{F}$ by continuity which follows from the continuity of Wick exponentials $\otimes\widehat{a}_{\bullet}^{\bullet}$ for the projective contractions $\widehat{a}_{\bullet}^{\bullet} \in \mathcal{E}$, and boundedness of $a_{+}^{\bullet} \in \mathcal{E}$, $a_{\bullet}^{-} \in \mathcal{E}'$. □

3. Let $\mathfrak{D}$ denote the Fréchet space $\mathcal{D} \otimes \mathfrak{F}$, generated by $\psi = \eta \otimes f^{\otimes}$, $\eta \in \mathcal{D}$, $f^{\bullet} \in \mathfrak{E}$. Assuming for simplicity the separability of the Itô algebra in the sense $\mathcal{E} \subseteq \ell^2$ such that $f^{\bullet} = (f^m)^{m\in\mathbb{N}}$, one can identify each $\psi' \in \mathfrak{D}'$ with a sequence of $\mathcal{D}'$-valued symmetric tensor-functions $\psi'_{m_1,\ldots,m_n}(t_1, \ldots t_n)$, $n = 0, 1, 2, \ldots$. Let $(\mathfrak{D}_t)_{t>0}$ be the natural filtration and $\left(\mathfrak{D}_{[t}\right)_{t>0}$ be the backward filtration of the subspaces $\mathfrak{D}_t = \mathcal{D}\otimes\mathfrak{F}_t$, $\mathfrak{D}_{[t} = \mathcal{D}\otimes\mathfrak{F}_{[t}$ generated by $\eta\otimes f^{\otimes}$ with $f^{\bullet} \in \mathfrak{E}_t$ and $f^{\bullet} \in \mathfrak{E}_{[t}$ respectively, where $\mathfrak{E}_t = L^2_{\mathcal{E}}[0,t)$, $\mathfrak{E}_{[t} = L^2_{\mathcal{E}}[t,\infty)$ are embedded into $\mathfrak{E}$. The spaces $\mathfrak{D}_t$, $\mathfrak{D}_{[t}$ of the restrictions $E_t\psi = \psi|\Gamma_t$, $E_{[t}\psi = \psi|\Gamma_{[t}$ onto $\Gamma_t = \{\tau_t = \tau \cap [0,t)\}$, $\Gamma_{[t} = \left\{\tau_{[t} = \tau \cap [t,\infty)\right\}$ are embedded into $\mathfrak{D}$ by the isometries $E_t^{\dagger} : \psi \mapsto \psi_t$, $E_{[t}^{\dagger} : \psi \mapsto \psi_{[t}$ as $\psi_t(\tau) = \psi(\tau_t)\,\delta_{\emptyset}\left(\tau_{[t}\right)$, $\psi_{[t}(\tau) = \delta_{\emptyset}(\tau_t)\,\psi\left(\tau_{[t}\right)$, where $\delta_{\emptyset}(\tau) = 1$ if $\tau = \emptyset$, otherwise $\delta_{\emptyset}(\tau) = 0$. The projectors $E_t, E_{[t}$ onto $\mathfrak{D}_t, \mathfrak{D}^t$ are extended onto $\mathfrak{D}'$ as the adjoints to $E_t^{\dagger}$, $E_{[t}^{\dagger}$. The time shift on $\mathfrak{D}'$ is defined by the semigroup $(T^t)_{t>0}$ of adjoint operators $T^t = T_t^*$ to $T_t\psi(\tau) = \psi(\tau + t)$, where $\tau{+}t = \{t_1 + t, ..., t_n + t\}$, $\emptyset{+}t = \emptyset$, such that $T^t\psi(\tau) = \delta_{\emptyset}(\tau_t)\,\psi\left(\tau_{[t} - t\right)$ are isometries for $\psi \in \mathfrak{D}$ onto $\mathfrak{D}_{[t}$. A family $(Z_t)_{t>0}$ of sesquilinear forms $\langle\psi|Z_t\psi\rangle$ given by linear operators $Z_t : \mathfrak{D} \to \mathfrak{D}'$ is called *adapted* (and $(Z^t)_{t>0}$ is called *backward adapted*) if

$$Z_t\left(\eta \otimes f^{\otimes}\right) = \psi' \otimes E_{[t} f^{\otimes} \quad \left(Z^t\left(\eta \otimes f^{\otimes}\right) = \psi' \otimes E_t f^{\otimes}\right), \quad \forall \eta \in \mathcal{D}, f^{\bullet} \in \mathfrak{E}, \tag{2.5}$$

where $\psi' \in \mathfrak{D}_t'$ $(\mathfrak{D}_{[t}')$ and $E_{[t}$ (E_t) are the projectors onto $\mathfrak{F}_{[t}$ $(\mathfrak{F}_t)$ correspondingly.

The (vacuum) *conditional expectation* on $\mathcal{B}(\mathfrak{D})$ with respect to the past up to a time $t \in \mathbb{R}_+$ is defined as a positive projector, $\epsilon_t(Z) \geq 0$, if $Z \geq 0$, $\epsilon_t = \epsilon_t \circ \epsilon_s, \forall s > t$, giving an adapted sesquilinear form $Z_t = \epsilon_t(Z)$ in (2.5) for each $Z \in \mathcal{B}(\mathfrak{D})$ by $\psi' = E_t Z E_t^{\dagger}\psi$, where $\psi = \eta \otimes E_t f^{\otimes}$. The time shift $\left(\theta^t\right)_{t>0}$ on $\mathcal{B}(\mathfrak{D})$ is uniquely defined by the covariance condition $\theta^t(Z)\,T^t = T^t Z$ as a backward adapted family $Z^t = \theta^t(Z), t > 0$ for each $Z \in \mathcal{B}(\mathfrak{D})$. As in the bounded case [7] between the maps ϵ_t and θ^t we have the relation $\theta^r \circ \epsilon_s = \epsilon_{r+s} \circ \theta^r$ which follows from the operator relation $T^r E_s = E_{r+s} T^r$. An adapted family $(M_t)_{t>0}$ of positive $\langle\psi|M_t\psi\rangle \geq 0, \forall\psi \in \mathfrak{D}$ Hermitian $M_t^{\dagger} = M_t$ forms $M_t \in \mathcal{B}(\mathfrak{D})$ is called *martingale* (*submartingale*) if $\epsilon_t(M_s) = M_t$ $(\epsilon_t(M_s) \leq M_t)$ for all $s \geq t \geq 0$. The bounded operator-valued martingales M_t were introduced in the case of the finite-dimensional HP-algebra in [29].

4. Let $\mathfrak{B}$ denote the space of all $Y \in \mathcal{B}(\mathfrak{D})$, commuting with all $X = \{X(t)\}$ in the sense

$$AY = YA, \quad \forall A \in \mathcal{A}, \quad YW(t,a) = W(t,a)\,Y, \quad \forall t > 0, a \in \mathfrak{a},$$

where $A(\eta\otimes\varphi)=A\eta\otimes\varphi$, $W(\eta\otimes\varphi)=\eta\otimes W\varphi$, and the unital $*$-algebra $\mathcal{B}\subseteq\mathcal{L}(\mathcal{H})$ be weakly dense in the commutant $\mathcal{A}^c$. The quantum filtration $(\mathfrak{B}_t)_{t>0}$ is defined as the increasing family of subspaces $\mathfrak{B}_t\subseteq\mathfrak{B}_s, t\leq s$ of the adapted sesquilinear forms $Y_t\in\mathfrak{B}$. The covariant shifts $\theta^t:Y\mapsto Y^t$ leave the space $\mathfrak{B}$ invariant, mapping it onto the subspaces of backward adapted sesquilinear forms $Y^t=\theta^t(Y)$.

The *quantum stochastic positive flow* over $\mathcal{B}$ is described by a one parameter family $\phi=(\phi_t)_{t>0}$ of linear w*-continuous maps $\phi_t:\mathcal{B}\to\mathfrak{B}$ satisfying

1. the causality condition $\phi_t(B)\subseteq\mathfrak{B}_t,\quad\forall B\in\mathcal{B},t\in\mathbb{R}_+$,
2. the complete positivity condition $[\phi_t(B_{kl})]\geq 0$ for each $t>0$ and for any positive definite matrix $[B_{kl}]\geq 0$ with $B_{kl}\in\mathcal{B}$,
3. the cocycle condition $\phi_r\circ\phi_s^r=\phi_{r+s},\forall t,s>0$ with respect to the covariant shift $\phi_s^r=\theta^r\circ\phi_s$.

Here the composition $\circ$ is understood as $\phi_r[\phi_s(B)]=\phi_{r+s}(B)$ in terms of the linear normal extensions of $\phi_t[B\otimes Z]=\overline{\phi}_t(B)Z^t$ to the CP maps $\mathfrak{B}\to\mathfrak{B}$, forming a one-parameter semigroup, where $B\in\overline{\mathcal{B}}$, $\overline{\phi}_t$ are the normal extensions of ϕ_t onto $\overline{\mathcal{B}}$, and $Z^t=\theta^t(Z)$, $Z\in\mathcal{B}(\mathfrak{F})$. These can be defined like in the classical case as $\phi_t[Y](\bar{f}^\bullet,f^\bullet)=\overline{\phi}_t\left(\bar{f}^\bullet,Y\left(\bar{f}_t^\bullet,f_t^\bullet\right),f^\bullet\right)$ with $f_t^\bullet(r)=f^\bullet(t+r)$ by the coherent matrix elements $Y(\bar{f}^\bullet,f^\bullet)=F^*YF$ for $Y\in\mathfrak{B}$ given by the continuous operators $F:\eta\mapsto\psi_f=\eta\otimes f^\otimes$, $\eta\in\mathcal{D}$ for each $f^\bullet\in\mathfrak{E}_t$ with the adjoints $F^*\psi'=\int_{\tau<t}f^\otimes(\tau)^*\psi'(\tau)\,\mathrm{d}\tau$ for $\psi'\in\mathfrak{D}'$.

The flow is called *(sub)-filtering*, if $R_t=\phi_t(I)$ is a (sub)-martingale with $R_0=I$, and is called contractive, if $I\geq R_t\geq R_s$ for all $0\leq t\leq s\in\mathbb{R}_+$.

Proposition 2.2. *The complete positivity for adapted linear maps $\phi_t:\mathcal{B}\to\mathcal{B}(\mathfrak{D})$ can be written as*

(2.6)
$$\sum_{f,h\in\mathfrak{E}_t}\sum_{B,C\in\mathcal{B}}\langle\xi_B^f|\phi_t\left(\bar{f}^\bullet,B^*C,h^\bullet\right)\xi_C^h\rangle:=\langle\eta^k|\phi_t\left(\bar{f}_k^\bullet,B_k^*B_l,h_l^\bullet\right)\eta^l\rangle\geq 0,\quad\forall t>0$$

(the usual summation rule over repeated cross-level indices is understood), where $\xi_B^f=\eta^k$ if $f^\bullet=f_k^\bullet$ and $B=B_k$ with $f_k^\bullet\in\mathfrak{E}_t,B_k\in\mathcal{B}$, $k=1,2,...$, otherwise $\xi_B^f=0$, and $\phi_t(B,f^\bullet)=\phi_t(B)F$, $\phi_t(\bar{f}^\bullet,B)=F^*\phi_t(B)$.

Proof. By definition the map ϕ into the sesquilinear forms is completely positive on $\mathcal{B}$ if $\left\langle\psi^k|\phi(B_{kl})\psi^l\right\rangle\geq 0$ whenever $\langle\eta^k|B_{kl}\eta^l\rangle\geq 0$, where η^k,ψ^k are arbitrary finite sequences. Approximating from below the latter positive forms by sums of the forms $\sum_{kl}\langle\eta^k|B_{ik}^*B_{il}\eta^l\rangle\geq 0$, the complete positivity can be tested only for the forms $\sum_{kl}\langle\eta^k|B_k^*B_l\eta^l\rangle\geq 0$ due to the additivity $\phi(\sum_i B_{ik}^*B_{il})=\sum_i\phi(B_{ik}^*B_{il})$. If ϕ_t is adapted, this can be written as

$$\sum_{B,C\in\mathcal{B}}\langle\chi_B|\phi(B^*C)\chi_C\rangle=\left\langle\psi^k|\phi(B_k^*B_l)\psi^l\right\rangle:=\sum_{k,l}\left\langle\psi^k|\phi(B_k^*B_l)\psi^l\right\rangle\geq 0,$$

where $\chi_B=\psi^k\in\mathfrak{D}_t$ if $B=B_k\in\mathcal{B}$, otherwise $\chi_B=0$. Because any $\psi\in\mathfrak{D}_t$ can be approximated by a $\mathcal{D}$-span $\sum_f\eta^f\otimes f^\otimes$of coherent vectors over $f_k^\bullet\in\mathfrak{E}_t$, it is sufficient to define the CP property only for such spans as

$$0\leq\sum_{f,h}\sum_{B,C}\left\langle\xi_B^f\otimes f^\otimes|\phi(B^*C)\left(\xi_C^h\otimes h^\otimes\right)\right\rangle=\sum_{f,h}\sum_{B,C}\left\langle\xi_B^f|\phi\left(\bar{f}^\bullet,B^*C,h^\bullet\right)\xi_C^h\right\rangle.$$

□

5. Note that the subfiltering (filtering) flows can be considered as quantum stochastic CP dilations of the quantum sub-Markov (Markov) semigroups $\theta = (\theta_t)_{t>0}$, $\theta_r \circ \theta_s = \theta_{r+s}$ in the sense $\theta_t = \epsilon \circ \phi_t$, where $\epsilon(Y)\eta = EY\psi_0$, $E\psi' = \psi'(\emptyset), \forall \psi' \in \mathfrak{D}'$, with $\theta_s(I) \leq \theta_t(I) \leq I$ $(\theta_t(I) = I)$, $\forall t \leq s$. The contraction $C_t = \theta_t(I)$ with $C_0 = I$ defines the probability $\langle \eta | C_t \eta \rangle \leq 1$, $\forall \eta \in \mathcal{H}, \|\eta\| = 1$ for an unstable system not to be demolished by a time $t \in \mathbb{R}_+$, and the conditional expectations $\langle \eta | A C_t \eta \rangle / \langle \eta | C_t \eta \rangle$ of the initial nondemolition observables $A \in \mathcal{A}$ in any state $\eta \in \mathcal{D}$, and thus in any initial state $\psi_0 \in \eta \otimes \delta_\emptyset$. The following theorem shows that the submartingale (or the contraction) $R_t = \phi_t(I)$ is also the density operator with respect to $\psi_0 = \eta \otimes \delta_\emptyset$, $\eta \in \mathcal{H}$ (or with respect to any $\psi \in \mathcal{H} \otimes \mathfrak{F}$) for the conditional state of the restricted nondemolition process $X_t = \{r \mapsto X(r) : r < t\}$.

Theorem 2.3. *Let $t \mapsto R_t \in \mathfrak{B}_t$ be a positive (sub)-martingale and $(\mathfrak{g}_t)_{t>0}$ be the increasing family of $\star$-semigroups $\mathfrak{g}_t$ of step functions $g : \mathbb{R}_+ \to \mathfrak{a}$, $g(s) = 0$, $\forall s \geq t$ under the $\star$-product*

$$(g_k \star g_l)(t) = g_l(t) + g_k(t)^\star g_l(t) + g_k(t)^\star \tag{2.7}$$

of $g_k^\star = g_k \star 0$ and $g_l = 0 \star g_l$. The generating function $\vartheta_t(g) = \epsilon[R_t W_t(g)]$ of the output state for the process $\Lambda(t)$, defined for any $g \in \mathfrak{g}_t$ and each $t > 0$ as

$$\langle \eta | \vartheta_t(g) \eta \rangle = \langle \psi_0 | R_t W_t(g) \psi_0 \rangle, \quad \psi_0 = \eta \otimes \delta_\emptyset, \tag{2.8}$$

is $\mathcal{B}^c$-valued, positive, $\vartheta_t \geq 0$ in the sense of positive definiteness of the kernel

$$\left\langle \eta^k | \vartheta_t(g_k \star g_l) \eta^l \right\rangle \geq 0, \quad \forall g_k \in \mathfrak{g}_t; \eta^k \in \mathcal{D}, \tag{2.9}$$

and $\vartheta_t \geq \vartheta_s|\mathfrak{g}_t$ in this sense for any $s \geq t$. If $R_0 = I$, then $\vartheta_0(0) = I \geq \vartheta_t(0)$, and if R_t is a martingale, then $\vartheta_t = \vartheta_s|\mathfrak{g}_t$ for any $s \geq t$, and $\vartheta_t(0) = I$ for all $t \in \mathbb{R}_+$. Any family $\vartheta = (\vartheta_t)_{t\geq 0}$ of positive-definite functions $\vartheta_t : \mathfrak{g}_t \to \mathcal{B}^c$, satisfying the above consistency and normalization properties, is the state generating function of the form (2.8) iff it is absolutely continuous in the following sense

$$\lim_{n\to\infty} \sum_{g \in \mathfrak{g}_t} \eta_n^g \otimes g_+^\otimes = 0 \Rightarrow \lim_{n\to\infty} \sum_{g,h \in \mathfrak{g}_t} \left\langle \eta_n^g | \vartheta_t(g \star h) \eta_n^h \right\rangle = 0, \tag{2.10}$$

where $g_+^\otimes(\tau) = \otimes_{t\in\tau} g_+^\bullet(t)$ and $\eta_n^g = 0$ for almost all g (i.e. except for a finite number of $g \in \mathfrak{g}_t$).

