THEORY OF THE CONTROL OF OBSERVABLE QUANTUM SYSTEMS

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Abstract. The fundamental mathematical definitions of the controlled Markov dynamics of quantum-mechanical systems are formulated with regard for the statistical reduction of quantum states in the course of quantum measurement in either discrete or continuous time. The concept of sufficient coordinates for the description of *a posteriori* quantum states in a given class is introduced and it is proved that they form a Markov process. The general problem of optimal control of a quantum-mechanical system is discussed and the corresponding Bellman equation in the space of sufficient coordinates is derived. The results are illustrated in the example of control of the semigroup dynamics of a quantum system that is observed at discrete times and evolves between measurement times according to the Schrődinger equation.

1. Introduction

The encouraging outlook for the application of coherent quantum optics (lasers) for communications and control has been recently stimulated by the steadily growing demands for greater accuracy of observation and monitoring, particularly under the "extreme" conditions of very faint signals at extremely great (astronomical) distances. On the other hand, instances of the successful exploitation of mathematical methods from information and control theory for the investigation of many physical phenomena in the microscopic world have also stimulated interest in the theoretical study, using general cybernetic principles, of the possibilities of dynamical systems described at the quantum-mechanical level [19][15][16][1][4]. It has been shown in [9] that it is natural to regard many physical problems as control problems for distributed systems described by standard quantum-mechanical equations. In particular, the possibility of the transition of a physical system from one microscopic state to another can be investigated [8] by the methods of the theory of controllability on Lie groups generated by the Schrődinger equation with a controlled Hamiltonian.

General problems in the theory of quantum dynamical systems with observation, control and feedback channels can be handled on the basis of the recent development [2] of an operational theory of open-loop quantum systems, for which the mathematical formalism was set down in [10] [13]. The investigation, undertaken in [3], of the dynamical observation and feedback control optimization problems for such systems has provided a means for solving these problems in the case of linear Markov systems of the boson type, in particular for a controllable and observable

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quantum oscillator [4]. This work was based on the multistage quantum-statistical decision theory originally described in [5][6] in application to problems in the optimal dynamical measurement and control of classical (i.e., non-quantum) Markov processes with quantum observation channels.

In the present article we describe a simplified problem of optimal feedback control of quantum dynamical systems which does not involve quantum-statistical decision theory. Here the observable subsystem at the output of the observable channel is regarded as classical and amendable to description at the macroscopic level, whereas the controlled entity remains a quantum dynamical system. In other words, we assume here, in contrast with [3][5][6], that the "instrument" at the output of the quantum-mechanical system is given, rather than to be optimized, and it is required only to find the optimal macroscopic feedback for a given performance criteria. The results obtained in this setting are special in relation to [4][2][3] as they correspond to the semiclassical case of commutativity of the algebra of output observables. They nonetheless deserve special consideration both from the methodological and from the practical point of view when the observation channels are given and cannot be optimized for the optimal feedback control purpose.

2. Controllable quantum dynamical systems with observation

Here we introduce the mathematical concept of controllable quantum system with observation channel on the basis of the operational theory on open-loop physical systems and quantum processes [2][10][13]. Such systems are open by the definition, and the necessarry concepts borrowed from the algebraic theory of open quantum systems are described in the Appendix.

Let $\mathcal{H}$ be the Hilbert space of representation of a certain quantum-mechanical system regarded as an observable and controllable system and let $\mathfrak{A}$ be the von Neumann algebra of admissable physical quantities $Q \in \mathfrak{A}$ which is generated (see Appendix 1) by the dynamical variables of this system, acting as operators in $\mathcal{H}$. The pair $\{\mathcal{H}, \mathfrak{A}\}$ plays the role of a measurable space $\{X, \mathcal{A}\}$ representing [14] the corresponding classical dynamical system in the phase space X of it's point states, endowed with the Borel σ-algebra $\mathcal{A}$ of admissible events $A \in \mathcal{A}$. The simple systems normally treated in traditional texts on quantum mechanics, for example [17], correspond to the algebras $\mathfrak{A} = \mathcal{B}(\mathcal{H})$ of all bounded operators in $\mathcal{H}$, but the models that emerge from quantum field theory and statistical mechanics [12] are described by the more general algebras $\mathfrak{A}$.

Normal states of the quantum-mechanical system at every time $t \in \mathbb{R}$ are determined by the linear functionals $\varrho_t : Q \mapsto \langle \rho_t, Q \rangle$ of the quantum-mechanical expectations $\langle Q \rangle_t = \langle \rho_t, Q \rangle$ of all the physical quantities $Q \in \mathfrak{A}$ for this system and are described by the densities ρ_t as the positive elements associated with the opposite (transposed) von Neumann algebra $\mathfrak{A}^\intercal$ (see Appendix 2). In the case of semifinite algebras [11], as in the simple case $\mathfrak{A} = \mathcal{B}(\mathcal{H})$, the states ϱ_t are usually represented with the trace one operators $\hat{\varrho}_t = \rho_t^\intercal$ in $\mathfrak{A}$ (or affiliated with $\mathfrak{A}$) as

$$\varrho_t(Q) = \mathrm{tr}\{\hat{\varrho}_t Q\} \equiv \langle \rho_t, Q \rangle, \tag{2.1}$$

and the Banch space $\mathcal{L}$ spanned by all such ϱ_t is identified with the space of trace class operators of $\mathfrak{A}$.

