RECONSTRUCTION THEOREM FOR A QUANTUM STOCHASTIC PROCESS

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Abstract. Statistically interpretable axioms are formulated that define a quantum stochastic process (QSP) as a causally ordered field in an arbitrary space–time region T of an open quantum system under a sequential observation at a discrete space-time localization. It is shown that to every QSP described in the weak sense by a self-consistent system of causally ordered correlation kernels there corresponds a unique, up to unitary equivalence, minimal QSP in the strong sense. It is shown that the proposed QSP construction, which reduces in the case of the linearly ordered discrete $T = \mathbb{Z}$ to the construction of the inductive limit of Lindblad's canonical representations [8], corresponds to Kolmogorov's classical reconstruction [12] if the order on T is ignored and leads to Lewis construction [14] if one uses the system of all (not only causal) correlation kernels, regarding this system as lexicographically preordered on $\mathbb{Z}\times T$. The approach presented encompasses both nonrelativistic and relativistic irreversible dynamics of open quantum systems and fields satisfying the conditions of local commutativity semigroup covariance. Also given are necessary and sufficient conditions of dynamicity (or conditional Markovianity) and regularity, these leading to the properties of complete mixing (relaxation) and ergodicity of the QSP.

0. Introduction

The problem of statistical foundation for the irreversible processes in open systems of quantum thermodynamics and measurement theory encountered in coherent optics, quantum communications and microelectronics [1]–[3] requires the development of a general theory of quantum stochastic processes (QSP) that contains the classical theory and the known QSP models as special cases. This theory must be operational and should admit a microscopically consistent statistical interpretation for the irreversible successive physical maps like quantum dynamical transformations and quantum measurement operations. On the phenomenological level such quantum operations were introduced already by von Neumann [4] and considered in a more general framework by Haag and Kastler [5] and Davies and Lewis [6]. A microscopically consistent operational approach to the general QSP theory which will be followed here, containing the classical and quantum statistical theories as special cases, was outlined in [7].

In physical applications, QSPs are usually described by special chronologically ordered correlation functions, the only functions which can be dynamically defined and tested on the basis of the statistics of successive measurements in real time, called causal. An axiomatic definition of QSPs based on causal correlation operators corresponding to a discrete time $T = \mathbb{Z}$ was given by Lindblad [8] for the case of a simple observable algebra $\mathcal{B} = \mathcal{B}(\mathcal{H})$, and by author [9] for a Boolean algebra $\mathcal{B}$ of measurable events $B \subseteq$ E. Such an approach leads to the description of

Date: Submitted June 15, 1981, revision submitted March 13, 1984.

Key words and phrases. Quantum stochastic process, Strong and weak equivalence, Relativistic causal sets, Quantum covariant measurements, Quantum strong and weak Markovianity.

This paper is translated from: *Teoreticheskaya i Matematicheskaya Fizika*,**62** (3) 409 – 431 (1985).

representations $\pi_t, t \in T$ of the algebra $\mathcal{B}$ in an expanding physical system, the mathematical definition of which lies in the basis of the operational approach [7] and will be developed here.

The aim of the present paper is threefold. First, we give a physically interpretable — in real time — definition of a QSP as a family of representations of the observable algebra $\mathcal{B}$ in a single (large) quantum system by indicating a universal method of constructing (reconstructing) such a system from *causal correlation kernels* described by the consistent axioms formulated in the paper. Second, we reconstruct the unitary representation of the irreversible *endomorphic dynamics* corresponding to the stationary QSP and encompass in a unified manner both nonrelativistic and relativistic covariant QSPs describing open quantum systems and fields in a causally ordered space-time region $T \subseteq \mathbb{R}^{d+1}$ with respect to a given semigroup of symmetries on T. And third, we derive the *principle of nondisturbance* (or the nondemolition principle) as microcausality of a given quantum subsystem with respect to the successive measurements of a QSP, assumed to be accessible for observations in the given space-time T.

In order to give an explicit statistical interpretation of a QSP as a process of chronologically ordered measurements at arbitrary times $\{t_1, \ldots, t_n\} \subset T$, we shall consider here only the QSP over an *event algebra*, that is a Boolean algebra $\mathcal{B}$, for which the probabilities of finite sequences of events $B_{t_1}, \ldots, B_{t_n} \in \mathcal{B}$, observed in 'space–time" $t \in T$ are determined directly by the diagonal values $\mu(B_{t_1}, \ldots, B_{t_n})$ of the corresponding correlation kernels. It will be shown that the self-consistent family of all causal correlation kernels that satisfy covariance conditions that generalize Lorentz covariance and Einstein causality on an arbitrary T determines a covariant QSP up to unitary equivalence uniquely if one requires the fulfillment for the family $(\pi_t)_{t\in T}$ of minimal conditions of normalization with respect to an initial state space (state vector). The corresponding theorem, which establishes the existence of the covariant QSP described axiomatically by the causal correlation kernels and its uniqueness up to equivalence 'almost everywhere' (i.e., up to unitary equivalence of minimal modifications of the representations of weakly equivalent processes), announced for arbitrary observable algebra $\mathcal{B}$ in [10], plays the same role in the operational statistical physics of open quantum systems as Wightman's reconstruction theorem [11] in quantum field theory or Kolmogorov's fundamental theorem [12] in the classical theory of random processes.

We also consider the question of determining a covariant QSP in the narrower sense recently suggested by Accardi [13] and Lewis [14] and also the relationship of the construction presented here with the noncausal reconstruction of [15] which uses all (not only chronologically ordered) correlation functions. We show that these functions, which in contrast to the causal functions form an infinite system even in the case of a finite set T, can also be formally defined as causal functions on the lexicographically ordered product $\mathbb{Z} \times T$. Although the process indexed by the set $\mathbb{Z} \times T$ does not have a direct physical interpretation in the real time T, its formal causal reconstruction leads to a covariant family of representations $\pi_{i,t} = \pi_t$ independent of $i \in \mathbb{Z}$, identical to the noncausal on T representations π_t of the reconstruction of [15]. Thus, the QSP reconstruction theorem in the narrow sense [15], like Kolmogorov's classical reconstruction [12], is a special case of the fundamental reconstruction theorem formulated and proved here.

1. Quantum Random Processes in the Narrow Sense

1. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{P}(\mathcal{H})$ denote the set of Hermitian projection operators, $P^* = P = P^2$, on $\mathcal{H}$, T be a linearly ordered set (e.g. the discrete or continuous time), and with each $t \in T$ there be associated a Boolean σ-semiring

$\mathcal{B}_t$. I.e. each $\mathcal{B}_t$ is a system of subsets $B \subseteq \mathrm{E}_t$ with the identity $\mathrm{E}_t \in \mathcal{B}_t$ and Boolean zero $\mathrm{O}_t \in \mathcal{B}_t$ – the empty subset $\mathrm{O}_t = \emptyset$ of E_t; $\mathcal{B}_t$ is invariant with respect to the operation of multiplication $BB' = B \cap B'$, and for any $B \in \mathcal{B}_t$ there exists a σ-partitioning $\{B^m\}_1^\infty \subseteq \mathcal{B}_t$ of the identity that contains B: $\mathrm{E}_t = \sum_{m=1}^\infty B^m$, $B \in \{B^m\}_1^\infty$ ($\sum_{m=1}^\infty B^m$ denotes the union $\cup_{m=1}^\infty B^m$ of the disjoint events $B^m \in \mathcal{B}_t$, $B^m B^{m'} = \mathrm{O}_t$ for $m \neq m'$). As $\mathcal{B}_t$ for each t one can take the smallest separating semiring of all single point subsets $B = \{x\}$ with the total set $B = \mathrm{E}_t$ of a discrete countable set $\mathrm{E}_t \subseteq \mathbb{N}$ or the largest one, the power set Boolean algebra $\wp(\mathrm{E}_t)$, or the semiring of intervals $B = [x, x')$ of $\mathrm{E}_t \subseteq \mathbb{R}$, or the whole Borel σ-algebra $\mathcal{B}(\mathbb{R})$ generated by these intervals. The elements $B \in \mathcal{B}_t$ describe the events '$x \in B$' at the time $t \in T$ with unit event E_t determining the readings $x \in \mathrm{E}_t$ of some measuring device with the scale E_t, which in general may depend on t. We shall assume that on T there acts a semigroup (or group) S of symmetries such that $st < st'$ if $t < t'$, this semigroup being represented on $\mathcal{H}$ by linear isometries (or unitaries) $V_s : \mathcal{H} \to \mathcal{H}$, $V_s^* V_s = \mathrm{I}$, $s \in S$, and on each E_t by measurable bijections $g_s : \mathrm{E}_t \to \mathrm{E}_{st}$, $g_s g_{s'} = g_{ss'}$, which determine Boolean homomorphisms $\mathcal{B}_{st} \ni B \mapsto B^s = g_s^{-1}(B)$ onto $\mathcal{B}_t$ for every $t \in T$.

Definition 1. *A quantum stochastic process over the family $\mathcal{B} = (\mathcal{B}_t)$ in the narrow sense is described on the Hilbert space $\mathcal{H}$ by a family $\pi = (\pi_t)$ of σ-homomorphisms $\pi_t : \mathcal{B}_t \to \mathcal{P}(\mathcal{H})$, $t \in T$ and a vector state specified by a normalized element $\xi \in \mathcal{H}$. The σ-homomorphisms are being defined as mappings of $B \in \mathcal{B}_t$ into the quantum logic of orthogonal projectors $P \in \mathcal{P}(\mathcal{H})$ by the conditions*

$$\mathrm{E}_t = \sum_{m=1}^\infty B^m, \quad B^m \in \mathcal{B}_t \Rightarrow \mathrm{I} = \sum_{m=1}^\infty P^m, \quad P^m = \pi_t(B^m) \tag{1.1}$$

holding for each $t \in T$ ($B^m B^{m'} = \mathrm{O}_t \Rightarrow P^m P^{m'} = 0$ for $m \neq m'$). The QSP is said to be S-stationary in the narrow sense if for every $s \in S$ and $t \in T$ the conditions

$$V_s \pi_t(B^s) = \pi_{st}(B) V_s, \quad \forall B \in \mathcal{B}_t, \quad t \in T$$

and $V_s \xi = \xi$, $s \in S$, are satisfied with respect to a representation $V = (V_s)$ of S on $\mathcal{H}$.

Note that without loss of generality, every QSP can be considered as S- stationary with respect to the given semigroup S if one admits the trivial action $t = st$ for all $s \in S$ on T and $V_s = \mathrm{I}$ on $\mathcal{H}$.

As an example of σ-homomorphisms describing an S-stationary QSP with respect to a given unitary representation $U = (U_s)$ with invariant state vector ξ, one can consider a family π of the projection-valued measures $\pi_t(B) = E_t(x') - E_t(x)$ for $B = [x, x')$, determined by spectral families $\{E_t(x), x \in \mathbb{R}\}$ of self-adjoint operators $X_t = \int x dE_t(x)$, $t \in T$ in $\mathcal{H}$ that transform covariantly with respect to a state vector ξ:

$$U_s X_t = X_{st} U_s \quad \forall t \in T, \;\; U_s \xi = \xi \quad \forall s \in S.$$

By such a family $X = (X_t)$ of unitarily equivalent (in the case of transitivity of S on T) operators $X_t = X_t^*$ one can specify any real QSP that is S-stationary with respect to the trivial representation $g_s(x) = x$ of the group S on $\mathrm{E}_t = \mathbb{R}$ $\forall t \in T$. However for time-like measurements, when $\mathrm{E}_t \subseteq T$, the stationarity condition must be determined with respect to nontrivial transformations $g_s(x) = sx$ from E_t into $\mathrm{E}_{st} = s\mathrm{E}_t$ as it is in the case of the translations $st = t + s$ on the additive group $T = \mathbb{R}$. For example, in the case $\mathrm{E}_t = \{x < t\}$, corresponding to the measurement at each t of the occurrence times $x \in \mathbb{R}$ of almost surely past events like the birth times of a historic phenomena or starting times of a continuous measurements, a QSP translationally invariant with respect to an additive semigroup $S \subseteq \mathbb{R}$ can be determined by the spectral decompositions of bounded from above selfadjoint operators $X_t < t\mathrm{I}$ that together with the vector $\xi \in \mathcal{H}$ satisfy the covariance condition

$$X_t + s\mathrm{I} = U_s^* X_{t+s} U_s \quad \forall t \in T, \ \ U_s \xi = \xi \quad \forall s \in S.$$

2. We denote by $\mathcal{F}$ the set of finite parts $\Lambda \subset T$, with each $\Lambda = \{t_1, \dots, t_n\} \equiv \Lambda_n$ being a chain $t_n > \cdots > t_1$ of length n, and we let $\mathcal{B}^\Lambda = \times_{t \in \Lambda} \mathcal{B}_t$ for every $\Lambda = \Lambda_n$ be the set of sequences $\mathbf{b} = (B_1, \dots, B_n)$ of events $\mathbf{b}(t_i) = B_i \in \mathcal{B}_{t_i}$. Each such $\mathbf{b} \in \mathcal{B}^\Lambda$ is determined by a unique function $b : T \ni t \mapsto b(t) \in \mathcal{B}_t$, $b\,(t) = \mathrm{E}_t$, $t \notin \Lambda$ as the restriction $\mathbf{b} = b|\Lambda$, and $\mathbf{e} = (\mathrm{E}_1, \dots, \mathrm{E}_n) \in \mathcal{B}^\Lambda$ is determined by the function $e(t) = \mathrm{E}_t$ for all $t \in T$. Note that the family $\{\mathcal{B}^\Lambda\}$ is inductive with respect to the extension $\mathbf{b} \in \mathcal{B}^\Lambda \mapsto \hat{b}_\mathrm{M} \in \mathcal{B}^\mathrm{M}$ such that $\hat{b}_\mathrm{M}(t) = \mathrm{E}_t$ for $t \in \mathrm{M} \backslash \Lambda$ on any $\mathrm{M} \supseteq \Lambda$, $\mathrm{M} \in \mathcal{F}$, defined by the restriction $\hat{b}_\mathrm{M} = \hat{b}|\mathrm{M}$ of the corresponding function $\hat{b}$ on T for $\mathbf{b} = \hat{b}|\Lambda$. The smallest element $\mathcal{B}^\emptyset$ of the inductive family $\{\mathcal{B}^\Lambda\}$ consists of a single element — the empty sequence which can be extended on any Λ as e_Λ by the identity function e on T.