Proof. Because the solutions $W_t(g)$ to the quantum stochastic equation (2.2) for a step function g are given by finite products of commuting exponential operators $W(t,a)$, they are multiplicative, $W_t(g_k)^* W_t(g_l) = W_t(g_k \star g_l)$, as the operators in (2.4) are. Then the positive definiteness of ϑ_t follows from their commutativity (2.7) with positive R_t:

$$\left\langle \eta^k | \vartheta_t(g_k \star g_l) \eta^l \right\rangle = \left\langle W_t(g_k) \left(\eta^k \otimes \delta_\emptyset\right) | R_t W_t(g_l) \left(\eta^l \otimes \delta_\emptyset\right) \right\rangle = \langle \psi_t | R_t \psi_t \rangle \geq 0.$$

It is $\mathcal{B}^c$-valued as

$$\langle \eta | \vartheta_t(g) B\eta \rangle = \langle \psi_0 | R_t W_t(g) B\psi_0 \rangle = \langle B\psi_0 | R_t W_t(g) \psi_0 \rangle = \langle B\eta | \vartheta_t(g) \eta \rangle, \forall B \in \mathcal{B}.$$

From $W_t(g_r) = W_r(g)$, $r < t$ and $W_t(0) = I$ as the case $g_0 = 0$ it follows that

$$\vartheta_t(g_r) = \langle \psi_0 | R_t W_t(g_r) \psi_0 \rangle = \langle \psi_0 | \epsilon_r(R_t) W_r(g) \psi_0 \rangle \leq \langle \psi_0 | R_r W_r(g) \psi_0 \rangle = \vartheta_r(g)$$

for any finite matrix $g = [g_k \star g_l]$, and $\vartheta_t(0) \leq 1 = \vartheta_0(0)$ if R_t is a submartingale with $R_0 = I$. This implies the normalization and compatibility conditions if R_t

is martingale. The continuity condition follows from the continuity of the forms $R_t \in \mathcal{B}(\mathfrak{D})$: if $\sum_g \left(\eta_n^g \otimes g_+^{\otimes}\right) \to 0$, then

$$\begin{aligned}\sum_{g,h} \left\langle \eta_n^g | \vartheta_t (g \star h) \eta_n^h \right\rangle &= \sum_{g,h} \left\langle W_t(g)(\eta_n^g \otimes \delta_\emptyset) | R_t W_t(h)\left(\eta_n^h \otimes \delta_\emptyset\right) \right\rangle \\ &= \left\langle \sum_g \eta_n^g \otimes g_+^{\otimes} | R_t \sum_g \eta_n^g \otimes g_+^{\otimes} \right\rangle \longrightarrow 0.\end{aligned}$$

Conversely, let $(\mathcal{E}, V_t, L)$ be the GNS triple, describing the decomposition $\vartheta_t(g) = L^* V_t(g) L$ for a positive-definite kernel-function ϑ_t. It is defined by the multiplicative $*$-representation $V_t(g \star h) = V_t(g)^* V_t(h)$ of $\mathfrak{g}_t$ on a pre-Hilbert space $\mathcal{E} \subseteq \mathcal{H}$ and by a linear operator $L : \mathcal{D} \to \mathcal{E}$. The correspondence $\pi_t(B) : V_t(g) L\eta \mapsto V_t(g) LB\eta$ for $B \in \mathcal{B}$ is extended to a $*$-representation $\pi_t : \mathcal{B} \to V_t(\mathfrak{g}_t)'$ on the linear combinations $\mathcal{E}_t^\circ = \{\sum_k V_t(g_k) L_t \eta_k : g_k \in \mathfrak{g}_t, \eta_k \in \mathcal{D}\}$ by virtue of the commutativity of $\vartheta_t(g)$ with $\mathcal{B}$:

$$\begin{aligned}\left\langle B^* \eta^g | \vartheta_t(g \star h) \eta^h \right\rangle &= \left\langle \pi_t(B^*) V_t(g) L_t \eta^g | V_t(h) L_t \eta^h \right\rangle \\ &= \left\langle V_t(g) L_t \eta^g | \pi_t(B) V_t(h) L_t \eta^h \right\rangle = \left\langle \eta^g | \vartheta_t(g \star h) B \eta^h \right\rangle.\end{aligned}$$

The linear correspondence $F_t : \sum_g \eta^g \otimes g_+^{\otimes} \mapsto \sum_g V_t(g) L\eta^g$ obviously intertwines this representation with $B \mapsto B \otimes I$ as well as the representation V_t with W_t on $\mathfrak{D}_t^\circ = \{\sum_k \eta_k \otimes W_t(g_k) \delta_\emptyset : g_k \in \mathfrak{g}_t, \eta_k \in \mathcal{D}\} \subseteq \mathfrak{D}_t$:

$$F_t\left(\eta \otimes W_t(f^\star) g_+^{\otimes}\right) = V_t(f \star g) L\eta = V_t(f^\star) V_t(g) L\eta = V_t(f^\star) F_t\left(\eta \otimes g_+^{\otimes}\right),$$

where $g_+^{\otimes} = W(g)\delta_\emptyset$. It is continuous operator with respect to the Hilbert space norm in $\mathcal{H}_t$ because if $\sum_g \eta_n^g \otimes g_+^{\otimes} \to 0$, then

$$\left\| F_t \sum_g \eta_n^g \otimes g_+^{\otimes} \right\|^2 = \left\| \sum_g V_t(g) L\eta_n^g \right\|^2 = \sum_{f,h} \left\langle \eta_n^f | \vartheta_t(f \star h) \eta_n^h \right\rangle \to 0$$

due to the strong absolute continuity of ϑ_t. Hence F_t can be continued to an intertwining operator $\mathfrak{D}_t^\circ \to \mathcal{H}$, and there exists the adjoint intertwining operator $F_t^* : \mathcal{E} \to \mathfrak{D}_t'$, $F_t^* V_t(g) = W_t(g) F_t^*$ such that

$$\langle \psi_0 | F_t^* F_t W_t(g) \psi_0 \rangle = \langle F_t \psi_0 | V_t(g) F_t \psi_0 \rangle = \langle L_t \eta | V_t(g) L_t \eta \rangle = \langle \eta | L_t^* V_t(g) L_t \eta \rangle.$$

The positive operators $F_t^* F_t \in \mathcal{B}(\mathfrak{D}_t)$ uniquely extended to the adapted ones $R_t : \mathfrak{D} \to \mathfrak{D}'$, commute with all $W_t(g), g \in \mathfrak{g}_t$. They define a submartingale (martingale) $R_t \in \mathfrak{B}_t$ due to the property $W_t = W_s | \mathfrak{g}_t$ for all $s \geq t$ and

$$\begin{aligned}&\left\langle W_t(g_k) \psi_0^k | \epsilon_t(R_s) W_t(g_l) \psi_0^l \right\rangle = \left\langle \psi_0^k | R_s W_s(g_k \star g_l) \psi_0^l \right\rangle = \left\langle \eta^k | \vartheta_s(g_k \star g_l) \eta^l \right\rangle \\ &\leq (=) \left\langle \eta^k | \vartheta_t(g_k \star g_l) \eta^l \right\rangle = \left\langle \psi_0^k | R_t W_t(g_k \star g_l) \psi_0^l \right\rangle = \left\langle W_t(g_k) \psi_0^k | R_t W_t(g_l) \psi_0^l \right\rangle\end{aligned}$$

if $\vartheta_s | \mathfrak{g}_t \leq (=) \vartheta_t$. It is normalized, $R_0 = F_0^* F_0 = I$, as $F_0 = I$ if $\vartheta_0(0) = I$. □

3. Generators of quantum CP dynamics

The quantum stochastically differentiable positive flow ϕ is defined as a weakly continuous function $t \mapsto \phi_t$ with CP values $\phi_t : \mathcal{B} \to \mathfrak{B}_t$, $\phi_0(B) = B \otimes I, \forall B \in \mathcal{B}$ such that for any product-vector $\psi_f = \eta \otimes f^{\otimes}$ given by $\eta \in \mathcal{D}$ and $f^\bullet \in \mathfrak{E}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \psi_f | \phi_t(B) \psi_f \right\rangle = \left\langle \psi_f | \phi_t\left(\lambda\left(\bar{f}^\bullet(t), B, f^\bullet(t)\right)\right) \psi_f \right\rangle, \qquad B \in \mathcal{B}, \tag{3.1}$$

where $\lambda(\bar{e}^\bullet, B, e^\bullet) = \lambda(B) + e_\bullet \lambda^\bullet(B) + \lambda_\bullet(B) e^\bullet + e_\bullet \lambda_\bullet^\bullet(B) e^\bullet$, $e_\bullet = \bar{e}^\bullet$ is the linear form on $\mathcal{E}$ with $e_\bullet^* = e^\bullet \in \mathcal{E}$ and $\langle \psi_f | \phi_0(B) \psi_f \rangle = \langle \eta | B\eta \rangle \exp \|f^\bullet\|^2$. The generator $\lambda(B) = \lambda(0, B, 0)$ of the quantum dynamical semigroup $\theta_t = \epsilon \circ \phi_t$ is a linear w*-continuous map $B \mapsto \lambda(B) \in \mathcal{A}^c$, $\lambda^\bullet = \lambda_\bullet^\dagger$ is a linear w*-continuous map given by the Hermitian adjoint values $\lambda_\bullet(B^*) = \lambda^\bullet(B)^\dagger$ in the continuous operators $\mathcal{E} \to \mathcal{A}^c$, and $\lambda_\bullet^\bullet : \mathcal{B} \to \mathcal{B}(\mathcal{D} \otimes \mathcal{E})$is a w*-continuous map with the values $\lambda_\bullet^\bullet(B)$ given by continuous operators $\mathcal{E} \otimes \mathcal{E} \to \mathcal{A}^c$.

1. The differential evolution equation (3.1) for the coherent vector matrix elements $\langle \psi_f | \phi_t(B) \psi_f \rangle$ corresponds to the Itô form [8] of the quantum stochastic equation

$$\mathrm{d}\phi_t(B) = \phi_t \circ \lambda_\nu^\mu(B)\, \mathrm{d}\Lambda_\mu^\nu := \sum_{\mu,\nu} \phi_t(\lambda_\nu^\mu(B))\, \mathrm{d}\Lambda_\mu^\nu, \qquad B \in \mathcal{B} \tag{3.2}$$

with the initial condition $\phi_0(B) = B$, for all $B \in \mathcal{B}$. Here λ_ν^μ are the flow generators $\lambda_+^- = \lambda$, $\lambda_+^\bullet = \lambda^\bullet$, $\lambda_\bullet^- = \lambda_\bullet$, $\lambda_\bullet^\bullet$, called the structural maps, and the summation is taken over the indices $\mu = -, \bullet$, $\nu = +, \bullet$ of the standard quantum stochastic integrators Λ_μ^ν. For simplicity we shall assume that the pre-Hilbert Fréchet space $\mathcal{E}$ is separable, $\mathcal{E} \subseteq \ell^2$. Then the index $\bullet$ can take any value in $\{1, 2, ...\}$ and $\Lambda_\mu^\nu(t)$ are indexed with $\mu \in \{-, 1, 2, ...\}$, $\nu \in \{+, 1, 2, ...\}$ as the standard time $\Lambda_-^+(t) = t\mathrm{I}$, annihilation $\Lambda_-^m(t)$, creation $\Lambda_n^+(t)$ and exchange-number $\Lambda_n^m(t)$ operator integrators with $m, n \in \mathbb{N}$. The infinitesimal increments $\mathrm{d}\Lambda_\nu^\mu(t) = \Lambda_\nu^{t\mu}(\mathrm{d}t)$ are formally defined by the HP multiplication table [8] and the $\star$ -property [16],

$$\mathrm{d}\Lambda_\mu^\alpha \mathrm{d}\Lambda_\beta^\nu = \delta_\beta^\alpha \mathrm{d}\Lambda_\mu^\nu, \qquad \Lambda^\star = \Lambda, \tag{3.3}$$

where δ_β^α is the usual Kronecker delta restricted to the indices $\alpha \in \{-, 1, 2, ...\}$, $\beta \in \{+, 1, 2, ...\}$ and $\Lambda_{-\nu}^{\star\mu} = \Lambda_{-\mu}^{\nu*}$ with respect to the reflection $-(-) = +$, $-(+) = -$ of the indices $(-, +)$ only.

The linear equation (3.2) of a particular type, (quantum Langevin equation) with bounded finite-dimensional structural maps λ_ν^μ was introduced by Evans and Hudson [2] in order to describe the $*$-homomorphic quantum stochastic evolutions. The constructed quantum stochastic $*$-homomorphic flow (EH-flow) is identity preserving and is obviously completely positive, but it is hard to prove these algebraic properties for the unbounded case. However the typical quantum filtering dynamics is not homomorphic or identity preserving, but it is completely positive and in the most interesting cases is described by unbounded generators λ_ν^μ. In the general content Eq. (3.2) was studied in [31], and the correspondent quantum stochastic, not necessarily homomorphic and normalized flow was constructed even for the infinitely-dimensional non-adapted case under the natural integrability condition for the chronological products of the generators λ_ν^μ in the norm scale (6.2). The EH flows with unbounded λ_ν^μ, satisfying certain analyticity conditions, have been recently constructed in the strong sense by Fagnola-Sinha in [30] for the non-Hilbert class L^∞ of test functions $f^\bullet$. Another type of sufficient analyticity conditions, which is related to the Hilbert scales of the test functions, is given in the Appendix. Here we will formulate the necessary differential conditions which follow from the complete positivity, causality, and martingale properties of the filtering flows, and which are sufficient for the construction of the quantum stochastic flows obeying these properties in the case of the bounded λ_ν^μ. As we showed in [9], the found properties are sufficient to define the general structure of the bounded generators, and this structure will help us in construction of the minimal completely positive weak solutions for the quantum filtering equations also with unbounded λ_ν^μ.

2. Obviously the linear w*-continuous generators $\lambda_\nu^\mu : \mathcal{B} \to \mathcal{A}^c$ for CP flows $\phi_t^* = \phi_t$, where $\phi_t^*(B) = \phi_t(B^*)^\dagger$, must satisfy the $\star$ -property $\lambda^\star = \lambda$, where $\lambda_{-\mu}^{\star\nu} = \lambda_{-\nu}^{\mu*}$, $\lambda_\nu^{\mu*}(B) = \lambda_\nu^\mu(B^*)^*$ and are independent of t, corresponding to cocycle property $\phi_s \circ \phi_r^s = \phi_{s+r}$, where ϕ_t^s is the solution to (3.2) with $\Lambda_\nu^\mu(t)$ replaced by $\Lambda_\nu^{s\mu}(t)$, and $\lambda_+^-(I) = 0$ if ϕ is a filtering flow, $\phi_t(I) = I$, as it is in the multiplicative case [2]. We shall assume that $\boldsymbol{\lambda} = (\lambda_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$ for each $B^* = B$ defines a continuous Hermitian form $\boldsymbol{b} = \boldsymbol{\lambda}(B)$ on the Fréchet space $\mathcal{D} \oplus \mathcal{D}_\bullet$,

$$\langle \boldsymbol{\eta} |\, \boldsymbol{b}\, \boldsymbol{\eta} \rangle = \sum_{m,n} \langle \eta^m | b_n^m \eta^n \rangle + \sum_m \left\langle \eta^m | b_+^m \eta \right\rangle + \sum_n \left\langle \eta | b_n^- \eta^n \right\rangle + \left\langle \eta | b_+^- \eta \right\rangle,$$

where $\eta \in \mathcal{D}$, $\eta^\bullet = (\eta^m)^{m\in\mathbb{N}} \in \mathcal{D}_\bullet = \mathcal{D} \otimes \mathcal{E}$. We say that an Itô algebra $\mathfrak{a}$, represented on $\mathcal{E}$, commutes in HP sense with a $\boldsymbol{b}$, given by the form-generator $\boldsymbol{\lambda}$ if $(I \otimes a_\bullet^\mu)\, b_\nu^\bullet = b_\bullet^\mu (I \otimes a_\nu^\bullet)$ (For simplicity the ampliation $I \otimes a_\nu^\mu$ will be written again as a_ν^μ.) Note that if we define the matrix elements a_ν^μ, b_ν^μ also for $\mu = +$ and $\nu = -$, by the extension

$$a_\nu^+ = 0 = a_-^\mu, \qquad \lambda_\nu^+(B) = 0 = \lambda_-^\mu(B), \quad \forall a \in \mathfrak{a}, B \in \mathcal{B},$$

the HP product (0.4) of $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written in terms of the usual matrix product $\mathbf{ab} = \left[a_\lambda^\mu b_\nu^\lambda\right]$ of the extended quadratic matrices $\mathbf{a} = [a_\nu^\mu]_{\nu=-,\bullet,+}^{\mu=-,\bullet,+}$ and $\mathbf{b} = \boldsymbol{b}\mathbf{g}$, where $\mathbf{g} = \left[\delta_{-\nu}^\mu\right]$. Then one can extend the summation in (3.2) so it is also over $\mu = +$, and $\nu = -$, such that $b_\nu^\mu \mathrm{d}\Lambda_\mu^\nu$ is written as the trace $\mathbf{b}{\cdot}\mathrm{d}\boldsymbol{\Lambda}$ over all μ, ν. By such an extension the multiplication table for $\mathrm{d}\Lambda(a) = \mathbf{a}{\cdot}\mathrm{d}\boldsymbol{\Lambda}$, $\mathrm{d}\Lambda(b) = \mathbf{b}{\cdot}\mathrm{d}\boldsymbol{\Lambda}$ can be represented as $\mathrm{d}\Lambda(a)\,\mathrm{d}\Lambda(b) = \mathbf{ab}{\cdot}\mathrm{d}\boldsymbol{\Lambda}$, and the involution $\mathbf{b} \mapsto \mathbf{b}^\star$, defining $\mathrm{d}\Lambda(b)^\dagger = \mathbf{b}^\star{\cdot}\mathrm{d}\boldsymbol{\Lambda}$, can be obtained by the pseudo-Hermitian conjugation $b_\alpha^{\star\nu} = g_{\alpha\mu} b_\beta^{\mu*} g^{\beta\nu}$ respectively to the indefinite Minkowski metric tensor $\mathbf{g} = [g_{\mu\nu}]$ and its inverse $\mathbf{g}^{-1} = [g^{\mu\nu}]$, given by $g^{\mu\nu} = \delta_{-\nu}^\mu I = g_{\mu\nu}$.

Now let us find the differential form of the normalization and causality conditions with respect to the quantum stationary process, with independent increments $\mathrm{d}X(t) = X(t+\Delta) - X(s)$ generated by an Itô algebra $\mathfrak{a}$ on the separable space $\mathcal{E}$.