The Markov controlled time evolution $t \mapsto \varrho_t$ of quantum-mechanical systems is described by the transformation of states

$$\varrho_t \in \mathcal{L} \mapsto \varrho_t \circ \mathrm{M}_t^\tau \left(u_t^\tau\right) \equiv \varrho_{t+\tau}\left(u_t^\tau\right) \in \mathcal{L} \qquad \forall t \in \mathbb{R}, \qquad \tau > 0. \tag{2.2}$$

which is determined by a family $\{\mathrm{M}_t^\tau\}_{t\in\mathbb{R},\ \tau>0}$ of controlled transfer operators (see Appendix 3) $\mathrm{M}_t^\tau\left(u_t^\tau\right) : \mathfrak{A} \mapsto \mathfrak{A}$. These operators satisfy a consistency condition analogous to the Chapman-Kolmogorov equation:

$$\mathrm{M}_t^\tau\left(u_t^\tau\right) \mathrm{M}_{t+\tau}^{\tau'}\left(u_{t+\tau}^{\tau'}\right) = \mathrm{M}_t^{\tau+\tau'}\left(u_t^{\tau+\tau'}\right) \qquad \forall t \in \mathbb{R}, \qquad \tau, \tau' > 0 \tag{2.3}$$

Here $u_t^\tau = \{u\left(t+\tau'\right)\}_{\tau'<\tau}$ is a segment of an admissible control function $u\left(t\right) \in U\left(t\right)$ of length $\tau > 0$ and $u_t^{\tau+\tau'}$ denotes the combination $\left(u_t^\tau, u_{t+\tau}^{\tau'}\right)$ of segments u_t^τ and $u_{t+\tau}^{\tau'}$ [the sets $U_t^\tau \subset \prod_{\tau'<\tau} U\left(t+\tau'\right)$ of admissible segments u_t^τ are assumed to be consistent in the sense of condition (2.4)]. For stationary (time-invariant) systems, where $U\left(t\right) = U$, (2.3) specifies the semigroup conditions for transfer operators $\mathrm{M}^\tau\left(u^\tau\right) = \mathrm{M}_t^\tau\left(u_t^\tau\right)$ independent of t.

In the special case of transfer operators $\mathrm{M}_t^\tau\left(u_t^\tau\right)$ specified by controllable propagators $\mathcal{H} \mapsto \mathcal{H}$

$$\mathrm{M}_t^\tau\left(u_t^\tau\right) Q = T_t^\tau\left(u_t^\tau\right)^\dagger Q T\left(u_t^\tau\right),$$

satisfying the condition corresponding to (2.2)

$$T_{t+\tau}^{\tau'}\left(u_{t+\tau}^{\tau'}\right) T_t^\tau\left(u_t^\tau\right) = T_t^{\tau+\tau'}\left(u_t^{\tau+\tau'}\right),$$

the dynamics $\varrho_t \mapsto \varrho_{t+\tau}\left(u_t^\tau\right)$ for the vector states $\left\langle \rho_{\psi_t}, Q\right\rangle = \langle\psi_t | Q\psi_t\rangle$ is described by the spatial transformation

$$\psi_t \in \mathcal{H} \mapsto T_t^\tau\left(u_t^\tau\right)\psi_t \equiv \psi_{t+\tau}\left(u_t^\tau\right) \in \mathcal{H} \qquad \forall t \in \mathbb{R}, \qquad \tau > 0.$$

This transformation is obtained, for example, as the fundamental solution of the time-dependant Schrődinger equation with a pertubating force $u\left(t\right)$, for which the isometric operators $T_t^\tau\left(u_t^\tau\right)$ are unitary and $\mathrm{M}_t^\tau\left(u_t^\tau\right)$ denotes Heisenberg transformations.

Open-loop quantum dynamical systems of the kind specified below by controllable systems with observation cannot, as a rule, be described in terms of propagators $T_t^\tau\left(u_t^\tau\right)$, because the measurements induce a reduction of quantum states, which is described by transfer operators of another type.

Let $\{U_t^\tau\}, \{V_t^\tau\}, t \in \mathbb{R}, \tau > 0$ be two-parameter families of sets of admissible segments $u_t^\tau \in U_t^\tau, v_t^\tau \in V_t^\tau$ of input and output signals (respectively), satisfying the consistency condition

$$U_t^\tau \times U_{t+\tau}^{\tau'} = U_t^{\tau+\tau'}, \qquad V_t^\tau \times V_{t+\tau}^{\tau'} = V_t^{\tau+\tau'}. \tag{2.4}$$

The sets U_t^τ are usually endowed with Hausdorff topologies, and the sets V_t^τ with Borel σ-algebras, which are consistent with the products (2.4) and in the time-invariant case are given by a shift $\tau \mapsto t+\tau$ of the initial sets $U^\tau = U_0^\tau$ and $V^\tau = V_0^\tau$.

Definition 1. *A controllable quantum dynamical system with observation is a family* $\{\Pi_t^\tau\}_{t\in\mathbb{R},\ \tau>0}$ *of controllable transfer-operator measures* $\Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right) : \mathfrak{A} \mapsto \mathfrak{A}$

(see Appendix 4) defined on spaces $U_t^\tau \ni u_t^\tau, V_t^\tau \supseteq dv_t^\tau$ and satisfying the condition

$$\Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right)\Pi_{t+\tau}^{\tau'}\left(u_{t+\tau}^{\tau'}, dv_{t+\tau}^{\tau'}\right) = \Pi_t^{\tau+\tau'}\left(u_t^{\tau+\tau'}, dv_t^{\tau+\tau'}\right) \tag{2.5}$$

for any $t \in \mathbb{R}$, $\tau, \tau' > 0$, where $u_t^{\tau+\tau'} = \left(u_t^\tau, u_{t+\tau}^{\tau'}\right)$, $dv_t^{\tau+\tau'} = dv_t^\tau \times dv_{t+\tau}^{\tau'}$.