In accordance with the statistical interpretation of quantum mechanics, the probability $\mu^\Lambda(\mathbf{b})$ of successive observation of the events $B = b(t)$, $t \in \Lambda$, in the corresponding chain Λ_n is determined by the family π and the vector ξ in agreement with the von Neumann projection postulate [4] as

$$\mu^\Lambda(\mathbf{b}) = \|P_n \cdots P_1 \xi\|^2 = \|\xi^\Lambda(\mathbf{b})\|^2. \tag{1.2}$$

Here $\xi^{\Lambda_n}(\mathbf{b}) = \pi_{t_n}(B_n) \cdots \pi_{t_1}(B_1)\xi$ is the result of the chronologically ordered action of the orthogonal projectors $P_i = \pi_{t_i}(B_i)$ on the state vector ξ. On the basis of these probabilities, which can be determined experimentally by counting the relative frequencies of the occurrences of the event sequences $\mathbf{b} = (B_1, \dots, B_n)$ when they are measured in a real flow of time $t \in \{t_1, \dots, t_n\}$ on each copy of the quantum ensemble, one can calculate different characteristics of the QSP and even attempt to reconstruct it by building a statistically equivalent mathematical model $(\mathcal{H}, \pi, \xi)$ of the real quantum system in which this process is observed.

Such QSP reconstruction problem, considered in the present paper, should play a key testing role for the foundation of any quantum dynamical theory. Its solution establishes necessary and sufficient conditions under which there exists — and is unique up to an equivalence — a minimal mathematical model of the physical system under consideration, the model correctly predicting the statistics of successive measurements of observations in this system as a quantum stochastic processes in the above narrow or in a wider sense.

In the framework of classical theory, this problem was solved by Kolmogorov's fundamental theorem [12], a necessary and sufficient condition for the applicability of this theorem being that the family $\{\mu^\Lambda\}$ must form a projective system of probability measures. It is readily verified that the family of positive normalized mappings $\mu^\Lambda : \mathcal{B}^\Lambda \to [0, 1]$ does indeed form a projective system, $\mu^\Lambda(\mathbf{b}) = \mu^\mathrm{M}(\mathbf{b})$ for $\Lambda \subseteq \mathrm{M} \in \mathcal{F}$, but these mappings are not in general additive, although they satisfy the condition of σ-additivity with respect to the last argument $B = \mathbf{b}(t)$, $t = \max \Lambda$. As with respect to the remaining arguments of $\mathbf{b}(t)$, $t < \max \Lambda$, the probabilities μ^Λ are not in general even finitely additive due to the possible noncommutativity of the orthoprojectors $\{P_i\}$ corresponding to different $t_i \in \Lambda$ in accordance with the causal dependence of the chronologically ordered events $\{B_i\}$. The absence of this additivity, observed experimentally in quantum interference processes, indicates the inadequacy of the classical probability theory dealing only with the additive measures corresponding to the compatible events which can always be represented by the commuting orthoprojectors, $P_i P_k = \pi\,(B_i B_k) = \pi\,(B_k B_i) = P_k P_i$, even if $t_i \neq t_k$. Thus in the framework of the classical theory it is impossible not only to construct an adequate mathematical model of the physical system capable of predicting the noncommutative QSP statistics, but even to formulate the problem of its reconstruction due to the nonadequacy of quantum probabilities μ^{Λ} to the (additive) probability measures.

3. For the QSP reconstruction, instead of the probabilities μ^{Λ}, it is necessary to use the correlation kernels (multikernels)

$$\kappa^{\Lambda}(\mathbf{b},\mathbf{b}')=(\xi^{\Lambda}(\mathbf{b})\mid\xi^{\Lambda}(\mathbf{b}')),\quad \mathbf{b},\ \mathbf{b}'\in\mathcal{B}^{\Lambda},\ \Lambda\in\mathcal{F}. \tag{1.3}$$

They are in principle determined by means of the polarization formulas from the diagonal values $\kappa^{\Lambda}(\mathbf{b},\mathbf{b})$ of these kernels, extended to the multi-sesquilinear forms on all possible operator sequences $\mathbf{b}$. The multikernels $\kappa^{\Lambda}:\mathcal{B}^{\Lambda}\times\mathcal{B}^{\Lambda}\to\mathbb{C}$ corresponding to the QSP in the narrow sense π on $\mathcal{B}$ obviously form a projective system: $\kappa^{\Lambda}(\mathbf{b},\mathbf{b}')=\kappa^{\mathrm{M}}(\mathbf{b},\mathbf{b}')$, if $\Lambda\subseteq\mathrm{M}\in\mathcal{F}$, $\mathbf{b},\ \mathbf{b}'\in\mathcal{B}^{\Lambda}$, and for every $\Lambda\in\mathcal{F}$ are readily described by the following properties 1–4, which are called, respectively, positivity, normalisability, σ-additivity, and factorisability with respect to the product $(B,B')\mapsto BB'$:

1. $\sum_{i,i'=1}^{m}\kappa^{\Lambda}(\mathbf{b}^{i},\mathbf{b}^{i'})\bar{c}_{i}c_{i'}\geq 0=,\ \forall\mathbf{b}^{i}\in\mathcal{B}^{\Lambda},\ c_i\in\mathbb{C},\ i\leq m{=}1,2,\ldots.$
2. $\kappa^{\Lambda}(\mathbf{e},\mathbf{e})=1$, in particular $\kappa^{\emptyset}=1$.
3. $\kappa^{\Lambda}(\mathbf{b},\mathbf{b})=\sum_{m=1}^{\infty}\kappa^{\Lambda}(\mathbf{b}^{m},\mathbf{b}^{m})$, where $\mathbf{b}^{m}(t)=\mathbf{b}(t)$, $t<t_n=\max\Lambda$, and
$$b(t_n)=\sum_{m=1}^{\infty}b^{m}(t_n)\quad(b^{m}(t_n)b^{m'}(t_n)=\mathrm{O}\ \text{ for }\ m\neq m').$$
4. $\kappa^{\Lambda}(\mathbf{b}B,\mathbf{b}')=\kappa^{\Lambda}(\mathbf{b},\mathbf{b}'B)$ for any $B\in\mathcal{B}_{tn}$, $\mathbf{b},\ \mathbf{b}'\in\mathcal{B}^{\Lambda}$, where $(\mathbf{b}B)(t)=\mathbf{b}(t)$, $t\neq t_n$, $(\mathbf{b}B)(t_n)=b(t_n)B$, $t_n=\max\Lambda$.

For a QSP that is S-stationary in the narrow sense, the multikernels κ^{Λ} also satisfy the condition of S-stationarity in the wide sense:

5. $\kappa^{s\Lambda}(\mathbf{b},\mathbf{b}')=\kappa^{\Lambda}(\mathbf{b}^{s},\mathbf{b}'^{s})$, $\forall\mathbf{b},\mathbf{b}'\in\mathcal{B}^{s\Lambda}$, $s\in S$, where $s\Lambda=\{st:t\in\Lambda\}$, $b^{s}(t)=b(st)^{s}$, $t\in\Lambda$.

As follows from Theorem 2 (see Sec.2), the natural conditions 1–5 also hold for the wider definition of QSPs described in Sec.2 by weakened conditions of normalization $\pi_t(\mathrm{E}_t)=E_t$, $E_t\xi=\xi$ instead of the condition $\pi_t(\mathrm{E}_t)=\mathrm{I}$, $\forall t\in T$, these holding with respect to a nondecreasing family $E=(E_t)_{t\in T}$ of orthogonal projectors $E_t\leq E_{t'}$, $t\leq t'$. Such widening of the QSP concept makes it possible to reconstruct the QSP from an arbitrary projective system $\{\kappa^{\Lambda}\}$ determined by Axioms 1–5 in a canonical way as it is formulated in the fundamental Theorem 3. Theorem 4 shows that this widening of QSPs, which requires fulfillment of the ordinary normalization condition only in the expanding system of subspaces $\mathcal{H}_t=E_t\mathcal{H}\ni\xi$, is necessary if the minimal process, defined in Sec.4, is to be described by the system $\{\kappa^{\Lambda}\}$ uniquely (up to unitary equivalence).

4. We now show how the canonical QSP reconstruction presented for an arbitrary preodered set T in Sec.3 makes it possible to obtain the canonical reconstruction of a QSP in the narrow sense as the particular case of the Theorem 3, using the much larger system of *all*, not only causal multikernels $\kappa_{\boldsymbol{x}}(\mathbf{b},\mathbf{b}')$, indexed by arbitrary finite sequences $\boldsymbol{x}\in\cup_{n=0}^{\infty}X^{n}$. Such reconstruction, recently suggested in [15], requires the considering of not necessarily chronological orderings $\pi_{x_1}(B_1)\cdots\pi_{x_n}(B_n)\xi=\xi_{\boldsymbol{x}}(\mathbf{b})$ of the events $\mathbf{b}=(B_1,\ldots,B_n)$ since the elements of a sequence $\boldsymbol{x}=(x_1,\ldots,x_n)\in X^n$ may not form a chain $x_1>\cdots>x_n$. Therefore such kernels cannot be not physically realized by the sequential measurements in real time unless points $x\in X$ are considered as causally equivalent, or merely disordered (space-like) coordinates $x\in\mathbb{R}^d$, in which case the families $\pi=(\pi_x)$ give

kinematic representation of quantum fields but not dynamic processes. The following construction should be therefore considered as a reconstruction of quantum fields rather than the processes.

Let $\mathrm{T} = \mathbb{Z} \times X$ be the set of pairs $\mathrm{t} = (l, x)$, $l = 0, \pm 1, \pm 2, \dots$, linearly preordered as $\mathrm{t} \lesssim \mathrm{t}' \Leftrightarrow l \leq l'$. It can be completely ordered by the lexicographic order $\mathrm{t} \leq \mathrm{t}' \Leftrightarrow x \leq x'$ if $l = l'$, or we can leave such points causally equivalent, $\mathrm{t} \sim \mathrm{t}'$ if $l = l'$, and identify with the corresponding points $x \sim x' \in X$ for $l = 0$. Let $S = \mathbb{Z}$ be the group of translations $\mathrm{t} = (l, x) \mapsto s\mathrm{t} = (l + s, x)$, acting on T quasitransitively in the following sense: For any pair $\mathrm{t} > \mathrm{t}'$, there exists $s \in \mathbb{Z}$ such that $\mathrm{t}' > s\mathrm{t}$ (it is necessary to take $s < l' - l$).

We consider the QSP $\pi_{l,x} = \pi_x$ over $\mathcal{B}_{l,x} = \mathcal{B}_x$, $(l, x) = \mathrm{t} \in \mathrm{T}$, determined by a QSP process $\pi_x : \mathcal{B}_x \to \mathcal{P}(\mathcal{H})$ on $X \ni x$ in the narrow sense on the Hilbert space $\mathcal{H}$ with state vector $\xi \in \mathcal{H}$. Such defined process $\pi_\mathrm{t}, \mathrm{t} \in \mathrm{T}$ is S-stationary in the narrow sense due to the condition of quasiconstancy

$$\pi_\mathrm{t}(B) = \pi_{s\mathrm{t}}(B), \quad \forall B \in \mathcal{B}_\mathrm{t}, \quad \mathrm{t} \in \mathrm{T}, \tag{1.4}$$

which is the S-covariance with respect to the trivial representations $g_s = \mathrm{id}$, $U_s = 1$ of the group S on $\mathrm{E}_\mathrm{t} = \mathrm{E}_x$ and $\mathcal{H}$. It is readily verified that this quasiconstant process induces a system of multikernels (1.3) indexed by pairs $\Lambda = (\boldsymbol{s}, \boldsymbol{x})$ of $\boldsymbol{s} = (s_1, \dots, s_n) \in \mathbb{Z}^n$, $s_1 \leq \ldots \leq s_n$, and $\boldsymbol{x} \in X^n$, satisfying the strengthened condition of S-stationarity (ultrastationarity):

5'. $\kappa^{\boldsymbol{s}}_{\boldsymbol{x}}(\mathbf{b}, \mathbf{b}') = \kappa^{\mathbf{0}}_{\boldsymbol{x}}(\mathbf{b}, \mathbf{b}')\ \forall \mathbf{b}, \mathbf{b}' \in \mathcal{B}^\Lambda = \times_{i=1}^n \mathcal{B}_{x_i}$, where $\mathbf{0} = (0, \dots, 0)$.

By virtue of condition $5'$, the projective system $\{\kappa^\Lambda = \kappa^{\boldsymbol{s}}_{\boldsymbol{x}}\}$, indexed by finite $\Lambda \subset \mathrm{T}$, is determined by a set $\{\kappa^{\mathbf{0}}_{\boldsymbol{x}} = \kappa_{\boldsymbol{x}}\}$ of arbitrarily ordered multikernels $\kappa_{\boldsymbol{x}}$ of the original process (field) π_x, $x \in X$, these being identical for any $\boldsymbol{x} = (x_1, \dots, x_n)$ to κ^Λ with $s_1 < \cdots < s_n$ and thus with $\Lambda = \{\mathrm{t}_1 < \cdots < \mathrm{t}_n\}$.

Theorem 1. *Let $\{\kappa^\Lambda\}$ be a projective system of multikernels $\kappa^\Lambda : \mathcal{B}^\Lambda \times \mathcal{B}^\Lambda \to \mathbb{C}$ that satisfy conditions 1–4 for every finite chain $\Lambda \subset \mathrm{T}$ and also condition $5'$ of ultrastationarity with respect to the quasitransitive action of the group $S = \mathbb{Z}$ on* T. *Then there exists a Hilbert space $\mathcal{H}$, a family of σ-homomorphisms $\pi_\mathrm{t} : \mathcal{B}_\mathrm{t} \to \mathcal{P}(\mathcal{H})$, $\pi_\mathrm{t}(\mathrm{E}_\mathrm{t}) = \mathrm{I}$, $\mathrm{t} \in \mathrm{T}$, and a state vector $\xi \in \mathcal{H}$ that determine an S-quasiconstant QSP $\pi = (\pi_\mathrm{t})$ in the narrow sense over $\mathcal{B} = (\mathcal{B}_\mathrm{t})$ which induces the system $\{\kappa^\Lambda\}$. On the minimal space $\mathcal{H}^*$ determined by the condition of separability of the commutant $\mathcal{A} = \cap \pi_l(\mathcal{B})'$ with respect to ξ, this process is described uniquely up to unitary equivalence.*

The proof of this theorem can be obtained as a corollary to Theorem 3 (see Sec. 3) by the construction of the canonical process π^*_t, $\mathrm{t} \in \mathrm{T}$, on the Hilbert space $\mathcal{H}$ that determines the minimal decomposition (1.3), as will be done in Sec. 3. By virtue of $5'$, this process is quasiconstant with respect to the quasitransitive action of S on T and has constant unit $E_l = \pi^*_l(\mathrm{E}_l) = E_0$ for any $l \in \mathbb{Z}$ due to $E_l \leq E_{l'} \leq E_l$ for any pair $l < l'$ of the ordered set $\mathbb{Z}$ and some by virtue of the monotonicity of the family E_l, $l \in \mathbb{Z}$. Therefore, $E_l = \mathrm{I}$ on the minimal space $\mathcal{H}$ on which this process is uniquely determined by virtue of Theorem 4. For the case $\mathrm{T} = \mathbb{Z} \times X$, the described canonical reconstruction determines from the family $\{\kappa_{\boldsymbol{x}}\} = \{\kappa^{\mathbf{0}}_{\boldsymbol{x}}\}$ the QSP $\pi_x = \pi_\mathrm{t}$, $\mathrm{t} = (l, x)$, in the narrow sense.

