GENERALIZED UNCERTAINTY RELATIONS AND EFFICIENT MEASUREMENTS IN QUANTUM SYSTEMS

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ABSTRACT. We consider two variants of a quantum-statistical generalization of the Cramér-Rao inequality that establishes an invariant lower bound on the mean square error of a generalized quantum measurement. The proposed complex variant of this inequality leads to a precise formulation of a generalized uncertainty principle for arbitrary states, in contrast to Helstrom's variant [1] in which these relations are obtained only for pure states. A notion of canonical states is introduced and the lower mean square error bound is found for estimating of the parameters of canonical states, in particular, the canonical parameters of a Lie group. It is shown that these bounds are globally attainable only for canonical states for which there exist efficient measurements or quasimeasurements.

1. Introduction

The development in recent years of the theory of generalized quantum measurements (see the review [2] and the literature cited there) has made it possible to introduce the concept of a quasimeasurement of incompatible observables described by noncommuting operators and, using this, to solve a number of problems of the quantum theory of information and communication [3, 4, 5, 6, 7, 8], give for pure states a precise formulation of a generalised Heisenberg uncertainty principle for quantities such as, for example, the time and energy, or phase and number of quanta [9], and to define precisely what is a measurement of the time and phase in quantum mechanics [8], [9]. In accordance with this theory, every quantum measurement in this generalised sense is described by a positive resolution of the identity operator $\hat{1}$ on the Hilbert space $\mathcal{H}$ of state-vectors $|\psi\rangle$ of the observed quantum system:

$$\hat{1} = \int \Pi\left(\mathrm{d}\varkappa\right). \tag{1.1}$$

Here $\Pi\left(\cdot\right)$ is an additive mapping (measure) on the Borel algebra $\mathfrak{B}\left(X\right)$ of a measurable space $X \ni \varkappa$ into the set of Hermitian-positive (i.e. nonnegative-definite Hermitian) operators in $\mathcal{H}$. Such normalized positive measure Π will be called *quantum probability measure (QPM)*, or simply quasimeasurement. If ϱ is a quantum state density operator, the probability $\Pr\left(B\right)$ of an event $\varkappa \in B$ in such a

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measurement is evaluated in accordance with the formula

$$\Pr(B) = \mathrm{Tr}\varrho\Pi(B), \; B \in \mathfrak{B}(X)$$

where Tr denotes the usual trace in $\mathcal{H}$. If the quantum measure Π in (1.1) is orthogonal, $\Pi(A)\Pi(C) = 0$ for every $A \cap C = \emptyset$, then it is a projerctor-valued measure. The generalised measurement in this case with $X = \mathbb{R}^n$ reduces to an ordinary measurement of the commuting self-adjoint operators

$$\hat{x}^j = \int \varkappa^j \Pi(\mathrm{d}\varkappa), \quad \varkappa^j \in \mathbb{R}, \tag{1.2}$$

For the nonorthogonal QPM, there is no one-to-one correspondence between (1.1) and (1.2). The corresponding generalized measurements, which are called henceforth approximate measurements of the operators (1.2), are not described as the measurements of these Hermitian operators even if they commute, though frequently they can be described uniquely by a single non-Hermitian (non-normal) operator (see Section 4) or, more generally, by a family of noncommuting Hermitian operators.

There is an intimate connection between the concept of a quasimeasurement as approximate measurement and the concept of an indirect quantum measurement (an indirect measurement is an ordinary measurement in an extended quantum system that includes the original system as a part [3]). This connection is a simple consequence of the Naimark's well-known theorem on the existence for every nonorthogonal QPM of an orthogonal one in an extended Hilbert space that compresses to the original QPM on the subspace $\mathcal{H}$.

One of the results of this paper is to show how the concept of a generalized measurement enables us to formulate precisely a generalised Heisenberg uncertainty principle for quantities such as the time and energy, phase and number of quanta, angle of rotation and angular momentum as a consequence of a quantum Cramer-Rao type inequality for the arbitrary states. The first members in each of these pairs – the time, phase, and angle – cannot, as is well known, be described by Hermitian operators in $\mathcal{H}$, though their measurement can be described as a statistical estimate of the corresponding parameters of quantum states. As Helstrom has shown in [9] by means of the symmetric quantum Cramér-Rao inequality which he introduced in [11], the variances of the results of any measurements to obtain such an estimate for pure states cannot be lower than a certain level that is inversely proportional to the variances of the generators of the unitary representations of the corresponding translation groups (i.e., the operators of the energy, number of quanta, or angular momentum). For example, if a pure state of a harmonic oscillator is known up to the oscillator phase, its state-vector is unitarily equivalent to a fixed vector $|\psi_0\rangle \in \mathcal{H}$ and can be described by the family

$$|\psi_\theta\rangle = \mathrm{e}^{\mathrm{i}\theta\hat{n}/\hbar}|\psi_0\rangle,$$

where $\hat{n}$, the operator of the number of quanta, is generator of the representation $\mathrm{e}^{\mathrm{i}\theta\hat{n}/\hbar}$ of the group of phase translations. If a QPM Π determines the probabilities

$$\Pr(\mathrm{d}\lambda|\theta) = \langle\psi_\theta|\Pi(\mathrm{d}\lambda)|\psi_\theta\rangle,$$

on $[-\pi, \pi]$ such that mean value of λ coincides with θ,

$$\mathsf{M}_\theta[\lambda] := \int \lambda \Pr(\mathrm{d}\lambda|\theta) = \theta,$$

it defines an unbiased estimate of the unknown value of the phase θ as the measurement result λ. The corresponding quasimeasurement, described by the induced QPM $\Pi(\mathrm{d}\lambda)$ on $(-\pi, \pi]$, is an approximate measurement of the "phase operator" $\hat{q} = \int \lambda \Pi(\mathrm{d}\lambda)$ .The mean quadratic error of the measurement approximation for such $\hat{q}$ is given by the quantum expectation $\langle\psi_\theta|\hat{\sigma}^2|\psi_\theta\rangle$ of the positive operator

$$\hat{\sigma}_\phi^2 = \int (\lambda - \hat{q}) \Pi_\phi(\mathrm{d}\lambda)(\lambda - \hat{q}),$$

and the total variance describing the estimation accuracy of θ,

$$R_\theta := \int (\lambda - \theta)^2 \Pr(\mathrm{d}\varkappa|\theta) \equiv \mathsf{M}_\theta \left[(\lambda - \theta)^2\right], \tag{1.3}$$

is the sum of this and mean square distance of the operator $\hat{q}$ and $\theta\hat{1}$:

$$R_\theta = \langle\psi_\theta| \left(\hat{\sigma}^2 + (\hat{q} - \theta)^2\right) |\psi_\theta\rangle .$$

The quantum Cramer-Rao inequality proves in this case that the second variance cannot be below the level $\hbar^2/4\left\langle(\hat{n} - n_\theta)^2\right\rangle_\theta$, and thus $R_\theta \geq \hbar^2/4G_\theta$, where

$$G_\theta = \langle\psi_\theta| (\hat{n} - n_\theta)^2 |\psi_\theta\rangle = \left\langle(\hat{n} - n_0)^2\right\rangle_0,$$

is Fisher information as the variance of $\hat{n}$ with $n_\theta = \langle\psi_\theta|\hat{n}|\psi_\theta\rangle = n_0$. This is Helstrom's precise formulation of the generalized Heisenberg's uncertainty principle for the conjugate quantities θ and $\hat{n}$, the first of which is described by a generalised measurement satisfying the unbiased condition (1.3).

In Section 2, we give the invariant formulation (2.4) of the Helstrom's Cramér-Rao inequality, and we also consider another generalization (2.7) of this inequality, which in contrast to Helstrom's can be naturally adapted to a complex situation and enables one to obtain straightforward a multidimensional generalization of the uncertainty relations (3.5) for not only pure but also mixed states. We also obtain the noncommutative generalization (3.12) of these relations for the generators and canonical parameters of unitary representations of an arbitrary Lie group. These generalizations are intimately related to the canonical families of states described in Section 3, whose particular role is disclosed in Section 4, in which it is shown that if the lower bounds for the mean square errors of a measurement are to be attainable, it is necessary and sufficient that the corresponding density operators have the canonical form (3.1).

2. Invariant Bounds of the Cramér-Rao Type in quantum Statistics

1. Let $\{\varrho_\vartheta, \vartheta \in M\}$ be a family of density operators ϱ_ϑ in $\mathcal{H}$ that describe the statistical state of a quantum system as a smooth function of unknown real parameters $\vartheta = \left(\vartheta^1, \ldots, \vartheta^m\right)$ in a given manifold $M \subseteq \mathbb{R}^m$. Every simultaneous measurement of these parameters can be described in $\mathcal{H}$ by QPM Π which defines a row-vector random variable $\lambda \in \mathbb{R}^m$ with probability distribution $\Pr(\mathrm{d}\lambda|\vartheta) = \mathrm{Tr}\varrho_\vartheta \Pi(\mathrm{d}\lambda)$ known up to ϑ. The mean quadratic errors of the measurement are determined by the components

$$R_\vartheta^{ik} = \mathsf{M}_\vartheta \left[\left(\lambda^i - \vartheta^i\right)\left(\lambda^k - \vartheta^k\right)\right]$$

of the covariance matrix $R_\vartheta = \left[R_\vartheta^{ik}\right]$ by means of expressions $\mathrm{tr} C^\intercal R \equiv c_{ik} R^{ik}$ given by a Hermitian-positive matrix $C = [c_{ik}]$ of the quadratic cost form

$$c(\lambda, \vartheta) = \left(\lambda^i - \vartheta^i\right) c_{ik} \left(\lambda^k - \vartheta^k\right)$$

which plays the role of a metric tensor. Here and in what follows the Einstein summation convention is assumed:

$$c_{ik} R_\vartheta^{ik} \equiv \sum_i \sum_k c_{ik} R_\vartheta^{ik}.$$

In what follows we shall consider only those measurements that satisfy the unbiased conditions $\mathsf{M}_\vartheta \left(\lambda^i\right) = \vartheta^i$, under which the matrix R_ϑ is the covariance matrix of the estimates ϑ_i, and the mean square error for fixed R_ϑ takes a minimal value.