The superoperators $\Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right)$ are assumed to be continuous in u_t^τ, σ-additive with respect to dv_t^τ (in the strong operator sense) and in the time-invariant case not to depend explicitly on t [the index t in the condition (2.5) specifying the semigroup dependance on τ can now be omitted].

On the basis of the positive quantities $Q \in \mathfrak{A}_+ \mapsto \Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right) Q \in \mathfrak{A}_+$ and the normalization condition $\Pi_t^\tau\left(u_t^\tau, V_t^\tau\right) = \mathrm{M}_t^\tau\left(u_t^\tau\right)$, the mappings Π_t^τ determine, for a given instantaneous state $\varrho_t \in \mathcal{L}$ and control function u_t^τ, the future ($\tau > 0$) states

$$\varrho_{t+\tau}\left(u_t^\tau, dv_t^\tau\right) = \varrho_t \circ \Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right) \tag{2.6}$$

of the quantum-mechanical process, normalized to the probabilities

$$\pi_t^\tau\left(u_t^\tau, dv_t^\tau\right) = \left\langle \varrho_t, \Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right) I\right\rangle \tag{2.7}$$

of the events $v_t^\tau \in dv_t^\tau$. The ratio of (2.6) to (2.7) determines conditional states normalized in the usual way, but depending non-linearly on $\varrho_t = \varrho$ in general,

$$\varrho_t^\tau = \frac{\varrho \circ \Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right)}{\left\langle \rho, \Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right) I\right\rangle} \tag{2.8}$$

with respect to measurable events $dv_t^\tau \subseteq V_t^\tau$ of non-zero probability (2.7), to which states at time $t+\tau$ the system transfers from the state $\varrho_t = \varrho$ as a result of the control action u_t^τ and the observation dv_t^τ on an interval of length τ. If the event $dv_t^\tau = V_t^\tau$ is certain, the states (2.8) are unconditional: $\varrho_t = \varrho_{t+\tau}\left(u_t^\tau\right)$ and coincide with the *a priori* states, whose controlled evolution is linear. This evolution is described by expression (2.2), in which $\mathrm{M}_t^\tau\left(u_t^\tau\right) = \Pi_t^\tau\left(u_t^\tau, V_t^\tau\right)$ denotes controllable transforms of the open-loop quantum-mechanical system, corresponding to the absence of observation. The process of precise measurement of the output signal v_t^τ on an interval of length $\tau > 0$ takes the quantum system from *a priori* state $\varrho = \varrho_t$ to the *a posteriori* state $\varrho_t^\tau = \varrho \mathrm{M}_{\varrho,t}^\tau\left(u_t^\tau, v_t^\tau\right)$, where the quasilinear mapping $\varrho \mapsto \varrho \mathrm{M}_{\varrho,t}^\tau\left(u_t^\tau, v_t^\tau\right)$ is given by expression (2.8) in the limit $dv_t^\tau \downarrow \{v_t^\tau\}$ almost everywhere with respect to the measure (2.7). For example, let the measures Π_t^τ have the density functions

$$\Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right) = \int_{dv_t^\tau} \mathrm{P}_t^\tau\left(u_t^\tau, v_t^\tau\right) \mu_t^\tau\left(u_t^\tau, dv_t^\tau\right), \tag{2.9}$$

where $\mathrm{P}_t^\tau\left(u_t^\tau, v_t^\tau\right) : \mathfrak{A} \mapsto \mathfrak{A}$ denotes completely positive superoperators [see the Appendix, (A.4)], say, of the form (3.10), continuous with respect to u_t^τ and integrable with respect to v_t^τ in the strong operator sense with respect to specified numerical measures μ_t^τ on V_t^τ. Then the *a posteriori* transfer operators $\mathrm{M}_{\varrho,t}^\tau\left(u_t^\tau, v_t^\tau\right)$ coincide, up to normalization, with $\mathrm{P}_t^\tau\left(u_t^\tau, v_t^\tau\right)$:

$$\mathrm{M}_{\varrho,t}^\tau\left(u_t^\tau, v_t^\tau\right) = \frac{\mathrm{P}_t^\tau\left(u_t^\tau, v_t^\tau\right)}{\left\langle \rho, \mathrm{P}_t^\tau\left(u_t^\tau, v_t^\tau\right) I\right\rangle}, \tag{2.10}$$

where the ratio is defined for those $u_t^\tau \in U_t^\tau$, $v_t^\tau \in V_t^\tau$ for which the densities

$$p_t^\tau\left(u_t^\tau, v_t^\tau\right) = \left\langle \rho, \mathrm{P}_t^\tau\left(u_t^\tau, v_t^\tau\right) I\right\rangle \tag{2.11}$$

of the probability measure (2.7) with respect to μ_t^τ are non-vanishing.

The following theorem states that the *a posteriori* mapping (2.10) in fact determines the classical Markov process introduced in the analogous situation for classical systems in [20], where it is called a secondary, or conditional (*a posteriori*) Markov process.