Proposition 3.1. *Let ϕ be a flow, satisfying the quantum stochastic equation (3.2), and $[W_t(g), \phi_t(B)] = 0$ for all $g \in \mathfrak{g}, B \in \mathcal{B}$. Then the coefficients $b_\nu^\mu = \lambda_\nu^\mu(B)$, $\mu = -, \bullet$, $\nu = +, \bullet$, where $\bullet = 1, 2, ...$, written in the matrix form $\boldsymbol{b} = (b_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$, commute in the sense of the HP product with $\boldsymbol{a} = (a_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$ for all $a \in \mathfrak{a}$ and $B \in \mathcal{B}$:*

$$[\boldsymbol{a}, \boldsymbol{b}] := (a_\bullet^\mu b_\nu^\bullet - b_\bullet^\mu a_\nu^\bullet)_{\nu=+,\bullet}^{\mu=-,\bullet} = 0. \tag{3.4}$$

Proof. Since $\epsilon_t(\phi_s(I) - \phi_t(I))$ is a negative Hermitian form,

$$\epsilon_t(\mathrm{d}\phi_t(I)) = \epsilon_t\left(\phi_t(\lambda_\nu^\mu(I))\,\mathrm{d}\Lambda_\mu^\nu\right) = \phi_t\left(\lambda_+^-(I)\right)\mathrm{d}t \le 0.$$

Since $Y_t = \phi_t(B)$ commutes with $W_t(g)$ for all B and $g(t) = a$, we have by virtue of quantum Itô's formula

$$\mathrm{d}[Y_t, W_t] = [\mathrm{d}Y_t, W_t] + [Y_t, \mathrm{d}W_t] + [\mathrm{d}Y_t, \mathrm{d}W_t] = 0.$$

Equations (2.2), (3.2) and commutativity of a^μ_ν with Y_t and W_t imply

$$\begin{aligned} & ([\phi_t(b^\mu_\nu), W_t] + [Y_t, a^\mu_\nu W_t] + \phi_t(b^\mu_\bullet) a^\bullet_\nu W_t - a^\mu_\bullet W_t \phi_t(b^\bullet_\nu)) \mathrm{d}\Lambda^\nu_\mu \\ = \; & W_t(\phi_t(b^\mu_\bullet) a^\bullet_\nu - a^\mu_\bullet \phi_t(b^\bullet_\nu)) \mathrm{d}\Lambda^\mu_\nu = W_t \phi_t(b^\mu_\bullet a^\bullet_\nu - a^\mu_\bullet b^\bullet_\nu) \mathrm{d}\Lambda^\nu_{\mu.} = 0. \end{aligned}$$

Thus $\boldsymbol{a} \bullet \boldsymbol{b} = \boldsymbol{b} \bullet \boldsymbol{a}$ by the argument [6]of independence of the integrators $\mathrm{d}\Lambda^\nu_\mu$. □

3. In order to formulate the CP differential condition we need the notion of *quantum stochastic germ* for the CP flow ϕ at $t = 0$. It was defined in [31, 11], for a quantum stochastic differential (3.2) with $\phi_0(B) = B, \forall B \in \mathcal{B}$ as $\gamma^\mu_\nu = \lambda^\mu_\nu + \imath^\mu_\nu$, where λ^μ_ν are the structural maps $B \mapsto \lambda^\mu_\nu(B)$ given by the generators of the quantum Itô equation (3.2) and $\imath^\mu_\nu : B \mapsto B\delta^\mu_\nu$ is the ampliation of $\mathcal{B}$. Let us prove that the germ-maps γ^μ_ν of a CP flow ϕ must be conditionally completely positive (CCP) in a degenerated sense as it was found for the finite-dimensional bounded case in [9, 12].

Theorem 3.2. *If ϕ is a completely positive flow satisfying the quantum stochastic equation (3.2) with $\phi_0(B) = B$, then the germ-matrix $\boldsymbol{\gamma} = (\lambda^\mu_\nu + \imath^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ is conditionally completely positive in the sense*

$$\sum_{B \in \mathcal{B}} \boldsymbol{\iota}(B) \boldsymbol{\zeta}_B = 0 \Rightarrow \sum_{B,C \in \mathcal{B}} \langle \boldsymbol{\zeta}_B | \boldsymbol{\gamma}(B^* C) \boldsymbol{\zeta}_C \rangle \geq 0.$$

Here $\boldsymbol{\zeta} \in \mathcal{D} \oplus \mathcal{D}_\bullet, \mathcal{D}_\bullet = \mathcal{D} \otimes \mathcal{E}$, and $\boldsymbol{\iota} = (\imath^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ is the degenerate representation $\imath^\mu_\nu(B) = B\delta^+_\nu \delta^\mu_-$, written both with $\boldsymbol{\gamma}$ in the matrix form as

$$\text{(3.5)} \qquad \boldsymbol{\gamma} = \begin{pmatrix} \gamma & \gamma_\bullet \\ \gamma^\bullet & \gamma^\bullet_\bullet \end{pmatrix}, \qquad \boldsymbol{\iota}(B) = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

where $\gamma = \lambda^-_+, \quad \gamma^m = \lambda^m_+, \quad \gamma_n = \lambda^-_n, \quad \gamma^m_n = \imath^m_n + \lambda^m_n$ with $\imath^m_n(B) = B\delta^m_n$ such that

$$\text{(3.6)} \qquad \gamma(B^*) = \gamma(B)^*, \qquad \gamma^n(B^*) = \gamma_n(B)^*, \qquad \gamma^m_n(B^*) = \gamma^n_m(B)^*.$$

If ϕ is subfiltering, then $D = -\lambda^-_+(I)$ is a positive Hermitian form, $\langle \eta | D\eta \rangle \geq 0$, for all $\eta \in \mathcal{D}$, and if ϕ is contractive, then $\boldsymbol{D} = -\boldsymbol{\lambda}(I)$ is positive in the sense $\langle \boldsymbol{\eta} | \boldsymbol{D}\boldsymbol{\eta} \rangle \geq 0$ for all $\boldsymbol{\eta} \in \mathcal{D} \oplus \mathcal{D}_\bullet$.

Proof. The CP condition in the form (2.6) for the adapted map ϕ_t can be obviously extended on all $f^\bullet \in \mathfrak{E}$ if the sesquianalytical function $f^\bullet \mapsto \phi_t(\bar{f}^\bullet, B, f^\bullet)$ is defined as the $\mathfrak{E}$-function

$$\text{(3.7)} \qquad \langle \eta | \phi_t(\bar{f}^\bullet, B, f^\bullet) \eta \rangle = \langle \eta \otimes f^\otimes | \phi_t(B) \eta \otimes f^\otimes \rangle \exp\left[-\int_t^\infty \|f^\bullet(s)\|^2 \mathrm{d}s \right],$$

where $\|f^\bullet(t)\|^2 = \sum_{n=1}^\infty |f^n(t)|^2$. It coincides with the former definition on $\mathfrak{E}_t$ and does not depend on $f^\bullet(s)$, $s > t$ due to the adaptiveness (2.5) of $Y_t = \phi_t(B)$. If the $\mathfrak{D}$-form $\phi_t(B)$ satisfies the stochastic equation (3.2), the $\mathcal{D}$-form $\phi_t(\bar{f}^\bullet, B, f^\bullet)$ satisfies the differential equation [8]

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi_t(\bar{f}^\bullet, B, f^\bullet) = \|f^\bullet(t)\|^2 \phi_t(\bar{f}^\bullet, B, f^\bullet) + \phi_t(\bar{f}^\bullet, \lambda^-_+(B), f^\bullet)$$

$$+\sum_{m=1}^{\infty}\overline{f}^{m}(t)\,\phi_t\left(\bar{f}^{\bullet},\lambda_{+}^{m}(B),f^{\bullet}\right)+\sum_{n=1}^{\infty}\phi_t\left(\bar{f}^{\bullet},\lambda_{n}^{-}(B),f^{\bullet}\right)f^{n}(t)$$
$$+\sum_{m,n=1}^{\infty}\overline{f}^{m}(t)\,\phi_t\left(\bar{f}^{\bullet},\lambda_{n}^{m}(B),f^{\bullet}\right)f^{n}(t)=\phi_t\left(\bar{f}^{\bullet},\gamma\left(\bar{f}^{\bullet}(t),B,f^{\bullet}(t)\right),f^{\bullet}\right).$$

The positive definiteness of (3.7) ensures the conditional positive definiteness $\sum_f\sum_B B\xi_B^f=0\Rightarrow$

$$\sum_{B,C}\sum_{f,h}\left\langle \xi_B^f\right|\gamma_t\left(\bar{f}^{\bullet},B^*C,h^{\bullet}\right)\xi_C^h\Big\rangle=\frac{1}{t}\sum_{B,C}\sum_{f,h}\left\langle \xi_B^f\right|\phi_t\left(\bar{f}^{\bullet},B^*C,h^{\bullet}\right)\xi_C^h\Big\rangle\geq 0$$

of the form, given by $\gamma_t\left(\bar{f}^{\bullet},B,f^{\bullet}\right)=\frac{1}{t}\left(\phi_t\left(\bar{f}^{\bullet},B,f^{\bullet}\right)-B\right)$ for each $t>0$. This holds also at the limit $\gamma_0\left(\bar{f}^{\bullet},B,f^{\bullet}\right)=\gamma\left(\bar{f}^{\bullet}(0),B,f^{\bullet}(0)\right)$, given at $t\downarrow 0$ by the $\mathcal{E}$-form

$$\gamma\left(\bar{e}^{\bullet},B,e^{\bullet}\right)=\sum_{m,n}\bar{e}^{m}\gamma_{n}^{m}(B)\,e^{n}+\sum_{m}\bar{e}^{m}\gamma^{m}(B)+\sum_{n}\gamma_{n}(B)\,e^{n}+\gamma(B),$$

where $e^{\bullet}=f^{\bullet}(0)\in\mathcal{E}$, $\bar{e}^{\bullet}=e_{\bullet}$ and the γ's are defined in (3.5). Hence the form

$$\sum_{B,C}\sum_{\mu,\nu}\langle\zeta_B^{\mu}|\,\gamma_{\nu}^{\mu}(B^*C)\,\zeta_C^{\nu}\rangle:=\sum_{B,C}\sum_{m,n}\langle\zeta_B^{m}|\,\gamma_{n}^{m}(B^*C)\,\zeta_C^{n}\rangle$$
$$+\sum_{B,C}\left(\sum_{n}\langle\zeta_B|\,\gamma_{n}(B^*C)\,\zeta_C^{n}\rangle+\sum_{m}\langle\zeta_B^{m}|\,\gamma^{m}(B^*C)\,\zeta_C\rangle+\langle\zeta_B|\,\gamma(B^*C)\,|\zeta_C\rangle\right)$$

with $\zeta=\sum_f\xi^f$, $\quad\zeta^{\bullet}=\sum_f\xi^f\otimes e_f^{\bullet}$, where $e_f^{\bullet}=f^{\bullet}(0)$, is positive if $\sum_B B\zeta_B=0$. The components ζ and $\zeta^{\bullet}$ of these vectors are independent because for any $\zeta\in\mathcal{D}$ and $\zeta^{\bullet}=\left(\zeta^1,\zeta^2,...\right)\in\mathcal{D}\otimes\mathcal{E}$ there exists such a function $e^{\bullet}\mapsto\xi^e$ on $\mathcal{E}$ with a countable support, that $\sum_e\xi^e=\zeta$, $\quad\sum_e\xi^e\otimes e^{\bullet}=\zeta^{\bullet}$, namely, $\xi^e=0$ for all $e^{\bullet}\in\mathcal{E}$ except $e^{\bullet}=0$ with $\xi^0=\zeta-\sum_{n=1}^{\infty}\zeta^n$ and $e^{\bullet}=e_n^{\bullet}$, the n-th basis element in ℓ^2, for which $\xi^e=\zeta^n$. This proves the complete positivity of the matrix form $\boldsymbol{\gamma}$, with respect to the matrix representation $\boldsymbol{\iota}$ defined in (3.5) on the ket-vectors $\boldsymbol{\zeta}=(\zeta^{\mu})$.

If $\epsilon(R_t)\leq I$, then $D=-\lambda(I)=\lim\frac{1}{t}\epsilon(I-R_t)\geq 0$, and we also conclude the dissipativity $\sum_{k,l}\left\langle\xi^k|D\left(\bar{k}^{\bullet},l^{\bullet}\right)\xi^l\right\rangle\geq 0$ from

$$0\leq\lim\frac{1}{t}\sum_{f,h}\left\langle\xi^f|e^{\int_0^t f_{\bullet}h^{\bullet}}I-\phi_t\left(\bar{f}^{\bullet},I,h^{\bullet}\right)\xi^h\right\rangle=-\left\langle\xi^f|\lambda\left(\bar{f}^{\bullet}(0),I,h^{\bullet}(0)\right)\xi^h\right\rangle$$

if $\phi_t(I)\leq I$, where $\lambda\left(\bar{e}^{\bullet},I,e^{\bullet}\right)=\gamma\left(\bar{e}^{\bullet},I,e^{\bullet}\right)-\left\|e^{\bullet}\right\|^2I=D\left(\bar{e}^{\bullet},e^{\bullet}\right)$. □

4. Obviously the CCP property for the germ-matrix $\boldsymbol{\gamma}$ is invariant under the transformation $\boldsymbol{\gamma}\mapsto\boldsymbol{\varphi}$ given by

$$\boldsymbol{\varphi}(B)=\boldsymbol{\gamma}(B)+\boldsymbol{\iota}(B)\,\boldsymbol{K}+\boldsymbol{K}^*\boldsymbol{\iota}(B),\tag{3.8}$$

where $\boldsymbol{K}=(K_{\nu}^{\mu})_{\nu=+,\bullet}^{\mu=-,\bullet}$ is an arbitrary matrix of $K_{\nu}^{\mu}\in\mathcal{L}(\mathcal{D})$ with $K_{-\nu}^{*\mu}=K_{-\mu}^{\nu*}$. As was proven in [9, 12] for the case of a finite-dimensional matrix $\boldsymbol{\gamma}$ of bounded γ_{ν}^{μ}, see also [13], the matrix elements K_{ν}^{-} can be chosen in such way that the matrix map $\boldsymbol{\varphi}=(\varphi_{\nu}^{\mu})_{\nu=+,\bullet}^{\mu=-,\bullet}$ becomes CP from $\mathcal{B}$ into the quadratic matrices of $\varphi_{\nu}^{\mu}(B)$. (The other elements can be chosen arbitrarily, say as $K_{+}^{\bullet}=0$, $K_{\bullet}^{\bullet}=\frac{1}{2}I_{\bullet}^{\bullet}$, because

(3.8) does not depend on $K_+^\bullet, K_\bullet^\bullet$.) Thus the generator $\boldsymbol{\lambda} = \boldsymbol{\gamma} - \boldsymbol{\imath}$ for a quantum stochastic CP flow ϕ can be written (at least in the bounded case) as $\boldsymbol{\varphi} - \imath \boldsymbol{K} - \boldsymbol{K}^* \imath$:

$$\lambda_\nu^\mu (B) = \varphi_\nu^\mu (B) - B \left(\tfrac{1}{2} \delta_\nu^\mu I + \delta_-^\mu K_\nu \right) - \left(\tfrac{1}{2} \delta_\nu^\mu I + K^\mu \delta_\nu^+ \right) B, \tag{3.9}$$

where $\varphi_\nu^\mu : \mathcal{B} \to \mathcal{B}(\mathcal{D})$ are matrix elements of the CP map $\boldsymbol{\varphi}$ and $K_\nu \in \mathcal{L}(\mathcal{D})$, $K^- = K_+^*$, $K^m = K_m^*$. Now we show that the germ-matrix of this form obeys the CCP property even in the general case of unbounded K_ν^-, $\varphi_\nu^\mu (B) \in \mathcal{B}(\mathcal{D})$.

Proposition 3.3. *The matrix map $\boldsymbol{\gamma} = (\gamma_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$ given in (3.8) by*

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi & \varphi_\bullet \\ \varphi^\bullet & \varphi_\bullet^\bullet \end{pmatrix}, \quad \text{and} \quad \boldsymbol{K} = \begin{pmatrix} K & K_\bullet \\ 0 & \frac{1}{2} I_\bullet^\bullet \end{pmatrix}, \; \boldsymbol{K}^* = \begin{pmatrix} K^* & 0 \\ K_\bullet^* & \frac{1}{2} I_\bullet^\bullet \end{pmatrix}, \tag{3.10}$$

with $\varphi = \varphi_+^-$, $\varphi^m = \varphi_+^m$, $\varphi_n = \varphi_n^-$ and $\varphi_n^m = \gamma_n^m$ is CCP with respect to the degenerate representation $\boldsymbol{\iota} = \left(\delta_-^\mu \delta_\nu^+ \iota \right)_{\nu=+,\bullet}^{\mu=-,\bullet}$, where $\iota (B) = B$, if $\boldsymbol{\varphi}$ is a CP map.

Proof. If $\boldsymbol{\iota} (B_k) \boldsymbol{\eta}^k = 0$, then

$$\langle \boldsymbol{\eta}^k | \boldsymbol{\iota} (B_k^* B_l) \boldsymbol{K} + \boldsymbol{K}^* \boldsymbol{\iota} (B_k^* B_l) \boldsymbol{\eta}^l \rangle$$
$$= 2 \operatorname{Re} \left\langle \boldsymbol{\iota} (B_k) \boldsymbol{\eta}^k | \boldsymbol{\iota} (B_l) \boldsymbol{K} \boldsymbol{\eta}^l \right\rangle = 0.$$

Hence the CCP for $\boldsymbol{\gamma}$ is equivalent to the CCP property for (3.8) and follows from its CP property:

$$\left\langle \boldsymbol{\eta}^k | \boldsymbol{\gamma} (B_k^* B_l) \boldsymbol{\eta}^l \right\rangle = \left\langle \boldsymbol{\eta}^k | \boldsymbol{\varphi} (B_k^* B_l) \boldsymbol{\eta}^l \right\rangle \geq 0$$

for such sequences $\boldsymbol{\eta}^k \in \mathcal{D} \oplus \mathcal{D}_\bullet$.

4. Construction of quantum CP flows

The necessary conditions for the stochastic generator $\boldsymbol{\lambda} = (\lambda_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$ of a CP flow ϕ at $t = 0$ are found in the previous section in the form of a CCP property for the corresponding germ $\boldsymbol{\gamma} = (\gamma_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$. In the next section we shall show, these conditions are essentially equivalent to the assumption (3.9), corresponding to

$$\gamma^m (B) = \varphi^m (B) - K_m^* B = \gamma_m^* (B), \quad \gamma (B) = \varphi (B) - K^* B - BK, \tag{4.1}$$

where $\boldsymbol{\varphi} = (\varphi_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$ is a CP map with $\varphi_n^m = \gamma_n^m$. Here we are going to prove under the following conditions for the operators $K, K_\bullet$ and the maps φ_ν^μ that this general form is also sufficient for the existence of the CP solutions to the quantum stochastic equation (3.2). We are going to construct the minimal quantum stochastic positive flow $B \mapsto \phi_t (B)$ for a given w*-continuous unbounded germ-matrix map of the above form, satisfying the following conditions.

(1) First, we suppose that the operator $K \in \mathcal{B}(\mathcal{D})$ generates the one parametric semigroup $\left(e^{-Kt} \right)_{t>0}$, $e^{-Kr} e^{-Ks} = e^{-K(r+s)}$ of continuous operators $e^{-Kt} \in \mathcal{L}(\mathcal{D})$ in the strong sense

$$\lim_{t \searrow 0} \frac{1}{t} \left(I - e^{-Kt} \right) \eta = K \eta, \quad \forall \eta \in \mathcal{D}.$$

(A contraction semigroup on the Hilbert space $\mathcal{H}$ if K defines an accretive $K + K^\dagger \geq 0$ and so maximal accretive form.)