2. Definition and Properties of a General QSP

1. Let T be an arbitrary set with elements $\mathrm{t} \in \mathrm{T}$ that is preordered by the reflexive and transitive relation $\leq$, which is called causality on T, and let S be a semigroup (or group) of causality preserving symmetries $s : \mathrm{T} \to \mathrm{T}$ as monotonic transformations $\mathrm{t} \leq \mathrm{t}' \Leftrightarrow s\mathrm{t} \leq s\mathrm{t}'$ (and therefore $\mathrm{t} \sim \mathrm{t}'$ if $s\mathrm{t} = s\mathrm{t}'$). We denote by

the symbol $\sim$ the equivalence relation $\mathrm{t} \sim \mathrm{t}' \Leftrightarrow \mathrm{t} \leq \mathrm{t}' \leq \mathrm{t}$, which means invertibility of the causal dependence of t and $\mathrm{t}' \in \mathrm{T}$, by $\bowtie$ the noncomparability relation $\mathrm{t} \bowtie \mathrm{t}' \Leftrightarrow \mathrm{t} \nleq \mathrm{t}' \nleq \mathrm{t}$, which determines causal independence of the corresponding t and t', and by $>$ the strict order (anticipation) relation $\mathrm{t} > \mathrm{t}' \Leftrightarrow \mathrm{t} \nsim \mathrm{t}' \leq \mathrm{t}$, and we shall say that a pair t, t' is mutually nonanticipatory if $\mathrm{t} \ngtr \mathrm{t}' \ngtr \mathrm{t}$, i.e., if $\mathrm{t} \sim \mathrm{t}'$, or $\mathrm{t} \bowtie \mathrm{t}'$.

Besides linearly ordered subsets $\mathrm{T} \subseteq \mathbb{R}$, it may be of interest to take as such T any space-time region of $(d+1)$-dimensional Minkowski space of $\mathrm{t} = (\tau, \mathbf{r})$, where $\tau \in \mathbb{R}$, $\mathbf{r} \in \mathbb{R}^d$, partially ordered by the relation $\mathrm{t} \leq \mathrm{t}' \Leftrightarrow c^{-1}|\mathbf{r} - \mathbf{r}'| \leq \tau' - \tau$ which has trivial equivalence $\mathrm{t} \sim \mathrm{t}' \Leftrightarrow \mathrm{t} = \mathrm{t}'$ and nonempty (for $d \neq 0$) Einsteinian causal independence $\mathrm{t} \bowtie \mathrm{t}' \Leftrightarrow |\mathbf{r} - \mathbf{r}'| > c|\tau - \tau'|$. As S one can consider any semigroup (subgroup) of the inhomogeneous Lorentz transformations $s\mathrm{t} = \Lambda\mathrm{t} + \mathrm{l}$ that preserves the order and leave the subset $\mathrm{T} \subseteq \mathbb{R}^{d+1}$ invariant. For example, if T is the future cone $\mathrm{t} \geq 0 : \ c\tau \geq |\mathbf{r}|$, then as S we can consider the semigroup of translations $\mathrm{t} \mapsto \mathrm{t} + \mathrm{l}$, $\mathrm{l} \in \mathrm{T}$, and the proper orthochronous Lorentz group $L_+^\uparrow$. Galilean causality, corresponding to the limit $c \to \infty$, is described, in contrast, by the preorder relation $\mathrm{t} \leq \mathrm{t}' \Leftrightarrow \tau \leq \tau'$ with nontrivial (for $d \neq 0$) equivalence $\mathrm{t} \sim \mathrm{t}' \Leftrightarrow \tau = \tau'$ and empty noncomparability relation $\bowtie$, i.e., it is a linear preorder given by the foliation of $\mathbb{R}^{d+1}$ into the hyperplanes $\tau = \mathrm{const}$. At the same time, any spatial region T corresponding to fixed $\tau \in \mathbb{R}$ is equipped with trivial causality: $\mathrm{t} \sim \mathrm{t}'$ for any pair t, $\mathrm{t}' \in \mathrm{T}$, in contrast to the relativistic case $c < \infty$, for which any spatial region has identical causality $\mathrm{t} \leq \mathrm{t}' \Rightarrow \mathrm{t} = \mathrm{t}'$.

We introduce the following notation: J is the class of all subsets $j \subset \mathrm{T}$ of pairwise *nonanticipatory* elements $\mathrm{t}, \mathrm{t}' \in j \Rightarrow \mathrm{t} \ngtr \mathrm{t}' \ngtr \mathrm{t}$; K is the subclass formed by the *finite* subsets $k \in J$; L is the subclass formed by the maximal subsets $l \in J$ such that $l \subseteq j \in J \Rightarrow l = j$; and $T \subset J$ is the factor-set of T formed by the maximal subsets $t \subset \mathrm{T}$ of pairwise *equivalent* elements $\mathrm{t}, \mathrm{t}' \in t \Rightarrow \mathrm{t} \sim \mathrm{t}'$. The class J, whose elements can be determined by the condition $j = \max j$, where $\max j$ is the subset of the elements $\mathrm{t} \in j$ which are maximal in j in the sense $\mathrm{t} \leq \mathrm{t}' \in j \Rightarrow \mathrm{t} \sim \mathrm{t}'$, contains together with $j \in J$ any subset of it, $j' \subseteq j$, and is a semilattice with respect to the operation $j \vee j' = \max j \cup j'$, which determines a strict order $j > j' \Leftrightarrow j \vee j' = j$, $j \cap j' = \emptyset$ of the partial preorder:

$$j \leq j' \Leftrightarrow \ \forall \mathrm{t} \in j \ \ \exists \mathrm{t}' \in j' : \mathrm{t} \leq \mathrm{t}'.$$

Both these properties are also inherited by the subclass K, whereas the subclass L is only a directed set: any pair $l, l' \in L$ has in L a majorant $l_+ \supseteq \max l \cup l'$ and a minorant $l_- \supseteq \min l \cup l'$. (It is assumed that for any $j \in J$ there exists $l \in L$ such that $j \subseteq l$). The factor-set T is in general an arbitrary partially ordered set. Subsets $j, j' \in J$ are said to be *equivalent*: $j \sim j'$, if $j \leq j' \leq j$; *independent*: $j \bowtie j'$, if $\mathrm{t} \bowtie \mathrm{t}'$ for all $\mathrm{t} \in j$, $\mathrm{t}' \in j'$; and *mutually nonanticipatory* if $j \vee j' = j \cup j'$. In these terms the empty subset $\emptyset \subset \mathrm{T}$ is the least element $\emptyset \in J : \ \emptyset \leq j \in J$; $j \sim \emptyset \Rightarrow j = \emptyset$; $j \bowtie \emptyset$ for any $j \in J$; and $j > \emptyset$, if $j \neq \emptyset$.

2. Let $\mathfrak{A} = (\mathfrak{A}_k)_{k \in K}$ be a nonincreasing family of $*$-subalgebras of the C*-algebra $\mathfrak{A}_\emptyset = \mathcal{B}(\mathcal{K})$ of bounded operators $a : \mathcal{K} \to \mathcal{K}$ of the Hilbert space $\mathcal{K}$ with common unit $1_k = 1 \in \mathfrak{A}_k$ such that $\mathfrak{A}_{k \vee k'} = \mathfrak{A}_k \cap \mathfrak{A}_{k'}$ for all $k, k' \in K$, and $\mathcal{B} = (\mathcal{B}_k)_{k \in K}$ be the family of Boolean semirings of subsets $B \subseteq \mathrm{E}_k$ with units E_k, $k \in K$, satisfying the conditions

$$k \ \sim \ k' \Rightarrow \mathcal{B}_{k \cup k'} \supseteq \mathcal{B}_k \cup \mathcal{B}_{k'}, \qquad \mathrm{E}_k = \mathrm{E}_{k'}, \tag{2.1}$$

$$k \ \bowtie \ k' \Rightarrow \mathcal{B}_{k \cup k'} = \mathcal{B}_k \otimes \mathcal{B}_{k'}, \qquad \mathrm{E}_{k \cup k'} = \mathrm{E}_k \times \mathrm{E}_{k'}, \tag{2.2}$$

where $\mathcal{B}_k \otimes \mathcal{B}_{k'}$ is a semiring of subsets $B \times B'$, $B \in \mathcal{B}_k$, $B' \in \mathcal{B}_{k'}$, of the Cartesian product $\mathrm{E}_k \times \mathrm{E}_{k'}$ ($\mathcal{B}_\emptyset$ is identified in accordance with (2.2) with the trivial Boolean algebra (O, E) of a single-point set $\mathrm{E}_\emptyset$). Denoting $\mathfrak{A}_t = \mathfrak{A}_{\{\mathrm{t}\}}$ for any single-point subset $\{\mathrm{t}\} \subseteq t \in T$ and $\mathcal{B}_t = \underset{k \subset t}{\cup} \mathcal{B}_k$, $\mathrm{E}_t = \mathrm{E}_{\{\mathrm{t}\}}$, we have for $k \neq \emptyset$: $\mathfrak{A}_k \subseteq \underset{t \in [k]}{\cap} \mathfrak{A}_t$, and $B = \underset{t \in [k]}{\times} b(t)$ for any $B \in \mathcal{B}_k$, where $\mathrm{E}_t \supseteq b(t) \in \mathcal{B}_t$, $[k] = \{t \in T : t \cap k \neq \emptyset\}$.

We assume that the semigroup S, acting on K by transformations $sk = \{s\mathrm{t} : \mathrm{t} \in k\}$, is represented on $\mathfrak{A}$ by self-consistent family of $*$-endomorphisms $a \mapsto a^s = U_s^* a U_s$, where $U = (U_s)_{s \in S}$ is a unitary representation of S on $\mathcal{K}$, the endomorphisms mapping each subalgebra $\mathfrak{A}_{sk}$ onto $\mathfrak{A}_k$, and we assume that the semigroup is also represented on $\mathcal{B}$ by σ-homomorphisms from each semiring $\mathcal{B}_{sk}$ onto $\mathcal{B}_k$, these being induced by measurable bijections $g_s : \mathrm{E}_t \to \mathrm{E}_{st}$.

Let $\mathcal{H}$ be a Hilbert space containing $\mathcal{K}$, and $\iota = (\iota_k)_{k \in \mathbb{Z}}$ be a family of faithful $*$-representations $\iota_k : \mathfrak{A}_k \to \mathcal{B}(\mathcal{H})$ that satisfy the self-consistency condition

$$\iota_k(a) = \iota_{k'}(a) I_k, \quad \forall a \in \mathfrak{A}_{k'},\ k \leq k' \in K, \tag{2.3}$$

where $I_k = \iota_k(1)$ are orthoprojectors in $\mathcal{H}$, $\iota_\emptyset$ is the identity representation on the subspace $\mathcal{K} = I_\emptyset \mathcal{H}$. We consider also an isometric representation $V = (V_s)_{s \in S}$ of the semigroup S on $\mathcal{H}$ with respect to which the condition of covariance of the representations ι is satisfied in the form

$$V_s \iota_k(a^s) = \iota_{sk}(a) V_s I_k \qquad \forall a \in \mathfrak{A}_{sk}\ , \tag{2.4}$$

so that $U_s = V_s I_\emptyset$ for any $s \in S$. Note that the family $I = (I_k)_{k \in K}$ of orthogonal projectors I_k determining essential subspaces $\mathcal{H}_k = I_k \mathcal{H}$ of the representations ι_k, is nondecreasing: $I_k = I_k I_{k'} \leq I_{k'}$ for any $k \leq k'$, and uniquely determines these representations in the case of a one-dimensional $\mathcal{K} \simeq \mathbb{C}$ (and therefore $\mathfrak{A}_k = \mathbb{C}$ for all $k \in K$): $\iota_k(c) = c I_k$, $c \in \mathbb{C}$ (at the same time $I_\emptyset$ is the one-dimensional orthogonal projector P_ξ corresponding to a state vector $\xi \in \underset{k > \emptyset}{\cap} \mathcal{H}_k$).

Definition 2. *An $\mathcal{H}$-process with respect to $(\mathfrak{A}, \iota)$ over $\mathcal{B}$ (or $\mathcal{H}$-QSP with respect to $I = (I_k)$ if $\mathcal{K} \simeq \mathbb{C}$) is a family $\pi = (\pi_k)_{k \in K}$ representations $\pi_k : \mathcal{B} \to \mathcal{P}(\mathcal{H})$ that satisfy the normalization conditions $\pi_k(E_k) I_k = I_k$, $\forall k \in K$, or even stronger conditions*

0) $\underset{k \subseteq l}{\wedge} P_k = \underset{k \subseteq l}{\vee} I_k$, $\forall l \in L$,

where $P_k = \pi_k(\mathrm{E}_k)$, $k \in K$, $P_\emptyset = \mathrm{I}$ is the identity in $\mathcal{H}$, $I_k = \iota_k(1)$, $k \in K$, $I_\emptyset = \mathrm{P}$ is the projection onto $\mathcal{K}$. These representations are described by the following axioms:

1) $k \sim k' \Rightarrow \pi_{k \cup k'}(BB') = \pi_k(B) \pi_{k'}(B')$, $\forall B \in \mathcal{B}_k$, $B' \in \mathcal{B}_{k'}$,
2) $k \bowtie k' \Rightarrow \pi_{k \cup k'}(B \times B') = \pi_k(B) \pi_{k'}(B') P_{k \cup k'}$,
3) $\mathrm{E}_k = \sum_{m=1}^{\infty} B^m$, $B^m \in \mathcal{B}_k \Rightarrow P_k = \sum_{m=1}^{\infty} \pi_k(B^m)$ $(B^m B^{m'} = \mathrm{O}_k,\ m \neq m')$,
4) $[\pi_k(B), \iota_k(a)] = 0$, $\forall B \in \mathcal{B}_k$, $a \in \mathfrak{A}_k =$, $k \in K$.
   The process $(\mathcal{B}, \pi)$ is said to be S-covariant with respect to the representation V if for any $s \in S$
5) $V_s \pi_k(B^s) = \pi_{sk}(B) V_s P_k$, $\forall B \in \mathcal{B}_{sk}$, $k \in K$.