Theorem 1. *The family* $\left\{\mathrm{M}_{\varrho,t}^\tau\right\}$ *of a posteriori transfer operators* $\mathrm{M}_{\varrho,t}^\tau\left(u_t^\tau, v_t^\tau\right)$ *satisfies, with respect to composition, the consistency condition*

$$\mathrm{M}_{\varrho,t}^\tau\left(u_t^\tau, v_t^\tau\right)\mathrm{M}_{\varrho',t+\tau}^{\tau'}\left(u_{t+\tau}^{\tau'}, v_{t+\tau}^{\tau'}\right) = \mathrm{M}_{\varrho,t}^{\tau+\tau'}\left(u_t^{\tau+\tau'}, v_t^{\tau+\tau'}\right) \tag{2.12}$$

almost everywhere under the measure (2.7), where $\varrho' = \varrho\mathrm{M}_{\varrho,t}^{\tau'}\left(u_t^\tau, v_t^\tau\right)$.

Proof. It is required to verify the property (2.12) for sublimiting conditional mappings (2.8), for which it follows at once from the definition and (2.5), and then to pass to the limit $dv_t^\tau \downarrow \{v_t^\tau\}$. In the case (2.9) condition (2.12) is verified by simply computing the product (2.12) of the *a posteriori* transfer operators (2.10); for this purpose it is necessary to invoke the corresponding condition

$$\mathrm{P}_t^\tau\left(u_t^\tau, v_t^\tau\right)\mathrm{P}_{t+\tau}^{\tau'}\left(u_{t+\tau}^{\tau'}, v_{t+\tau}^{\tau'}\right) = \mathrm{P}_t^{\tau+\tau'}\left(u_t^{\tau+\tau'}, v_t^{\tau+\tau'}\right), \tag{2.13}$$

which it is sufficient to require on $V_t^{\tau+\tau'}$ almost everywhere $(\mathrm{mod}\,\mu_t^\tau)$ and which guarantees the satisfaction of condition (2.4) if

$$\mu_t^\tau\left(u_t^\tau, dv_t^\tau\right)\mu_{t+\tau}^{\tau'}\left(u_{t+\tau}^{\tau'}, dv_{t+\tau}^{\tau'}\right) = \mu\left(u_t^{\tau+\tau'}, dv_t^{\tau+\tau'}\right).$$

□

Remark 1. *If the superoperator densities* P_t^τ *of the transition measures (2.9) preserve unity:* $\mathrm{P}_t^\tau\left(u_t^\tau, v_t^\tau\right)I = p_t^\tau\left(u_t^\tau, v_t^\tau\right)I$*, the ratio (2.10) determines* ρ*-independent transfer operators* $\mathrm{M}_t^\tau\left(u_t^\tau, v_t^\tau\right)$ *describing the controllable quantum dynamics of a system with inputs* u *and* v*, the second of which is an observable stochastic process with probability measures* $\pi_t^\tau\left(u_t^\tau, v_t^\tau\right) = p_t^\tau\left(u_t^\tau, v_t^\tau\right)\mu_t^\tau\left(u_t^\tau, v_t^\tau\right)$ *independent of the state of the system. The a posteriori mappings (2.8) in this case are linear:* $\varrho_t^\tau = \varrho\mathrm{M}_t^\tau\left(u_t^\tau, v_t^\tau\right)$ *almost everywhere under the measure* π_t^τ.

3. Sufficient coordinates of quantum-mechanical systems

The description of the dynamics of simple closed-loop quantum-mechanical systems for a certain class of initial states is known to be often reducible to the determination of the time evolution of certain coordinates, the role of which can be taken, for example, by vectors $\psi \in \mathcal{H}$, if only vector initial states are considered. Aspects of the controllability of closed-loop quantum-mechanical systems described by a sufficient coordinate $\psi_t \in \mathcal{H}$, satisfying the controlled Schrődinger equation have been investigated previously [8].

The concept of sufficient coordinates, which is introduced below for general controllable quantum dynamical systems with observation and is intimately related to the classical notion of sufficient statistics [20], plays an even greater role for quantum control theory than the analogous concept in stochastic control theory, because it permits control problems for quantum-mechanical systems to be reduced to classical control problems with lumped or distributed parameters.

Definition 2. *Let X be a Borel space*[1] *and let $\{\rho_{x,t}\}_{x\in X, t\in\mathbb{R}}$ be a family of states that generates, for every $t \in \mathbb{R}$, a measurable mapping $x \mapsto \varrho_{x,t}$ of the space X into the space of states $\varrho_{x,t}$ of a quantum-mechanical system at time t, where the controlled evolution (2.5) of the system during an observation leaves this family invariant:*

$$\varrho_{x,t}\Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right) = \pi_{x,t}^\tau\left(u_t^\tau, dv_t^\tau\right)\varrho_{f_{x,t}^\tau\left(u_t^\tau, v_t^\tau\right), t+\tau} \tag{3.1}$$

Then X is called the space of sufficient coordinates $x \in X$ with respect to $\{\varrho_{x,t}\}$, the controlled statistical evolution of which $x \in X \mapsto f_{x,t}^\tau\left(u_t^\tau, v_t^\tau\right) \equiv x_t^\tau \in X$ is described by mappings $f_{x,t}^\tau : U_t^\tau \times V_t^\tau \to X$ continuous with respect to $u_t^\tau \in U_t^\tau$ and measurable with respect to $v_t^\tau \in V_t^\tau$ almost everywhere under the measure

$$\pi_{x,t}^\tau\left(u_t^\tau, dv_t^\tau\right) = \left\langle \rho_x, \Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right) I\right\rangle.$$