(2) Second, we suppose that the solution $S_t^n, n \in \mathbb{N}$ to the recurrence

$$S_t^{n+1} = S_t^\circ - \int_0^t S_{t-r}^\circ \sum_{m=1}^\infty K_m S_r^n \mathrm{d}\Lambda_-^m, \quad S_t^0 = S_t^\circ,$$

where $S_t^\circ = e^{-Kt} \otimes T_t \in \mathcal{L}(\mathfrak{D})$ is the contraction given by the shift co-isometries $T_t : \mathfrak{F} \to \mathfrak{F}$, strongly converges to a continuous operator $S_t \in \mathcal{L}(\mathfrak{D})$ at $n \longrightarrow \infty$ for each $t > 0$.

(3) Third, we suppose that the solution $R_t^n, n \in \mathbb{N}$ to the recurrence

$$R_t^{n+1} = S_t^* S_t + \int_0^t \mathrm{d}\Lambda_\mu^\nu \left(r, S_r^* \varphi_\nu^\mu \left(R_{t-r}^n\right) S_r\right), \quad R_t^0 = S_t^* S_t,$$

where the quantum stochastic non-adapted integral is understood in the sense [31] (see the Appendix), weakly converges to a continuous form $R_t \in \mathcal{B}(\mathfrak{D})$ at $n \longrightarrow \infty$ for each $t > 0$.

The first and second assumptions are necessary to define the existence of the free evolution semigroup $S^\circ = (S_t^\circ)_{t>0}$ and its perturbation $S = (S_t)_{t>0}$ on the product space $\mathfrak{D} = \mathcal{D} \otimes \mathfrak{F}$ in the form of multiple quantum stochastic integral

(4.2)
$$S_t = S_t^\circ + \sum_{n=1}^\infty (-1)^n \int\cdots\int_{0<t_1<\ldots<t_n<t} K_{m_n}(t-t_n)\cdots K_{m_1}(t_2-t_1) S_{t_1}^\circ \mathrm{d}\Lambda_-^{m_1}\cdots \mathrm{d}\Lambda_-^{m_n},$$

iterating the quantum stochastic integral equation

$$S_t = S_t^\circ - \int_0^t \sum_{m=1}^\infty K_m(t-r) S_r \mathrm{d}\Lambda_-^m, \quad S_0 = I, \tag{4.3}$$

where $K_m(t) = S_t^\circ (K_m \otimes I)$. A sufficient analyticity condition under which this iteration strongly coverges in $\mathfrak{D}$ is given in the Appendix. The third assumption supplies the weak convergence for the series

(4.4)
$$R_t = S_t^* S_t + \sum_{n=1}^\infty \int\cdots\int_{0<t_1<\ldots<t_n<t} \mathrm{d}\Lambda_{\mu_1\ldots\mu_n}^{\nu_1\ldots\nu_n}\left(t_1,\ldots,t_n, \varphi_{\nu_1\ldots\nu_n}^{\mu_1\ldots\mu_n}\left(t_1,\ldots,t_n, S_{t-t_n}^* S_{t-t_n}\right)\right)$$

of non-adapted n-tuple integrals, i.e. for the multiple quantum stochastic integral (see the definition in the Appendix) with

$$\varphi_{\nu_1\ldots\nu_n}^{\mu_1\ldots\mu_n}(t_1,\ldots,t_n) = \varphi_{\nu_1\ldots\nu_{n-1}}^{\mu_1\ldots\mu_{n-1}}(t_1,\ldots,t_{n-1}) \circ \varphi_{\nu_n}^{\mu_n}(t_n - t_{n-1}), \tag{4.5}$$

where $\varphi_\nu^\mu(t, B) = S_t^* \varphi_\nu^\mu(B) S_t$. A sufficient analyticity condition for this convergence is also given in the Appendix.

The following theorem gives a characterization of the evolution semigroup S in terms of cocycles with unbounded coefficients, characterized by Fagnola [33] in the isometric and unitary case.

Proposition 4.1. *Let the family $V^\circ = (V_t^\circ)_{t>0}$ be a quantum stochastic adapted cocycle, $V_r^\circ T_s V_s^\circ = T_s V_{r+s}^\circ$, satisfying the HP differential equation*

$$\mathrm{d}V_t^\circ + KV_t^\circ \mathrm{d}t + \sum_{m=1}^\infty K_m V_t^\circ \mathrm{d}\Lambda_-^m + \sum_{n=1}^\infty V_t^\circ \mathrm{d}\Lambda_n^n = 0, \quad V_0^\circ = I. \tag{4.6}$$

Then $S_t = T_t V_t^\circ$ is a semigroup solution, $S_r S_s = S_{r+s}$ to the non-adapted integral equation (4.3) such that $S_t \psi_f = S_t(f^\bullet)\eta \otimes \delta_\varnothing, \forall \eta \in \mathcal{D}$ on $\psi_f = \eta \otimes f^\otimes$ with

$f^{\bullet} \in \mathfrak{E}_t$. *Conversely, if* $S = (S_t)_{t>0}$ *is the non-adapted solution (4.2) to the integral equation (4.3), then* $V_t^{\circ} = T_t^{*} S_t$ *is the adapted solution to (4.6), defined as* $V_t^{\circ}\psi_f = S_t(f^{\bullet})\eta \otimes f^{\otimes}, \forall \eta \in \mathcal{D}$, *where* $S_t(f^{\bullet}) = F^{*} S_t F$ *is given by* $F\eta = \eta \otimes f^{\otimes}$ *with* $f^{\bullet} \in \mathfrak{E}_t$.

Proof. First let us show that Eq. (4.6) is equivalent to the integral one

$$V_t^{\circ} = e^{-Kt} \otimes I_t - \int_0^t \sum_{m=1}^{\infty} \left(e^{-K(t-r)} \otimes I_t(r)\right) K_m V_r^{\circ} \mathrm{d}\Lambda_-^m, \quad V_0^{\circ} = I,$$

where $I_t = T_t^{\dagger} T_t$ is the orthoprojector onto $\mathfrak{F}_{[t}$ and $I_t(r) = \theta^r(I_{t-r})$. Indeed, multiplying both parts of the integral equation from the left by $e^{K(t-s)}$ and differentiating the product $e^{K(t-s)} V_t^{\circ}$ at $t = s$, we obtain (4.6) by taking into account that $\mathrm{d}I_t + \sum_{n=1}^{\infty} I_t \mathrm{d}\Lambda_n^n = 0$ and $\mathrm{d}\Lambda_n^n \mathrm{d}\Lambda_-^m = 0$. Conversely, the integral equation can be obtained from (4.6) by the integration:

$$\begin{aligned} V_t^{\circ} - e^{-Kt} \otimes I_t &= \int_0^t \mathrm{d}\left(\left(e^{-K(t-r)} \otimes I_t(r)\right) V_r^{\circ}\right) \\ &= \int_0^t \left(e^{-K(t-r)} \otimes I_t(r)\right)\left(\mathrm{d}V_r^{\circ} + K V_r^{\circ}\mathrm{d}r + V_r^{\circ}\mathrm{d}\Lambda_{\bullet}^{\bullet}\right) \\ &= -\int_0^t \left(e^{-K(t-r)} \otimes I_t(r)\right) K_{\bullet} V_r^{\circ} \mathrm{d}\Lambda_-^{\bullet}, \end{aligned}$$

where we used that $\mathrm{d}I(r) = I(r)\mathrm{d}\Lambda_{\bullet}^{\bullet}$ and $\mathrm{d}(I(r)V_r) = \mathrm{d}I(r)V_r + I(r)\mathrm{d}V_r$ due to the non-adapted Itô formula [31]. The non-adapted equation (4.3) is obtained by applying the operator $T_t = T_{t-r}T_r$ to both parts of this integral equation and taking into account the commutativity of $e^{K(r-t)}K_m$ with T_r. Moreover, due to the adaptiveness of V_t°, $S_t\psi_f = T_t\left(E_t V_t^{\circ}\psi_f \otimes E_{[t} f^{\otimes}\right) = S_t(f^{\bullet})\eta \otimes f_t^{\otimes}$, where $f_t^{\otimes} = T_t f^{\otimes}$, and $S_t(f^{\bullet}) = E V_t^{\circ} F$ is the solution to the equation

$$S_t(f^{\bullet}) = e^{-Kt} + \int_0^t e^{-K(t-r)} K_{\bullet} f^{\bullet}(r) S_r(f^{\bullet})\mathrm{d}r, \qquad S_0(f^{\bullet}) = I.$$

Hence $S_t F = E^{*} S_t(f^{\bullet})$ if $f^{\bullet} \in \mathfrak{E}_t$, and $F^{*} S_t F = S_t(f^{\bullet})$ as $EF = I$. Since this equation is equivalent to the differential one

$$\text{(4.7)} \qquad \frac{\mathrm{d}}{\mathrm{d}t} S_t(f^{\bullet})\eta + (K_{\bullet} f^{\bullet}(t) + K) S_t(f^{\bullet})\eta = 0, \quad S_0(f^{\bullet})\eta = \eta, \qquad \forall \eta \in \mathcal{D},$$

the function $t \mapsto S_t(f^{\bullet})$, $f^{\bullet} \in \mathfrak{E}$ is a strongly continuous cocycle,

$$S_r(f_s^{\bullet}) S_s(f^{\bullet}) = S_{r+s}(f^{\bullet}), \ \forall r, s > 0, \qquad f_s^{\bullet}(t) = f^{\bullet}(t+s), \quad S_0(f^{\bullet}) = I.$$

As was proved in [31], the multiple integral (4.2) gives a solution to the integral equation (4.3), and so the multiple integral for $V_t^{\circ}\psi_f = S_t(f^{\bullet})\eta \otimes f^{\otimes}$,

$$S_t(f^{\bullet}) = e^{-Kt} + \sum_{n=1}^{\infty} (-1)^n \int_{0<t_1<\ldots<t_n<t} \cdots \int K(t, t_n) \cdots K(t_2, t_1) e^{-Kt_1} \mathrm{d}t_1 \cdots \mathrm{d}t_n,$$

where $K(t, r) = e^{-K(t-r)} K_{\bullet} f^{\bullet}(r)$, corresponding to the iteration of the integral equation for V_t° on ψ_f, satisfies the HP equation (4.6). □

The following theorem reduces the problem of solving differential evolution equations to the problem of iteration of integral equations similar to the nonstochastic case [34, 35].

Proposition 4.2. *Let $S_t = T_t V_t^\circ$, where $V_t^\circ \in \mathcal{L}(\mathfrak{D})$ are continuous operators defining the adapted cocycle solution to Eq. (4.6). Then the linear stochastic evolution equation (3.2) is equivalent to the quantum non-adapted (in the sense of* [31]*) integral equation*

$$\phi_t(B) = S_t^* B S_t + \int_0^t \mathrm{d}\Lambda_\mu^\nu \left(r, \phi_r \left[\varphi_\nu^\mu \left(S_{t-r}^* B S_{t-r}\right)\right]\right) \tag{4.8}$$

with $\phi_0(B) = B \in \mathcal{B}$, where φ_ν^μ are extended onto $\mathfrak{B}$ by w^*-continuity and linearity as $\varphi_\nu^\mu(B \otimes Z) = \varphi_\nu^\mu(B) \otimes Z$ for $B \in \mathcal{B}$, $Z \in \mathcal{B}(\mathfrak{F})$.

Proof. The non-adapted equation (4.8) is understood in the coherent form sense as

$$\left\langle \psi_f | \phi_t(B) \psi_f \right\rangle = \left\langle S_t \psi_f | B S_t \psi_f \right\rangle + \int_0^t \left\langle \psi_f | \phi_r \left[\varphi\left(\bar{f}^\bullet(r), S_{t-r}^* B S_{t-r}, f^\bullet(r)\right)\right] \psi_f \right\rangle \mathrm{d}r,$$

where $\varphi(\bar{e}^\bullet, B, e^\bullet) = \sum_{m,n} \bar{e}^m \varphi_n^m(B) e^n + \sum_m \bar{e}^m \varphi^m(B) + \sum_n \varphi_n(B) e^n + \varphi(B)$. Due to the adaptiveness of ϕ_t this can be written for $\psi_f = \eta \otimes f^\otimes = F\eta$ with $f^\bullet \in \mathfrak{E}_t$ as

$$\begin{aligned} \phi_t\left(\bar{f}^\bullet, B, f^\bullet\right) &= S_t^*\left(\bar{f}^\bullet\right) B S_t\left(f^\bullet\right) \\ &+ \int_0^t \phi_r\left(\bar{f}^\bullet, \varphi\left(r, S_{t-r}^*\left(\bar{f}_r^\bullet\right) B S_{t-r}\left(f_r^\bullet\right)\right), f^\bullet\right) \mathrm{d}r, \end{aligned} \tag{4.9}$$

where $S_t^*\left(\bar{f}^\bullet\right) = S_t\left(f^\bullet\right)^*$, $f_r^\bullet(t) = f^\bullet(t+r)$. Here we take into account that due to adaptiveness $F^* \phi_r[Y] F = \phi_r\left(\bar{f}^\bullet, F_r^* Y F_r, f^\bullet\right)$, where $F_r = T_r F$, and therefore

$$F^* \phi_r\left[\varphi\left(r, S_{t-r}^* B S_{t-r}\right)\right] F = \phi_r\left(\bar{f}^\bullet, F_r^* \varphi\left(r, S_{t-r}^* B S_{t-r}\right) F_r, f^\bullet\right) =$$

$$\phi_r\left(\bar{f}^\bullet, \varphi\left(r, F_r^* S_{t-r}^* B S_{t-r} F_r\right), f^\bullet\right) = \phi_r\left(\bar{f}^\bullet, \varphi\left(r, S_{t-r}^*\left(\bar{f}_r^\bullet\right) B S_{t-r}\left(f_r^\bullet\right)\right), f^\bullet\right)$$

as $F_t^* \varphi(t, B) F_t = \varphi(t, F_t^* B F_t)$ for $\varphi(t, B) = \varphi\left(\bar{f}^\bullet(t), B, f^\bullet(t)\right)$ and $S_{t-r} F_r = F_t S_{t-r}\left(f_r^\bullet\right)$, where $F_t \eta = \eta \otimes \delta_\emptyset$ for any $f^\bullet \in \mathfrak{E}_t$.

Let us prove that the operator-valued function $t \mapsto S_s\left(t, f^\bullet\right) := S_{s-t}\left(f_t^\bullet\right)$ satisfies the backward evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t} S_s\left(t, f^\bullet\right) \eta = S_s\left(t, f^\bullet\right)\left(K_\bullet f^\bullet(t) + K\right) \eta, \quad S_0\left(f_s^\bullet\right) \eta = \eta \quad \forall t \in [0, s).$$

Indeed, taking into account the forward equation (4.7), we obtain it at $r = t$ from the cocycle property $S_s\left(t, f^\bullet\right) S_t\left(r, f^\bullet\right) = S_s\left(r, f^\bullet\right)$:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(S_s\left(r, f^\bullet\right) \eta\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} S_s\left(t, f^\bullet\right) - S_s\left(t, f^\bullet\right)\left(K_\bullet f^\bullet(t) + K\right)\right) S_t\left(r, f^\bullet\right) \eta.$$

Now, replacing B in (4.9) by $Y_s\left(\bar{f}^\bullet, t, f^\bullet\right) = S_s^*\left(t, \bar{f}^\bullet\right) B S_s\left(t, f^\bullet\right)$, we can write

$$\begin{aligned} &\phi_t\left(\bar{f}^\bullet, S_s^*\left(t, \bar{f}^\bullet\right) B S_s\left(t, f^\bullet\right), f^\bullet\right) \\ = \;& S_s^*\left(\bar{f}^\bullet\right) B S_s\left(f^\bullet\right) + \int_0^t \phi_r\left(\bar{f}^\bullet, \varphi\left(r, S_s^*\left(r, \bar{f}^\bullet\right) B S_s\left(r, f^\bullet\right)\right), f^\bullet\right) \mathrm{d}r. \end{aligned}$$

Calculating the total derivative $\frac{\mathrm{d}}{\mathrm{d}t} \phi_t\left(\bar{f}^\bullet, S_s^*\left(t, \bar{f}^\bullet\right) B S_s\left(t, f^\bullet\right), f^\bullet\right)$ by taking into account the backward equation, we obtain the differential equation at $s = t$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi_t\left(\bar{f}^\bullet, B, f^\bullet\right) + \phi_t\left(\bar{f}^\bullet, K(t)^* B + B K(t), f^\bullet\right) = \phi_t\left(\bar{f}^\bullet, \varphi\left(\bar{f}^\bullet(t), B, f^\bullet(t)\right), f^\bullet\right)$$

where $K(t) = K + K_\bullet f^\bullet(t)$. This equation written for $\langle \eta | \phi_t(\bar{f}^\bullet, B, f^\bullet)\eta\rangle$ coincides with the coherent matrix form (3.1) for the quantum stochastic equation (3.2) with $\psi_f = F\eta$.

The converse is easy to show by integrating the equation for $\phi_t(\bar{f}^\bullet, B, f^\bullet)$ with B replaced by $Y(t) = S_s^*(t, \bar{f}^\bullet) BS_s(t, f^\bullet)$:

$$\phi_s(\bar{f}^\bullet, B, f^\bullet) - S_s^*(\bar{f}^\bullet) BS_s(f^\bullet) = \int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\phi_t(\bar{f}^\bullet, S_s^*(t, \bar{f}^\bullet) BS_s(t, f^\bullet), f^\bullet)\,\mathrm{d}t$$

$$= \int_0^s \left(\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(\bar{f}^\bullet, Y(r), f^\bullet) + \phi_r\left(\bar{f}^\bullet, \frac{\mathrm{d}}{\mathrm{d}t}Y(t), f^\bullet\right)\right)_{r=t}\mathrm{d}t$$

$$= \int_0^s \phi_t(\bar{f}^\bullet, \varphi(t, S_s^*(t, \bar{f}^\bullet) BS_s(t, f^\bullet)), f^\bullet)\,\mathrm{d}t,$$

whereas $\frac{\mathrm{d}}{\mathrm{d}t}Y(t) = (K_\bullet f^\bullet(t) + K)^* Y(t) + Y(t)(K_\bullet f^\bullet(t) + K)$. □

Theorem 4.3. *Let $\boldsymbol{\varphi}$ be a w^*-continuous CP-map, and $S_t = T_t V_t^\circ$ be given by the solution to the quantum stochastic equation (4.6). Then the solutions to the evolution equation (3.2) with the generators, corresponding to (4.1), have the CP property, and satisfy the submartingale (contractivity) condition $\phi_t(I) \leq \epsilon_t[\phi_s(I)]$ for all $t < s$ if $\varphi(I) \leq K + K^\dagger$ ($\phi_t(I) \leq \phi_s(I)$ if $\boldsymbol{\varphi}(I) \leq \boldsymbol{K} + \boldsymbol{K}^\dagger$). The minimal solution can be constructed in the form of a multiple quantum stochastic integral in the sense* [31] *as the series*

(4.10)
$$\phi_t(B) = \sum_{n=0}^{\infty} \int\cdots\int_{0<t_1<\ldots<t_n<t} \mathrm{d}\Lambda^{\nu_1\cdots\nu_n}_{\mu_1\cdots\mu_n}\left(t_1, \ldots, t_n, \varphi^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_n}\left(t_1, \ldots, t_n, S^*_{t-t_n} BS_{t-t_n}\right)\right)$$

of non-adapted n-tuple CP integrals with $S_t^* BS_t$ at $n = 0$ and

$$\varphi^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_n}(t_1, \ldots, t_n) = \varphi^{\mu_1}_{\nu_1}(t_1) \circ \varphi^{\mu_2}_{\nu_2}(t_2 - t_1) \circ \ldots \circ \varphi^{\mu_n}_{\nu_n}(t_n - t_{n-1}),$$

where $\varphi^\mu_\nu(t, B) = S_t^* \varphi^\mu_\nu(B) S_t$. If $\boldsymbol{\varphi}$ is bounded, then the solution to the equation is unique, and $\phi_t(I) = \epsilon_t[\phi_s(I)]$ for all $t < s$ if $K + K^\dagger = \varphi(I)$ ($\phi_t(I) = I$ if $\boldsymbol{K} + \boldsymbol{K}^\dagger = \boldsymbol{\varphi}(I)$).