3. By virtue of the uniqueness of the partitioning $\Lambda = \cup k_i$ of any finite subset $\Lambda \subset T$ into elements $k_i \in K$ that form a chain $k_n > \cdots > k_1 > \emptyset$ of a length $n \leq |\Lambda|$, we can uniquely associate with every function $[\Lambda] \ni t \mapsto b(t) \in \mathcal{B}_{\Lambda \cap t}$, defined on a subset $[\Lambda] \subset T$ of equivalence classes $t = [\mathrm{t}]$ such that $t \cap \Lambda \neq \emptyset$, a sequence $\mathbf{b} = (B_1, \ldots, B_n)$ of events $B_i = \prod_{t \in [k_i]} b(t) = \mathbf{b}(k_i)$. Here $k_i = \max \Lambda_i$, $\Lambda_i \subseteq \Lambda$ are determined recursively: $\Lambda_{i-1} = \Lambda_i \backslash k_i$, $\Lambda_n = \Lambda$, and $n = n(\Lambda)$ is such that $\Lambda_1 = k_1$, $\Lambda_0 = \emptyset$. For any $\Lambda \in \mathcal{F}$ such that $\Lambda > k$ for a $k \in K$, we denote by $\mathcal{B}_k^\Lambda$ the set of sequences $\mathbf{b}$ as functions $B_i = \mathbf{b}(k_i)$ on the chain $\{k_n > \cdots > k_1\} > k$ corresponding to such Λ, and we determine an induction mapping $\mathbf{b} \in \mathcal{B}_j^\Lambda \mapsto \hat{\mathbf{b}} \in$

$\mathcal{B}_k^{\mathrm{M}}$ for any $j \in K$, $k \leq j$, $\Lambda \subseteq \mathrm{M} \in \mathcal{F}$, by extending the function $b(t)$ corresponding to the sequence $\mathbf{b}$ to the function on M as $\hat{b}(t) = b(t)$ for $t \in [\Lambda]$ and $\hat{b}(t) = \mathrm{E}_t$ for $t \in [\mathrm{M} \backslash \Lambda]$. This function determines the sequence $\hat{\mathbf{b}} : \hat{\mathbf{b}}(k_i) = \underset{t \in [k_i]}{\times} \hat{b}(t)$ with $i = 1, \ldots, n$, $n = n(\mathrm{M})$ such that $\hat{\mathbf{b}}(k_i) = \mathbf{b}(k_i) \times \mathrm{E}_{\bar{\Lambda} k_i}$, where $\mathbf{b}(k_i) = \underset{t \in [\Lambda k_i]}{\times} b(t)$, $\Lambda k_i = \Lambda \cap k_i$.

With each pair $\mathbf{b}, \mathbf{b}' \in \mathcal{B}_k^\Lambda$ we associate the correlation operator

$$\kappa_k^\Lambda(\mathbf{b}, \mathbf{b}') = F_k^\Lambda(\mathbf{b})^* F_k^\Lambda(\mathbf{b}'), \tag{2.5}$$

where

$$F_k^{\Lambda_n}(\mathbf{b}) = \pi_{k_n}(B_n) \cdots \pi_{k_1}(B_1) I_k \tag{2.6}$$

is the 'Feynman integral" over the paths in $\{b\,(t)\}$ determined by the $\mathcal{H}$-process π over $\mathcal{B}$. The following theorem establishes the properties of the operator-valued multikernels κ_k^Λ, these properties when $k = \emptyset$, $\mathcal{K} \simeq \mathbb{C}$ being identical to the corresponding properties of the scalar kernels (1.3) of the QSP with respect to $\mathfrak{A}_t = \mathbb{C}$, $t \in T$, defined in the narrow sense.

Theorem 2. *Let $(\mathcal{B}, \pi)$ be an $\mathcal{H}$-process with respect to $(\mathfrak{A}, \iota)$ that is S-covariant with respect to V. Then Eq.(2.5) determines mappings $\kappa_k^\Lambda : \mathcal{B}_k^\Lambda \times \mathcal{B}_k^\Lambda \to \iota_k(\mathfrak{A}_l)'$, where $\Lambda > k$, $l = \max \Lambda$, that satisfy the following conditions:*

0) *(self-consistency) for any $\mathbf{b}, \mathbf{b}' \in \mathcal{B}_j^\Lambda$*

$$I_k \kappa_j^\Lambda(\mathbf{b}, \mathbf{b}') I_k = \kappa_k^{\mathrm{M}}(\hat{\mathbf{b}}, \hat{\mathbf{b}}'), \ \ k \leq j \in K, \ \Lambda \subseteq \mathrm{M} \in \mathcal{F};$$

1) *(positivity) for any $\mathbf{b}^i \in \mathcal{B}_k^\Lambda$, $\eta_i \in \mathcal{H}_k$, $i \leq m$*

$$\sum_{i,i'=1}^{m} (\eta_i \mid \kappa_k^\Lambda(\mathbf{b}^i, \mathbf{b}^{i'}) \eta_{i'}) \geq 0, \quad m = 1, 2, \ldots; \Lambda > k;$$

2) *(normalizability) $\kappa_k^\Lambda(\mathbf{e}, \mathbf{e}) = I_k$, $\Lambda > k \in K$;*
3) *(σ-additivity [In the weak operator topology and also in the sense of order convergence.])*

$$\mathbf{b}(l) = \sum_{m=1}^{\infty} \mathbf{b}^m(l), \quad l = \max \Lambda \Rightarrow \kappa_k^\Lambda(\mathbf{b}, \mathbf{b}) = \sum_{m=1}^{\infty} \kappa_k^\Lambda(\mathbf{b}^m, \mathbf{b}^m),$$

where $\mathbf{b}^m(k_i) = \mathbf{b}(k_i)$, $k_i \subseteq \Lambda \backslash l$;

4) *(factorizability) $\kappa_k^\Lambda(\mathbf{b}B, \mathbf{b}') = \kappa_k^\Lambda(\mathbf{b}, \mathbf{b}'B)$ where*

$$(\mathbf{b}B)(l) = \mathbf{b}(l)B, (\mathbf{b}B)(k_i) = \mathbf{b}(k_i), k_i \subseteq \Lambda \backslash l \ ;$$

5) *(covariance) for any $s \in S$, $\mathbf{b}, \mathbf{b}' \in \mathcal{B}_k^\Lambda$*

$$V_{k,s}^* \kappa_{sk}^{s\Lambda}(\mathbf{b}, \mathbf{b}') V_{k,s} = \kappa_k^\Lambda(\mathbf{b}^s, \mathbf{b}'^s),$$

where $s\Lambda = \{st : t \in \Lambda\}$, $\mathbf{b}^s(l) = \mathbf{b}(sl)^s$, $\mathbf{b}'^s(l) = \mathbf{b}'(sl)^s$, and $V_{k,s} : \mathcal{H}_k \to \mathcal{H}_{sk}$ are isometric operators, made consistent by the condition $V_{k,s} | \mathcal{H}_{k'} = V_{k',s}$, $k' \leq k$, which form a representation, $V_{sk,s'} V_{k,s} = V_{k,s's}$, of the semigroup S on the family $\mathcal{H}_k = I_k \mathcal{H}$, $k \in K$, with unitary subrepresentation $U_s = V_{\emptyset,s}$ on the invariant subspace $\mathcal{K} = \mathcal{H}_\emptyset$.

Proof. Bearing in mind that $I_{k_n} > \cdots > I_{k_1} > I_k$, and also $[\pi_k(B), I_k] = 0$, we represent the operators $F_k^\Lambda(\mathbf{b})$ determined by (2.5) in the form

$$F_k^\Lambda(\mathbf{b}) = I_{k_n} \pi_{k_n}(B_n) \cdots I_{k_1} \pi_{k_1}(B_1) I_k, \qquad k_n \leq \cdots \leq k_1 \leq k. \tag{2.7}$$

Hence, in accordance with (2.3), we obtain $\iota_l(a) F_k^\Lambda(\mathbf{b}) = F_k^\Lambda(\mathbf{b}) \iota_l(a)$, where $l = \max \Lambda$, $a \in \mathfrak{A}_l$, whence we have the commutativity of $\kappa_k^\Lambda(\mathcal{B}_k^\Lambda, \mathcal{B}_k^\Lambda) \subseteq \iota_k(\mathfrak{A}_l)'$.

The properties 1–4 are direct consequences of the corresponding conditions of Definition 2 and can be directly verified for the compositions (2.5). covariance holds for (2.5), as can be seen from condition 5, with respect to the restrictions $V_{k,s} = V_s I_k$, which map the subspaces $\mathcal{H}_k = I_k\mathcal{H}$ (as follows from (2.4)) to $\mathcal{H}_{sk}$: $V_s I_k = I_{sk} V_k I_k$. At the same time, the operators $U_s = V_{\emptyset,s} = V_s \mathrm{P}$, which leave $\mathcal{K} = \mathrm{P}\mathcal{H}$ invariant, are unitary, $U_s U_s^* = \mathrm{P} = U_s^* U_s$, by virtue of the fact, which follows from (2.4), that the orthogonal projectors $U_s U_s^*$ commute with any of the operators $a \in \mathfrak{A}_\emptyset = \mathcal{B}(\mathcal{K})$.

We now prove the self-consistency (0) of the family $\{\kappa_k^\Lambda\}$. The condition 0 obviously holds, $\Lambda = \emptyset$, since $\kappa_j^\emptyset = I_j$, and the empty sequence, of which $\mathcal{B}_j^\emptyset$ consists, induces the sequence $\mathbf{e} \in \mathcal{B}_k^{\mathrm{M}}$, for which $\kappa_k^{\mathrm{M}}(\mathbf{e},\mathbf{e}) = I_k$ by virtue of the normalization condition $P_{k_i} \leq I_{k_i}$ for any $\mathrm{M} \in \mathcal{F}$, $\mathrm{M} > k$.

Now suppose $\Lambda_m = \Lambda_{m-1} \cup j_m$, where $m \geq 1$, $j_m = \max \Lambda_m$, $\mathbf{b}_m = (\mathbf{b}_{m-1}, B_m)$ is the sequence $\mathbf{b}_m \in \mathcal{B}_j^{\Lambda_m}$ determined by the subsequence $\mathbf{b}_{m-1} \in \mathcal{B}_j^{\Lambda_{m-1}}$, $\Lambda_{m-1} = \Lambda_m \backslash j_m$, and $\mathrm{M} = \mathrm{M}_n \supseteq \Lambda_m$ is the finite subset $\mathrm{M}_n = \cup_{i=1}^n k_i$, $n \geq m$, represented by the chain $k_n > \cdots > k_1 > \emptyset$ of elements $k_i \in K$. We assume that the property 0 holds for $\Lambda = \Lambda_{m-1}$, $m \geq 1$, i.e.,

$$F_j^{\Lambda_{m-1}}(\mathbf{b}_{m-1}) I_k = \pi_{k_n}(B_n^{\Lambda_{m-1}}) \pi_{k_{n-1}}(B_{n-1}^{\Lambda_{m-1}}) \cdots \pi_{k_1}(B_1^{\Lambda_{m-1}}) I_k, \tag{2.8}$$

where $(B_1^{\Lambda_{m-1}}, \ldots, B_{n-1}^{\Lambda_{m-1}}, B_n^{\Lambda_{m-1}}) \equiv \mathbf{b}_{m-1}^{\Lambda_{m-1}}$ is determined by the induction $\mathbf{b}_{m-1} \mapsto \mathbf{b}_{m-1}^{\Lambda_{m-1}}$ from $\mathcal{B}^{\Lambda_{m-1}}$ to $\mathcal{B}^{\mathrm{M}_n}$: $B_i^{\Lambda_{m-1}} = B_i^{m-1} \times \mathrm{E}_{\bar{\Lambda}_{m-1} k_i}$, $i \leq n$. Note that $B_n = \mathrm{E}_{k_n}$, since $\Lambda_{m-1} k_n = \emptyset$ ($\Lambda_{m-1} < j_m \leq k_n$), and, taking into account conditions 0, 1, and 2, we obtain from (2.8), using (2.7),

$$F_j^{\Lambda_{m-1}}(\mathbf{b}_{m-1}) I_k = \pi_{\Lambda_{m-1} k_{n-1}}(B_{n-1}^{m-1}) \cdots \pi_{\Lambda_{m-1} k_1}(B_1^{m-1}) I_k. \tag{2.9}$$

We now write $F_j^{\Lambda_m}(\mathbf{b}_m) I_k = \pi_{j_m}(B_m) F_j^{\Lambda_{m-1}}(\mathbf{b}_{m-1}) I_k$ in the form

$$F_j^{\Lambda_m}(\mathbf{b}_m) I_k = \pi_{j_m k_n}(B_{m_n}) \pi_{j_m k_{n-1}}(B_{m_{n-1}}) \cdots \pi_{j_m k_1}(B_{m_1}) F_j^{\Lambda_{m-1}}(\mathbf{b}_{m-1}) I_k \tag{2.10}$$

again using 0–2 and (2.7), where the elements $B_{mi} \in \mathcal{B}_{j_m k_i}$ are determined by the representation $B_m = \mathop{\times}_{i=1}^{n} B_{mi}$ which corresponds to the decomposition $j_m = \bigcup_{i=1}^{n} j_m k_i$. Substituting (2.9) in (2.10) and bearing in mind that $j_m k_i \bowtie \Lambda_{m-1} k_{i'}$ for $i \leq i'$, and also that $j_m k_n = \Lambda_m k_n$, $j_m k_{n-1} \cup \Lambda_{m-1} k_{n-1} = \Lambda_m k_{n-1}, \ldots, j_m k_1 \cup \Lambda_{m-1} k_1 = \Lambda_m k_1$, we obtain by virtue of the same conditions in the representation (2.7)

$$F_j^{\Lambda_m}(\mathbf{b}_m) I_k = \pi_{\Lambda_m k_n}(B_n^m) \pi_{\Lambda_m k_{n-1}}(B_{n-1}^m) \cdots \pi_{\Lambda_m k_1}(B_1^m) I_k, \tag{2.11}$$

where $B_n^m = B_{mn}$, $B_{n-1}^m = B_{mn-1} \times B_{n-1}^{m-1}, \ldots, B_1^m = B_{m1} \times B_1^{m-1}$. The representation (2.7), like (2.9), is equivalent to (2.8), and therefore

$$F_j^{\Lambda_m}(\mathbf{b}_m) I_k = \pi_{k_n}(B_n^{\Lambda_m}) \pi_{k_{n-1}}(B_{n-1}^{\Lambda_m}) \cdots \pi_{k_1}(B_1^{\Lambda_m}) I_k. \tag{2.12}$$