Helstrom established [1] for the covariance matrix R_ϑ of such measurements a lower bound by using the concept of operators $\hat{g}_i$ of symmetrized logarithmic derivatives of the function ϱ_ϑ with respect to ϑ^i. He defined these $\hat{g}_i$ by means of the equations

$$\hat{g}_i \varrho_\vartheta + \varrho_\vartheta \hat{g}_i = 2\partial_i \varrho_\vartheta, \ \partial_i := \frac{\partial}{\partial \vartheta^i}. \tag{2.1}$$

As in the classical case [11], this bound is determined by the matrix $\mathsf{G}_\vartheta = [G_{ik}(\vartheta)]$ of the covariances of the logarithmic derivatives $\hat{g}_i = \hat{g}_i(\vartheta)$ of Eqs. (2.1), defined in the symmetrized form as

$$G_{ik}(\vartheta) = \frac{1}{2} \langle \hat{g}_i(\vartheta) \hat{g}_k(\vartheta) + \hat{g}_k(\vartheta) \hat{g}_i(\vartheta) \rangle_\vartheta \tag{2.2}$$

(Note that due to $\mathrm{Tr} \partial^i \varrho_\vartheta = 0$

$$\langle \hat{g}_i(\vartheta) \rangle_\vartheta := \mathrm{Tr} \varrho_\vartheta \hat{g}_i(\vartheta) = 0$$

for all ϑ). The corresponding inequality has the form

$$R_\vartheta \geq \mathsf{G}_\vartheta^{-1}, \quad \vartheta \in M, \tag{2.3}$$

and is understood in the sense of nonnegative definiteness of the matrix $\left[R_\vartheta^{ik} - G_\vartheta^{ik}\right]$, where G_ϑ^{ik} are the components of the inverse matrix $\mathsf{G}_\vartheta^{-1} : G^{ij} G_{jk} = \delta_k^i$. The inequality (2.3) establishing an uncertainty relation between the variances of estimation and the variances of the corresponding logarithmic derivatives, is a quantum analog of the Cramér-Rao inequality [11]. The matrix G_ϑ which we call symmetric quantum Fisher information, or more fair, Helstrom information, is one of possible generalizations of classical Fisher information. It plays the role of a metric tensor that locally defines the geodesic distance

$$s_{\mathsf{G}}(\vartheta, \vartheta + \mathrm{d}\vartheta) = \left(G_{ik}(\vartheta) \mathrm{d}\vartheta^i \mathrm{d}\vartheta^k\right)^{1/2}$$

in the parameter space $M \subseteq \mathbb{R}^m$; this is analogous to Fisher information distance in classical statistics.

2. Further, we shall consider a slightly more general situation in which the state parameters are not the measured parameters ϑ^i but local coordinates $\alpha = \left(\alpha^1, \ldots, \alpha^n\right)$ of a smooth manifold $\mathfrak{S}$ parametrizing the unknown $\varrho = \varrho(\alpha)$. The measured parameters are assumed to be known smooth functions $\vartheta_i(\alpha)$ of the unknown parameters α^k. The corresponding generalization of Helstrom's inequality (2.3) is a lower bound for the matrix $\mathsf{R}(\alpha) = [R_{ik}(\alpha)]$ of the covariances

$$R_{ik}(\alpha) = \mathsf{M}\left[(\lambda_i - \vartheta_i)(\lambda_k - \vartheta_k) | \alpha\right]$$

of the estimates λ_i in the form

$$\mathsf{R}(\alpha) \geq \mathsf{D}(\alpha)\,\mathsf{G}(\alpha)^{-1}\,\mathsf{D}(\alpha)^{\intercal}\,. \tag{2.4}$$

that is invariant under the choice of the state coordinates $\alpha = \left(\alpha^k\right)$. Here $\mathsf{D}(\alpha)$ is the matrix $[D_{ik}(\alpha)]$ of the partial derivatives $D_{ik}(\alpha) = \partial\vartheta_i/\partial\alpha^k$, $\mathsf{D}^{\intercal} = [\partial_i\vartheta_k]$, and $\mathsf{G}(\alpha) = [G_{ik}(\alpha)]$ is the symmetric quantum Fisher information corresponding to the coordinates α, that is a matrix of the covariances

$$G_{kl}(\alpha) = \mathrm{Tr}\left[\hat{g}_k(\alpha)\cdot\hat{g}_l(\alpha)\,\varrho(\alpha)\right],\ \hat{g}_k\cdot\hat{g}_l = \frac{1}{2}\left(\hat{g}_k\hat{g}_l + \hat{g}_l\hat{g}_k\right),$$

of the Helstrom's logarithmic derivatives

$$\varrho(\alpha)\,\hat{g}_k + \hat{g}_k\varrho(\alpha) = 2\frac{\partial}{\partial\alpha^k}\varrho(\alpha)$$

with respect to the coordinates α^k.

The inequality (2.4) reduces to the classical Cramér-Rao inequality only when the family $\{\varrho(\alpha)\}$ is commutative. For noncommutative families, one can have other quantum generalizations [4], [5] of the Cramér-Rao inequality based on other definitions of the logarithmic derivatives; these lead to other lower bounds for R that may differ from Helstrom's invariant bound $\mathsf{D}\mathsf{G}^{-1}\mathsf{D}^{\intercal}$. Moreover, in the noncommutative case it makes sense to consider also the complex-valued parameters as any quantum state has the natural complex coordinatization $\varrho = \alpha\alpha^*/\mathrm{Tr}\alpha^*\alpha$ in terms of the complex Hilbert-Schmidt operators α with is the adjoint operators α^*as their complex conjugated. In the case of complex parameters $\vartheta_i \in \mathbb{C}$ represented by analytic functions $\vartheta_i(\alpha,\alpha^*)$ the particular importance is acquired by the following invariant generalization of the Cramér-Rao inequality based on the right and left logarithmic derivatives which were proposed independently by the author [3] and Yuen and Lax [5].

3. Suppose the parameters α^k are given in pairs $\left(\gamma^k,\theta^k\right) \in \mathbb{R}^2$ which are complexified as $\frac{1}{2}\gamma^k + \mathrm{i}\theta^k \equiv \beta^k$. Such parameters $\alpha \in \mathbb{R}^{2n}$, considered as complex n-columns, will often be denoted as $\beta = \left(\beta^k\right) \in \mathbb{C}^n$, with $\gamma = \beta + \bar{\beta} \in \mathbb{R}^n$ and $\theta = \mathrm{Im}\,\beta \in \mathbb{R}^n$. The partial derivatives $\partial_k = \partial/\partial\beta^k$, $\bar{\partial}_k = \partial/\partial\bar{\beta}^k$ are defined by means of the partial derivatives $\partial/\partial\gamma^i$, $\partial/\partial\theta^i$ in the usual manner:

$$\frac{\partial}{\partial\beta^k} = \left(\frac{\partial}{\partial\gamma^k} + \tfrac{\mathrm{i}}{2}\frac{\partial}{\partial\theta^k}\right),\quad \frac{\partial}{\partial\bar{\beta}^k} = \left(\frac{\partial}{\partial\gamma^k} - \tfrac{\mathrm{i}}{2}\frac{\partial}{\partial\theta^k}\right)$$

such that $\partial_k\beta^l = \delta^l_k = \bar{\partial}_k\beta^l$ and $\partial_k\bar{\beta}^l = 0 = \bar{\partial}_k\beta^l$.

The estimated parameters $\vartheta_i,\ i = 1,\ldots,m$ as functions of complex $\alpha,\bar{\alpha}$ can still be real functions of γ and θ. They are not assumed to be analytic with respect to α, but differentiable independently with respect to α and $\bar{\alpha}$ (e.g. given by bi-analytic functions $\vartheta_i(\alpha,\alpha')$ at $\alpha' = \bar{\alpha}$). We define the non-Hermitian right and left logarithmic derivatives of the density operator $\varrho(\alpha,\bar{\alpha})$ by the relations

$$\varrho\hat{h}_k = \frac{\partial\varrho}{\partial\bar{\alpha}^k},\quad \hat{h}^*_k\varrho = \frac{\partial\varrho}{\partial\alpha^k},\quad k = 1,\ldots,n. \tag{2.5}$$

The operators $\hat{h}_k = \hat{h}_k(\alpha,\bar{\alpha})$ of the right derivatives with respect to $\bar{\alpha}^k$ are Hermitian conjugate at each α to the operators $\hat{h}^*_k = \hat{h}_k(\alpha,\bar{\alpha})^*$ of the left derivatives with respect to α^k, and they both have zero expectations

$$\mathrm{Tr}\hat{h}_k(\alpha,\bar{\alpha})\,\varrho(\alpha,\bar{\alpha}) = 0 = \mathrm{Tr}\hat{h}_k^*(\alpha,\bar{\alpha})\,\varrho(\alpha,\bar{\alpha}).$$

The corresponding quantum Fisher information is given by the matrix $\mathsf{H} = [H_{kl}]$ of covariances

$$H_{kl}(\alpha,\bar{\alpha}) = \mathrm{Tr}\left[\hat{h}_k(\alpha,\bar{\alpha})\,\hat{h}_l(\alpha,\bar{\alpha})^*\,\varrho(\alpha,\bar{\alpha})\right]. \tag{2.6}$$

Obviously this matrix is Hermitian-positive, and under the assumption of its non-degeneracy it defines a positive-definite metric

$$ds^2_{\mathsf{H}} = H_{kl}d\bar{\alpha}^k d\alpha^l$$

in some complex domain $\mathcal{O} \subset \mathbb{C}^n$ of the unknowns $\alpha \in \mathcal{O}$.