Proof. The existence and uniqueness of the solutions $\phi_t(B)$ to the quantum stochastic equations (3.2) with the bounded generators $\lambda^\mu_\nu(B) = \gamma^\mu_\nu(B) - B\delta^\mu_\nu$ and the initial conditions $\phi_0(B) = B$ in an operator algebra $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ was proved in [31]. The CP property of the solution to this equation with the generators, corresponding to the conditionally positive germ-matrix (4.1), can be proven in the form (4.10), which is obtained by the iteration

$$\phi_t^{n+1}(B) = S_t^* BS_t + \int_0^t \mathrm{d}\Lambda^\nu_\mu\left(r, \phi^n_r\left(\varphi^\mu_\nu\left(S^*_{t-r} BS_{t-r}\right)\right)\right), \quad \phi^0_t(B) = S_t^* BS_t$$

of the equivalent non-adapted integral equation (4.8). Indeed, in order to prove the complete positivity of the solution, written in this form, one should prove the positive definiteness of the iteration

$$\phi_t^{n+1}(\bar{f}^\bullet, B, f^\bullet) = S_t^*(\bar{f}^\bullet) BS_t(f^\bullet) + \int_0^t \phi^n_r\left(\bar{f}^\bullet, \varphi\left(\bar{f}^\bullet(r), S^*_{t-r}(\bar{f}^\bullet_r) BS_{t-r}(f^\bullet_r), f^\bullet(r)\right), f^\bullet\right)\mathrm{d}r$$

of the integral equation (4.9) with the CP $\phi_t^0(B) = S_t^* B S_t$. Thus, we have to test the positive definiteness of the forms

$$\sum_{B,C}\sum_{f,h}\left\langle \xi_B^f \right| \phi_t^{n+1}\left(\bar{f}^\bullet, B^*C, h^\bullet\right)\xi_C^h\left.\right\rangle = \sum_{B,C}\sum_{f,h}\left\langle BS_t\left(f^\bullet\right)\xi_B^f | CS_t\left(h^\bullet\right)\xi_C^h\right\rangle$$

$$+\int_0^t \sum_{B,C}\sum_{f,h}\left\langle \boldsymbol{\eta}_B^f(r) | \phi_s^n\left(\bar{f}^\bullet, \boldsymbol{\varphi}\left(S_{t-r}^*\left(\bar{f}_r^\bullet\right)B^*CS_{t-r}\left(h_r^\bullet\right)\right), h^\bullet\right)\boldsymbol{\eta}_C^h(r)\right\rangle,$$

where $\boldsymbol{\eta}_B^f(r) = \sum_{f(r)}\xi_B^f \otimes \boldsymbol{f}(r)$, and $\boldsymbol{f}(r) = 1 \oplus f^\bullet(r)$. It is a consequence of the CP condition for $\boldsymbol{\varphi}$ and the CP property for $\phi_{t_n}^n, \forall t_n < t$, which obviously follows from the positive definiteness of $\phi_r^{n-1}, r < t_n$, and so on up to $\phi_r^0, r < t_1$. The direct iteration of this integral recursion with the initial CP condition $\phi_t^0(B) = S_t^* B S_t$ gives at the limit $n \to \infty$ the solution in the form of a series

$$\phi_t\left(\bar{f}^\bullet, B, f^\bullet\right) = \sum_{n=0}^{\infty}\int\cdots\int_{0<t_1<\ldots<t_n<t}\varphi\left(t_1,\ldots,t_n; S_{t-t_n}^*\left(\bar{f}_{t_n}^\bullet\right)BS_{t-t_n}\left(f_{t_n}^\bullet\right)\right)\mathrm{d}t_1\cdots\mathrm{d}t_n$$

of n-tuple integrals on the interval $[0,t)$ with $S_t^*\left(\bar{f}^\bullet\right)BS_t\left(f^\bullet\right)$ at $n = 0$. The positive definite kernels

$$\varphi\left(t_1,\ldots,t_n\right) = \varphi^0\left(t_1\right)\circ\varphi^{t_1}\left(t_2\right)\circ\ldots\circ\varphi^{t_{n-1}}\left(t_n\right),$$

where $\varphi^r(t,B) = S_{t-r}^*\left(\bar{f}_r^\bullet\right)\varphi\left(\bar{f}^\bullet(t), B, f^\bullet(t)\right)S_{t-r}\left(f_r^\bullet\right)$, are obtained by the recurrence

$$\varphi\left(t_1,\ldots,t_n\right) = \varphi\left(t_1,\ldots,t_{n-1}\right)\circ\varphi^{t_{n-1}}\left(t_n\right), \qquad \varphi(t) = \varphi^0(t),$$

corresponding to (4.5). This proves the CP property for the series (4.10), which converges to a $0 \le Y_t \le \kappa R_t$ for any positive bounded $0 \le B \le \kappa I$ because of the increase $Y_t^n \le Y_t^{n+1}$ for $Y_t^n = \phi_t^n(B)$ and the boundedness $Y_t^n \le \kappa R_t^n$, $R_t^n \le R_t$, where R_t is the continuous sesquilinear form (4.4).

As follows from the exponential estimate [31] for the solutions to the quantum stochastic equations (3.2) with the bounded generators, $R_t = \phi_t(I)$ might be unbounded, but strongly continuous in the Fock scale $\mathfrak{F}$. In the case of unbounded generators the solution to (4.4) might not be unique, and the iterated series (4.10) gives obviously the minimal one, which is unique among such solutions. Let us prove the submartingale property for the sesquilinear form R_t, given by the weakly convergent series (4.4). R_s for a $s > t$ is defined as the iterated solution $Y_s = R_s := \lim R_s^n$ to the backward integral equation

$$Y_s = S_s^* B S_s + \int_0^s \mathrm{d}\Lambda_\mu^\nu\left(r, S_r^*\varphi_\nu^\mu\left(Y_{s-r}\right)S_r\right)$$

for the series $Y_s = \phi_s(B)$ with $B = I$. It satisfies the integral equation

$$R_s = S_t^* R_{s-t} S_t + \int_0^t \mathrm{d}\Lambda_\mu^\nu\left(r, S_r^*\varphi_\nu^\mu\left(R_{s-r}\right)S_r\right),$$

where we used the semigroup property $S_{s-t}S_t = S_s$ and that

$$\int_t^s \mathrm{d}\Lambda_\mu^\nu\left(r, S_r^*\varphi_\nu^\mu\left(Y_{s-r}\right)S_r\right) = V_t^{\circ *}\int_t^s \mathrm{d}\Lambda_\mu^\nu\left(r, T_t^* S_{r-t}^*\varphi_\nu^\mu\left(Y_{s-r}\right)S_{r-t}T_t\right)V_t^\circ,$$

$$\int_t^s \mathrm{d}\Lambda_\mu^\nu\left(r-t, S_{r-t}^*\varphi_\nu^\mu\left(Y_{s-r}\right)S_{r-t}\right) = \int_0^{s-t}\mathrm{d}\Lambda_\mu^\nu\left(r, S_r^*\varphi_\nu^\mu\left(Y_{s-t-r}\right)S_r\right).$$

This can be written in terms of the coherent matrix elements $R_s\left(\bar{f}^\bullet, f^\bullet\right) = F^* R_s F$, $f^\bullet \in \mathfrak{E}_s$ as

$$\begin{aligned} R_s\left(\bar{f}^\bullet, f^\bullet\right) &= S_t^*\left(\bar{f}^\bullet\right) R_{s-t}\left(\bar{f}^\bullet, f_t^\bullet\right) S_t\left(f^\bullet\right) \\ &+ \int_0^t S_r^*\left(\bar{f}^\bullet\right) \varphi\left(\bar{f}^\bullet(r), R_{s-r}\left(\bar{f}^\bullet, f_r^\bullet\right), f_r^\bullet\right) S_r\left(f^\bullet\right) \mathrm{d}r. \end{aligned}$$

The coherent matrix elements $Y_t\left(\bar{f}^\bullet, f^\bullet\right)$ of the conditional expectation $Y_s = \epsilon_t\left(R_s\right)$ coincide with $R_s\left(\bar{f}^\bullet, f^\bullet\right)$ if $f^\bullet \in \mathfrak{E}_t$. Hence, they satisfy the integral equation

$$\begin{aligned} Y_t\left(\bar{f}^\bullet, f^\bullet\right) &= S_t^*\left(\bar{f}^\bullet\right) P_{s-t} S_t\left(f^\bullet\right) \\ &+ \int_0^t S_r^*\left(\bar{f}^\bullet\right) \varphi\left(\bar{f}^\bullet(r), Y_{t-r}\left(\bar{f}^\bullet, f_r^\bullet\right), f_r^\bullet\right) S_r\left(f^\bullet\right) \mathrm{d}r, \end{aligned}$$

corresponding to the non-adapted backward equation

$$Y_t = S_t^* P_{s-t}^\circ S_t + \int_0^t \mathrm{d}\Lambda_\mu^\nu\left(r, S_r^* \varphi_\nu^\mu\left(Y_{t-r}\right) S_r\right),$$

where $P_s^\circ = P_s \otimes I$, $P_s = R_s(0,0)$, as $f_t^\bullet(r) = f^\bullet(r+t) = 0, \forall r \in \mathbb{R}_+$ if $f^\bullet \in \mathfrak{E}_t$. The operators $P_s = \epsilon\left(R_s\right) = \theta_s(I)$ are given by the Markov semigroup $\theta_s = \epsilon \circ \phi_s$ as the decreasing solutions to the integral equation

$$P_s = e^{-Ks*} e^{-Ks} + \int_0^s e^{-Kr*} \varphi\left(P_{s-r}\right) e^{-Kr} \mathrm{d}r,$$

and $P_t \leq I$ if $K + K^\dagger \leq 0$. (See, for example, [34].) Thus, the difference $\widetilde{R}_t = R_t - \epsilon_t\left(R_s\right) = R_t - Y_t$ satisfies the same equation

$$\widetilde{R}_t = S_t^* \widetilde{I}_{s-t} S_t + \int_0^t \mathrm{d}\Lambda_\mu^\nu\left(r, S_r^* \varphi_\nu^\mu\left(\widetilde{R}_{t-r}\right) S_r\right)$$

as R_t with $\widetilde{I}_s = I - P_s^\circ$ instead of I. The iteration of this equation defines it as the weak limit $\widetilde{R}_t = \lim \widetilde{R}_t^n$ in the form of the series (4.10) with $B = I - P_{s-t} \geq 0$. Hence $\widetilde{R}_t = \phi_t\left(I - P_{s-t}\right)$ is a positive sesquilinear form on $\mathfrak{D}$ for any $s \geq t$ due to the positivity of ϕ_t. The proof of contractivity $\phi_t(I) \leq \phi_s(I)$ for $t < s$ is similar to that one, without the vacuum averaging of R_t. □

5. The structure of the generators and flows

First, let us prove the structure (3.9) for the (unbounded) form-generator of CP flows over the algebra $\mathcal{B} = \mathcal{L}(\mathcal{H})$ of all bounded operators. This algebra contains the one-dimensional operators $|\eta'\rangle\langle\eta^0| : \eta \mapsto \langle\eta^0|\eta\rangle\, \eta'$ given by the vectors $\eta^0, \eta' \in \mathcal{H}$.

1. Let us fix a vector $\boldsymbol{\eta}^0 \in \mathcal{D} \oplus \mathcal{D}_\bullet$ with the unit projection $\eta^0 \in \mathcal{D}$, $\left\|\eta^0\right\| = 1$, and make the following assumption of the weak continuity for the linear operator $\eta' \mapsto \boldsymbol{\gamma}\left(|\eta'\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0$.

0) The sequence $\boldsymbol{\eta}_n' = \boldsymbol{\gamma}\left(|\eta_n'\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0 \in \mathcal{D}' \oplus \mathcal{D}_\bullet'$ of anti-linear forms

$$\boldsymbol{\eta} \in \mathcal{D} \oplus \mathcal{D}_\bullet \mapsto \langle\boldsymbol{\eta}|\boldsymbol{\eta}_n'\rangle := \left\langle\boldsymbol{\eta}|\boldsymbol{\gamma}\left(|\eta_n'\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle$$

converges for each sequence $\eta_n' \in \mathcal{H}$ converging in $\mathcal{D}' \supseteq \mathcal{H}$.

Proposition 5.1. *Let the CCP germ-matrix $\boldsymbol{\gamma}$ satisfy the above continuity condition for a given $\boldsymbol{\eta}^0$. Then there exist strongly continuous operators $K \in \mathcal{L}(\mathcal{D}), K_\bullet : \mathcal{D}_\bullet \to \mathcal{D}$ defining the matrix operator $\boldsymbol{K}$ in (3.9), such that the matrix map (3.8) is CP, and there exists a Hilbert space $\mathcal{K}$, a $*$-representation $\jmath : B \mapsto B \otimes J$ of*

$\mathcal{B}=\mathcal{L}(\mathcal{H})$ on the Hilbert product $\mathcal{G}=\mathcal{H}\otimes\mathcal{K}$, given by an orthoprojector J in $\mathcal{K}$, such that

$$\varphi(B)=\left(L^{\mu}\jmath(B)L_{\nu}\right)_{\nu=+,\bullet}^{\mu=-,\bullet}=\boldsymbol{L}^{*}\jmath(B)\boldsymbol{L}. \tag{5.1}$$

Here $\boldsymbol{L}=(L,L_\bullet)$ is a strongly continuous operator $\mathcal{D}\oplus\mathcal{D}_\bullet\to\mathcal{G}$ with $L=L_+$, $L^-=L^*$, $L^\bullet=L_\bullet^*$ which is always possible to make

$$\left\langle\eta^0\otimes e|\boldsymbol{L}\boldsymbol{\eta}^0\right\rangle=0,\qquad\forall e\in\mathcal{K}_1, \tag{5.2}$$

where $\mathcal{K}_1=J\mathcal{K}$. If $D=-\lambda(I)\geq 0$, then one can make $L^*L=K+K^\dagger$ in a canonical way, and in addition $L^*L_\bullet=K_\bullet$, $L_\bullet^*L_\bullet=I_\bullet^\bullet$, where $I_\bullet^\bullet=I\delta_\bullet^\bullet$, if $\boldsymbol{D}=-\boldsymbol{\lambda}(I)\geq 0$.

Proof. Define the linear operator $\boldsymbol{A}:\mathcal{H}\to\mathcal{D}'\oplus\mathcal{D}'_\bullet$ by the relation

$$\langle\boldsymbol{\eta}|\boldsymbol{A}\eta'\rangle=\left\langle\boldsymbol{\eta}|\boldsymbol{\gamma}\left(|\eta'\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle$$

for all $\boldsymbol{\eta}\in\mathcal{D}\oplus\mathcal{D}_\bullet$ and $\eta'\in\mathcal{H}$. By the weak continuity it can be extended on $\mathcal{D}'$ and its dual operator $\boldsymbol{A}^*=(A^*,A_\bullet^*)$ into $\mathcal{D}$ is strongly continuous on $\mathcal{D}\oplus\mathcal{D}_\bullet$. The operators

$$K=\tfrac{1}{2}\left\langle\boldsymbol{\eta}^0|\boldsymbol{\gamma}\left(|\eta^0\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle I-A^*,\quad K_\bullet=-A_\bullet^*$$

define the matrix-map (3.8) in the form

$$\begin{aligned}\langle\boldsymbol{\eta}|\boldsymbol{\varphi}(B)\boldsymbol{\eta}\rangle &= \langle\boldsymbol{\eta}|\boldsymbol{\gamma}(B)\boldsymbol{\eta}\rangle+\left\langle\boldsymbol{\eta}^0|\boldsymbol{\gamma}\left(|\eta^0\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle\langle\eta|B\eta\rangle-\\ &\quad\left\langle\boldsymbol{\eta}|\boldsymbol{\gamma}\left(B|\eta\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle-\left\langle\boldsymbol{\eta}^0|\boldsymbol{\gamma}\left(|\eta^0\rangle\langle\eta|B\right)\boldsymbol{\eta}\right\rangle,\end{aligned}$$

where $\eta\in\mathcal{D}$ is the natural projection of $\boldsymbol{\eta}\in\mathcal{D}\oplus\mathcal{D}_\bullet$ onto $\mathcal{D}$. Let us prove that this is a CP map, i.e.