Proceeding from (3.1) taken in the limit $dv_t^\tau \downarrow \{v_t^\tau\}$, we note that a sufficient coordinate $x = \varrho$ with respect to the family $\{\varrho \in \mathcal{L}\}$ of all states ϱ on $\mathfrak{A}$ is specified, for example, by the *a posteriori* mapping $f_{\varrho,t}^\tau\left(u_t^\tau, v_t^\tau\right) = \varrho \mathrm{M}_{\varrho,t}^\tau\left(u_t^\tau, v_t^\tau\right)$, provided only [as in the case (2.9)] that there exists the derivative

$$\mathrm{M}_{\varrho,t}^\tau\left(u_t^\tau, v_t^\tau\right) = \frac{\Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right)}{\pi_{\varrho,t}^\tau\left(u_t^\tau, dv_t^\tau\right)}. \tag{3.2}$$

Theorem 2. *The mapping $f_{x,t}^\tau$ determined in (3.1) generates for $\varrho_t \in \{\varrho_{x,t}\}_{x\in X}$ and for a certain fixed $t \in \mathbb{R}$ a sufficient statistic $x_t^\tau = f_{x,t}^\tau\left(u_t^\tau, v_t^\tau\right)$, in terms of which are described the a posteriori states $\varrho_t^\tau = \varrho\mathrm{M}_{\varrho,t}^\tau\left(u_t^\tau, v_t^\tau\right)$ for $\varrho_t = \varrho_{x,t}$ in correspondence with the formula $\varrho_t^\tau = \varrho_{x,t,t+\tau}^\tau \forall\tau > 0$. Here the transition probabilities $x_t = x \to dx' \ni x_t^\tau$ defined by the formula*

$$\pi_{x,t}^\tau\left(u_t^\tau, dx'\right) = \left\langle \rho_x, \Pi_{x,t}^\tau\left(u_t^\tau, dx'\right)\right\rangle, \tag{3.3}$$

where

$$\Pi_{x,t}^\tau\left(u_t^\tau, dx'\right) = \Pi_t^\tau\left(u_t^\tau, f_{x,t}^{-1}\left(u_t^\tau, dx'\right)\right),$$

$$f_{x,t}^{-1}\left(u_t^\tau, dx'\right) = \left\{v_t^\tau : f_{x,t}^\tau\left(u_t^\tau, v_t^\tau\right) \in dx'\right\}, \tag{3.4}$$

satisfy the Chapman-Kolmogorov equation

$$\int_{x'\in X} \pi_{x,t}^\tau\left(u_t^\tau, dx'\right)\pi_{x',t+\tau}^{\tau'}\left(u_{t+\tau}^{\tau'}, dx''\right) = \pi_{x,t}^{\tau+\tau'}\left(u_t^{\tau+\tau'}, dx''\right) \tag{3.5}$$

for all $t \in \mathbb{R}$, $\tau, \tau' > 0$, $u_t^\tau, u_{t+\tau}^{\tau'}$, so that the sufficient statistics form a controllable Markov process.

Proof. The existence of the sufficient statistic is determined by the *a posteriori* mapping, which for $\varrho = \varrho_{x,t}$ according to expression (2.8) in correspondence with (3.1) gives

$$\varrho_t^\tau = \varrho\mathrm{M}_{\varrho,t}\left(u_t^\tau, v_t^\tau\right) = \varrho_{f_{x,t}^\tau\left(u_t^\tau, v_t^\tau\right), t+\tau'} \tag{3.6}$$

[1]Usually standard, i.e. a complete seperable metric space, also known as a Polish space (for example, $\mathbb{R}^n, \mathbb{C}^n$, or any countable set).

thus proving the first statement of Theorem 2. The transition mappings $x \mapsto f\left(u_t^\tau, v_t^\tau\right)$ in correspondence with Theorem 1 satisfy the following semigroup property with respect to their composition:

$$f_{t+\tau}^{\tau'}\left(u_{t+\tau}^{\tau'}, v_{t+\tau}^{\tau'}\right) \circ f_t^\tau\left(u_t^\tau, v_t^\tau\right) = f_t^{\tau+\tau'}\left(u_t^{\tau+\tau'}, v_t^{\tau+\tau'}\right), \tag{3.7}$$

which, according to (2.5), yields the equation

$$\int\limits_{x' \in X} \Pi_{x,t}^\tau\left(u_t^\tau, dx'\right) \Pi_{x',t+\tau}^{\tau'}\left(u_t^{\tau'}, dx''\right) = \Pi_{x,t}^{\tau+\tau'}\left(u_t^{\tau+\tau'}, dx''\right) \tag{3.8}$$

for the transfer-operator measures specified in (3.3) and (3.4) for the transitions $x \mapsto dx'$. For states in the class $\left\{\varrho_{x,t}\right\}$, (3.8) is equivalent to the Chapman-Kolmogorov equation (3.5), because in accordance with (3.1),

$$\varrho_{x,t}\Pi_{x,t}^\tau\left(u_t^\tau, dx'\right) = \pi_{x,t}^\tau\left(u_t^\tau, dx'\right)\varrho_{x',t+\tau}. \tag{3.9}$$