4. Suppose a simultaneous measurement of the parameters ϑ_i is described by a QPM Π on X that determines the estimates λ_i of ϑ_i as complex-valued random variables of $\varkappa \in X$ with respect to the distribution

$$\Pr[\mathrm{d}\lambda \mid \alpha,\ \bar{\alpha}] = \mathrm{Tr}\Pi(\mathrm{d}\lambda)\,\varrho(\alpha,\bar{\alpha})$$

parametrized by α.

The mean quadratic errors of the measurement are determined by the matrix $\mathsf{R}(\alpha,\bar{\alpha}) = [R_{ij}(\alpha,\bar{\alpha})]$ of covariances

$$R_{ij}(\alpha,\bar{\alpha}) = \mathsf{M}\left[(\lambda_i - \vartheta_i)\left(\bar{\lambda}_j - \bar{\vartheta}_j\right)|\alpha,\bar{\alpha}\right]$$

which can be written as the sum $R_{ij} = \left\langle\hat{\sigma}^2_{ij}\right\rangle + Q_{ij}$ of two kind errors. The first one is given by the Hermitian-positive matrix of the elements

$$\left\langle\hat{\sigma}^2_{ij}\right\rangle(\alpha,\bar{\alpha}) = \mathrm{Tr}\left[\hat{\sigma}^2_{ij}\varrho(\alpha,\bar{\alpha})\right]$$

as the quantum expectation of the covariance operators

$$\hat{\sigma}^2_{ij} = \int(\lambda_i - \hat{q}_i)\,\Pi(\mathrm{d}\lambda)\,(\lambda_i - \hat{q}_j)$$

for the quantum estimates

$$\hat{q}_i = \int \lambda_i \Pi(\mathrm{d}\lambda)$$

The second forms the mean quadratic error matrix $\mathsf{Q} = [Q_{ij}]$

$$Q_{ij}(\alpha,\bar{\alpha}) = \left\langle(\hat{q}_i - \vartheta_\iota)\left(\hat{q}_j^* - \bar{\vartheta}_j\right)\right\rangle(\alpha,\bar{\alpha})$$

for the the operators $\hat{q}_i$ "estimating" the parameters ϑ_i.

Assuming the convergence of the integral defining $\hat{q}_i$, the unbiasness condition

$$\mathsf{M}[\vartheta_i|\alpha,\bar{\alpha}] := \int \lambda_i \Pr(\mathrm{d}\lambda \mid \alpha,\ \bar{\alpha}) = \vartheta_i(\alpha,\bar{\alpha}),$$

for the estimates λ_i can be written in the form of quantum unbiasness

$$\langle\hat{q}_i\rangle(\alpha,\bar{\alpha}) = \mathrm{Tr}\hat{q}_i\varrho(\alpha,\bar{\alpha}) = \vartheta_\iota(\alpha,\bar{\alpha}).$$

Under this assumption the matrix $\mathsf{Q}(\alpha,\bar{\alpha}) = [Q_{ij}(\alpha,\bar{\alpha})]$ is the covariance matrix of the operators $\hat{q}_i$, and as it is shown in the Appendix, it has lower bound $\mathsf{Q} \geq \mathsf{D}\mathsf{H}^{-1}\mathsf{D}^\dagger$, and therefore

$$\mathsf{R}(\alpha,\bar{\alpha}) \geq \mathsf{D}(\alpha,\bar{\alpha})\,\mathsf{H}(\alpha,\bar{\alpha})^{-1}\,\mathsf{D}(\alpha,\bar{\alpha})^\dagger, \tag{2.7}$$

where $\mathsf{D} = \mathsf{D}(\alpha,\bar{\alpha})$, as in (2.4), is the matrix $[d_{ik}]$ of the derivatives $\partial\vartheta_i/\partial\alpha^k$, and $\mathsf{D}^\dagger = \left[\overline{\partial_k\vartheta_i}\right]$ is the Hermitian adjoint matrix.

As we shall see, even in the real case $\vartheta^i = \bar{\vartheta}^i$, the bound (2.7) may lead to a lower bound that differs from Helstrom's bound (2.4). We shall say that (2.7) is the right lower bound. Besides this bound, we can consider other bounds, for example, the "left" bound, which is based on the left logarithmic derivatives with respect to $\bar{\alpha}$. All these bounds are proved in the same way as (2.7) see the Appendix. Note that the right bound in (2.7) is invariant under the change of variables $\left(\alpha^k\right) \mapsto (\vartheta_i)$ by replacing the derivatives with respect to α^k by derivatives with respect to the new variables $\vartheta_i = \vartheta_i(\alpha, \bar{\alpha})$ only under the analyticity condition $\partial\vartheta_i/\partial\bar{\alpha}^k = 0$ of the transforming functions $\vartheta_i(\alpha, \bar{\alpha}) = \vartheta_i(\alpha)$ and the condition of nondegeneracy of the matrix of the derivatives $\partial\vartheta_i/\partial\alpha^k$. Therefore, the inequality (2.7) and its noninvariant form $\mathsf{R} \geq \mathsf{H}^{-1}$ are not equivalent unless not only the nondegeneracy of the matrix D but also the analyticity condition $\partial\vartheta_i/\partial\bar{\alpha}^k = 0$ (i.e., the condition that the functions $\vartheta_i(\alpha, \bar{\alpha})$ are independent of $\bar{\alpha}$) hold.

3. Canonical States and Uncertainty Relations

In classical mathematical statistics, a particular role is played by canonical, or exponential, families of probability distributions, for which the Cramér-Rao bound is attainable for a special choice of the parameters ϑ. In Section 4, we shall show that in quantum statistics an analogous role is played by the density operators of the form

$$\varrho\left(\beta, \bar{\beta}\right) = \chi\left(\beta, \bar{\beta}\right)^{-1} \mathrm{e}^{\beta^k \hat{x}_k^*} \varrho_0 \mathrm{e}^{\bar{\beta}^k \hat{x}_k}, \tag{3.1}$$

where $\hat{x}_k,\ k = 1, \ldots, n$ are linearly independent operators in $\mathcal{H}$, which may be non-Hermitian: $\hat{x}_k^* \neq \hat{x}_k$, and even need not commute with the conjugates $\hat{x}_i\hat{x}_k^* \neq \hat{x}_k^*\hat{x}_i$. We shall assume that the generating function

$$\chi\left(\beta, \bar{\beta}\right) = \mathrm{Tr}\varrho_0 \mathrm{e}^{\bar{\beta}^k \hat{x}_k} \mathrm{e}^{\beta_k \hat{x}_k^*}, \tag{3.2}$$

of the moments of these operators in the state $\varrho = \varrho_0$ is defined in an open neighborhood of the origin $\beta = 0$ of the complex space $\mathbb{C}^n$ with finite first and second moments

$$\frac{\partial}{\partial\bar{\beta}_i}\chi|_{\beta=0} = 0 = \frac{\partial}{\partial\beta_k}\chi|_{\beta=0},\ \langle\hat{x}_i\hat{x}_k^*\rangle_0 = \frac{\partial}{\partial\bar{\beta}_i}\frac{\partial_k}{\partial\beta_k}\ln\chi|_{\beta=0}$$

(the operators $\hat{x}_k$ in (3.1) can always be chosen to have zero expectations $\langle\hat{x}_i\rangle_0 = \partial_i \ln\chi|_{\beta=0} = 0$ in the state ϱ_0). We shall call that the family of density operators (3.1) canonical, with the parameters β^k canonically conjugate to the quantum variables $\hat{x}_k$. In contrast to the classical case, even for selfadjoint $\hat{x}_k$ one can meaningfully consider complex values of the conjugate parameters β^k.

Particular interest attaches to the case, which does not have a classical analog, of the canonical states (3.1) when β^k are imaginary, $\beta^k = i\theta^k$, and $\hat{x}_k$ are selfadjoint, $\hat{x}_k = \hat{s}_k = \hat{x}_k^*$. The parameters $\theta_k = \hbar\theta^k$ ($\hbar$ is Planck's constant) then take the dimension and meaning of the classical variables which are dynamically conjugate to their shift generators $\hat{s}_k$. For example, [9], if $\hat{s}$ is the Hamiltonian, then θ is the time, if $\hat{s}$ is the momentum then θ is the position, and if $\hat{s}$ is the number of quanta, or angular momentum, then θ is the phase, or polar coordinate. For $\beta^k = \mathrm{i}\hbar^{-1}\theta_k \equiv \beta_\theta^k$ the canonical states $\varrho_\theta = \rho\left(\beta_\theta, \bar{\beta}_\theta\right)$ (3.1) become unitary equivalent

$$\varrho_\theta = \mathrm{e}^{\mathrm{i}\theta^k \hat{s}_k/\hbar} \varrho_0 \mathrm{e}^{-\mathrm{i}\theta^k \hat{s}_k/\hbar} \tag{3.3}$$

to the state $\varrho_0 = \varrho(0, 0)$ corresponding to the zero value $\theta = 0$.