$$\sum_{B,C\in\mathcal{B}}\langle\boldsymbol{\xi}_B|\boldsymbol{\varphi}(B^*C)\boldsymbol{\xi}_C\rangle\geq 0$$

for all $\boldsymbol{\xi}_B=0$ except for a finite number of $B=B_k\in\mathcal{B},k=1,2,\ldots$, for which $\boldsymbol{\xi}_B=\boldsymbol{\eta}^k$. Indeed,

$$\begin{aligned}\left\langle\boldsymbol{\eta}^k|\boldsymbol{\varphi}(B_k^*B_l)\boldsymbol{\eta}^l\right\rangle &= \left\langle\boldsymbol{\eta}^k|\boldsymbol{\gamma}(B_k^*B_l)\boldsymbol{\eta}^l\right\rangle+\left\langle\boldsymbol{\eta}^0|\boldsymbol{\gamma}\left(|\eta^0\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle\left\langle\eta^k|B_k^*B_l\eta^l\right\rangle-\\ &\quad\left\langle\boldsymbol{\eta}^k|\boldsymbol{\gamma}\left(B_k^*B_l|\eta^l\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle-\left\langle\boldsymbol{\eta}^0|\boldsymbol{\gamma}\left(|\eta^0\rangle\langle\eta^k|B_k^*B_l\right)\boldsymbol{\eta}^l\right\rangle\\ &= \sum_{k,l\geq 0}\left\langle\boldsymbol{\eta}^k|\boldsymbol{\gamma}(B_k^*B_l)\boldsymbol{\eta}^l\right\rangle,\end{aligned}$$

where $B_0=-\sum_{B\in\mathcal{B}}B|\xi_B\rangle\langle\eta^0|$, and $\boldsymbol{\eta}^k=\boldsymbol{\xi}_B$ for $B=B_k,k=1,2,\ldots$. Because $\sum_{k\geq 0}B_k\eta^k=B_0\eta^0+\sum_{B\in\mathcal{B}}B\xi_B=0$, this form is positive, as it is written as a conditionally positive form

$$\sum_{B,C\in\mathcal{B}}\langle\boldsymbol{\zeta}_B|\boldsymbol{\gamma}(B^*C)\boldsymbol{\zeta}_C\rangle\geq 0,$$

with $\sum_{B\in\mathcal{B}}\boldsymbol{\iota}(B)\boldsymbol{\zeta}_B=0$, where $\boldsymbol{\zeta}_B=\boldsymbol{\eta}^k=\boldsymbol{\xi}_B$ if $B=B_k\neq B_0$, and $\boldsymbol{\zeta}_B=\boldsymbol{\xi}_B+\boldsymbol{\eta}^0$ for $B=B_0$, otherwise $\boldsymbol{\zeta}_B=0$. Moreover,

$$\begin{aligned}\left\langle\boldsymbol{\eta}|\boldsymbol{\varphi}\left(|\eta'\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle &= \left\langle\boldsymbol{\eta}|\boldsymbol{\gamma}\left(|\eta'\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle+\left\langle\boldsymbol{\eta}^0|\boldsymbol{\gamma}\left(|\eta^0\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle\langle\eta|\eta'\rangle-\\ &\quad\left\langle\boldsymbol{\eta}|\boldsymbol{\gamma}\left(|\eta'\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle-\left\langle\boldsymbol{\eta}^0|\boldsymbol{\gamma}\left(|\eta^0\rangle\langle\eta|\eta'\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle=0.\end{aligned}$$

Thus, the form-generator over $\mathcal{B}=\mathcal{L}(\mathcal{H})$ has the form (3.9), where the CP map $\boldsymbol{\varphi}$ can always be chosen to satisfy $\boldsymbol{\varphi}\left(|\eta'\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0=0$ for all $\eta'\in\mathcal{H}$ and a given vector

$\eta^0 \in \mathcal{D}$. The Steinspring dilation (5.1) of the CP map $\boldsymbol{\varphi}$ into the continuous forms $\boldsymbol{\varphi}(B) \in \mathcal{B}(\mathcal{D} \oplus \mathcal{D}_\bullet)$ is given by a continuous operator $\boldsymbol{L} : \mathcal{D} \oplus \mathcal{D}_\bullet \to \mathcal{G}$ with the dual $\boldsymbol{L}^* : \mathcal{G} \to \mathcal{D}' \oplus \mathcal{D}'_\bullet$ because

$$\|\boldsymbol{L}\boldsymbol{\eta}_n\|^2 = \langle \boldsymbol{\eta}_n | \boldsymbol{\varphi}(I)\, \boldsymbol{\eta}_n \rangle \longrightarrow 0$$

if $\boldsymbol{\eta}_n \longrightarrow 0$ strongly in $\mathcal{D} \oplus \mathcal{D}_\bullet$. The w*-representation $\jmath : \mathcal{B} \to \mathcal{L}(\mathcal{G})$ of $\mathcal{B} = \mathcal{L}(\mathcal{H})$ is always an ampliation $\jmath(B) = B \otimes J$, where J is an orthoprojector onto a subspace $\mathcal{K}_1 \subseteq \mathcal{K}$, corresponding to the minimal dilation in $\mathcal{G}_1 = \mathcal{H} \otimes \mathcal{K}_1$. The property (5.2) follows from the inequality $|e\rangle\langle e| \leq J$ if $Je = e$:

$$\begin{aligned} \left|\left\langle \eta^0 \otimes e | \boldsymbol{L}\boldsymbol{\eta}^0 \right\rangle\right|^2 &\leq \left\langle \boldsymbol{L}\boldsymbol{\eta}^0 | \jmath\left(|\eta^0\rangle\langle\eta^0|\right) \boldsymbol{L}\boldsymbol{\eta}^0 \right\rangle \\ &= \left\langle \boldsymbol{\eta}^0 | \boldsymbol{\varphi}\left(|\eta^0\rangle\langle\eta^0|\right) \boldsymbol{\eta}^0 \right\rangle = 0. \end{aligned}$$

If $K + K^\dagger \geq \varphi(I)$, $L^1 = (I \otimes J)\, L$ is the operator of the minimal dilation $\varphi(B) = L_1 \jmath(B)\, L^1$, so that $\varphi(I) = L_1 L^1$ with respect to the adjoint $L_1 : \mathcal{G}_1 \to \mathcal{D}'$, and L^0 is an operator on $\mathcal{D}$ into a Hilbert product $\mathcal{G}_0 = \mathcal{H} \otimes \mathcal{K}_0$, satisfying the condition $L_0 L^0 = D$ with respect to the adjoint $L_0 : \mathcal{G}_0 \to \mathcal{D}'$, then $L^\circ : \eta \mapsto L^0\eta \oplus L^1\eta$ defines the canonical dilation in $\mathcal{G}_\circ = \mathcal{H} \otimes \mathcal{K}_\circ$ having the property $L_\circ L^\circ = \varphi(I) + D = K + K^\dagger$, where $\mathcal{K}_\circ = \mathcal{K}_0 \oplus \mathcal{K}_1$ and $L_\circ : \mathcal{G}_\circ \to \mathcal{D}'$ is the adjoint to $L^\circ : \mathcal{D} \to \mathcal{G}_\circ$. Moreover, if

$$\boldsymbol{D} = \left(\delta^\mu_\nu I - \varphi^\mu_\nu(I) + \delta^\mu_- K_\nu + K^\mu \delta^+_\nu\right)^{\mu=-,\bullet}_{\nu=+,\bullet} = \boldsymbol{K} + \boldsymbol{K}^\dagger - \boldsymbol{\varphi}(I) \geq 0,$$

the operator $\boldsymbol{L}^\circ : \eta \mapsto \boldsymbol{L}^0\eta \oplus \boldsymbol{L}^1\eta$ defines the canonical dilation with the property $\boldsymbol{L}^*\boldsymbol{L} = \boldsymbol{K} + \boldsymbol{K}^\dagger$:

$$L^\mu_\circ L^\circ_\nu = L^\mu_0 L^0_\nu + L^\mu_1 L^1_\nu = D^\mu_\nu + \varphi^\mu_\nu(I) = \delta^\mu_- K_\nu + K^\mu \delta^+_\nu + \delta^\mu_\nu I,$$

where $\boldsymbol{L}^0 : \boldsymbol{\mathcal{D}} \to \mathcal{G}_0$ are operators $\left(L^0, L^0_\bullet\right)$ with the adjoints $L^\mu_0 = L^{0*}_{-\mu}$, satisfying the conditions $L^\mu_0 L^0_\nu = D^\mu_\nu$, and $\boldsymbol{L}^1_\nu = (I \otimes J)\, \boldsymbol{L}_\nu$ are the operators of the minimal dilation $\varphi^\mu_\nu(B) = L^\mu_1 \jmath(B)\, L^1_\nu$. □

2. Thus we have proved that Eq. (3.2) for completely positive quantum stochastic flows over $\mathcal{B} = \mathcal{L}(\mathcal{H})$ has the following general form:

$$\mathrm{d}\phi_t(B) + \phi_t(K^*B + BK - L^*\jmath(B)\, L)\,\mathrm{d}t = \sum_{m,n=1}^{\infty} \phi_t(L^*_m \jmath(B)\, L_n - B\delta^m_n)\,\mathrm{d}\Lambda^n_m$$

$$+ \sum_{m=1}^{\infty} \phi_t(L^*_m \jmath(B)\, L - K^*_m B)\,\mathrm{d}\Lambda^+_m + \sum_{n=1}^{\infty} \phi_t(L^*\jmath(B)\, L_n - BK_n)\,\mathrm{d}\Lambda^n_-,$$

generalizing the Lindblad form [1] for the semigroups of completely positive maps. This can be written in the tensor notation form as

$$\mathrm{d}\phi_t(B) = \phi_t\left(L^{\star\mu}_\alpha \jmath^\alpha_\beta(B)\, L^\beta_\nu - \imath^\mu_\nu(B)\right)\mathrm{d}\Lambda^\nu_\mu = \phi_t(\mathbf{L}^\star \mathbf{\jmath}(B)\, \mathbf{L} - \mathbf{\imath}(B)) \cdot \mathrm{d}\boldsymbol{\Lambda}, \tag{5.3}$$

where the summation is taken over all $\alpha, \beta = -, \circ, +$ and $\mu, \nu = -, \bullet, +$, $\jmath^-_-(B) = B = \jmath^+_+(B)$, $\jmath^\circ_\circ(B) = \jmath(B)$, $\jmath^\alpha_\beta(B) = 0$ if $\alpha \neq \beta$, $\imath^\mu_\nu(B) = B\delta^\mu_\nu$, and $\mathbf{L}^\star = [L^{\star\mu}_\alpha]^{\mu=-,\bullet,+}_{\alpha=-,\circ,+}$ is the triangular matrix, pseudoadjoint to $\mathbf{L} = \left[L^\beta_\nu\right]^{\beta=-,\circ,+}_{\nu=-,\bullet,+}$ with $L^-_- = I = L^+_+$,

$$L^\circ_\bullet = L_\bullet, \quad L^\circ_+ = L, \quad L^-_\bullet = -K_\bullet, \quad L^-_+ = -K.$$

(All other L^β_ν are zero.) If the Hilbert space $\mathcal{H} \otimes \mathcal{G}$ is embedded into the direct sum $\mathcal{H} \oplus \mathcal{H} \oplus \ldots$ of copies of the initial Hilbert space $\mathcal{H}$ such that $J = \left[\delta^i_l\right]$ for a subset $i,l\notin\mathbb{N}_0\subseteq\mathbb{N}$, this equation can be resolved as $\phi_t(B)=V_t^*(B\otimes I_t)V_t$, where $V=(V_t)_{t>0}$ is an (unbounded) cocycle on the product $\mathcal{D}\otimes\mathfrak{F}$ with Fock space $\mathfrak{F}$ over the Hilbert space $L^2(\mathbb{N}\times\mathbb{R}_+)$ of the quantum noise, and I_t is the solution to the stochastic equation $\mathrm{d}I_t+\sum_{n\in\mathbb{N}_0}I_t\mathrm{d}\Lambda_n^n$ with $I_0=I$ in $\mathfrak{F}$. The cocycle V satisfies the quantum stochastic equation $\mathrm{d}V_t=(L_\nu^\mu-I\delta_\nu^\mu)V_t\mathrm{d}\Lambda_\mu^\nu$ of the form

(5.4)
$$\mathrm{d}V_t+KV_t\mathrm{d}t+\sum_{n=1}^{\infty}K_nV_t\mathrm{d}\Lambda_-^n=\sum_{m,n=1}^{\infty}(L_n^m-I\delta_n^m)V_t\mathrm{d}\Lambda_m^n+\sum_{m=1}^{\infty}L^mV_t\mathrm{d}\Lambda_m^+,$$

where L_n^i and L^i are the operators in $\mathcal{D}$, defining

$$\begin{aligned}\varphi_n^m(B)&=\sum_{l\notin\mathbb{N}_0}L_m^{l*}BL_n^l,\qquad \varphi(B)=\sum_{l\notin\mathbb{N}_0}L^{l*}BL^l\\ \varphi^m(B)&=\sum_{l\notin\mathbb{N}_0}L_m^{l*}BL^l,\qquad \varphi_n(B)=\sum_{l\notin\mathbb{N}_0}L^{l*}BL_n^l\end{aligned}\tag{5.5}$$

with $\sum_{i=1}^{\infty}L^{i*}L^i=K+K^\dagger$ if $K+K^\dagger\geq\varphi(I)=\sum_{l\notin\mathbb{N}_0}L^{i*}L^i$. The formal derivation of Eq. (5.4) from (5.3) is obtained by a simple application of the HP Itô formula. The martingale M_t, describing the density operator for the output state of $\Lambda(t,a)$, is then defined as $M_t=V_t^*V_t$.

3. The following theorem ensures the existence of a $*$-representation $\iota:\Lambda(t,a)\mapsto\Lambda(t,i(a)):=i_\beta^\alpha(a)\Lambda_\alpha^\beta(t)$ of the quantum stochastic process (0.2), commuting with $Y_t=\phi_t(B)$ for all $a\in\mathfrak{a},B\in\mathcal{L}(\mathcal{H})$, in the form

$$\Lambda(t,i(a))=i_\circ^\circ(a)\Lambda_\circ^\circ(t)+i_+^\circ(a)\Lambda_\circ^+(t)+i_\circ^-(a)\Lambda_-^\circ(t)+i_+^-(a)\Lambda_-^+(t).$$

Here $\boldsymbol{i}=\left(i_\beta^\alpha\right)_{\beta=+,\circ}^{\alpha=-,\circ}$ is a $\star$-representation

$$i_\beta^\alpha(a^\star a)=i_\circ^\alpha(a^\star)i_\beta^\circ(a),\quad i_{-\beta}^\alpha(a^\star)=i_{-\alpha}^\beta(a)^*$$

of the Itô algebra $\mathfrak{a}$ in the operators $i_\beta^\alpha(a):\mathcal{K}_\beta\to\mathcal{K}_\alpha$, with a domain $\mathcal{K}_\circ\subseteq\mathcal{K}$, $\mathcal{K}_-=\mathbb{C}=\mathcal{K}_+$, and $\Lambda_\alpha^\beta(t)$ are the canonical quantum stochastic integrators in the Fock space $\Gamma(\mathfrak{K})$ over $\mathfrak{K}=L_\mathcal{K}^2(\mathbb{R}_+)$, the space of $\mathcal{K}$-valued square-integrable functions on $\mathbb{R}_+$.We shall extend $\boldsymbol{i}$ to the triangular matrix representation $\mathbf{i}=\left[i_\beta^\alpha\right]_{\beta=-,\circ,+}^{\alpha=-,\circ,+}$ on the pseudo-Hilbert space $\mathbb{C}\oplus\mathcal{K}\oplus\mathbb{C}$ with the Minkowski metrics tensor $\mathbf{g}=\left[\delta_{-\beta}^\alpha\right]=\mathbf{g}^{-1}$, by $i_\beta^+(a)=0=i_-^\alpha(a)$, for all $a\in\mathfrak{a}$, as it was done for $\mathbf{a}=[a_\nu^\mu]_{\nu=-,\bullet,+}^{\mu=-,\bullet,+}$, and denote the ampliation $I\otimes i_\beta^\alpha(a)$ again as $i_\beta^\alpha(a)$ by omitting the index $\circ$. Note that if the stochastic generator of the form (3.9) is restricted onto an operator algebra $\mathcal{B}\subseteq\mathcal{L}(\mathcal{H})$ with the weak closure $\bar{\mathcal{B}}=\mathcal{A}^c$, and all the sesquilinear forms $\gamma_\nu^\mu(B)$, $B\in\mathcal{B}$ commute with the $*$-algebra $\mathcal{A}\subset\mathcal{L}(\mathcal{D})$, then $\lambda_\nu^\mu(B)\in\bar{\mathcal{B}}$.

Proposition 5.2. *Let $\boldsymbol{b}=\boldsymbol{\gamma}(B)-\boldsymbol{\imath}(B)$ satisfy the commutativity conditions (3.4) for all $a\in\mathfrak{a}$, $B\in\mathcal{L}(\mathcal{H})$. Then there exists a $\star$-representation $a\mapsto\boldsymbol{i}(a)$ of the Itô algebra $\mathfrak{a}$, defining the operators $i_\beta^\alpha(a):\mathcal{K}_\beta\to\mathcal{K}_\alpha$, with $i_\beta^\alpha(a)^*\mathcal{K}_\alpha\subseteq\mathcal{K}_\beta$, where $\mathcal{K}_-=\mathbb{C}=\mathcal{K}_+$, such that $L_\mu^\alpha(I\otimes a_\nu^\mu)=\left(I\otimes i_\beta^\alpha(a)\right)L_\nu^\beta$ for all $a\in\mathfrak{a}$:*

$$\begin{aligned}L_\bullet a_\bullet^\bullet&=i(a)L_\bullet,\qquad a_+^- - K_\bullet a_+^\bullet=i^-(a)L+i_+^-(a),\\ L_\bullet a_+^\bullet&=i(a)L+i_+(a),\qquad a_\bullet^- - K_\bullet a_\bullet^\bullet=i^-(a)L_\bullet.\end{aligned}\tag{5.6}$$

If $[A, \gamma^\mu_\nu(B)] = 0$ *for all* $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$*, where* $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ *is a* $*$*-algebra of bounded operators, and* $\bar{\mathcal{B}} = \mathcal{A}^c$*, then there exists a triangular* $\star$*-representation* $\mathbf{j} = \left[j^\alpha_\beta\right]^{\alpha=-,\circ,+}_{\beta=-,\circ,+}$ *of the operator algebra* $\mathcal{A}$ *with* $j^\circ_\circ(I) = J$ *such that*

(5.7)

$$\mathbf{JLA} = \mathbf{j}(A)\mathbf{L}, \quad [\mathbf{j}(A), \mathbf{i}(a)] = 0, \quad [\mathbf{j}(A), \jmath(B)] = 0, \quad \forall A \in \mathcal{A}, a \in \mathfrak{a}, B \in \mathcal{B}.$$

Proof. Let $\boldsymbol{\mathcal{G}} = \mathcal{G}_- \oplus \mathcal{G} \oplus \mathcal{G}_+$ be the pseudo-Hilbert space, where $\mathcal{G}_+ = \mathcal{D} = \mathcal{G}_-$, $\mathcal{G} \subseteq \mathcal{H} \otimes \mathcal{K}$ is the linear span of $\{\jmath(B)\boldsymbol{L\eta} | B \in \mathcal{B}, \boldsymbol{\eta} \in \mathcal{D} \oplus \mathcal{D}_\bullet\}$ and the indefinite metrics is defined by

$$\left\langle \xi^\alpha | g_{\alpha\beta}\xi^\beta \right\rangle = \|\xi\|^2 + 2\,\mathrm{Re}\left\langle \xi^+ | \xi^- \right\rangle, \quad \xi^\alpha \in \mathcal{G}_\alpha, \xi^\circ = \xi \in \mathcal{G}.$$