Equation (3.5) determines the Markov stochastic evolution $\widehat{x}(t)$ of the sufficient coordinates $x(t) \in X$ generating, according to a theorem of Kolmogorov (see, e.g. [10], p. 48) for a standard space X, a Markov measure in the functional Borel space of trajectories $\{x(t)\}$. This completes the proof. □

We now discuss in more detail the most interesting case, in which the transition measures (2.9) have the superoperator densities

$$\mathrm{P}_t^\tau\left(u_t^\tau, v_t^\tau\right)Q = F_t^\tau\left(u_t^\tau, v_t^\tau\right)^\dagger Q F_t^\tau\left(u_t^\tau, v_t^\tau\right) \tag{3.10}$$

under a consistent family of measures $\mu_t^\tau\left(u_t^\tau, dv_t^\tau\right)$. Here $F_t^\tau\left(u_t^\tau, v_t^\tau\right)$ denotes operators belonging to the Hilbert space $\mathcal{H}$ and satisfying the normalization condition

$$\int F_t^\tau\left(u_t^\tau, v_t^\tau\right)^\dagger F_t^\tau\left(u_t^\tau, v_t^\tau\right)\mu_t^\tau\left(u_t^\tau, dv_t^\tau\right) = I \tag{3.11}$$

as well as the condition

$$F_{t+\tau}^{\tau'}\left(u_{t+\tau}^{\tau'}, v_{t+\tau}^{\tau'}\right) F_t^\tau\left(u_t^\tau, v_t^\tau\right) = F_t^{\tau+\tau'}\left(u_t^{\tau+\tau'}, v_t^{\tau+\tau'}\right), \tag{3.12}$$

which guarantees the fulfillment of (2.13).

It is seen at once that the *a posteriori* transfer operators (2.10) preserve the vectorial property of the vector states $\langle\rho, Q\rangle = \langle\psi|Q\psi\rangle$: $\langle\rho_t^\tau, Q\rangle = \langle\psi_t^\tau|Q\psi_t^\tau\rangle$, where $\psi_t^\tau = T_{\psi,t}^\tau\left(u_t^\tau, v_t^\tau\right)\psi$ for any $\psi \in \mathcal{H}$, $\tau > 0$ and

$$T_{\psi,t}^\tau\left(u_t^\tau, v_t^\tau\right) = \frac{F_t^\tau\left(u_t^\tau, v_t^\tau\right)}{\left\|F_t^\tau\left(u_t^\tau, v_t^\tau\right)\psi\right\|}. \tag{3.13}$$

Corollary 1. *The set X of all vectors $\psi \in \mathcal{H}$, for which $\|\psi\| = 1$ forms, in the case (3.10) with respect to the family $\left\{\varrho_\psi\right\}_{\psi \in X}$ of vector states $\left\langle\rho_\psi, Q\right\rangle = \langle\psi|Q\psi\rangle$ the space of sufficient coordinates specified by an a posteriori mapping $f_{\psi,t}^\tau\left(u_t^\tau, v_t^\tau\right) = T_{\psi,t}^\tau\left(u_t^\tau, v_t^\tau\right)\psi$ of the quasilinear form (3.13).*

We note that the *a posteriori* propagators $T_{\psi,t}^\tau\left(u_t^\tau, v_t^\tau\right)$ satisfy the semigroup property (3.7):

$$T_{\psi',t+\tau}^{\tau'}\left(u_{t+\tau}^{\tau'}, v_{t+\tau}^{\tau'}\right) T_{\psi,t}^\tau\left(u_t^\tau, v_t^\tau\right) = T_t^{\tau+\tau'}\left(u_t^{\tau+\tau'}, v_t^{\tau+\tau'}\right), \tag{3.14}$$

where $\psi' = T_{\psi,t}^\tau\left(u_t^\tau, v_t^\tau\right)\psi$ and in contrast with the operators $F_t^\tau\left(u_t^\tau, v_t^\tau\right)$ preserve the norm in $\mathcal{H}$, but they are non-linear. Only in the case discussed at the end of Section 3, where $F_t^\tau\left(u_t^\tau, v_t^\tau\right)^\dagger F_t^\tau\left(u_t^\tau, v_t^\tau\right) = p_t^\tau\left(u_t^\tau, v_t^\tau\right) I$ are the operators (3.13) ψ-independent isometries: $T_t^\tau\left(u_t^\tau, v_t^\tau\right) = F_t^\tau\left(u_t^\tau, v_t^\tau\right) / \sqrt{p_t^\tau\left(u_t^\tau, v_t^\tau\right)}$. We note, however, that the *a priori* transfer operators

$$\mathrm{M}_t^\tau\left(u_t^\tau\right) Q = \int F_t^\tau\left(u_t^\tau, v_t^\tau\right)^\dagger Q F\left(u_t^\tau, v_t^\tau\right) \mu_t^\tau\left(u_t^\tau, dv_t^\tau\right) \tag{3.15}$$

determining the controllable Markov dynamics of the quantum system (2.9), (3.10) in the absence of observations are not described by the propagators $T_t^\tau\left(u_t^\tau\right)$, with the exception of the degenerate case in which the *a posteriori* states coincide mod μ_t^τ with *a priori* states, i.e., actually do not depend on the results of the observations v_t^τ.