Now we shall see that the inequality (2.7) with $\theta_k = \hbar \operatorname{Im} \beta^k$ applied to the canonical family (3.3) with commuting $\hat{s}_k = \hat{s}_k^*$ immediately provides the precise formulation of a generalised Heisenberg uncertainty principle for an unbiased estimation of θ. In this the right and left logarithmic derivatives with respect to $\bar{\beta}$ and β for the family (3.1) are equal to the symmetric logarithmic derivatives $\hat{g}_k(\gamma) = \hat{s}_k - \mu_k(\gamma)$ with respect to $\gamma = \beta + \bar{\beta}$:

$$\hat{h}_k = \hat{s}_k - \frac{\partial}{\partial \bar{\beta}_i} \chi\left(\beta + \bar{\beta}\right) = \hat{g}_k = \hat{s}_k - \frac{\partial}{\partial \beta_k} \chi\left(\beta + \bar{\beta}\right) = \hat{h}_k^*.$$

This implies that the Fisher informations $\mathsf{H}\left(\beta, \bar{\beta}\right)$ and $\mathsf{G}(x)$ coincide with the covariance matrix $\mathsf{S}\left(\beta + \bar{\beta}\right)$ of the commutative family $\hat{s}_k$ given at the state (3.1) by

$$S_{ik} = \chi\left(\beta + \bar{\beta}\right)^{-1} \left\langle \left(\hat{s}_i - \mu_i\right) \mathrm{e}^{\left(\beta + \bar{\beta}\right)^j \hat{s}_j} \left(\hat{s}_k - \mu_k\right) \right\rangle_0 . \tag{3.4}$$

While the complex quantum Cramer Rao bound (2.7) for the unbiased estimation of real parameters $\vartheta_i(\gamma)$ with $\gamma^k = \beta^k + \bar{\beta}^k$ coincides in this case with the Helstrom's invariant bound (2.4), it also gives immediately the uncertainty relation

$$\mathsf{R}_\theta \geq \frac{1}{4} \hbar^2 \mathsf{S}_0^{-1}, \quad \mathsf{S}_0 = \mathsf{S}(0) \tag{3.5}$$

for the unbiased estimation of $\theta_i = \hbar \theta^i$ based on the imaginary parts $\theta^i = \operatorname{Im} \beta^i$ for the canonical coordinates β with the fixed $\gamma = 0$.

Indeed, setting $\theta_i = \hbar \operatorname{Im} \beta^i$ such that $\partial \theta_i / \partial \beta^k = \hbar \delta_{ik} / 2\mathrm{i}$, we obtain from (2.7) the generalised Heisenberg uncertainty relation in the form

$$\mathsf{R}\left(\beta, \bar{\beta}\right) \geq \frac{1}{4} \hbar^2 \mathsf{S}^{-1}\left(\beta + \bar{\beta}\right).$$

Here R is the mean quadratic error matrix

$$\mathsf{R} = \mathsf{M}\left[\left(\lambda_i - \hbar \operatorname{Im} \beta^i\right)\left(\lambda_j - \hbar \operatorname{Im} \beta^j\right) | \beta, \bar{\beta}\right]$$

of unbiased estimates γ_i and S is the matrix of the covariances (3.4) defining the uncertainty relation (3.5) at $\gamma = \beta + \bar{\beta} = 0$.

The uncertainty relation (3.5) acquires the following matrix meaning: The covariance matrix $\mathsf{R}_\theta = \mathsf{R}\left(\beta_\theta, \bar{\beta}_\theta\right)$ of the unbiased estimates for the canonical parameters θ_i of the translation group represented in $\mathcal{H}$ by the unitary transformations (3.3) with the selfadjoint generators $\hat{s}_k$ is in the canonical uncertainty relation with the covariance matrix of these generators,

$$S_{ik}(\theta) := \mathrm{Tr} \varrho_\theta \hat{s}_i \hat{s}_k = \mathrm{Tr} \varrho_0 \hat{s}_i \hat{s}_k \equiv S_{ik}(0),$$

in the initial (and any other transformed) state $\varrho_0 = \varrho(0)$.

This uncertainly relation holds for all commuting Hermitian operators $\hat{s}_i$, not only for those like momenta which have dynamically conjugate observables $\hat{s}_i$. Helstrom derived this generalized uncertainty relation (3.5) in one dimensional version from his bound for the particular case of pure states $\varrho_0 = |\varphi_0\rangle \langle \varphi_0|$ [9]. However for this purpose the symmetric inequality (2.3) is inappropriate, and this is why his derivation involved so complicated matrix elements calculations.

The uncertainty relations naturally correspond to not symmetric but antisymmetric logarithmic derivatives, defined as the Hermitian solutions $\hat{p}_k = \hat{p}_k(\theta)$ of the von Neumann equations

$$[\varrho_\theta, \hat{p}_k] := \varrho_\theta \hat{p}_k - \hat{p}_k \varrho_\theta = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial \theta_k} \varrho_\theta$$

For the canonical family (3.3) we have the solutions $\hat{p}_k(\theta) = \hat{s}_k$ which are uniquely defined by the condition $\mathrm{Tr}\varrho_\theta \hat{p}_k(\theta) = 0$. Assuming that the solutions $\hat{p}_k(\theta)$ exist for an arbitrary parametric family ϱ_θ, one can derive the generalized uncertainty relation for the covariance matrix R_θ of the unbiased estimates

$$\mathsf{M}_\theta[\theta_i] = \langle \hat{q}_i \rangle_\theta = \theta_i, \quad \hat{q}_i = \int \theta_i \Pi(\mathrm{d}\theta),$$

in terms of the new quantum Fisher information matrix $\mathsf{S}_\theta = [S_{ik}(\theta)]$ given by the symmetric covariances

$$S_{ik}(\theta) = \mathrm{Tr}\hat{p}_i(\theta) \cdot \hat{p}_k(\theta) \varrho_\theta.$$

It simply follows form of matrix inequality

$$\mathsf{R}_\theta \geq \mathsf{Q}_\theta \geq \frac{\hbar^2}{4} \mathsf{S}_\theta^{-1},$$

where $\mathsf{Q}_\theta = [Q_{ik}(\theta)]$ is the matrix of covariances

$$Q_{ik}(\theta) = \langle (\hat{q}_i - \theta_i)(\hat{q}_k - \theta_k) \rangle_\theta$$

with $\mathsf{R}_\theta - \mathsf{Q}_\theta = \int [(\lambda_i - \hat{q}_i) \Pi(\mathrm{d}\lambda)(\lambda_k - \hat{q}_k)] \geq 0$.

Indeed, due to the unbiasness $\langle \hat{q} \rangle_\theta = \theta$ we have *mean canonical commutation relations*

$$\langle [\hat{q}_i, \hat{p}_k(\theta)] \rangle_\theta = \mathrm{Tr}\hat{q}_i [\hat{p}_k(\theta), \varrho_\theta] = \mathrm{i}\hbar \frac{\partial}{\partial \theta_k} \langle \hat{q}_i \rangle_\theta = \mathrm{i}\hbar \delta_{ik}.$$

From this and $\langle [\hat{q}, \hat{p}] \rangle_\theta = 2\,\mathrm{Im}\,\langle \tilde{q}\tilde{p} \rangle_\theta$, where $\tilde{q} = \hat{q} - \theta$, $\tilde{p} = \hat{p} - \mu$, we derive $\mathsf{Q} \geq \hbar^2 S^{-1}/4$ by Schwarz inequality and $|\langle \tilde{q}\tilde{p} \rangle_\theta| \geq |\mathrm{Im}\,\langle \tilde{q}\tilde{p} \rangle_\theta|$:

$$\langle \tilde{q}^2 \rangle_\theta \langle \tilde{p}^2 \rangle_\theta \geq |\langle \tilde{q}\tilde{p} \rangle_\theta|^2 \geq \frac{1}{4} |\langle [\hat{q}, \tilde{p}] \rangle_\theta|^2 = \left(\frac{\hbar}{2} \right)^2.$$

Note that Heisenberg's uncertainty principle is usually proved only for a single state $\varrho = \varrho_0$ in the form of the Robertson inequality $\mathsf{R}_0 \geq \hbar^2 \mathsf{S}_0^{-1}/4$ for the variances R_0 and S_0 of the dynamically conjugate variables described by the canonical operators $\hat{q}_i$ and $\hat{p}_k$ in $\mathcal{H}$ which satisfy the exact canonical commutation relations

$$[\hat{q}_i, \hat{q}_k] = 0, \quad [\hat{q}_i, \hat{p}_k] = \mathrm{i}\hbar \delta_{ik} I, \quad [\hat{p}_i, \hat{p}_k] = 0.$$

A more precise matrix multidimensional generalization of the Robertson inequality in terms of the covariances of estimates of an arbitrary family of noncommuting operators is proposed in [7]. Note that Robertson inequality implies the uncertainty relation

$$\mathsf{R}_\theta \geq \hbar^2 \mathsf{S}_0^{-1}/4, \quad \mathsf{S}_0 = [\mathrm{Tr}\hat{s}_i \hat{s}_k \varrho_0] = \mathsf{S}_\theta$$

for the unbiased measurements of the unknown expectations $\theta_i = \langle \hat{q}_i \rangle_\theta$ in the canonical states (3.3) with $\int \lambda_i \Pi(\mathrm{d}\lambda) = \hat{q}_i$, where $\hat{q}_i$ satisfy the canonical commutation relations with the canonically conjugated $\hat{s}_k = \hat{p}_k$. In this case the unbiasness

$$\mathsf{M}_\theta[\theta] = \langle \hat{q} \rangle_\theta = \mathrm{Tr}\varrho_0 \hat{q}(\theta) = \langle \hat{q} \rangle_0 + \theta = \theta,$$

simply means that $\langle \hat{q}_i \rangle_0 = 0$ for the state ϱ_0 as

$$\hat{q}(\theta) = \mathrm{e}^{-\mathrm{i}\theta_k \hat{s}^k} \hat{q} \mathrm{e}^{\mathrm{i}\theta_k \hat{s}^k} = \hat{q} + \theta$$

Every such unbiased measurement has the variance $\mathsf{R}_\theta \geq \mathsf{Q}_\theta$, and among such measurements there is an optimal one corresponding to $\mathsf{R}_\theta = \mathsf{Q}_\theta$. It is realized by the direct measurement of all $\hat{q}_i$ described by the orthogonal spectral measure $\Pi(\mathrm{d}\lambda) = E(\mathrm{d}\lambda)$ of the commutative family $\hat{q}_i = \int \lambda_i E(\mathrm{d}\lambda)$. Note that in this case $\hat{p}(\theta) = \hat{p}$, and both $\mathsf{S}_\theta = \mathsf{S}_0$ and $\mathsf{Q}_\theta = \mathsf{Q}_0$ do not depend on θ in any state $\varrho = \varrho_\theta$.