The algebra $\mathcal{B} = \mathcal{L}(\mathcal{H})$ is represented on $\boldsymbol{\mathcal{G}}$ by the ampliation $\jmath(B) = B \otimes \mathbf{J}$, where $\mathbf{J} = 1 \oplus J \oplus 1$, and $\jmath(B)\mathbf{Lg}\boldsymbol{\eta} \in \boldsymbol{\mathcal{G}}_\circ$, where the pre-Hilbert space $\mathcal{D} \oplus \mathcal{D}_\bullet$ is isometrically embedded into $\mathcal{D} \oplus \mathcal{D}_\bullet \oplus \mathcal{D}$ as $\mathbf{g}(\eta \oplus \eta^\bullet) = 0 \oplus \eta^\bullet \oplus \eta$. We define the representations $\mathbf{i}$ and $\mathbf{j}$ on $\boldsymbol{\mathcal{G}}$ by intertwining

$$\mathbf{i}(a)\mathbf{L} = \mathbf{La}, \quad \mathbf{i}(a)\jmath(B)\mathbf{L} = \jmath(B)\mathbf{La}, \quad \mathbf{j}(A)\jmath(B)\mathbf{L} = \jmath(B)\mathbf{LA},$$

the operators $\mathbf{a} = I \otimes \boldsymbol{a}\mathbf{g}$ and $\mathbf{A} = A \otimes \mathbf{I}$. Such a definition is correct, because if $\jmath(B_k)\mathbf{L}\boldsymbol{\zeta}^k = 0$ for a finite family of non-zero $\boldsymbol{\zeta}^k \in \mathcal{D} \oplus \mathcal{D}_\bullet$, then

$$\left(\jmath(B)\mathbf{Lg}\boldsymbol{\eta} | \mathbf{i}(a)\jmath(B_k)\mathbf{Lg}\boldsymbol{\zeta}^k\right) = \left(\jmath(B)\mathbf{Lg}\boldsymbol{\eta} | \jmath(B_k)\mathbf{Lag}\boldsymbol{\zeta}^k\right) =$$

$$\left\langle \boldsymbol{\eta} | \boldsymbol{\gamma}(B^* B_k)\mathbf{g}\boldsymbol{a}\boldsymbol{\zeta}^k \right\rangle = \left\langle \boldsymbol{\eta} | \boldsymbol{a}\mathbf{g}\boldsymbol{\gamma}(B^* B_k)\boldsymbol{\zeta}^k \right\rangle =$$

$$\left\langle \boldsymbol{a}^\star\boldsymbol{\eta} | \mathbf{g}\boldsymbol{\gamma}(B^* B_k)\boldsymbol{\zeta}^k \right\rangle = \left(\jmath(B)\mathbf{La}^\star\mathbf{g}\boldsymbol{\eta} | \jmath(B_k)\mathbf{Lg}\boldsymbol{\zeta}^k\right) = 0$$

for all $\boldsymbol{\eta} \in \mathcal{D} \oplus \mathcal{D}_\bullet$ and $B \in \mathcal{L}(\mathcal{H})$, and so $\mathbf{i}(a)\jmath(B_k)\mathbf{L}\boldsymbol{\zeta}^k = 0$. Here we used the condition

$$\boldsymbol{\gamma}(B)\mathbf{g}\boldsymbol{a} = (\imath(B) + \mathbf{b})\mathbf{ag} = \mathbf{a}(\imath(B) + \mathbf{b})\mathbf{g} = \boldsymbol{a}\mathbf{g}\boldsymbol{\gamma}(B),$$

as $\mathbf{ab} = \mathbf{ba}$ due to the HP commutativity $\boldsymbol{a}\mathbf{g}\boldsymbol{b} = \boldsymbol{b}\mathbf{g}\boldsymbol{a}$ of $\boldsymbol{b} = \boldsymbol{\lambda}(B)$ where $\boldsymbol{\gamma} = (\gamma^\mu_\nu)$ is extended to all indexes as $\gamma^\mu_\nu(B) = B\delta^\mu_\nu + C^\mu_\nu$ with $C^\mu_\nu = 0$ if $\mu = +$ or $\nu = -$. In the same way, the operators $\mathbf{j}(A)$ are correctly defined for $B \in \mathcal{A}^c$ if $\boldsymbol{\gamma}(B)\imath(A) = \imath(A)\boldsymbol{\gamma}(B)$. This also proves that $\mathbf{i}(a)^\star = \mathbf{i}(a^\star)$, and $\mathbf{j}(A)^\star = \mathbf{j}(A^*)$. (The multiplicativity of $\mathbf{i}, \mathbf{j}$ as well as the commutativity properties (5.7) directly follow from the definition of these operators.) Note that $\mathbf{j}(I) = I \otimes \mathbf{J}$, and if the dilation is minimal, $\mathbf{j}(I) = I \otimes \mathbf{I}$. If it is not, the unital property can still be achieved for the canonical dilations in $\mathcal{K}_\circ$, by adding $j^\circ_\circ(A)(\eta \otimes e_0) = A\eta \otimes e_0$ for all $e_0 \in \mathcal{K}$ with $Je_0 = 0$. □

4. Now we are going to construct the quantum stochastic dilation for the flow $\phi_t(B)$ and the quantum state generating function $\vartheta^a_t = \epsilon[R_t W(t,a)]$ of the output process $\boldsymbol{\Lambda}(t,a)$ in the form

$$\phi_t(B) = V^*_t(I_t \otimes B)V_t, \quad \vartheta_t(g) = \epsilon\left[V^*_t(W^a_t \otimes I)V_t\right], \quad \forall B \in \mathcal{L}(\mathcal{H}), a \in \mathfrak{a},$$

where V_t is an operator on $\mathcal{D} \otimes \mathfrak{F}$ into $\Gamma(\mathfrak{K}) \otimes \mathcal{D} \otimes \mathfrak{F}$, intertwining the Weyl operators $W(t,a)$ with the operators $W^a_t = W(t, i(a))I_t$ in the Fock space $\Gamma(\mathfrak{K})$,

$$\mathrm{d}W(t, i(a)) = W(t, i(a))\,\mathrm{d}\Lambda(t, i(a)), \quad W(0, i(a)) = I,$$

and $I_t \geq I_s, \forall t \leq s$ is a decreasing family of orthoprojectors.

In order to prove the existence of the Fock space dilation we need the following assumptions in addition to the continuity assumptions of this and previous sections.

1) The minimal quantum stochastic CP flow over the algebra $\mathcal{A}$, resolving the quantum Langevin equation

$$\mathrm{d}\tau_t(A) = \tau_t(\mathbf{j}(A) - \imath(A)) \cdot \mathrm{d}\boldsymbol{\Lambda}, \quad \tau_0(A) = I \otimes A, \quad A \in \mathcal{A}, \tag{5.8}$$

where $\mathbf{j}(I) = I \otimes \mathbf{J}$, $\imath(A) = A \otimes \mathbf{I}$, is the multiplicative flow, satisfying the condition $\tau_t(I) = I_t \otimes I$, where I_t is the solution to the stochastic equation $\mathrm{d}I_t = (J - I)^\circ_\circ I_t \mathrm{d}\Lambda^\circ_\circ$ with $I_0 = I$.

2) Let us assume the strong continuity of the operators $L(\bar{e}) : \mathcal{D} \to \mathcal{D}$, $L_\bullet(\bar{e}) : \mathcal{D}_\bullet \to \mathcal{D}$, given for all $e \in \mathcal{K}$ as $L_\nu(\bar{e}) = (I \otimes e^*) L_\nu$ by $\langle L_\nu(\bar{e})\eta|\eta'\rangle = \langle L_\nu \eta|\eta' \otimes e\rangle \quad \forall \eta \in \mathcal{D}, \eta' \in \mathcal{D}'$. This is necessary for the definition of the operators $V_t(\sigma)$ for each subset $\sigma \subset [0, t)$ of a finite cardinality $|\sigma| \in \mathbb{N}$ by the recurrence

$$V_t(\sigma)\psi = V_t^\circ(s)\left(LV_s(\sigma\backslash s)\psi + \sum_m L_m V_r(\sigma\backslash s)\psi^m(s)\right), \quad s = \max\sigma,$$

with $V_t(\emptyset) = V_t^\circ$. Here $V_t^\circ(s) = T_t^* S_{t-s} T_s$, V_t° is the solution to Eq. (4.3) in $\mathcal{D}\otimes\mathfrak{F}$ acting as $I_\circ^{\otimes|\sigma|} \otimes V_t^\circ$ on $\mathcal{K}^{\otimes|\sigma|} \otimes \mathcal{D} \otimes \mathfrak{F}$, the operators $L_\nu : \mathcal{D} \to \mathcal{K} \otimes \mathcal{H}$ act on $\mathcal{K}^{\otimes|\sigma\backslash s|} \otimes \mathcal{H} \otimes \mathfrak{F}$ as $I_\circ^{\otimes|\sigma\backslash s|} \otimes L \otimes I$ (s is identified with the single point subset $\{s\}$ such that $\sigma\backslash\max\sigma$ is the σ without its maximum,) and $\psi^\bullet(s) \in \mathcal{K} \otimes \mathcal{H} \otimes \mathfrak{F}$ is given as $\psi^\bullet(\tau, s) = \psi(\tau \sqcup s)$ of $\psi \in \mathcal{H} \otimes \mathfrak{F}$, where $\tau \sqcup s$ is defined for almost all s ($s \notin \sigma$) as the disjoint union of the single point $\{s\}$ with a finite subset $\tau \in \mathbb{R}_+$.

3) The operator-valued function $\sigma \mapsto V_t(\sigma)$, defined for all such $\sigma \in \Gamma_t$, is weakly square integrable for each t with respect to the measure $\mathrm{d}\sigma = \prod_{s\in\sigma} \mathrm{d}s$ in the sense

$$\int_{\Gamma_t} \|V_t(\sigma)\psi\|^2 \mathrm{d}\sigma := \sum_{n=0}^\infty \int_{0<s_1\dots s_n<t}\!\!\dots\int \|V_t(s_{1,\dots,}s_n)\psi\|^2 \mathrm{d}s_1\dots\mathrm{d}s_1 < \infty,$$

for all $\psi \in \mathcal{D}\otimes\mathfrak{F}$. Thus the operators V_t can be extended to the Fock space ones $V_t : \mathcal{D}\otimes\mathfrak{F} \to \Gamma(\mathfrak{K})\otimes\mathcal{D}\otimes\mathfrak{F}$, say, by letting $V_t(\sigma) = V_t(\sigma_t)\otimes\delta_\emptyset\left(\sigma_{[t}\right)$ for all finite $\sigma \subset \mathbb{R}_+$ if $\sigma_{[t} = \sigma \cap [t, \infty) \neq \emptyset$. Obviously they form a cocycle, $V_{t-r}^r(\sigma) V_r(\sigma) = V_t(\sigma)$, where $V_s^r(\sigma) = I_\circ^{\otimes|\sigma_r|} \otimes T_r^* V_s\left(\sigma_{[r} - r\right) T_r$ with $\sigma_r = \sigma \cap [0, r)$.

Theorem 5.3. *Under the given assumptions 0), 1), 2), 3) there exist:*

(i) *A cocycle dilation $V_t : \mathcal{D} \otimes \mathfrak{F} \to \Gamma(\mathfrak{K}) \otimes \mathcal{D} \otimes \mathfrak{F}$ of the minimal CP flow ϕ, intertwining the Weyl operator $W(t, a)$ with W_t^a:*

$$V_t(I \otimes W(t, a)) = (W_t^a \otimes I) V_t, \quad \phi_t(B) = V_t^*(I_t \otimes B) V_t, \quad \forall a \in \mathfrak{a}, B \in \mathcal{L}(\mathcal{H}), \tag{5.9}$$

where $I_t \leq I_s$, $\forall t < s$ are orthoprojectors in $\Gamma(\mathfrak{K})$.

(ii) *A $*$-multiplicative flow $\tau = (\tau_t)$ over $\mathcal{A}$ in $\Gamma(\mathfrak{K}) \otimes \mathcal{H}$ with the properties $\tau_t(I) = I_t$,*

$$V_t A = \tau_t(A) V_t, \quad [\tau_t(A), W_t^a] = 0, \quad [\tau_t(A), I \otimes B] = 0, \quad \forall A \in \mathcal{A}, a \in \mathfrak{a}, B \in \mathcal{B}. \tag{5.10}$$

(iii) *If* $\lambda(I) \leq 0$*, then one can make* $M_t = V_t^* V_t$ *martingale, and, if* $\boldsymbol{\lambda}(I) \leq 0$*, one can make* V_t *isometric,* $V_t^* V_t = I$*.*

(iv) *Moreover, let* $U = (U_t)_{t>0}$ *be a one parametric weakly continuous cocycle of unitary operators on* $\Gamma(\mathfrak{K}) \otimes \mathcal{H} \otimes \Gamma(\mathfrak{E})$ *, giving the unique solution to the quantum stochastic equation*

$$\begin{aligned}(5.11)\qquad & \mathrm{d}U_t + \left(K\mathrm{d}t + K_\bullet^- \mathrm{d}\Lambda_-^\bullet + K_\circ^- \mathrm{d}\Lambda_-^\circ\right) U_t \\ = & \left(L_+^\circ \mathrm{d}\Lambda_\circ^+ - I_\bullet^\bullet \mathrm{d}\Lambda_\bullet^\bullet + J_\bullet^\circ \mathrm{d}\Lambda_\circ^\bullet + J_\circ^\bullet \mathrm{d}\Lambda_\bullet^\circ + (J_\circ^\circ - I_\circ^\circ)\,\mathrm{d}\Lambda_\circ^\circ\right) U_t\end{aligned}$$

with $U_0 = I$ *and the necessary differential unitarity conditions*

$$K + K^\dagger = L_\circ^- L_+^\circ, K_\bullet^- = L_\circ^- J_\bullet^\circ, J_\circ^\bullet J_\bullet^\circ = I_\bullet^\bullet, K_\circ^- = L_\circ^- J_\circ^\circ, J_\circ^\circ = I_\circ^\circ - J_\bullet^\circ J_\circ^\bullet,$$

where $L_\circ^- = L_+^{\circ *}$*,* $J_\circ^\bullet = J_\bullet^{\circ *}$*. If* $\lambda(I) \leq 0$*, and* $L_+^\circ = L^\circ$ *is the canonical operator in the dilation (5.1), then*

$$(5.12)\qquad \langle\psi|\,(A \otimes I)\,\phi_t^a(B)\,\psi\rangle = \langle U_t(\delta_\emptyset \otimes \psi)\,|\,(\tau_t^a(A)\,(I \otimes B))\,U_t(\delta_\emptyset \otimes \psi)\rangle$$

for all $A \in \mathcal{A}, a \in \mathfrak{a}, B \in \mathcal{B}$*, where* ψ *is any initial state* $\psi_0 = \eta \otimes \delta_\emptyset$*,* $\eta \in \mathcal{D}$*, and*

$$\phi_t^a(B) = (I \otimes W(t,a))\,\phi_t(B)\,, \quad \tau_t^a(A) = (W_t^a \otimes I)\,\tau_t(A)\,.$$

If $\boldsymbol{\lambda}(I) \leq 0$*, and in addition* $J_\bullet^\circ = L_\bullet^\circ$ *is the canonical isometry in (5.1), this unitary cocycle dilation is valied also for any vector-state* $\psi \in \mathcal{D} \otimes \mathfrak{F}$ *.*

Proof. (Sketch). The cocycle $V = (V_t)_{t>0}$ is recurrently constructed due to the above assumptions 0)–3). It obviously intertwines the Weyl operators (2.4) with the operators W_t^a, acting in the same way in $\Gamma(\mathfrak{K})$, by virtue of the property (5.6).

Let us denote by $\mathfrak{K}_1 = L^2_{\mathcal{K}}(\mathbb{R}_+)$ the functional Hilbert space corresponding to the minimal dilation (5.1) sub-space $\mathcal{K} = \mathcal{K}_1$ for the CP map $\boldsymbol{\varphi}$, given by the orthoprojector $J = J_1$ in the space $\mathcal{K}_\circ$ of the canonical dilation, and $\mathfrak{K}_0$ its orthogonal compliment, corresponding to $\mathcal{K}_0 = J_0 \mathcal{K}_\circ$, where $J_0 = I - J_1$. Representing $\Gamma(\mathfrak{K}_0 \oplus \mathfrak{K}_1)$ as $\Gamma(\mathfrak{K}_0) \otimes \Gamma(\mathfrak{K}_1)$, let us denote by I_t the survival orthoprojectors $I_t \chi(\sigma^0, \sigma^1) = \delta_\emptyset(\sigma_t^0)\,\chi(\sigma^0, \sigma^1)$, $\sigma_t = \sigma \cap [0,t)$, where $\chi(\sigma^0, \sigma^1) = \chi(\sigma^0 \sqcup \sigma^1) \in \mathcal{K}^{\otimes|\sigma^0|} \otimes \mathcal{K}^{\otimes|\sigma^1|}$ is the set function, representing a $\chi \in \Gamma(\mathfrak{K}_0 \oplus \mathfrak{K}_1)$. The decreasing family $(I_t)_{t>0}$ defines the decay orthoprojectors $E_t = I - I_t$ in $\Gamma(\mathfrak{K}_\circ)$ satisfying the quantum stochastic equation $\mathrm{d}E_t = E_t J_0 \cdot \mathrm{d}\Lambda_\circ^\circ$ with $E_0 = 0$, and $\Lambda_\circ^\circ$ is the number integrator in the Fock space $\Gamma(\mathfrak{K}_\circ)$ over $\mathfrak{K}_\circ = \mathfrak{K}_0 \oplus \mathfrak{K}_1$. Then one easily find that the minimal CP flow (4.10) can be represented as $\phi_t(B) = V_t^*(I_t \otimes B)\,V_t$.

We may also construct the minimal quantum stochastic $*$-flow [2] over the operator algebra $\mathcal{A}$, resolving the quantum Langevin equation (5.8) by its iteration as it was done in Sect 4 for the flow ϕ, and then prove its $*$-multiplicativity under certain conditions as in [30]. However, we can directly construct the representations τ_t with the property $\tau_t(I) = I_t$ in a similar way as it was done for the representation $\mathbf{j}$, and then prove that it satisfies the Langevin equation. Then the properties (5.10) follow from the definition of the operators V_t, and can be checked recurrently by use of (5.6) and (5.7).