Since (2.8) holds for $m = 1$ ($\Lambda_0 = \emptyset$), we obtain by induction on m the validity of (2.12), for any m, from which it follows that property 0 holds for all $j < \Lambda \in \mathcal{F}$, which is what we wanted to prove. □

3. Existence and Reconstruction of QSP

For every $l \in L$ we denote by $\mathfrak{A}_l = \bigcap_{k \leq l} \mathfrak{A}_k$, $T_l = \{t \in T : t \leq l\}$, and $\mathfrak{B}_l = \bigcup_{\Lambda \leq l} \mathcal{B}^\Lambda$ is the inductive limit of the family $\mathcal{B}^\Lambda = \mathcal{B}_\emptyset^\Lambda$, $\Lambda \in \mathcal{F}$, identified, as the set of cylindric subsets $\times_{t \leq l} b(t)$, with the set of functions $b : t \in T_l \mapsto b(t) \in \mathcal{B}_t$ equal to the unit E_t outside some finite $[\Lambda] \subseteq T_l$. The sets $\mathfrak{B}_l$ are embedded into $\mathfrak{B} = \bigcup_{\Lambda \in \mathcal{F}} \mathcal{B}^\Lambda$ by the induction $b \in \mathfrak{B}_l \mapsto b_l \in \mathfrak{B}$, $b_l(t) = \mathrm{E}_t$ for $t \not\leq l$, which extends $b = b_l|T_l$ to the function b_l on the complete factor-set T, and $\mathfrak{B}^s$, $s \in S$, will denote the set of functions $b \in \mathfrak{B}$ that satisfy the condition $b(t) = \mathrm{E}_t$ if $t \notin sT$.

Let $\kappa : \mathfrak{B}\times\mathfrak{B} \to \mathcal{B}(\mathcal{K})$ be the mapping with operator values uniquely determined by the projective system $\{\kappa^\Lambda\}$ of multikernels $\kappa^\Lambda : \mathcal{B}^\Lambda \times \mathcal{B}^\Lambda \to \mathcal{B}(\mathcal{K})$,

$$\kappa^\Lambda(\mathbf{b}, \mathbf{b}') = \kappa(\hat{\mathbf{b}}, \hat{\mathbf{b}}'), \qquad \mathbf{b}, \mathbf{b}' \in \mathcal{B}^\Lambda, \quad \Lambda \in \mathcal{F}. \tag{3.1}$$

The functional kernel κ corresponding to the system of operator-valued multikernels $\kappa^\Lambda(\mathbf{b}, \mathbf{b}') = \kappa^\Lambda_\emptyset(\mathbf{b}, \mathbf{b}')$ determined by (2.5) can be described by the following simple system of axioms, which are equivalent to conditions 1–5 of Theorem 2 for the case $k = \emptyset$.

Definition 3. *A $\mathcal{K}$-process with respect to $\mathfrak{A}$ in the wide sense over $\mathcal{B}$ (or simply a QSP in the wide sense over $\mathcal{B}$ if $\mathcal{K} \simeq \mathbb{C}$) is a projective system of multikernels $\kappa = \{\kappa^\Lambda : \Lambda \in \mathcal{F}\}$ described by the mapping $\kappa : \mathfrak{B} \times \mathfrak{B} \to \mathcal{B}(\mathcal{K})$ with values $\kappa(b_l, b'_l) \in \mathfrak{A}'_l$, $l \in L$, that satisfy the following axioms:*

1^0 $\sum_{i,i'=1}^m (\zeta_i \mid \kappa(b^i, b^{i'})\zeta_{i'}) \geq 0$, $\forall b^i \in \mathfrak{B}$, $\zeta_i \in \mathcal{K}$, $i \leq m = 1, 2, \ldots$.

2^0 $\kappa(e, e) = 1$, $e(t) = \mathrm{E}_t$, $\forall t \in T$.

3^0 $\kappa(b_l, b_l) = \sum_{m=1}^m \kappa(b^m_l, b^m_l)$, *if* $b(t) = \sum_{m=1}^m b^m(t)$, $t \subseteq l$, $b^m(t) = b(t)$, $t \not\subseteq l$.

4^0 $\kappa(b_l B, b'_l) = \kappa(b_l, b'_l B)$ *for every* $B \in \mathcal{B}_t$, $t \subseteq l$, *where* $b, b' \in \mathfrak{B}_l$, $(bB)(t) = b(t)B$, *and* $(bB)(t) = b(t)$ *for* $t \not\subseteq l$. *The process κ in the wide sense is said to be S-covariant if*

5^0 $U^*_s \kappa(b, b')U_s = \kappa(b^s, b'^s)$, $\forall b, b' \in \mathfrak{B}^s$, *where* $b^s(t) = b(st)^s$, $b'^s(t) = b'(st)^s$, $t \in T$, $s \in S$, *with respect to the isometric representation* $U = (U_s)_{s\in S}$ *of the semigroup S on $\mathcal{K}$. Every $\mathcal{H}$-process $(\mathcal{B}, \pi)$ with respect to the system $(\mathfrak{A}, \iota)$ on $\mathcal{H}$ that determines a decomposition $\kappa^\Lambda = \kappa^\Lambda_\emptyset$ of the projective system of multikernels (3.1) in the form (2.5) is called a realization of the process κ.*

The existence of realizations for an arbitrary process κ, ensured for the nonfunctional case of a single-point set T by Naimark's reconstruction theorem [16], is established by the following (fundamental) theorem, which serves, in the case $\mathcal{K} \simeq \mathbb{C}$, as the noncommutative analog of Kolmogorov's reconstruction theorem taking into account the causality relation on T.

Theorem 3. *Let $\{\kappa^\Lambda\}$ be the projective system (3.1) of mappings $\kappa^\Lambda : \mathcal{B}^\Lambda \times \mathcal{B}^\Lambda \to \mathcal{B}(\mathcal{K})$ with values in $\mathfrak{A}'_l$ for any $l \supseteq \max\Lambda$, determining in accordance with the conditions $1^0 - 5^0$ an S-covariant $\mathfrak{A}$ process with respect to $\mathcal{K}$ over $\mathcal{B}$ in the wide sense. Then there exists a canonical system π^* of σ-representations of the family $\mathcal{B}$ which is described by conditions 1–5 of Definition 2 with respect to a representation system $(\mathfrak{A}, \iota)$ on $\mathcal{H}$ and an isometric S-representation V as S-covariant $\mathcal{H}$-process that determines the decomposition $\kappa^\Lambda = \kappa^\Lambda_\emptyset$ (2.5) and satisfies the condition $\pi^*_k(\mathrm{E}_k) = \bigvee_{l\supseteq k'} E_l \equiv P^*_k$ for all $k \in K$, where $E_l = \bigwedge_{k\subseteq l} P^*_k$, $l \in L$. If in addition $\mathfrak{A}_k = \bigvee_{l\subseteq k} \mathfrak{A}_l$, $k \neq \emptyset$, and for any $b \in \mathfrak{B}$, $\zeta \in \mathcal{K}$*

$$\lim_{l\downarrow\emptyset} \sup_{\{\zeta_i, b^i\}} \left| \sum_i (\zeta_i|\kappa(b^i, b) - \kappa(b^i, e)\kappa(e, b)|\zeta) \right| = 0, \tag{3.2}$$

where $\zeta_i \in \mathcal{K}$, $b^i \in \mathfrak{B}_l$, $i \in \mathbf{N}$, $\sum(\zeta_i|\kappa(b^i, b^{i'})\zeta_{i'}) \leq 1$, then there exists a realizing $\mathcal{H}$-process $(\mathcal{B}, \pi)$ with respect to $*$-representations ι^* of the family $\mathfrak{A}$ that satisfy the conditions $\iota^*_k(1) = \bigwedge_{l\supseteq k} E_l$ for all $k \in K$. Moreover, if for each sequence $B_i =$

$\underset{t\subseteq l_i}{\times} b(t)b'(t)$, $i = 1,\ldots,n$, *corresponding to a chain* $l_1 \le \cdots \le l_n$ *the kernel* κ *is determined by a regression*

$$\kappa(b,b') = \rho_{l_1}(\pi_{l_1}(B_1)\theta_{l_1l_2}(\pi_{l_2}(B_2)\cdots\theta_{l_{n-1}l_n}(\pi_{l_n}(B_n))\cdots)), \tag{3.3}$$

where $\rho_l : \mathcal{A}_l \to \mathcal{B}(\mathcal{K})$, $l \in L$ *and* $\theta_{ll'} : \mathcal{A}_{l'} \to \mathcal{A}_l$ *are normal mappings determined for all* $l \le l' \in L$ *on the* W^**-algebra* $\mathcal{A} = \pi_l(\mathcal{B}_l)\vee\iota_l(\mathfrak{A}_l)$ *generated by representations* $\pi_l : \mathcal{B}_l \to \mathcal{P}(\mathcal{H}_l)$, $\iota_l : \mathfrak{A}_l \to \pi_l(\mathcal{B}_l)'$ *on some Hilbert spaces* $\mathcal{H}_l$, $l \in L$, *then the canonical realization* π^* *has the conditional Markov property*

$$E_l\pi^*_{l'}(\mathcal{B}_{l'})E_l \subseteq \pi^*_l(\mathcal{B}_l) \vee \iota^*_l(\mathfrak{A}_l), \qquad \forall l \le l' \in L. \tag{3.4}$$

We divide the proof into five stages.

1. **Construction of the space** $\mathcal{H}$. We consider the $\mathcal{K}$-hull of the set $\mathfrak{B}$, which is defined as the linear space of formal sums ζ_ib^i $(= \sum_i \zeta_ib^i)$ with finite number of nonvanishing coefficients $\zeta_i \in \mathcal{K}$ with respect to the componentwise defined operations

$$\zeta_ib^i + \zeta'_ib^i = (\zeta_i + \zeta'_i)b^i, \quad c(\zeta_ib^i) = (c\zeta_i)b^i, \quad c \in \mathbb{C}.$$

Let $\mathcal{E}$ be the factorisation of this $\mathcal{K}$-hull with respect to the non-negative definite (by virtue of 1^0) Hermitian form

$$(\zeta_ib^i \mid \zeta_{i'}b^{i'}) = (\zeta_i \mid \kappa(b^i,b^{i'})\zeta_{i'} =) \equiv \|\zeta_ib^i\|^2, \tag{3.5}$$

and $\mathcal{H} = \bar{\mathcal{E}}$ be the completion of $\mathcal{E}$ with respect to the norm $\|\cdot\|$. We denote by $|\zeta b) = \{\zeta_ib^i : \zeta_ib^i - \zeta b \in \mathcal{N}\}$ the equivalence classes of elements ζb generating $\mathcal{E}$, determined by the kernel $\mathcal{N} = \{\zeta_ib^i : \|\zeta_ib^i\| = 0\}$ of the form (3.1), retaining it as notation of the scalar product of the bra $(\zeta_ib^i|$ and ket $|\zeta_ib^i) = (\zeta_ib^i|^*$ vectors. The space $\mathcal{K}$ can be embedded into $\mathcal{H}$ by identification of each $\zeta \in \mathcal{K}$ and vector $|\zeta e)$ isometric to it by virtue of 2^0. Let $\mathcal{E}_j \subseteq \mathcal{E}$ for each $j \in J$ be the subspace generated by the vectors $|\zeta b)$, $\zeta \in \mathcal{K}$, $b \in \mathfrak{B}_j = \underset{\Lambda\le j}{\cup}\mathcal{B}^\Lambda$, and E_j be the orthogonal projector onto the completions $\mathcal{H}_j = \bar{\mathcal{E}}_j$, which obviously satisfies the condition $E_j = \underset{k\subseteq j}{\vee}E_k$, where $k \in K$, $j \in J$, $E_\emptyset = \mathrm{P}$ is the orthogonal projector onto the subspace $\mathcal{K} \subseteq \mathcal{H}$. Let $P^*_j = \underset{l\supseteq j}{\vee}E_l$, $I^*_j = \underset{l\supseteq j}{\wedge}E_l$ be the orthogonal projectors onto the completions $\mathcal{K}^*_j = \bar{\mathcal{E}}_{j[}$, $\mathcal{H}^*_j = \bar{\mathcal{E}}_{j]}$ of the subspaces $\mathcal{E}_{j[} = \underset{l\supseteq j}{\cup}\mathcal{E}_l$, $\mathcal{E}_{j]} = \underset{l\supseteq j}{\cap}\mathcal{E}_{l'}$ generated by the vectors $|\zeta b)$, $\zeta \in \mathcal{K}$, respectively, for $b \in \mathfrak{B}_{j[} := \underset{l\supseteq j}{\cup}\mathfrak{B}_l$ and $b \in \mathfrak{B}_{j]} := \underset{l\supseteq j}{\cap}\mathfrak{B}_l$. Because $\underset{k\subseteq l}{\cap}\mathfrak{B}_{k[} = \mathfrak{B}_l$ for any $l \in L$ and $\mathfrak{B}_{\emptyset[} = \mathfrak{B}$, we obtain condition 0 for $P_k = P^*_k$ and $I_k = E_k$, $k \in K$: $\underset{k\supseteq j}{\wedge}P^*_k = E_l$, $P^*_\emptyset = \mathrm{I}$. Since $\mathfrak{B}_{k[} \subseteq \mathfrak{B}_l$ for $k \subseteq l$, we also have $\underset{k\subseteq l}{\cup}\mathfrak{B}_{k]} = \mathfrak{B}_l$ and $\underset{k\subseteq l}{\vee}I^*_k = E_l$ for $l \in L$. We show that $I^*_\emptyset = \mathrm{P}$, i.e., $\underset{l\in L}{\wedge}E_l = \lim_{l\downarrow\emptyset}E_l = \mathrm{P}$ (by virtue of the monotonicity of E_t) when the regularity condition (3.2) is satisfied. Since $E_l \ge \mathrm{P}$,

$$(\zeta b|\,(E_l - \mathrm{P})\,|\zeta b) = (\zeta.b^\cdot|E_l|\zeta.b^\cdot) = \|E_l\zeta.b^\cdot\|^2,$$

where $|\zeta.b^\cdot) = |\zeta b) - \mathrm{P}|\zeta b) = |\zeta b) - |\kappa(e,b)\zeta e)$. By virtue of the fact that the family $|\zeta_ib^i)$, $\zeta_i \in \mathcal{K}$, $b^i \in \mathfrak{B}_l$, is dense in the subspace $\mathcal{H}_l = E_l\mathcal{H}$ and by the definition of the norm (3.5), we have

$$\begin{aligned}\|E_l\zeta.b^\cdot\|^2 &= \sup\{|(\zeta_ib^i|\zeta.b^\cdot)|^2 : \|\zeta_ib^i\|^2 \le 1\}\\ &= \sup\{|(\zeta_i|\kappa(b^i,b^\cdot)\zeta.)|^2 : (\zeta_i|\kappa(b^i,b^{i'})\zeta_{i'}) \le 1\}\\ &= \sup_{\{b^i,\zeta_i\}}|(\zeta_i|\kappa(b^i,b) - \kappa(b^i,e)\kappa(e,b)|\zeta)|^2 \downarrow 0\end{aligned}$$

for $l \downarrow \emptyset$, $\zeta_i \in \mathcal{K}$, $b^i \in \mathfrak{B}_l$, $(\zeta_i|\kappa(b^i,b^{i'})\zeta_{i'}) \le 1$. Thus, $E_l \downarrow \mathrm{P}$ for $l \downarrow \emptyset$, $l \in L$ in the strong operator topology.