Our analysis extends the Heisenberg uncertainty principle to any unbiased measurement satisfying $\langle \hat{q} \rangle_\theta = \theta$. Note that without unbiasness the uncertanicy relation doesn't hold for such dynamically conjugate variables as polar coordinate described by the bounded selfadjoint operator $-\pi\hat{1} \leq \hat{q} \leq \pi\hat{1}$ and the discrete angular momentum $\hat{s}$. In this case one can find a state ϱ_0 (e.g. the eigen state of angular momentum for which the uncertancy relation is obviously not true as $\mathsf{S}_0 = 0$ and $\mathsf{Q}_0 \leq \pi^2$). There is no good operator $\hat{q}$ in $\mathcal{H}$ satisfying the unbiasness condition $\langle \hat{q} \rangle_\theta = \theta$.

We now consider the general case of the non-commuting generators $\hat{x}_k$ in the canonical family (3.1). Differentiating (3.1) with respect to $\bar{\beta}^k$ and comparing the result with (2.5), we obtain

$$\hat{h}_k = \mathrm{e}^{-\bar{\beta}^k \hat{x}_k} \chi \frac{\partial}{\partial \bar{\beta}^k} \chi^{-1} \mathrm{e}^{\bar{\beta}^k \hat{x}_k} = \hat{x}_k\left(\bar{\beta}\right) - \mu_k, \tag{3.6}$$

where $\hat{x}_k\left(\bar{\beta}\right) = \mathrm{e}^{-\bar{\beta}^k \hat{x}_k} \frac{\partial}{\partial \bar{\beta}^k} \mathrm{e}^{\bar{\beta}^k \hat{x}_k}$, and $\mu_k = \mu_k\left(\beta, \bar{\beta}\right)$ is the expectation value of $\hat{x}_k\left(\bar{\beta}\right)$ at the state $\varrho = \varrho\left(\beta, \bar{\beta}\right)$:

$$\mu_k = \mathrm{Tr} \hat{x}_k\left(\bar{\beta}\right) \varrho\left(\beta, \bar{\beta}\right) = \frac{\partial}{\partial \bar{\beta}^k} \ln \chi\left(\beta, \bar{\beta}\right).$$

The right Fisher information matrix (2.6) is therefore the matrix of the covariances

$$h_{ik} = \mathrm{Tr}\left(\hat{x}_i\left(\bar{\beta}\right) - \mu_i\right)\left(\hat{x}_k\left(\bar{\beta}\right) - \mu_k\right)^* \varrho\left(\beta, \bar{\beta}\right) = \frac{\partial^2 \ln \chi}{\partial \bar{\beta}^i \partial \beta^k}\left(\beta, \bar{\beta}\right) \tag{3.7}$$

of the operators $\hat{x}_k\left(\bar{\beta}\right)$ depending analytically on $\bar{\beta}$ (but with not necessarily analytic expectations x_k at $\varrho\left(\beta, \bar{\beta}\right)$). The inequality (2.7) in the neighborhood of the point $\beta = 0$ can therefore be expressed in the form of the uncertainty relation

$$\mathsf{R}\left(\beta, \bar{\beta}\right) \gtrsim \mathsf{D}\left(\beta, \bar{\beta}\right) \mathsf{S}\left(\beta, \bar{\beta}\right)^{-1} \mathsf{D}\left(\beta, \bar{\beta}\right)^{\dagger}, \tag{3.8}$$

which establishes an inverse proportionality between the matrix $\mathsf{S} = \left[S_{ik}\left(\beta, \bar{\beta}\right)\right]$ of the covariances

$$S_{ik} = \mathrm{Tr} \varrho\left(\beta, \bar{\beta}\right)\left(\hat{x}_i - \mu_i\right)\left(\hat{x}_k - \mu_k\right)^* \tag{3.9}$$

for the operators $\hat{x}_k = \hat{x}_k(0)$ with the expectations $\mu_k = \mathrm{Tr} \hat{x}_k \varrho\left(\beta, \bar{\beta}\right)$ and the covariance matrix $\mathsf{R}\left(\beta, \bar{\beta}\right)$ of the estimates λ_i for the functions $\vartheta_i\left(\beta, \bar{\beta}\right)$ of the canonical parameters β^k.

Let us consider the case when the operators $\hat{x}_k$ are the generators of a Lie algebra. Suppose the operators $\hat{x}_k$ satisfy a Lie algebra commutation relations

$$\hat{x}_i \hat{x}_k - \hat{x}_k \hat{x}_i = C_{ik}^j \hat{x}_j, \tag{3.10}$$

where C_{ik}^j are the structure constants. In this case, the operators $\hat{x}_i\left(\bar{\beta}\right)$ in 3.6) are linear combinations of the generators $\hat{x}_i = \hat{x}_i(0)$ [12]:

$$\hat{x}_i\left(\bar{\beta}\right) = \mathsf{K}^{-1}\left(\bar{\beta}\right)_i^j \hat{x}_j, \tag{3.11}$$

where $\mathsf{K}\left(\bar{\beta}\right) = \bar{\beta}^k \mathsf{C}_k \left(\mathrm{e}^{\bar{\beta}^k \mathsf{C}_k} - \mathsf{I}\right)^{-1}$ is an $n \times n$ matrix which exists in, at least, a certain neighborhood $\mathcal{O} \subset \mathbb{C}^n$ of the origin $\beta = 0$, and $\mathsf{C}_k = \left[C^j_{ik}\right]$ are the generators of the adjoint matrix representation

$$\mathsf{C}_i \mathsf{C}_k - \mathsf{C}_k \mathsf{C}_i = C^j_{ik} \mathsf{C}_j$$

of the commutation relations (3.10). Expressing the covariance matrix H of the operators (3.11) in terms of the covariances (3.9) of the generators $\hat{x}_i$, we obtain in place of 3.8) the exact inequality

$$\mathsf{R}\left(\beta, \bar{\beta}\right) \geq \left(\mathsf{D}\mathsf{K}^\dagger \mathsf{S}^{-1} \mathsf{K}\mathsf{D}^\dagger\right)\left(\beta, \bar{\beta}\right). \tag{3.12}$$

In the case (3.3), the family ϱ_θ is unitarily homogeneous with respect to the Lie group having Hermitian generators $\hat{x}_k = \hat{s}_k = \hat{x}^*_k$ and canonical parameters θ_i. As in the case of (3.5), we obtain a generalized uncertainty relation

$$\mathsf{R}_\theta \geq \frac{\hbar^2}{4} \mathsf{K}^\intercal_\theta \mathsf{S}_0^{-1} \mathsf{K}_\theta, \tag{3.13}$$

where $\mathsf{K}_\theta = \mathrm{i}\theta_k \mathsf{C}^k \left(\mathrm{e}^{\mathrm{i}\theta_k \mathsf{C}^k} - \mathbf{1}\right)^{-1}$ and $\mathsf{C}^k = \hbar^{-1} \mathsf{C}_k$. In the domain $\Theta \subseteq \mathbb{R}^n$ of convergence of the series

$$\left(\mathsf{I} - \mathrm{e}^{\mathrm{i}\theta_k \mathsf{C}^k}\right)^{-1} = \sum_{m=1}^{\infty} \mathrm{e}^{\mathrm{i}m\theta_k \mathsf{C}^k},\ \theta \in \Theta,$$

the inequality (3.13) determines the lower bound of the mean quadratic error of measurement of the canonical parameters for the unitary representation $\mathrm{e}^{\mathrm{i}\theta_k \hat{s}^k}$ of the Lie group generated by the selfadjoint $\hat{s}^k = \hbar^{-1} \hat{s}_k$.