The cocycle $U = (U_t)$ is constructed to satisfy the HP quantum stochastic equation (5.11). It can be represented in the form of the stochastic multiple integral of the chronologically ordered products of the coefficients of the quantum differential equation under the integrability conditions given in the Appendix.

If $K + K^\dagger \geq \varphi(I)$, the HP unitarity condition [8] is satisfied for the canonical choice $L_+^\circ = L^\circ$, where $L^\circ = L^0 + L^1$, and arbitrary isometric operator $J_\bullet^\circ$, $J_\circ^\bullet J_\bullet^\circ =$

$I_\bullet^\bullet$ with $K_\bullet^- = L_\circ J_\bullet^\circ$, $K_\circ^- = L_\circ J_\circ^\circ$, $J_\circ^\circ = I_\circ^\circ - J_\bullet^\circ J_\circ^\bullet$. In addition if $\boldsymbol{K}+\boldsymbol{K}^\dagger \geq \boldsymbol{\varphi}(I)$, we make the choice $J_\bullet^\circ = L_\bullet^\circ$ from the canonical dilation, $L_\bullet^\circ = L_\bullet^0 + L_\bullet^1$, and so $K_\bullet^- = L_\circ L_\bullet^\circ = K_\bullet$, where $L_\circ^* = L^\circ$, $J_\circ^\bullet = J_\bullet^{\circ *}$. In the first, subfiltering case $\lambda(I) \leq 0$ such a choice gives $U_t(\delta_\emptyset \otimes \psi_0) = V_t \psi_0$ for any $\psi_0 = \eta \otimes \delta_\emptyset$, $\eta \in \mathcal{D}$ and therefore $\|V_t\psi_0\| = \|\psi_0\|$. Thus $M_t = V_t^* V_t$ is a martingale and the condition (5.12) is satisfied for any initial ψ_0. In the second, contractive case $\boldsymbol{\lambda}(I) \leq 0$ the canonical choice gives $U_t(\delta_\emptyset \otimes \psi) = V_t\psi$ and therefore $\|V_t\psi\| = \|\psi\|$ for any $\psi \in \mathcal{D}\otimes\mathfrak{F}$. Thus $V_t^*V_t = I$ and the condition (5.12) is satisfied for any ψ. □

6. Appendix

Here we give a resume on the sufficient analytical conditions for the quantum multiple integration [5] of stochastic linear differential equations in Hilbert spaces, based on the noncommutative analysis in the Fock scale [31].

1. Let $\|e^\bullet\|^2(\xi) = \xi\|e^\bullet\|^2, \xi > 0$ as in [31], so that the projective limit $\mathcal{E}$ and the dual space $\mathcal{E}'$ coincide with the Hilbert space $\mathcal{K}$ with the norm $\|e\|^2$. The projective Fock space $\mathfrak{F} = \cap_\xi \Gamma(\mathfrak{K},\xi)$ over $\mathfrak{K} = L^2_{\mathcal{K}}(\mathbb{R}_+)$ with respect to the exponential scale (2.1), where $\|f^\otimes\|^2(\xi) = \exp\left[\xi\|f^\bullet\|^2\right]$, $f^\bullet \in \mathfrak{K}$, is the natural domain for the quantum stochastic integration [5], and $\mathfrak{F}' = \cup_\xi \Gamma\left(\mathfrak{K},\xi^{-1}\right)$. If $\mathcal{D} = \cap\mathcal{H}_p$ is the projective limit of an increasing family of the dense Hilbert subspaces $\mathcal{H}_p \subseteq \mathcal{H}_{p-1}$, the π-product $\mathfrak{D} = \mathcal{D}\otimes\mathfrak{F}$ of the Fréchet spaces $\mathcal{D}$ and $\mathfrak{F}$ is the projective limit of the directed family of the spaces $\mathfrak{H}_p(\xi) = \mathcal{H}_p \otimes \Gamma(\mathfrak{K},\xi)$ and $\mathfrak{D}' = \mathcal{D}'\otimes\mathfrak{F}'$ is given as $\cup\mathfrak{H}_{-p}\left(\xi^{-1}\right)$, where $\mathcal{H}_{-p}$ denote the duals $\mathcal{H}_p'$ to the Hilbert spaces $\mathcal{H}_p$, with respect to the standard pairing in the Hilbert product $\mathfrak{H}$ of $\mathcal{H} = \mathcal{H}_0$ and $\Gamma(\mathfrak{K}) = \Gamma(\mathfrak{K},1)$. Following [31, 5], we define the multiple quantum stochastic integral $Y_t = \Lambda^\otimes_{[0,t)}(B)$ of a function $B(\boldsymbol{\tau})$ of the quadruple $\boldsymbol{\tau} = (\tau^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ of finite subsets $\tau^\mu_\nu \subset [0,t)$ with values in the nonadapted kernels $\mathfrak{D}\otimes\mathcal{K}^{\otimes|\tau\cup\tau^-|} \to \mathfrak{D}'\otimes\mathcal{K}^{\otimes|\tau\cup\tau_+|}$, as the sesquilinear form $\langle\psi|Y_t\psi\rangle =$

$$\int_{\Gamma_t}\int_{\Gamma_t}\int_{\Gamma_t}\int_{\Gamma_t}\left\langle \overset{\bullet}{\psi}(\tau\cup\tau_+)\,|B\begin{pmatrix}\tau_+^- & \tau^- \\ \tau_+ & \tau\end{pmatrix}\overset{\bullet}{\psi}\left(\tau\cup\tau^-\right)\right\rangle \mathrm{d}\tau\mathrm{d}\tau^-\mathrm{d}\tau_+\mathrm{d}\tau_+^-, \tag{6.1}$$

where $\overset{\bullet}{\psi}(\sigma,\tau) = \psi(\sigma\cup\tau)$, given by the quadruple of the multiple integrals

$$\int_{\Gamma_t} B(\tau)\,\mathrm{d}\tau = \sum_{n=0}^{\infty}\int\cdots\int_{0<t_1,<\ldots<t_n<t} B(t_1,\ldots,t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n.$$

The function B is integrable up to a $t>0$ if $Y_t \in \mathcal{B}(\mathfrak{D})$, and it is strongly integrable if $Y_t \in \mathcal{L}(\mathfrak{D})$. The natural criterion of multiple integrability was formulated in [31] in terms of the norms $\|B\|'_{p,q}(\xi,\zeta) =$

(6.2)

$$\int_{\Gamma_t}\left(\int_{\Gamma_t}\int_{\Gamma_t}\left(\frac{1}{\zeta}\right)^{|\tau^-|}\left(\frac{1}{\xi}\right)^{|\tau_+|}\sup_{\tau\in\Gamma_t}\left(\left(\frac{1}{\xi\zeta}\right)^{\frac{|\tau|}{2}}\|B(\boldsymbol{\tau})\|'_{p,q}\right)^2\mathrm{d}\tau_+\mathrm{d}\tau^-\right)^{\frac{1}{2}}\mathrm{d}\tau_+^-$$

as $\|B\|'_{p,q}(\xi,\zeta)<\infty$ for some $p,q,\xi,\zeta<\infty$. The function B is strongly integrable if $\|B\|'_{p,q}(\xi,\zeta)<\infty$ for any $p<0,\xi<1$ and some q,ζ. Here

$$\left\|B\begin{pmatrix}\tau_+^- & \tau^- \\ \tau_+ & \tau\end{pmatrix}\right\|'_{p,q}(\xi,\zeta)=\sup_{\psi,\chi}\frac{\left|\left\langle\psi_{n,n_+}|B(\boldsymbol{\tau})\chi_{n,n^-}\right\rangle\right|}{\left\|\psi_{n,n_+}\right\|_p(\zeta)\left\|\chi_{n,n^-}\right\|_q(\zeta)} \tag{6.3}$$

denotes the norm of the kernel $B(\boldsymbol{\tau}):\mathfrak{H}_q(\zeta)\otimes\mathcal{K}^{\otimes n}\otimes\mathcal{K}^{\otimes n_-}\to\mathfrak{H}_{-p}\left(\xi^{-1}\right)\otimes\mathcal{K}^{\otimes n}\otimes\mathcal{K}^{\otimes n_+}$, where $n_\nu^\mu=|\tau_\nu^\mu|$ are the cardinalities of τ_ν^μ. The norms (6.2) define the estimate for the integral by virtue of the inequality [31]

$$\left\|\Lambda_{[0,t)}^{\otimes}(B)\right\|'_{p,q}(3\xi,3\zeta)\le\|B\|'_{p,q}(\xi,\zeta)$$

so that $\Lambda_{[0,t)}^{\otimes}(B)\in\mathcal{B}(\mathfrak{D})$ (or $\in\mathcal{L}(\mathfrak{D})$) is defined as a bounded kernel $Y_t:\mathfrak{H}_q(\zeta)\to\mathfrak{H}_{-p}\left(\xi^{-1}\right)$ if $\|B\|'_{p,q}\left(\frac{1}{3}\xi,\frac{1}{3}\zeta\right)<\infty$ (or if for each p,ξ the norm is finite for some q,ζ).

2. Let us consider the case when the operator-valued function $B(\boldsymbol{\tau})$ is relatively bounded in the following sense

$$\left\|B\begin{pmatrix}\tau_+^- & \tau^- \\ \tau_+ & \tau\end{pmatrix}\right\|'_{p,q}\le\frac{(n+n_++n^-+n_+^-)!}{n!\sqrt{n_+!n^-!}}c_{p,q}\left(n+n_++n^-+n_+^-\right), \tag{6.4}$$

where $c_{p,q}(n)$ are positive constants such that $\sum_{n=0}^\infty c_{p,q}(n)\rho^n<\infty$ for a strictly positive number ρ and sufficiently large ξ or ζ. Integrating $\int\cdots\int_{0<t_1,<\ldots<t_n<t}\mathrm{d}t_1\cdots\mathrm{d}t_n=t^n/n!$ three times, one can find that $\|B\|'_{p,q}(\xi,\zeta)\le$

$$\sum_{n_+^-}\frac{\left(t\sqrt{\xi\zeta}\right)^{n_+^-}}{n_+^-!}\left(\sum_{n^-}\sum_{n_+}\sup_n\left(\frac{(t\xi)^{\frac{n^-}{2}}(t\zeta)^{\frac{n_+}{2}}n!c_{p,q}(n)}{(\xi\zeta)^{\frac{n}{2}}n_+!n^-!\left(n-n_+-n^--n_+^-\right)!}\right)^2\right)^{\frac{1}{2}}$$

$$\le\sum_n\sum_{n_+}\sum_{n^-}\sum_{n_+^-}\frac{(t\zeta)^{\frac{n_+}{2}}(t\xi)^{\frac{n^-}{2}}(t\xi t\zeta)^{\frac{n_+^-}{2}}n!}{(\xi\zeta)^{\frac{n}{2}}n_+!n^-!n_+^-!\left(n-n_+-n^--n_+^-\right)!}c_{p,q}(n),$$

where the supremum and summation is taken over $n\ge n_++n^-+n_+^-$. Then the function B is integrable up to a $t<\rho$ as it has the finite estimate

$$\|B\|'_{p,q}(\xi,\zeta)\le\sum_{n=0}^\infty\left(\frac{\left(1+\sqrt{t\xi}\right)\left(1+\sqrt{t\zeta}\right)}{\sqrt{\xi\zeta}}\right)^n c_{p,q}(n)\le\sum_{n=0}^\infty\rho^n c_{p,q}(n), \tag{6.5}$$

and so $\|Y_t\|'_{p,q}(3\xi,3\zeta)\le\sum_{n=0}^\infty\rho^n c_{p,q}(n)$ if $\sqrt{\xi\zeta}>1/\rho$ and

$$\sqrt{\xi\zeta t}<\left(\xi\zeta\rho+\frac{1}{4}\left(\sqrt{\xi}-\sqrt{\zeta}\right)^2\right)^{\frac{1}{2}}-\frac{1}{2}\left(\sqrt{\xi}+\sqrt{\zeta}\right).$$

In particular, the integral Y_t is defined as a continuous operator $\mathfrak{D}\to\mathfrak{H}$ into the Hilbert space $\mathfrak{H}$ if this analytical estimate is valid for $p=0$, $\xi=1/3$ and some q,ζ, and it is a strongly continuous operator, $Y_t\in\mathcal{L}(\mathfrak{D})$, if it is also valid for any $p<0,\xi<1/3$.

3. Let us apply this estimate to the multiple integral of the chronological products

$$B(\boldsymbol{\tau}) = L(\boldsymbol{t}_n)\cdots L(\boldsymbol{t}_1) = L(|\boldsymbol{\tau}|), \tag{6.6}$$

defined by the unique decomposition $\boldsymbol{\tau} = \boldsymbol{t}_1 \cup \cdots \cup \boldsymbol{t}_n$ of the set table $\boldsymbol{\tau}$ into the single point tables

$$\boldsymbol{t}_+^- = \begin{pmatrix} t & \varnothing \\ \varnothing & \varnothing \end{pmatrix}, \boldsymbol{t}_\bullet^- = \begin{pmatrix} \varnothing & t \\ \varnothing & \varnothing \end{pmatrix}, \boldsymbol{t}_+^\bullet = \begin{pmatrix} \varnothing & \varnothing \\ t & \varnothing \end{pmatrix}, \boldsymbol{t}_\bullet^\bullet = \begin{pmatrix} \varnothing & \varnothing \\ \varnothing & t \end{pmatrix},$$

with $L(\boldsymbol{t}_\nu^\mu) = L_\nu^\mu \in \mathcal{L}(\mathcal{D})$ and $t_1 < \ldots < t_n$, $n = |\cup\tau_\nu^\mu|$. If the norms (6.2) of the chronological product (6.6) with

$$\left\| B \begin{pmatrix} \tau_+^- & \tau^- \\ \tau_+ & \tau \end{pmatrix} \right\|_{p,q}' = \sup_{\eta_{n,n_+} \in \mathcal{H}_p \otimes \mathcal{K}^{\otimes n_+}, \eta_{n,n^-} \in \mathcal{H}_q \otimes \mathcal{K}^{\otimes n^-}} \frac{\left| \left\langle \eta_{n,n_+} | B(\boldsymbol{\tau}) \eta_{n,n^-} \right\rangle \right|}{\left\| \eta_{n,n_+} \right\|_p \left\| \eta_{n,n^-} \right\|_q} \tag{6.7}$$

are finite for some $\xi\zeta \geq 1$ and a $t = T$, the multiple integral $V_t = \Lambda_{[0,t)}^{\otimes}(B)$ satisfies the quantum linear differential equation $\mathrm{d}V_t = L_\nu^\mu V_t \mathrm{d}\Lambda_\mu^\nu$, $t \leq T$ with $V_0 = I$, see Theorem 1 in [31]. Thus the estimate (6.4) for the chronological products (6.6) with

$$L_+^- = -K, \quad L_\bullet^- = -K^-, \quad L_+^\bullet = L, \quad L_\bullet^\bullet = J - I$$

gives a sufficient condition for the existence of the unique solution to Eq. (1.1), (1.3) of the type (1.6) in the form of the stochastic chronologically ordered operator-valued exponents V_t.

4. A similar estimate

$$\|L(\boldsymbol{t}_n)\cdots L(\boldsymbol{t}_1)\|_{p,q}' \leq \frac{n!}{\sqrt{n^-!}} c_{p,q}(n) \tag{6.8}$$

for the chronological products $B(\tau_+^-, \tau^-)$ of $L(\boldsymbol{t}_+^-) = -K$ and $L(\boldsymbol{t}^-) = -K_\bullet$ gives the sufficient condition

$$\begin{aligned} \|B\|_{p,q}'(\xi,\zeta) &= \left(\int_{\Gamma_t} \left(\frac{1}{\zeta}\right)^{|\tau^-|} \left(\int_{\Gamma_t} \left\| B(\tau_+^-, \tau^-) \right\|_{p,q}' \mathrm{d}\tau_+^- \right)^2 \mathrm{d}\tau^- \right)^{\frac{1}{2}} \\ &\leq \sum_n \left(\frac{1}{\sqrt{\zeta}} + \sqrt{t} \right)^n c_{p,q}(n) < \infty, \quad \forall \xi^{-1} \leq \zeta \end{aligned} \tag{6.9}$$

of the integrability $V_t^\circ = \int_{\Gamma_t} \int_{\Gamma_t} B(\tau_+^-, \tau^-) \mathrm{d}\tau_+^- \Lambda^-(\mathrm{d}\tau^-)$ for the quantum stochastic equation (4.6) with $t \leq \left(\rho - 1/\sqrt{\zeta}\right)^2$. Thus the iteration $S_t = T_t V_t^\circ$ of the nonadapted integral equation 4.3 has the estimates $\|S_t\|_{p,q}(\xi,\zeta) \leq \sum_{n=0}^\infty \rho^n c_{p,q}(n)$ for all $2^{-1}\zeta \geq \max\left\{\rho^{-2}, 2\xi^{-1}\right\}$, and $S_t \in \mathcal{L}(\mathfrak{D})$ if the chronological products B satisfy the analyticity condition (6.8) for each $p < 0$ and some $q > 0$.

5. In order to formulate an analyticity condition for the weak convergence of (4.4) in terms of the structural maps λ_ν^μ, let us represent this multiple integral in the equivalent form (see Theorem 2 in [31]) as the adapted one $R_t = \Lambda_{[0,t)}^{\otimes}(L)$, for the integrand

$$L(\boldsymbol{\tau}) = \lambda(\boldsymbol{\tau}, I), \quad \boldsymbol{\tau} = \cup_{i=1}^n \boldsymbol{t}_i, \tag{6.10}$$

giving the solution to the equivalent equation (3.2) for $B = I$. The integrant L for such a representation is given by the chronological composition

$$\lambda(\boldsymbol{\tau}) = \lambda(\boldsymbol{t}_1) \circ \ldots \circ \lambda(\boldsymbol{t}_n)$$

of $\lambda(\boldsymbol{t}_\nu^\mu, \cdot) = \lambda_\nu^\mu(\cdot)$, see [31]. If this integrant $B = L$ has the estimate (6.4) for some $p, q > 0$ and $\xi, \zeta > 1$, then the series (4.4) converges to the continuous sesquilinear form $R_t \in \mathcal{B}(\mathfrak{D})$ with $\|R_t\|_{p,q}(3\xi, 3\zeta) < \infty$. Another analyticity condition, corresponding to a smaller (L^∞) space of test functions $f^\bullet \in L^2$ and a stronger (L^2) integrability, is given in [30]. In the next paper, which will be published elsewhere, we generalize the sufficient integrability condition (6.4) to the $L^{1+\xi}$ spaces of test functions, $\xi \geq 1$, and will show that our method gives more precise estimates in the limit case $\xi \longrightarrow \infty$.

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