2. **Construction of the representation** ι. For any $a \in \mathfrak{A}_j = \bigcap_{k \subseteq j} \mathfrak{A}_k$, we set

$$\lambda_j(a)|\zeta_i \hat{b}^i) = |a\zeta_i \hat{b}^i), \qquad \zeta_i \in \mathcal{K}, \quad b^i \in \mathcal{B}^\Lambda, \quad \Lambda \leq j \in J. \tag{3.6}$$

The operator on $\mathcal{E}_j$ defined in this manner is linear and identical to $\lambda_{j'}(a)|\mathcal{E}_j$ for $j \leq j'$, and this definition is correct by virtue of the inequality

$$\|a\zeta_i \hat{b}^i\|^2 = (a\zeta_i|\kappa^\Lambda(b^i, b^{i'})a\zeta_{i'}) \leq \|a\|^2 \|\zeta_i \hat{b}^i\|^2,$$

which follows from the condition of commutativity $\kappa^\Lambda(b, b') \in \mathfrak{A}'_j$ with $a \in \mathfrak{A}_j$ for $b' \in \mathcal{B}^\Lambda$, $\Lambda \leq j$. The mapping $a \mapsto \lambda_j(a)$ is obviously linear and multiplicative with respect to $a \in \mathfrak{A}_j$, and $\lambda_j(a^*) = \lambda_j(a)^*|\mathcal{E}_j$, and $\lambda_j(1) = \mathrm{I}|\mathcal{E}_j$. Denoting by $\iota_j(a^*) = \lambda_j(a)^*$ the continuous extension of the bounded operators $\lambda_j(a)^*$ to the whole of $\mathcal{H}$ determined by the condition $\iota_j(a^*)|\mathcal{E}_j^\perp = 0$ on the orthogonal complement $\mathcal{E}_j^\perp = \mathcal{H}_j^\perp$, we obtain the family of $*$-representations $\iota_j : \mathfrak{A}_j \to \mathcal{B}(\mathcal{H})$, which satisfy the condition (2.3) with respect to $\iota_j(1) = I_j = E_j$, $j \in J$, and $\iota_\emptyset$ is obviously the identity representation of the C*-algebra $\mathfrak{A}_\emptyset = \mathcal{B}(\mathcal{K})$ on the subspace $\mathcal{K} \subseteq \mathcal{H}$.

Denoting $\lambda_j^l(a) = \lambda_l(a)|\mathcal{E}_{j]}$ for fixed $j \leq l \in L$ and taking into account the invariance with respect to $\lambda_l(a)$ of the subspaces $\mathcal{E}_{l'}$, $l' \leq l$, whose intersection for $l' \supseteq j$ is $\mathcal{E}_{j]}$, we obtain a family of self-consistent $*$-representations $\lambda_j^l : \mathfrak{A}_l \to \mathcal{B}(\mathcal{E}_{j]})$, $\lambda_j^{l'}|\mathfrak{A}_l = \lambda_j^l$ for any $j \leq l' \leq l$, that determine the projective limit $\lambda_j^* : \mathfrak{A}_j^* \to \mathcal{B}(\mathcal{E}_{j]})$ by $\lambda_j^*|\mathfrak{A}_l = \lambda_j^l$ for $l \supseteq j$, this representing the $*$-algebra $\mathfrak{A}_j^* = \bigvee_{l \supseteq j} \mathfrak{A}_l$ on $\mathcal{E}_{j]}$ with $\lambda_j^*(1) = \mathrm{I}|\mathcal{E}_{j]}$. The continuous extension $\iota_j^*(a^*) = \lambda_j^*(a)^*$, equal to zero on $\mathcal{E}_{j]}^\perp$, also satisfies the condition (2.3) with respect to $\iota_j^*(1) = I_j^* = \bigwedge_{l \supseteq j} E_l$, determining self-consistent $*$-representations $\iota_j^* : \mathfrak{A}_j^* \to \mathcal{B}(\mathcal{H})$ that are identical on the subspace $\mathcal{H}_j \subseteq \mathcal{H}_j^*$ to the representations ι_j restricted to the $*$-subalgebras $\mathfrak{A}_j^* \subseteq \mathfrak{A}_j$ (for $j \in L$ $\mathfrak{A}_j^* = \mathfrak{A}_j$, $\mathcal{H}_j^* = \mathcal{H}_j$, and $\iota_j^* = \iota_j$). If $\mathfrak{A}_k^* = \mathfrak{A}_k$ for any $k \in K$, $k \neq \emptyset$, then as the required family of representations $\mathfrak{A} = (\mathfrak{A}_k)_{k \in K}$ we can also choose $\iota^* = (\iota_k^*)_{k \in K}$.

3. **Construction of the representation** π. Identifying the elements $b \in \mathfrak{B}_l$ with functions $b \in \mathfrak{B}$ equal to unity for $t \not\leq l$, we set for each $B \in \mathcal{B}_j \subseteq \bigotimes_{t \in [j]} \mathcal{B}_t$, $j \subseteq l \in L$,

$$\rho_l^j(B)|\zeta_i b^i) = |\zeta_i b^i B), \qquad \forall \zeta_i \in \mathcal{K}, \quad b^i \in \mathfrak{B}_l, \tag{3.7}$$

where the product $b^i B \in \mathfrak{B}$ is the function equal to $(b^i B)(t) = b^i(t) b(t)$ when $t \in [j]$ and $(b^i B)(t) = b^i(t)$ when $t \notin [j]$ for $B = \bigtimes_{t \in [j]} b(t)$. Since $(b^i B)B = b^i B$, the operator $\rho_l^j(B)$ is idempotent on $\mathcal{E}_l$ and satisfies the condition

$$(\zeta_i b^i B|\zeta_{i'} b^{i'}) = (\zeta_i b^i|\zeta_{i'} b^{i'} B) = \|\zeta_i b^i B\|^2,$$

which defines it correctly on $\mathcal{E}_l$ as a linear symmetric projector:

$$\rho_l^j(B)^*|\mathcal{E}_l = \rho_l^j(B) = \rho_l^j(B)^2|\mathcal{E}_l,$$

identical to $\rho_{l'}^j(B)|\mathcal{E}_l$ for any $l' \in L$ such that $l \leq l'$, $j \subseteq l'$. The mapping $B \mapsto \rho_l^j(B)$ satisfies on $\mathcal{E}_l$ the following conditions for $j \subseteq l \in L$, these following directly from the definition (3.7) for all $\zeta_i \in \mathcal{K}$, $b^i \in \mathfrak{B}_l$:

0) $\rho_l^j(\mathrm{E}_j)|\zeta_i b^i) = |\zeta_i b^i \mathrm{E}_j) = |\zeta_i b^i) \qquad (\mathrm{E}_j = \bigtimes_{t \in [j]} \mathrm{E}_t)$;

1) $j \sim j' \Rightarrow \rho_l^{j \cup j'}(BB')|\zeta_i b^i) = \rho_l^j(B)\rho_l^{j'}(B')|\zeta_i b^i)$;

2) $j \bowtie j' \Rightarrow \rho_l^{j \cup j'}(B \times B')|\zeta_i b^i) = \rho_l^j(B)\rho_l^{j'}(B')|\zeta_i b^i)$, where $l \supseteq j \cup j'$, $B \in \mathcal{B}_j$, $B' \in \mathcal{B}_{j'}$;

3) for any decomposition $\mathrm{E}_j = \sum_{m=1}^{\infty} B^m$ it follows from 3^0 that

$$(\zeta_i b^i|\zeta_{i'} b^{i'}) = \sum_{m=1}^{\infty} (\zeta_i b^i B^m|\zeta_{i'} b^{i'} B^m) = \sum_{m=1}^{\infty} (\zeta_i b^i|\rho_l^j(B^m)|\zeta_{i'} b^{i'});$$

4) $[\iota_j(a), \rho_l^j(B)]|\zeta_i b^i) = 0$, $\forall a \in \mathfrak{A}_j$, $B \in \mathcal{B}_j$ (and similarly for $\iota_j^*(a)$, $a \in \mathfrak{A}_j^*$), since the projectors $\rho_l^j(B)$ commute with both $\iota_j(\mathfrak{A}_j)$ and $\iota_j^*(\mathfrak{A}_j^*)$ on the invariant subspaces $\mathcal{E}_j$ and $\mathcal{E}_{j]} \subseteq \mathcal{E}$. The symmetric projectors $\rho^j(B)$, which are uniquely determined on $\mathcal{E}_{j[} = \bigcup_{l \supseteq j} \mathcal{E}_l$ by the condition $\rho^j(B)|\mathcal{E}_l = \rho_l^j(B)$, $l \supseteq j$, also have properties 0–4 and, therefore, can be extended by continuity to the Hilbert subspace $\mathcal{K}_j^* = \bar{\mathcal{E}}_{j[}$ to Hermitian projectors $\pi_j^*(B) = \rho^j(B)^*$, $\pi_j^*(B)|\mathcal{E}_{j[}^{\perp} = 0$, satisfying conditions 0–4 of Definition 2. Noting that $\pi_j^*(\mathrm{E}_j)$ is an orthogonal projector P_j^* onto $\mathcal{K}_j^*$, we obtain $P_j^* = P_{j'}^*$, if $j \sim j'$, $P_j^* \wedge P_{j'}^* = P_{j \cup j'}^*$, if $j \bowtie j'$, and $P_\emptyset^* = \mathrm{I}$.

4. **Construction of the representation** V. For any $s \in S$, we set

$$U^s|\zeta_i b^i) = |U_s^* \zeta_i b^{is}), \qquad \forall \zeta_i \in \mathcal{K}, \quad b^i \in \mathfrak{B}^s, \tag{3.8}$$

where $b^s(t) = b(st)^s$ is determined by functions $b \in \mathfrak{B}^s$ equal to identity $b(t) = \mathrm{E}_t$ for $t \notin [s\mathrm{T}]$. The operator U^s is correctly defined on the pre-Hilbert subspace $\mathcal{E}^s$ generated by the vectors $|\zeta b)$ for $\zeta \in \mathcal{K}$, $b \in \mathfrak{B}^s$, by virtue of condition 5^0, which reduces to the condition that the operator be isometric:

$$(\zeta_i b^i|U^{s*} U^s|\zeta_{i'} b^{i'}) = (U_s^* \zeta_i|\kappa(b^{is}, b^{i's}) U_s^* \zeta_{i'}) = (\zeta_i|U_s \kappa(b^{is}, b^{i's}) U_s^*|\zeta_{i'}) = (\zeta_i b^i|\zeta_{i'} b^{i'}),$$

where we have used the fact that $U_s U_s^* \zeta = \zeta$ for $\zeta \in \mathcal{K}$. By virtue of the assumed surjectivity of the mappings $B \in \mathcal{B}_{sj} \mapsto B^s \in \mathcal{B}_j$, the operators U^s map every subspace $\mathcal{E}_{sj}^s = \mathcal{E}^s \cap \mathcal{E}_{sj}$ isometrically on $\mathcal{E}_j$, i.e., are unitary on these subspaces and are made self-consistent by the condition $U^s U^{s'}|\mathcal{E}^{ss'} = U^{ss'}$, $\forall s, s' \in S$. Using the definitions (3.6) and (3.7) of the representations λ_j and ρ^j, we obtain for $a \in \mathfrak{A}_{sj}$, $b^i \in \mathfrak{B}_{sj}^s$

$$\lambda_j(a^s) U^s|\zeta_i b^i) = |a^s \zeta_i b^{is}) = U^s \lambda_{sj}(a)|\zeta_i b^i)$$

and similarly for λ_j^* when $a \in \mathfrak{A}_{sj}^*$, $b^i \in \mathfrak{B}_{sj]}^s$, and also

$$\rho^j(B^s) U^s|\zeta_i b^i) = |\zeta_i^s b^{is} B^s) = U^s \rho^{sj}(B)|\zeta_i b^i)$$

for $b^i \in \mathfrak{B}_{sj[}^s$, $B \in \mathcal{B}_{sj}$. Thus, the operators U^s intertwine the representations $\iota_j^s(a) = \iota_j(a^s)$ on $\mathcal{E}_j$ with $\iota_{sj}(a)$ on $\mathcal{E}_{sj}^s \subseteq \mathcal{E}_{sj}$, $\iota_j^{*s}(a) = \iota_j^*(a^s)$ on $\mathcal{E}_{j]}$ with $\iota_{sj}^*(a)$ on $\mathcal{E}_{sj]}^s \subseteq \mathcal{E}_{sj]}$, and $\pi_j^{*s}(B) = \pi_j^*(B^s)$ on $\mathcal{E}_{j[}$ with $\pi_{sj}^*(B)$ on $\mathcal{E}_{sj[}$. Denoting by V_s the inverse unitary operators $\mathcal{E} \to \mathcal{E}^s$, extended by continuity to linear isometries $V_s : \mathcal{H} \to \mathcal{H}$ that map $\mathcal{H}$ onto $\mathcal{H}^s = \bar{\mathcal{E}}^s$, we obtain from this, bearing in mind that $V_s^*|\mathcal{E}^s = U^s$, the required representation $V = (V_s)_{s \in S}$, which determines for the constructed representations $\iota = (\iota_k)_{k \in K}$ or for ι^* and $\pi^* = (\pi_k^*)_{k \in K}$ the covariance conditions (2.5) and condition 5 of Definition 2 and is identical to $U = (U_s)_{s \in S}$ on the subspace $\mathcal{H}_\emptyset = \mathcal{K}$.