4. Efficient Measurements and Quasimeasurements

1. In classical statistics, estimates whose covariance matrix attains the minimal value, transforming the Cramér-Rao inequality locally or globally into an equality, are said to be efficient (locally or globally, respectively). In quantum statistics, because of the nonunique generalization of the Cramér-Rao inequality, the concept of efficiency, introduced by analogy with the classical concept, loses its universality, and the definitions of locally efficient estimates [1], [4], [5] based on the different variants of this generalization are not equivalent. Therefore, we shall distinguish efficient measurements (or estimates), for which the invariant Helstrom's bound (2.4) is attained, from efficient measurements corresponding to the right bound(2.7), calling the former Helstrom efficient and the latter right efficient. As we shall show here, the concept of right efficiency is more universal: Measurements that are globally Helstrom efficient are also right efficient, but not vice versa. We show first that Helstrom efficient estimates exist globally for the canonical families of density operators (3.1) if the operators $\hat{x}_k$ are commuting self-adjoint operators $\hat{s}_k$, and the estimated parameters $\vartheta\left(\gamma\right)$ are taken to be their expectations

$$\vartheta_i\left(\gamma\right) = \mathrm{Tr}\hat{s}_i \varrho\left(\gamma\right) = \mu_k\left(\gamma\right) \tag{4.1}$$

as the derivatives $\mu_k = \partial \ln \chi / \partial \gamma^k$ for the moment generating function $\chi\left(\gamma\right) = \mathrm{Tr}\varrho_0 \mathrm{e}^{\gamma^k \hat{s}_k}$ of the canonical states

$$\varrho\left(\gamma\right) = \chi^{-1}\left(\gamma\right) \mathrm{e}^{\gamma^k \hat{s}_k/2} \varrho_0 \mathrm{e}^{\gamma^k \hat{s}_k/2} \tag{4.2}$$

corresponding to zero imaginary parts $\operatorname{Im}\beta^k = 0$ in (3.1) with $\chi\left(\beta, \bar{\beta}\right) = \chi\left(\beta + \bar{\beta}\right)$. Differentiating the operator-function (4.2) we find the symmetrized logarithmic derivatives $\hat{g}_k = \hat{s}_k - \mu_k$ with respect to γ^k. Thus, the symmetric Fisher information (2.2) in this case is the matrix of covariances

$$S_{ik} = \operatorname{Tr}\varrho\left(\gamma\right)\left(\hat{s}_i - \mu_i\right)\left(\hat{s}_k - \mu_k\right) = \frac{\partial^2 \ln \chi}{\partial\gamma^i \partial\gamma^k}. \tag{4.3}$$

for the operators $\hat{s}_k$. However these covariances as the second derivatives of $\ln\chi$ are the derivatives $\partial\mu_i/\partial\gamma^k = \partial\mu_k/\partial\gamma^i$ of (4.1). That defines the matrix $\mathsf{D} = \left[\partial\vartheta_i/\partial\gamma^k\right]$ in (2.4) as

$$\mathsf{D}\left(\gamma\right) = \left[\partial\mu_i\left(\gamma\right)/\partial\gamma^k\right] = \mathsf{S}\left(\gamma\right).$$

The inequality (2.4) therefore takes the form $\mathsf{Q}\left(\gamma\right) \geq \mathsf{S}\left(\gamma\right)$, i.e. $\left[\mathsf{Q}_{ik} - \mathsf{S}_{ik}\right] \geq 0$, where $\mathsf{Q}\left(\gamma\right) = \mathsf{R}\left(\gamma\right)$ is the covariance matrix of the operators $\hat{q} = \hat{s}$ realizing the unbiased estimates by the joint measurement of $\hat{s}_i$. One can take the spectral QPM Π of the family $\hat{s}_i = \int \varkappa_i \Pi\left(\mathrm{d}\varkappa\right)$ and define these estimates as spectral values $\varkappa_k$ for $\hat{s}_k$. The covariance matrix $\mathsf{R}\left(\gamma\right)$ of such estimates obviously achieves its minimal value

$$\mathsf{R} = \mathsf{M}_\vartheta\left[\left(\lambda_i - \vartheta_i\right)\left(\lambda_k - \vartheta_k\right)\right] = \mathsf{M}_\vartheta\left[\left(\varkappa_i - \mu_i\right)\left(\varkappa_k - \mu_k\right)\right] = \mathsf{S}.$$

Thus, for the canonical families (4.2) with commuting self-adjoint $\hat{s}_k$ there exists a Helstrom-efficient estimation $\lambda = \varkappa$ of the functions (4.1) defined by the canonical parameters μ_k, and this is realized by an a simultaneous measurement of the commuting observables $\hat{s}_k$. The domain of this efficiency obviously coincides with the domain $\mathcal{O} \subset \mathbb{R}^n$ in which $\chi\left(\gamma\right) < \infty$ is twice differentiable. It can be shown that the opposite assertion holds in the following sense.

Suppose that the estimates λ_i (i.e., the results of a measurement) have, in a certain domain, differentiable mean values $\vartheta_i\left(\alpha\right)$ and the covariances $R_{ik}\left(\alpha\right)$, and suppose the matrices $\mathsf{R} = \left[R_{ik}\left(\alpha\right)\right]$ and $\mathsf{D} = \left[\partial\vartheta_i/\partial\alpha^k\right]$ satisfy the following regularity conditions

$$\frac{\partial}{\partial\alpha^i}\left(\mathsf{R}^{-1}\mathsf{D}\right)^j_k = \frac{\partial}{\partial\alpha^k}\left(\mathsf{R}^{-1}\mathsf{D}\right)^j_i \tag{4.4}$$

(which are trivial in one-dimensional case). Then one can introduce the canonical parameters γ^k by setting $\gamma^k\left(\alpha_0\right) = 0$ for an α_0 at which $\vartheta\left(\alpha_0\right) = 0$.

It is readily verified that for a family of density operators $\varrho\left(\gamma\right)$ of the canonical form (4.2) the regularity conditions are satisfied for the efficient measurement of $\vartheta_k = \mu_k\left(\gamma\right)$ as in this case

$$\mathsf{R}\left(\gamma\right) = \mathsf{S}\left(\gamma\right), \quad \mathsf{D}\left(\gamma\right) = \mathsf{S}\left(\gamma\right)$$

and therefore $\left(\mathsf{R}^{-1}\mathsf{D}\right) = \mathsf{I}$. The proof of the opposite assertion, that if the regularity conditions are satisfied, global Helstrom efficiency holds only for the canonical families 4.2), is given in the Appendix for the more general complex situation. Thus,

Theorem 1. *Under the above regularity condition the inequality (2.4) becomes an equality in the domain $\mathcal{O} \subset \mathbb{R}^n$ iff the density operators $\varrho\left(\alpha\right)$ have the canonical form (4.2), where $\hat{s}_k$, $k = 1, \ldots, n$, are Hermitian commuting operators in $\mathcal{H}$, and the canonical coordinates γ are functions of the parameters α defined by the equations*

$$\frac{\partial}{\partial\gamma^k}\ln\chi\left(\gamma\right) = \vartheta_k\left(\alpha\right), \; k = 1, \ldots, n.$$

The optimal estimation in this case reduces to the measurement of the Hermitian operators $\hat{s}_k$ described by their joint spectral resolution of identity, and the minimal mean square error is determined by the matrix of their covariances (4.3).

2. Suppose that in a domain $\mathcal{O} \subset \mathbb{C}^n$ of some complex coordinates $\alpha = (\alpha^k)$ the unbiased estimates λ_k have mathematical expectations $\vartheta_k(\alpha)$ and covariances $R_{ik}(\alpha, \bar{\alpha})$ satisfying the regularity conditions (4.4)

$$\frac{\partial}{\partial \alpha^i} \left(\mathsf{R}^{-1}\mathsf{D}\right)^j_k = \frac{\partial}{\partial \alpha^k} \left(\mathsf{R}^{-1}\mathsf{D}\right)^j_i, \quad \frac{\partial}{\partial \bar{\alpha}^k} \mathsf{R}^{-1}\mathsf{D} = 0. \tag{4.5}$$

(which simply means in one-dimensional case the analyticity $\partial \mathsf{R}^{-1}\mathsf{D}/\partial \bar{\alpha} = 0$). Then, as in the real case, one can introduce the canonically conjugate parameters $\beta^k = \beta^k(\alpha)$ as analytic functions satisfying the the equations

$$\partial \beta^i / \partial \alpha^k = \left(\mathsf{R}^{-1}\mathsf{D}\right)^i_k, \;\; \beta^k(\alpha_0) = 0.$$

and the functions $\beta^k(\alpha)$ are analytic by virtue of the condition (4.5).

Theorem 2. *Under the above formulated regularity conditions, the inequality (2.6) becomes an equality if and only if the family $\{\varrho(\alpha, \bar{\alpha}), \alpha \in \mathcal{O}\}$ has the canonical form (3.1), where $\varrho_0 = \varrho(0,0)$, the operators $\hat{x}_k$, $k = 1, \ldots, n$, have simultaneously in $\mathcal{H}$ the right eigen QPM*

$$\hat{1} = \int \Pi(\mathrm{d}\varkappa), \quad \hat{x}_k \Pi(\mathrm{d}\varkappa) = \varkappa_k \Pi(\mathrm{d}\varkappa), \quad \varkappa = (\varkappa_1, \ldots \varkappa_n) \in \mathbb{C}^n, \tag{4.6}$$

and the canonical parameters β^k, $k = 1, \ldots, n$ are defined by the equations

$$\frac{\partial \ln \chi\left(\beta, \bar{\beta}\right)}{\partial \bar{\beta}^k} = \vartheta_k(\alpha, \bar{\alpha}), \quad \alpha \in O. \tag{4.7}$$

The optimal estimation in this case reduces to a quasimeasurement of the non-Hermitian operators $\hat{x}_k$ described by the resolution of the identity (4.6), and the minimal mean square error is determined by the matrix of the covariances

$$H_{ik} = \mathrm{Tr}\varrho\left(\hat{x}_i - \vartheta_i\right)\left(\hat{x}_k - \vartheta_k\right)^*. \tag{4.8}$$