4. Optimal quantum control

We now discuss the optimal control of a quantum dynamical system with observation $\{\Pi_t^\tau\}$, the performance of which as a function of the initial time $t \in \mathbb{R}$ is determined by the mathematical expectation $\langle \rho_t, Q_t\left(u_t, dv_t\right)\rangle$ of a certain physical quantity $Q_t\left(u_t, dv_t\right) \in \mathfrak{A}$ depending on the input state $u_t = \{u\left(t+\tau\right)\}_{\tau \geq 0}$ continuously[2] and on the output event $dv_t = d\{v\left(t+\tau\right)\}_{\tau > 0}$ according to the equation in t

$$Q_t\left(u_t, dv_t\right) = \Pi_t^\tau\left(u_t^\tau, dv_t^\tau\right) Q_{t+\tau}\left(u_{t+\tau}, dv_{t+\tau}\right) + S_t^\tau\left(u_t^\tau, dv_t^\tau\right). \tag{4.1}$$

Here $S_t^\tau\left(u_t^\tau, dv_t^\tau\right) \in \mathfrak{A}$ denotes Hermitian operators having the integral form[3]

$$S_t^\tau\left(u_t^\tau, dv_t^\tau\right) = \int_0^\tau \Pi_t^{\tau'}\left(u_t^{\tau'}, dv_t^{\tau'}\right) S\left(u\left(t+\tau'\right), t+\tau'\right) d\tau' \tag{4.2}$$

where $S\left(u,t\right) = S\left(u,t\right)^\dagger$ denotes Hermitian operator functions completely determining (4.1) for a certain boundary condition $Q_T\left(u_T, dv_T\right) = Q$ at the final time $T > t$, corresponding to the specification of a terminal risk $\langle \rho_T, Q\rangle$ ($Q = Q^\dagger$ is a certain Hermitian operator).

Definition 3. *A measurable mapping $v_t \mapsto u_t\left(v_t\right) \in U_t$ is called a non-advanced control strategy if its components $u\left(t+\tau,\cdot\right) : v_t \mapsto u\left(t+\tau\right)$ are determined by functions independent of $v_{t+\tau}$ and it is called a retarded control strategy if all $u\left(t+\tau,\cdot\right)$ are determined by functions $v_t^{\tau'} \mapsto u\left(t+\tau, v_t^{\tau'}\right)$ for some measurable $\tau' = \tau'\left(t+\tau\right) < \tau$. A non-advanced strategy $u_t\left(\cdot\right)$ is called admissible if the integral*

$$Q_t\left[u\left(\cdot\right)\right] = \int Q_t\left(u_t\left(v_t\right), dv_t\right)$$

exists in strong operator topology and it is called optimal for an initial state $\varrho_t = \varrho$ if it realizes the extremum

$$q\left(\varrho, t\right) = \inf_{u_{t(\cdot)} \in U_t(\cdot)} \langle \rho, Q_t\left[u_t\left(\cdot\right)\right]\rangle, \tag{4.3}$$

[2] In strong operator topology

[3] The conditions for the existence of the integral 4.2, its continuous dependence on u_t^τ, and its σ-additivity with respect to dv_t^τ, requiring of the operator function $(u,t) \to S(t,u) \in \mathsf{A}$ continuity in $u \in U$ and measurablilty with respect to $t \in \mathbb{R}$ under strong operator topology, are presumed to be fulfilled.

where $U_t(\cdot)$ is a certain set of admissible strategies $u_t(\cdot)$ [ε-optimal if $\langle \rho, Q_t[u_t(\cdot)]\rangle$ exceeds (4.3) at most by ε].

We note that in accordance with (4.1), a strategy $u_t(\cdot)$ is admissible with respect to $Q_t(\cdot,\cdot)$ if and only if its segments $u_{t+\tau}(\cdot)$ for fixed v_t^τ are admissible strategies with respect to $Q_{t+\tau}(\cdot,\cdot)$ and for the segments $u_t^\tau(\cdot)$ there exist measures

$$\Pi_t^{u,\tau}(dv_t^\tau) = \int_{dv_t^\tau} \Pi_t^\tau\left(u_t^\tau(v_t^\tau), dv_t^\tau\right), \tag{4.4}$$

specifying operator-valued integrals

$$S_t^\tau[u_t^\tau(\cdot)] = \int_0^\tau \int_{v_t^\tau} \Pi_t^{u,\tau'}\left(dv_t^{\tau'}\right) S\left(t+\tau', u\left(t+\tau', v_t^{\tau'}\right)\right). \tag{4.5}$$

The latter holds for any delayed strategy that is admissible for a given boundary condition $Q_T(\cdot,\cdot) = Q$.

Theorem 3. *Let the sets $U_t^\tau(\cdot)$ of segments of admissible strategies satisfy the condition*

$$U_t^\tau(\cdot) \times U_{t+\tau}^{\tau'}(\cdot) \subseteq U_t^{\tau+\tau'}(\cdot) \qquad \forall t \in \mathbb{R}, \qquad \tau, \tau' > 0 \tag{4.6}$$

Then the minimum risk (4.3) as a function of the state ϱ and the time t satisfies the functional equation

$$q(\varrho, t) = \inf_{u_t^\tau(\cdot)\in U_t^\tau(\cdot)} \left[\langle \rho, S_t^\tau[u_t^\tau(\cdot)]\rangle + \int \pi_{\varrho t}^{u\tau}(dv_t^\tau)\, q(\varrho, t+\tau)\right], \tag{4.7}$$

where $\pi_{\varrho t}^{u\tau}(\cdot) = \langle \rho, \Pi_t^{u\tau}(\cdot) I\rangle$, $\widehat{\varrho} = \varrho \Phi_{\varrho t}^\tau(u_t^\tau(v_t^\tau), v_t^\tau)$ denotes the probability measures (2.7) and a posteriori states (2.8) corresponding to an admissible strategy $u = u_t^\tau(\cdot)$ and an initial state $\varrho = \varrho_t$.