5. **Decomposition of the multikernel** κ. Since any pair $b, b' \in \mathfrak{B}$ of functions $t \mapsto b(t)$, $b(t') \in \mathcal{B}_t$ has common set $\Lambda \in \mathcal{F}$, for which $b(t) = E_t = b'(t)$ for $t \notin [\Lambda]$, there exists a finite chain $\Lambda_n = \{k_1, \dots, k_n\}$, $k_n > \dots > k_1 > \emptyset$, $\bigcup_{i=1}^n k_i = \Lambda$, and a pair of sequences $\mathbf{b} = (B_1, \dots, B_n)$, $\mathbf{b}' = (B_1', \dots, B_n')$ determined by functions on $[\Lambda]$ having continuations $b = \hat{b}$ and $b' = \hat{b}'$ on T that represent the corresponding value of the kernel κ in the form

$$\kappa(b, b') = \kappa(\hat{b}, \hat{b}'). \tag{3.9}$$

We define the chronologically ordered product $F^{\Lambda_n}(\mathbf{b})$ by means of the constructed representation π, acting successively on an arbitrary vector $\zeta \in \mathcal{K}$:

$$F^{\Lambda_n}(\mathbf{b})\zeta = \pi_{k_n}(B_n)\cdots\pi_{k_1}(B_1)|\zeta e) = |\zeta b^n),$$

where $b^n \in \mathfrak{B}$ is a function of $t \in [T] \mapsto \mathcal{B}_t$ determined recursively as follows: $b^n(t) = b^{n-1}(t)$ for $t \notin [k_n]$, $b^n(t) = b_n(t)$, $t \in [k_n]$ with the initial function $b^0(t) = e(t) = \mathrm{E}_t$ and the functions b_n defining the representation $B_n = \times_{t\in[k_n]} b_n(t)$, $n = 1, 2, \ldots$, this being obviously identical to $b = \hat{b}$. Therefore, we obtain from (3.1) and (3.5)

$$(\zeta|F^{\Lambda_n}(b)^* F^{\Lambda_n}(b)|\zeta') = (\zeta b|\zeta' b') = (\zeta|\kappa^{\Lambda_n}(b, b')|\zeta').$$

Thus, the obtained family π^*, which represents $\mathcal{B}$ on $\mathcal{H}$, determines the decomposition (2.5) and satisfies all the conditions of Definition 2 with respect to the family ι and the representations of $\mathfrak{A}$, and also with respect to ι^* for $\mathfrak{A}^* = \mathfrak{A}$. This proves the existence of a realization of the process defined in the wide sense.

We now show that when the conditions of the regression (3.3) are satisfied, the constructed representations π_l^*, ι_l^* $(= \iota_l)$ for $l \in L$ have the property (3.4) with respect to the orthogonal projectors $\iota_l^*(1) = E_l = \pi_l^*(\mathrm{E}_l)$. Using the definition (3.7) of the representations $\pi_l^*(B)|\mathcal{E}_{l[} = \rho_l^l(B)$, $B \in \mathcal{B}_l$, we have

$$E_l \pi_{l'}^*(B)|\zeta b) = \lambda_l \circ \nu_{ll'}(B)|\zeta b), \quad l \le l', \quad \zeta \in \mathcal{K}, \quad b \in \mathfrak{B}_l,$$

where $\nu_{ll'} = \theta_{ll'} \circ \pi_{l'}$, $B \in \mathcal{B}_{l'}$, and $\lambda_l : \mathcal{A}_l \to \mathcal{B}(\mathcal{H})$ is the W*-representation uniquely determined by the condition $\lambda_l(AP) = \iota_l^*(a)\pi_l^*(B)$ for all $A = \iota_l(a)$, $a \in \mathfrak{A}_l$ and $P = \pi_l(B)$, $B \in \mathcal{B}_l$. Since obviously $\lambda_l(AP)A' = A'\lambda_l(AP)$ for any $A \in \iota_l(\mathfrak{A}_l)$ and $P \in \pi_l(\mathcal{B}_l)$, $A' \in \pi_l^*(\mathcal{B}_l)' \cap \iota_l^*(\mathfrak{A}_l)'$, we obtain $A'E_l\pi_l^*(B)E_l = E_l\pi_l^*(B)E_lA'$, i.e., the condition (3.4). Thus, the reconstruction theorem, Theorem 3, is proved.

4. Equivalence and Interpretation of QSP

1. Let $\mathfrak{A}_l = \bigcap_{k\subseteq l} \mathfrak{A}_k$ be a nonincreasing family of $*$-subalgebras $\mathfrak{A}_l \subseteq \mathcal{B}(\mathcal{K})$, $\mathfrak{A}_l \supseteq \mathfrak{A}_{l'}$, if $l \le l'$, indexed by the maximal subsets $l \subset T$ of pairwise nonanticipatory elements $t \in T$, and $\mathcal{B}_l = \times_{t\in l} \mathcal{B}_t$ be Boolean semirings of cylindrical subsets $B = \times_{t\in l} b(t)$, determined by functions $b : t \mapsto b(t) \in \mathcal{B}_t$ equal to E_t outside a certain finite subset of equivalence classes $t = [t] \subseteq l$. For every $\mathcal{H}$-process $(\mathcal{B}_k, \pi_k)_{k\in K}$, defined with respect to the system $(\mathfrak{A}_k, \iota_k)_{k\in K}$, we denote by ι_l, π_l for each $l \in L$, the $*$-representation $\mathfrak{A}_l \to \mathcal{B}(\mathcal{H})$ and σ-representation $\mathcal{B}_l \to \mathcal{P}(\mathcal{H})$ uniquely determined, respectively, by the conditions

$$\iota_l(a)I_k = \iota_k(a), \quad \pi_l(B_k) = \pi_k(B)E_l, \quad \forall l \in L, \quad k \subseteq l, \tag{4.1}$$

where $a \in \mathfrak{A}_l$, $B_k = B \times \mathrm{E}_{l\setminus k}$, $B \in \mathcal{B}_k$, $E_l = \bigwedge_{k\subseteq l} P_k$. By virtue of condition 4 of Definition 2, the representations ι_l and π_l have a common nondecreasing unit $\iota_l(1) = E_l = \pi_l(\mathrm{E}_l)$, $E_l \le E_{l'}$, if $l \le l'$ and are self-consistent,

$$\iota_l(a) = \iota_{l'}(a)E_l, \quad \pi_l(B_k) = \pi_{l'}(B'_k)E_l, \quad l < l' \in L, \tag{4.2}$$

where $a \in \mathfrak{A}_{l'}$, $B'_k = B \times \mathrm{E}_{l'\setminus k}$, $B \in \mathcal{B}_k$, $k \subseteq l \cup l'$. In accordance with condition 4 of Definition 2 they satisfy the commutation condition $\pi_l(\mathcal{B}_l) \subseteq \iota_l(\mathfrak{A}_l)'$ for every $l \in L$, determining uniquely through the relations (4.1) the original $\mathcal{K}$-process if it satisfies the conditions

$$\bigvee_{l\supseteq k} E_l = P_k, \qquad \bigwedge_{l\supseteq k} E_l = I_k, \qquad l \in L, \quad k \in K, \tag{4.3}$$

and if $\mathfrak{A}_k = \bigvee_{l \supseteq k} \mathfrak{A}_l$. The latter condition in accordance with Theorem 3 is sufficient for the existence of the reconstruction of the process possessing the properties (4.3) for $k \neq \emptyset$; the necessary and sufficient condition for (4.3) to hold for $k = \emptyset$ is the condition (3.2) of asymptotic noncorrelation with the distant past relative to $\mathcal{K}$. It follows from the inequality

$$(\zeta b| (E_l - \mathrm{P}) |\zeta b) \geq \left| \sum (\zeta_i | \kappa(b^i, b) - \kappa(b^i, e)\kappa(e, b)|\zeta) \right|^2, \tag{4.4}$$

as $E_l \to \mathrm{P}$ if $l \to \emptyset$, where $\zeta_i \in \mathcal{K}$, $b^i \in \mathfrak{B}_l$ and $\sum(\zeta_i|\kappa(b^i, b^{i'})\zeta_{i'}) \leq 1$. This inequality is obtained in the the proof (Sec. 3.1) of the sufficiency of the condition (3.2) if one uses only the property $E_l|\xi_i b^i) = |\xi_i b^i)$ equivalent to the condition $E_l F(b) = F(b)$ for all $b \in \mathfrak{B}_l$, which is obvious in the representation (2.7) for the "Feynman integral" $F(\hat{\mathbf{b}}) = F_{\emptyset}^{\Lambda}(\mathbf{b})$, $\Lambda \leq l$. Note that if the semigroup S acts on T quasitransitively in the sense that for each $\mathrm{t} \geq \mathrm{t}'$ there exists $s \in S$ such that $s\mathrm{t} \leq \mathrm{t}'$, and the conditions of S-covariance

$$V_s \iota_l(a^s) = \iota_{sl}(a) V_s E_l, \qquad V_s \pi_l(a^s) = \pi_{sl}(a) V_s E_l, \tag{4.5}$$

following from the local conditions (2.5) and condition 5 of Definition 2, are satisfied, the regularity $\bigwedge_{l \in L} E_l = \mathrm{P}$ ensures generation of the whole space $\mathcal{H}$ by the action of the operators $U^s = V_s^*$, $s \in S$, on any "neighborhood" $\mathcal{H}_l = E_l \mathcal{H}$ of the "infinitely distant initial subspace" $\mathcal{K} = \mathrm{P}\mathcal{H}$. In the scalar case $\mathrm{P} = \mathrm{P}_\xi$ ($\mathcal{K} \simeq \mathbb{C}$) the condition $\bigwedge_{l \in L} E_l = \mathrm{P}_\xi$ is a noncommutative analog of the Kolmogorov regularity condition (law "0 or 1") of a classical stochastic process in the sense [12].

Definition 4. *An $\mathcal{H}$-process $(\mathcal{B}, \pi)$ with respect to $(\mathfrak{A}, \iota)$, is said to be canonical if $\bigvee_{l \supseteq k} E_l = P_k$, $k \in K$, and regular if $\bigwedge_{l \supseteq k} E_l = I_k$, $k \in K$, where*

$$\bigwedge_{k \subseteq l} P_k = E_l = \bigvee_{k \subseteq l} I_k, \quad l \in L, \quad P_\emptyset = \mathrm{I}, \quad I_\emptyset = \mathrm{P}.$$

2. A model of a $\mathcal{H}$-process $(\mathcal{B}, \pi)$ with respect to $(\mathfrak{A}, \iota)$ is any $\mathcal{H}^1$-process $(\mathcal{B}, \pi^1)$, with respect to $(\mathfrak{A}, \iota^1)$ on some Hilbert space $\mathcal{H}^1 \supseteq \mathcal{K}$ for which there exists an isometry $U : \mathcal{H}^1 \to \mathcal{H}$, $U^*U = I^1$, intertwining the corresponding representations:

$$U\iota_k^1(a) = \iota_k(a) U I_k^1, \quad U\pi_k^1(B) = \pi_k(B) U P_k^1, \quad k \in K. \tag{4.6}$$

Note that an arbitrary model of a canonical or regular S-covariant process is in general neither canonical, nor regular, nor S-covariant; however, it satisfies the condition $\bigwedge_{l \in L} E_l = \mathrm{P}$ of ‘regularity at the origin" $k = \emptyset$, and will be S-covariant if the condition $UV_s^1 = V_s U$ determines for every $s \in S$ an operator $V_s^1 : \mathcal{H}^1 \to \mathcal{H}^1$ for which

$$V_s^1 I_k^1 = I_{sk}^1 V_s^1 I_k^1, \quad V_s^1 P_k^1 = P_{sk}^1 V_s^1 P_k^1, \quad k \in K. \tag{4.7}$$

This always holds for a unitarily equivalent model determined by the condition of its being able to be modeled by the original process with respect to the adjoint operator U^* as isometry $\mathcal{H} \to \mathcal{H}^1$, i.e., for $I_k = U I_k^1 U^*$ and $P_k = U P_k^1 U^*$ for all $k \in K$.

By a subprocess of a $\mathcal{H}$-process $(\mathcal{B}, \pi)$ with respect to $(\mathfrak{A}, \iota)$ we understand any model of it determined by subrepresentations

$$\iota_k^1(a) = \iota_k(a) I_k^1, \quad \pi_k^1(B) = \pi_k(B) P_k^1, \quad k \in K, \tag{4.8}$$

on some subspace $\mathcal{H}^1 \subseteq \mathcal{H}$ (U is injection, e.g. $U = \mathrm{I}$, if $\mathcal{H}^1 = \mathcal{H}$). It necessary for the subspace $\mathcal{H}^1$ to be invariant with respect to V_s, $s \in S$, if the subprocess of the S-covariant process is to remain S-covariant. This invariance holds in particular for the minimal subprocess determined by the least of the orthogonal projectors I_k^1, P_k^1 satisfying the conditions $I_k^1 F(b) = F(b)$, $b \in \mathfrak{B}_k$ and $P_k^1 F(b) = F(b)$, $b \in \mathfrak{B}_{k[}$ (this is a canonical process, called the *minimal modification*), and for the minimal regular subprocess determined for every regular process by the orthogonal projectors (4.3) as canonical regular process with respect to the least of the orthogonal projectors E^1_l that satisfy the conditions $E^1_l F(b) = F(b)$, $b \in \mathfrak{B}_l$, $l \in L$. Note that the (minimal) subrepresentations ι^*_l are normal and can be uniquely extended to von Neumann algebras $\mathfrak{A}''_l$, and similarly the representations π^1_l can be uniquely extended to σ-representations of the σ-algebras $\sigma(\mathcal{B}_l)$, which are generated on E_l by the semirings $\mathcal{B}_l$ by σ-additive extension of the original σ-representations π_l, $l \in L$ to $\sigma(\mathcal{B}_l)$.