The sufficiency is proved as in Section 2. Using the invariance of the right bound (2.7) under the analytic transformations $\alpha \mapsto \beta$, we choose as the displaced α^k determining this bound the canonical parameters β^k of the family of density operators (3.1). The elements $\partial \vartheta_i / \partial \beta^k$ of the matrix D with allowance for $\vartheta_\iota = \partial \ln \chi / \partial \bar{\beta}^i$ then coincide with the elements (3.7) of the matrix H. Since the operators $\hat{x}_k$ commute in accordance with (4.6),

$$\hat{x}_i \hat{x}_k = \int \varkappa_i \varkappa_k \Pi(\mathrm{d}\varkappa) = \hat{x}_k \hat{x}_i,$$

we have $\vartheta_k = \mu_k$, $\mathsf{H} = \mathsf{S}$, where μ_k are the mathematical expectations of $\hat{x}_k$ and S is the covariance matrix (3.9) of these operators. Therefore, the inequality (2.7) takes the form $\mathsf{R} \geq \mathsf{S}$. It remains to show that the measurement described by the resolution of the identity (4.6) leads to an estimation for which $\mathsf{R} = \mathsf{S}$ even in the case when the operators $\hat{x}_k$ do not commute with their Hermitian conjugates: $\hat{x}_i^* \hat{x}_k \neq \hat{x}_k^* \hat{x}_i$ (which is the case for a nonorthogonal resolution (4.6)). For this, it is sufficient to take into account the representation

$$\hat{x}_i = \int \varkappa_i \Pi(\mathrm{d}\varkappa), \quad \hat{x}_i \hat{x}_k^* = \int \varkappa_i \bar{\varkappa}_k \Pi(\mathrm{d}\varkappa), \tag{4.9}$$

obtained by integrating the equations in (4.6) $x \in \mathbb{C}^n$ and also the conjugate equation $\Pi(\mathrm{d}\varkappa)\hat{x}_k^* = \bar{\varkappa}_k \Pi(\mathrm{d}\varkappa)$. Because of (4.9), the covariances

$$R_{ik} = \int (\varkappa_i - \vartheta_i)\left(\bar{\varkappa}_k - \bar{\vartheta}_k\right) \mathrm{Tr}\varrho\Pi(\mathrm{d}\varkappa) \tag{4.10}$$

of the estimates $\lambda_k = \varkappa_k$ obtained on the basis of the quasimeasurement of the operators $\hat{q}_k = \hat{x}_k$ coincide with the covariance H_{ik} of these operators, which proves that this generalized measurement is efficient for the density operators (3.1). The proof of the opposite assertions of Theorem 2 follows from the very derivation of the inequality (2.7) and is given in the Appendix.

3. Thus, the condition of (right) efficiency requires the existence of commuting operators that have a joint right spectral resolution and play the role of sufficient statistics, which we call right-efficient. At the same time, it is sufficient to restrict the study of these operators to the minimal subspace generated by the domains $\varrho\left(\beta, \bar{\beta}\right)\mathcal{H}$ with density operators $\varrho\left(\beta, \bar{\beta}\right)$ for $\beta \in \mathcal{O}$. further, if one considers only real values of the parameters $\vartheta_k\left(\beta, \bar{\beta}\right)$, the optimal estimation can be described by non-Hermitian and noncommuting (with the conjugate) operators of right-efficient statistics and is not therefore Helstrom efficient. However, estimates that are Helstrom efficient correspond, in accordance with Theorem 1, to the special case of right efficiency for which the operators $\hat{x}_k$ are Hermitian. If the operators $\hat{x}_k$ in (3.1) are not Hermitian but commute with the Hermitian conjugates, the right efficient estimates also coincide with the complexified Helstrom efficient estimates However, the commutativity $\hat{x}_k\hat{x}_i^* = \hat{x}_i^*\hat{x}_k$ need not hold.

Example. Suppose $\hat{x}_k = \varphi_k(\hat{a})$, where φ_k are entire functions $\mathbb{C}^r \to \mathbb{C}$, $\hat{a} = (\hat{a}_i, \ldots, \hat{a}_r)$ are boson annihilation operators satisfying the commutation relations

$$[\hat{a}_i, \hat{a}_j] = 0, \quad [\hat{a}_j, \hat{a}_i^*] = \delta_{ij}\hat{1}.$$

It is well known that the operators $\hat{a}$ have right eigenvectors $|\alpha\rangle \in \mathcal{H}, \alpha \in \mathbb{C}^r$, that define a nonorthogonal resolution of the identity:

$$\hat{1} = \int |\alpha\rangle\langle\alpha| \prod_{i=1}^{r} \frac{1}{\pi} \mathrm{d}\,\mathrm{Re}\,\alpha_i \mathrm{d}\,\mathrm{Im}\,\alpha_i, \quad \hat{a}_i|\alpha\rangle = \alpha_i|\alpha\rangle.$$

Obviously, the operators $\hat{x} = \varphi(\hat{a})$ also have a right eigen resolution of the identity (4.5), where

$$\Pi(\mathrm{d}\varkappa) = \int \delta(\mathrm{d}\varkappa, \varphi(\alpha))|\alpha\rangle\langle\alpha| \prod_{i=1}^{r} \frac{1}{\pi} \mathrm{d}\,\mathrm{Re}\;\alpha_i \mathrm{d}\,\mathrm{Im}\,\alpha_i$$

($\delta(\mathrm{d}\varkappa, \lambda)$ is the Dirac delta measure of unite mass at the point λ). Therefore, the optimal estimation of the parameters $\vartheta_k = \partial \ln \chi / \partial \bar{\beta}^k$ of the density operators (3.1) for $\hat{x} = \varphi(\hat{a})$ is right efficient and reduces to a coherent measurement and extension of the estimate $\vartheta = \varphi(\alpha)$ with respect to the result α. For the special case when the function $\varphi(\alpha)$ is linear and the state ϱ_0 is Gaussian, this fact was established in [5].

Note that besides right and left lower bounds one can also consider other, combined bounds by means of the factorization [10] $\vartheta = \vartheta_+ + \vartheta_-$, defining right derivatives with respect to ϑ_+ and left derivatives with respect to ϑ_-. An interesting question is this: Is the class of efficient statistics exhausted by statistics for which at least one such bound can be attained?

4. In conclusion, let us consider the question of the (right) efficiency of the estimation of the parameters β^k themselves of the canonical families (3.1). The inequality (2.7) corresponding to this case $\vartheta^k = \beta^k$ has the form $\mathsf{R} \geq \mathsf{H}^{-1}$ where H is the matrix of the derivatives (3.7). Without loss of generality, we shall assume that $\mathrm{Tr}\hat{x}_k \varrho_0 = 0$.

Theorem 3. *The inequality* $\mathsf{R} \geq \mathsf{H}^{-1}$ *becomes an equality if and only if the operators* $\hat{x}_k$ *in (3.1) have a right joint spectral measure (4.5), the generating function of the moments (3.2) of these operators in the state* ϱ_0 *is Gaussian:* $\chi\left(\beta, \bar{\beta}\right) = \exp\left\{\bar{\beta}^i H_{ik} \beta^k\right\}$, *where* H_{ik} *does not depend on* β *and* $\bar{\beta}$, *and the unbiased estimates* $\lambda^k = \lambda^k(\varkappa)$ *are taken to be linear functions* $\lambda^k = H^{ki} \varkappa_i$ *of the results* $\varkappa_k$ *of simultaneous quasimeasurement of the observables* $\hat{x}_k$.

The proof of the sufficiency of these conditions for the existence of the right efficient estimation is obvious: From the fact that the matrix H coincides with the covariance matrix S of the operators $\hat{x}_k$ it follows that the covariance matrix $\mathsf{R} = \mathsf{H}^{-1}\mathsf{H}\mathsf{H}^{-1}$ is equal to H^{-1}.

The necessity follows from the necessary conditions of right efficiency of Theorem 2, according to which the family $\varrho\left(\beta, \bar{\beta}\right)$ must also have the form

$$\varrho\left(\beta, \bar{\beta}\right) = \chi^{-1} \mathrm{e}^{\beta_k \hat{x}^{k*}} \varrho_0 \mathrm{e}^{\bar{\beta}_k \hat{x}^k}, \tag{4.11a}$$

where $\chi\left(\beta, \bar{\beta}\right) = \mathrm{Tr}\varrho_0 \mathrm{e}^{\beta_k \hat{x}^k} \mathrm{e}^{\bar{\beta}_k \hat{x}^{k*}}$, $\frac{\partial}{\partial \bar{\beta}_k} \ln \chi = \beta^k$, and the operators $\hat{x}^k$ have the joint right resolution of the identity

$$\hat{1} = \int \Pi(\mathrm{d}\varkappa), \quad \hat{x}^k \Pi(\varkappa) = \varkappa^k \Pi(\mathrm{d}\varkappa), \quad \varkappa = \left(\varkappa^k\right) \in \mathbb{C}^n.$$

Comparing (4.2) and (3.1), we obtain $\bar{\beta}_k \hat{x}^k = \bar{\beta}^k \hat{x}_k$, whence

$$\beta_k = H_{ki}\beta^i, \quad \chi\left(\beta, \bar{\beta}\right) = \bar{\beta}^i H_{ik} \beta^k, \quad \hat{x}^k = H^{ki}\hat{x}_i.$$

Theorem 3 has been proved.