The physical equivalence, according to which two quantum stochastic processes are equivalent if they are statistically indistinguishable in due course of all possible successive measurements performed on the quantum model in the causal order is much weaker than the strong unitary equivalence. This leads to the concept of *wide equivalence* of quantum stochastic processes described by the same families $\kappa = \{\kappa^\Lambda\}$ of correlation kernels, i.e., coinciding as the processes in the wide sense. Two processes π and π^1 over the same family $\mathcal{B}$ that are represented with respect to the systems $(\mathfrak{A}, \iota)$ and $(\mathfrak{A}, \iota^1)$, respectively, on $\mathcal{H}$ and $\mathcal{H}^1$ are said to be equivalent in the wide sense with respect to the family $\mathfrak{A}$ if they are described by the identical multikernels as operator-valued functions on the same initial Hilbert space $\mathcal{K}$, i.e. if $\kappa(b, b') = \kappa^1(b, b')$ for all $b, b' \in \mathfrak{B}$.

3. The relationship between equivalences in the narrow and wide senses is established by the following theorem, from which follows the uniqueness up to unitary equivalence of the minimal (regular) process determining the canonical reconstruction of Sec. 3.

Theorem 4. *Let $(\mathcal{B}, \pi)$ be a $\mathcal{H}$-process with respect to the system $(\mathfrak{A}, \iota)$ and $(\mathcal{B}, \pi^1)$ be its isometric model on $\mathcal{H}^1 \supseteq \mathcal{H}$ with respect to $(\mathfrak{A}, \iota^1)$. Then these processes are equivalent in the wide sense. Conversely: The minimal (regular) modification of the $\mathcal{H}$-process $(\mathcal{B}, \pi)$ with respect to $(\mathfrak{A}, \iota)$ is an isometric model of any other equivalent in the wide sense (regular) $\mathcal{H}$-process $(\mathcal{B}, \pi^1)$ with respect to $(\mathfrak{A}, \iota^1)$ and is unitarily equivalent to this process if the latter is minimal (regular).*

Proof. Let U be the isometry $\mathcal{H}^1 \to \mathcal{H}$ that determines the modeling relation (4.6), and $F^1(b) = \pi^1_{k_n}(B_n) \cdots \pi^1_{k_1}(B_1)\mathrm{P}^1$ be the 'Feynman integral" that determines the decomposition $\kappa^1(b, b') = F^1(b)^* F^1(b')$. Representing $F^1(b)$ in the form (2.7), and taking into account (4.6), we obtain $UF^1(b) = \pi_{k_n}(B_n) \cdots \pi_{k_1}(B_1)U\mathrm{P}^1$, and since for $k \neq \emptyset$ $Ua = aU\mathrm{P}^1$ for any $a \in \mathfrak{A}_\emptyset = \mathcal{B}(\mathcal{K})$, the isometric operator U on the subspace $\mathcal{K} = \mathrm{P}^1\mathcal{H}^1$ is a multiple of the unit: $U\mathrm{P}^1 = c\mathrm{P}^1$, where $|c| = 1$. It follows that $UF^1(b) = cF(b)$ and $\kappa^1(b, b') = \kappa(b, b')$ for all $b, b' \in \mathfrak{B}$, i.e., the relation of isometric modeling of the processes implies their equivalence in the wide sense.

Conversely: Suppose the processes π and π^1 have the same kernels κ and κ^1. By virtue of the transitivity of the modeling relation, it is sufficient to prove unitary equivalence of the minimal (regular) modification π^* of the process π and the minimal (regular) process π^1. Taking into account the isometry

$$(F^1(b)\zeta \mid F^1(b')\zeta') = (F^*(b)\zeta \mid F^*(b')\zeta'), \quad \forall b, b' \in \mathfrak{B}, \quad \zeta, \zeta' \in \mathcal{K}$$

due to $\kappa^1 = \kappa$ of the vector systems $F^1(b)\zeta$ and $F(b)\zeta = F^*(b)\zeta$ that generate the minimal subspaces $\mathcal{H}^1$ and $\mathcal{H}^* \subseteq \mathcal{H}$, which are invariant with respect to the corresponding processes π^1 and π^*, we can extend the one-to-one correspondence $F^1(b)\zeta \mapsto F^*(b)\zeta$ of these systems to a unitary operator $U : \mathcal{H}^1 \to \mathcal{H}^*$. We then obtain for any $B \in \mathcal{B}_l$, $l \in L$,

$$U\pi^1_l(B)F^1(b)\zeta = F(bB)\zeta = \pi^*_l(B)UF^1(b)\zeta, \quad \forall b \in \mathfrak{B}_l$$

and accordingly for any $a \in \mathfrak{A}_l$, $l \in L$,

$$U\iota^1_l(a)F^1(b)\zeta = F(b)a\zeta = \iota^*_l(a)UF^1(b)\zeta, \quad \forall b \in \mathfrak{B}_l,$$

from which there follows the unitary equivalence of the systems of representations π^1 and π^* and, respectively, ι^1 and ι^*. The proof is completed. $\square$

4. We shall say that a $\mathcal{H}$-process is conditionally Markov, or simply *dynamic* if it has the property

$$E_l \pi_{l'}(\mathcal{B}_{l'}) E_l \subseteq \pi_l(\mathcal{B}_l) \vee \iota_l(\mathfrak{A}_l), \quad \forall l \le l' \in L, \tag{4.9}$$

which is obviously inherited by any isometric model of it.

Note that from (4.9) we immediately deduce the regression property (3.3) with respect to the W*-algebras $\mathcal{A}_l = \pi_l(\mathcal{B}_l) \vee \iota_l(\mathfrak{A}_l)$ on $\mathcal{H}_l = E_l\mathcal{H}$ for $\rho_l(A) = \mathrm{P}A\mathrm{P}$, $A \in \mathcal{A}_l$, and $\theta_{ll'}(A) = E_l A E_l$, $A \in \mathcal{A}_{l'}$. By virtue of the fundamental theorem, Theorem 3, the regression condition (3.3) is also sufficient for existence of the dynamic representation (4.9), and the minimal canonical realization reconstructing the process satisfying this weak Markovianity condition is dynamic, i.e. conditionally Markov in the strong sense. The dynamic mappings $\rho_l : \mathcal{A}_l \to \mathcal{B}(\mathcal{K})$, $\theta_{ll'} : \mathcal{A}_{l'} \to \mathcal{A}_l$ obviously satisfy the conditions of self-consistency

$$\rho_l \circ \theta_{ll'} = \rho_{l'}, \qquad \theta_{l_0 l} \circ \theta_{ll'} = \theta_{l_0 l'}$$

for $l_0 \le l \le l'$ and are S-covariant

$$\rho_l(A^s) = \rho_{sl}(A)^s, \quad \theta_{ll'}(A^s) = \theta_{sl,sl'}(A)^s,$$

where $A^s = V_s^* A V_s$. The condition of regularity $E_l \to \mathrm{P}$ at the origin $l \to \emptyset$ of the dynamic process $(\rho_l, \theta_{ll'})$ is obviously equivalent to the relaxation, or mixing property $\theta_{l_0 l}(A) \to \rho_l(A)$, $A \in \mathcal{A}_l$, $l_0, l \in L$ for $l_0 \to \emptyset$. This allows the dependence of the limit relaxed states

$$\omega_l(A) = \lim_{l_0 \to \emptyset} (\eta \mid \theta_{l_0 l}(A)\eta) = (\xi \mid \rho_l\,(A)\,\xi)$$

on $\mathcal{A}_l$ only on the initial states on the controlling C*-algebra $\mathfrak{A}_\emptyset = \mathcal{B}(\mathcal{K})$ determined by vectors $\xi = \mathrm{P}\eta \in \mathcal{K}$ for each $\eta \in \mathcal{H}$ but not on the initial states on $\mathcal{B}_{l_0}$.

An example of a dynamic $\mathcal{H}$-process gives the model of a quantum coherent filter [17] describing the irreversible process of an optimal indirect measurements (filtration) of the amplitude of an open (relaxing) quantum oscillator in a thermostat. Note that the dynamicity condition (4.9) leads for quantum processes of observation to their weak commutativity $\nu_{ll'}(\mathcal{B}_{l'}) \subseteq \pi_l(\mathcal{B}_l)'$ for $l \le l'$, where $\nu_{ll'}(B) = E_l \pi_{l'}(B) E_l$ and we have noted that $\iota_l(\mathfrak{A}_l) \subseteq \pi_l(\mathcal{B}_l)' \supseteq \pi_l(\mathcal{B}_l)$, by virtue of which there is complete factorization $\kappa(b, b') = \mu(b \cdot b')$, where $(b \cdot b')(t) = b(t)b(t')$ for all $t \in T$. This means that for quantum dynamic processes defined in the narrow sense ($E_l = \mathrm{I}$) ordinary commutativity must hold,

$$\pi_{l'}(\mathcal{B}_{l'}) \subseteq \pi_l(\mathcal{B}_l) \vee \iota_l(\mathfrak{A}_l) \subseteq \pi_l(\mathcal{B}_l)', \quad \forall l \le l' \in L, \tag{4.10}$$

which leads to the strong compatibility condition of the observed events corresponding to all l and $l' \in L$ by virtue of the directionality of the set L. Such representation corresponds to the dynamic treatment of quantum measurements in the usual, narrow sense as a strongly Markov process in a "measurement apparatus", described for every maximal region of simultaneous observation $l \in L$ by a $*$-subalgebra $\mathfrak{A}_l$, which corresponds to the part of the physical medium which will be causally connected to the observed output process only in the "future" in accordance with the condition $\pi_l(\mathcal{B}_l) \subseteq \iota_{l'}(\mathfrak{A}_l)'$ if $l \le l'$. This latter explains why the $*$-algebras $\mathfrak{A}_l$ are to be assumed nonincreasing, however their representations in $\mathcal{H}$ are essential not on the whole Hilbert space but on the expanding subspaces $\mathcal{H}_l \subseteq \mathcal{H}$ containing only the state vectors to which the large system can pass from the original state $\xi \in \mathcal{K}$ in a result of the successive measurements in the observed subsystem. Such weaker description of the "apparatus" corresponds to the admission of the irreversibility that is introduced in the process of *indirect* successive quantum measurements of noncommuting observables which can be realized in even a "larger system" without change of the "future state of the thermostat". This leads to the weakening (4.9) of the dynamicity condition (4.10), corresponding to weakening of the normalization condition $\pi_l(\mathrm{E}_l) = \mathrm{I}$ and the assumption of non-Hamiltonian (conditionally Markovian) evolution. The condition (4.9), like (4.10), determines independence of the future statistics from the past if the current state of the observed subsystem and the controlling thermostat is known, but, in contrast to (4.10), it also permits the description of irreversible relaxation processes in quantum systems.

5. In conclusion, we show that Kolmogorov reconstructions of classical stochastic processes and fields arise in the reconstruction process described in Sec.3 as the minimal processes corresponding to the neglect of the causality relation on $T = \mathrm{T}$, or the assumption of only symmetric causality, i.e. an equivalence on T. Note that every classical process can be represented as a quantum one by σ-homomorphisms $\pi_k : \mathcal{B}_k \mapsto \mathcal{P}(\mathcal{H})$ over σ-algebras $\mathcal{B}_k$ with commuting values in the commutant of the Gel'fand–Naimark–Segal representation $(\mathcal{H}, \iota, \xi)$ of a $*$-algebra $\mathfrak{A}$ with a state ω. In the case of trivial, or symmetric causality the set L contains a single element $l = T$ – the factor-set $T = [\mathrm{T}]$, and the set $\mathfrak{B}$ of functions $b : t \mapsto \mathcal{B}_t$ can be identified with the semiring $\mathcal{B}_l = \mathcal{B}_T$ of cylindrical subsets on $\mathrm{E}_l = \mathrm{E}_T$. Therefore the functional kernel $\kappa(b, b') = \kappa_T(b, b')$ is completely determined by an operator-valued (positive, normalized, and σ-additive) distribution $\mu : \mathcal{B}_T \to \mathcal{P}(\mathcal{H})$ as $\kappa_T(b, b') = \mu_T(b \cdot b')$, commuting with a certain $*$-algebra $\mathfrak{A}_T \subseteq \mathcal{B}(\mathcal{K})$, $\mathfrak{A}_T = \mu_T(\mathcal{B}_T)'$ say. The canonical reconstruction of the process described in the wide sense by such a distribution μ_T reduces to the construction of a σ-representation $\pi_T : \mathcal{B}_T \to \mathcal{P}(\mathcal{H})$ on the Hilbert $\mathcal{K}$-hull $\mathcal{H} = L^2_{\mathcal{K}}(\mathcal{B}_T)$ of the Boolean semiring $\mathcal{B}_T = \otimes_{t \in T} \mathcal{B}_t$ such that $\mu_T(B) = \mathrm{P}\pi_T(B)\mathrm{P}$. It commutes with the corresponding $*$-representation $\iota_T : \mathfrak{A}_T \to \mathcal{B}(\mathcal{H})$ of the $*$-algebra $\mathfrak{A}_T$ and determines the resolution of the identity $\iota_T(1) = \mathrm{I} = \pi_T(\mathrm{E}_T)$. The existence and uniqueness (up to unitary equivalence) of such a representation, realizing a commutative process with operator-valued probability distribution μ_T determined by the self-consistent family $\mu^\Lambda(b) = \kappa^\Lambda(b, b)$, $\Lambda \in \mathcal{F}$, are direct consequences of Theorems 3 and 4. Extending by σ-additivity the distribution μ_T to a σ-measure ρ_T on the generated σ-algebra $\mathcal{F}_T = \sigma(\mathcal{B}_T)$ of the space of trajectories $\mathrm{E}_T = \times_{t \in T} \mathrm{E}_t$, we obtain a Kolmogorov canonical process $(\mathrm{E}_T, \mathcal{F}_T, \rho_T)$ with operator-valued probability measure $\rho_T : \mathcal{F}_T \to \iota_T(\mathfrak{A}_T)'$. The possibility of this extension for a self-consistent family of operator-valued distributions μ^Λ and standard Borel spaces E_t, $t \in T$ with discrete $T = \mathbb{Z}$ was shown by Benioff in [18].

Note that the condition of regularity "at the origin" with respect to the subspace $\mathcal{K}$ as introduced here leads in the case of a one-dimensional $\mathcal{K} \simeq \mathbb{C}$ and quasitransitive action of S on T to the property of complete mixing and ergodicity of the QSP, which is analogous to K mixing [19] of classical quasiregular systems [20].

I thank V. V. Anshelevich, R. L. Dobrushin, V. P. Maslov, and Ya. G. Sinai for discussions and interest in the work.

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