5. Appendix

1. Let us proof the inequality (2.7). First consider the one-dimensional case. Let $\hat{q}$ be an operator in $\mathcal{H}$ for which

$$\mathrm{Tr}\hat{q}\varrho(\alpha, \bar{\alpha}) = \vartheta(\alpha, \bar{\alpha}). \tag{5.1}$$

Differentiating (5.1) with respect to α and using the definition (2.5) and the normalization condition $\mathrm{Tr}\varrho(\alpha, \bar{\alpha}) = 1$, due to which $\mathrm{Tr}\varrho\hat{h}^* = 0$, we obtain

$$\frac{d\vartheta}{d\alpha} = \mathrm{Tr}\varrho(\hat{q} - \vartheta)\hat{h}^*.$$

Since the covariance $\mathrm{Tr}\varrho(\hat{q} - \vartheta)\hat{h}^*$ satisfies the Schwarz inequality

$$\left|\mathrm{T}\left[\mathrm{r}\varrho(\hat{q} - \vartheta)\hat{h}^*\right]\right|^2 \leq \mathrm{Tr}\left[\varrho(\hat{q} - \vartheta)(\hat{q} - \vartheta)^*\right] \mathrm{Tr}\left[\varrho\hat{h}\hat{h}^*\right], \tag{5.2}$$

which is the condition of non-negativity of the determinant of the 2×2 matrix of covariances $\mathrm{Tr}\varrho\hat{h}_i\hat{h}_k^*$, $i = 0, 1$, where $\hat{h}_0 = (\hat{q} - \vartheta)$, $\hat{h}_1 = \hat{h}$, we can write

$$\mathrm{Tr}\varrho(\hat{q} - \vartheta)\left(\hat{q}^* - \bar{\vartheta}\right) \geq \left|\frac{d\vartheta}{d\alpha}\right|^2 \Big/ \mathrm{Tr}\varrho\hat{h}\hat{h}^*, \tag{5.3}$$

This inequality obviously establishes a lower bound for the variance of the estimation of the parameter $\vartheta = \vartheta(\alpha, \bar{\alpha})$ in the class of ordinary measurements described by normal operators $\hat{q}$. However since the normality condition $\hat{q}\hat{q}^* = \hat{q}^*\hat{q}$ was not used in the derivation of (5.3), this bound gives a lower bound for the variance of any unbiased estimation of ϑ. Indeed, if $\Pi(\mathrm{d}\lambda), \lambda \in \mathbb{C}$ is a QPM describing the unbiased estimation as a generalized measurement in $\mathcal{H}$, then the operator $\hat{q} = \int \lambda \Pi(\mathrm{d}\lambda)$ satisfy the condition (5.1). From the Hermitian positivity

$$(\lambda - \hat{q})\,\Pi(\mathrm{d}\lambda)\,(\lambda - \hat{q})^* \geq 0 \quad (\Pi \geq 0) \tag{5.4}$$

it follows that $\int |\lambda|^2 \Pi(\mathrm{d}\lambda) \geq \hat{q}\hat{q}^*$, and

$$\int |\lambda - \hat{q}|^2 \,\Pi(\mathrm{d}\lambda) \geq (\lambda - \hat{q})(\lambda - \hat{q})^* . \tag{5.5}$$

Taking the mathematical expectation of both sides of (5.4) and bearing in mind that the variance $R = \mathsf{M}_\vartheta\left[|\lambda - \vartheta|^2\right]$ of the estimation ϑ is

$$R = \mathrm{Tr}\varrho \int |\lambda - \vartheta|^2 \,\Pi(\mathrm{d}\lambda),$$

we obtain in conjunction with (5.3)

$$R \geq \mathrm{Tr}\varrho\,(\hat{q} - \vartheta)(\hat{q} - \vartheta)^* \geq |d|^2/g, \tag{5.6}$$

where we have denoted $d = d\vartheta/d\alpha,\ g = \mathrm{Tr}\varrho\hat{h}\hat{h}^*$. Thus, for the one-dimensional case the inequality (2.7) has been proved.

2. Equality can be attained in (5.5) if, first, the expectations of the two sides of (5.5) coincide and, second, the Schwarz inequality becomes an equality. The first condition actually establishes equality in (5.4). More precisely:

Lemma 1. *Suppose the ranges $\varrho(\alpha, \bar{\alpha})\mathcal{H}$ of density operators $\{\varrho(\alpha, \bar{\alpha}) : \alpha \in \mathcal{O}\}$ generate the whole of $\mathcal{H}$. Then the equality $\mathrm{Tr}\varrho R = 0$ for any non-negative definite operator R in $\mathcal{H}$ and all $\alpha \in \mathcal{O}$ implies that $R = 0$.*

It is sufficient to show that in $\mathcal{H}$ there is no vector $|\chi\rangle$ of the form $|\chi\rangle = \varrho^{1/2}|\psi\rangle$ for which $\langle\chi| R |\chi\rangle \neq 0$. But this follows from the inequality

$$\mathrm{Tr}\varrho^{1/2}R\varrho^{1/2} \geq \langle\psi|\,\varrho^{1/2}R\varrho^{1/2}\,|\psi\rangle .$$

which holds for any non-negative R when $\langle\psi \mid \psi\rangle = 1$.

Applying this result to the operator R equal to the difference of the right- and left-hand sides of (5.5), we find, under the conditions of the lemma, that equality holds in (5.5) only if

$$(\lambda - \hat{q})\,\Pi(\mathrm{d}\lambda)\,(\lambda - \hat{q})^* = 0, \text{ or } \hat{q}\Pi(\mathrm{d}\lambda) = \lambda\Pi(\mathrm{d}\lambda).$$

This proves that for the existence of right efficient unbiased estimation in some domain $\mathcal{O} \ni \alpha$ it is necessary to have an operator $\hat{q}$ with a right-eigen QPM in the subspace generated by the subspaces $\varrho(\alpha, \bar{\alpha})\mathcal{H}$, with $\mathrm{Tr}\hat{q}\varrho(\alpha, \bar{\alpha}) = \lambda$. In the case of real spectrum $\lambda \in \mathbb{R}$ such an operator $\hat{q}$ is obviously selfadjoint.

The second condition of equality in (5.6) is equivalent to the condition of linear dependence $\varrho(\hat{q} - \lambda) = t\varrho\hat{h}$, where $t = d/g$ is a constant. Setting

$$t\hat{s} = \hat{q} - \vartheta(0)$$

we obtain the equations

$$\partial\varrho/\partial\bar{\alpha} = \varrho(\hat{s} - \mu), \quad \partial\varrho/\partial\alpha = (\hat{s} - \mu)^*\varrho$$

where $t\mu = \vartheta(\alpha) - \vartheta(0)$. Its solution of these equations with the boundary condition $\varrho(0,0) = \varrho_0$ has the canonical form (3.1). The the operator $\hat{q} = t\hat{s} + \vartheta(0)$ should have right-eigen QPM, so the operator $\hat{s}$ should. This proves for the one-dimensional case, the necessity of the canonicity of the density operators $\varrho(\alpha, \bar{\alpha})$ for the existence of the right efficient estimation formulated in Theorem 2. In the Hermitian case $\hat{x}^* = \hat{x}$, this also proves the necessity of Theorem 1.

3. A multidimensional generalization is obtained from the one-dimensional case by taking

$$\hat{q} - \lambda = (\hat{q}_i - \lambda_i)\,\bar{\eta}^i, \quad \hat{h} = \hat{h}_k \bar{\xi}^k,$$

where $\eta^i,\ i = 1, \dots, m,\ \alpha^k,\ k = 1, \dots, n$, are arbitrary complex numbers. Remembering that then

$$\mathrm{Tr}\varrho\,(\hat{q} - \lambda)\,\hat{h}^* = \bar{\eta}^i \frac{\partial \vartheta_i}{\partial \alpha^k} \xi^k,$$

we obtain from (5.2) for $\xi^k = \left(\mathsf{H}^{-1}\mathsf{D}^\dagger\right)_i^k \eta^i$ the second of the inequalities

$$R_{ik}\bar{\eta}^i\eta^k \geq \mathrm{Tr}\varrho\,(\hat{q}_i - \lambda_i)\,(\hat{q}_k - \lambda_k)^*\,\bar{\eta}^i\eta^k \geq \left(\mathsf{D}\mathsf{H}^{-1}\mathsf{D}^\dagger\right)_{ik}\bar{\eta}^i\eta^k,$$

which holds for arbitrary $\hat{q}_i$ for which $\mathrm{Tr}\varrho\hat{q}_i = \vartheta_i$. Setting

$$\hat{q}_i = \int \lambda_i \Pi\,(\mathrm{d}\lambda)\,, \text{ where } \int \Pi\,(\mathrm{d}\lambda) = \hat{1},\ \lambda \in \mathbb{C}^m,$$

is the resolution of the identity describing the estimator $\lambda_i = \varkappa_i$, and applying the inequality (5.5) for $\hat{q} = \hat{q}_i\bar{\eta}^i,\ \lambda = \lambda_i\bar{\eta}^i$, we obtain for the matrix R of the covariances of ϑ_i satisfying the first of the inequalities (5.6), whence (2.7) follows because η^i is arbitrary.

The inequality (2.6) becomes an equality for $\alpha \in \mathcal{O}$ only if

$$\hat{q}_i\Pi\,(\mathrm{d}\lambda) = \lambda_i\Pi\,(\mathrm{d}\lambda)\,, \text{ and } \varrho\,(\hat{q}_i - \lambda_i) = t_k^i \partial\varrho/\partial\bar{\alpha}_k, \text{ where } t_k^i = \left(\mathsf{D}\mathsf{H}^{-1}\right)_k^i,$$

whence with allowance for $\mathsf{T} = \left[T_k^i\right]$ to be constants nondegenerated matrix we obtain (3.1).

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