Proof. The proof of (4.7), which generalizes the Bellman equation [7], is reducible to the substitution of (4.1) into (4.3) and the transition from minimization on $u_t(\cdot)$ to the successive minimization of (4.7), first on $u_{t+\tau}(\cdot)$ and then on $u_t^\tau(\cdot)$, which by condition (4.6) yields the same result as (4.3). Since the integral (4.5) does not depend on $u_{t+\tau}(\cdot)$ and by definition,

$$\varrho \Pi_t^{u\tau}(dv_t^\tau) = \pi_{\varrho t}^{u\tau}(dv_t^\tau)\, \varrho_{\varrho t}^{u\tau}, \tag{4.8}$$

the first minimization entails finding the second term of the minimized sum (4.7):

$$\inf_{u_{t+\tau}(\cdot)\in U_{t+\tau}(\cdot)} \int \pi_{\varrho t}^{u\tau}(dv_t^\tau) \left\langle \widehat{\rho}, \int Q_t(u_t(v_t), dv_t)\right\rangle = \int \pi_{\varrho t}^{u\tau}(dv_t^\tau)\, q(\widehat{\varrho}, t+\tau).$$

□

In the case of a given boundary condition $q(\varrho, t) = \langle \rho, Q\rangle$ the theorem proved above provides a constructive method of synthesizing an optimal or ε-optimal strategy $u_{\varrho t}^{T-t}\left(v_t^{T-t}\right)$ by the successive minimization of (4.7) in reverse time. In this case it is sufficient to restrict the discussion to Markov admissible strategies described by segments $u_{\varrho t'}^\tau(v_{t'}^\tau)$, $\tau = T - t$, depending on the *a priori* history v_t^τ only through the agency of their dependence on the *a posteriori* state $\widehat{\varrho} = \varrho_t^{t'-t}$ for any $t' > t$. Accordingly, the determination of the *a posteriori* quantum states ϱ_t^τ, which generate an *a posteriori* Markov process, enables us to reduce the optimal quantum control problem to the classical problem of stochastic control theory [20][7] with numerical transition and final risk functions

$$s(\varrho, t, u) = \langle \rho, S(t, u) \rangle \, q(\varrho, T) = \langle \rho, Q \rangle ,$$

determined by the operators of the corresponding physical quantum variables $S(t, u)$ and Q.

We distinguish cases in which the quantum states ϱ are considered in a certain class $\{\varrho_{xt}\}$ for which sufficient coordinates exist.

Corollary 2. *Let $f_t^\tau : U_t^\tau \times V_t^\tau \mapsto X$ denote mappings satisfying the conditions of Theorem 2. Then in problem (4.3) for $\varrho \in \{\varrho_{xt}\}$ it is sufficient to restrict the discussion to Markov strategies described by measurable mappings $u_t^\tau : X \times V_t^\tau \mapsto U_t^\tau$ satisfying the consistency condition*

$$\left(u_{xt}^\tau(v_t^\tau), u_{x't+\tau}^{\tau'}\left(v_{t+\tau}^{\tau'}\right)\right) = u_{xt}^{\tau+\tau'}\left(v_t^{\tau+\tau'}\right), \tag{4.9}$$

where $x' = f_{xt}^{u\tau}(v_t^\tau) = f_{xt}^\tau(u_t^\tau(v_t^\tau), v_t^\tau)$. In particular, the instantaneous control functions $u_x(\tau)$ for any $\tau \in [t, T)$ are determined by functions $u(\tau, x)$ of the instantaneous state x in accordance with the equation

$$u_x(t+\tau, v_t^\tau) = u(t+\tau, f_{xt}^{u\tau}(v_t^\tau)). \tag{4.10}$$

The foregoing assertion, which follows directly for the "maximum" sufficient coordinate $\widehat{x}(t) = \widehat{\varrho}(t)$ from the optimality equation (4.7), is readily proved on the basis of the properties formulated for sufficient coordinates in Theorem **??**.

The further simplification of problem (4.3) entails utilizing the specific properties of the Markov process $\widehat{x}(t)$, the role of which is logically assigned to sufficient coordinates of the fewest possible dimensions.

For example, in the case where for the transition probabilities $\pi_{xt}^\tau(dx')$, the t-continuous strong limit

$$L(t) q(t)(x) = \lim_{\tau \downarrow 0} \int_x \pi_{xt}^\tau(u_t^\tau, dx') \left(q(x', t+\tau) - q(x, t+\tau)\right) \tag{4.11}$$

exists on some set $\mathcal{D}(X)$ of bounded and measurable functions $x \mapsto q(x, t)$, depending continuously on t, the optimality equation (4.3) is written in the infinitesimal form

$$-\frac{\partial}{\partial t} q(x, t) = \inf_{u \in U} \left(s(x, t, u) + L(t, u) q(t)(x)\right), \tag{4.12}$$

where $s(x, t, u) = \langle \rho_{xt}, S(t, u) \rangle$. Equation (4.7), which represents the standard Bellman equation for controlled Markov processes in continuous time, can be used for $q(x, T) = \langle \rho_{xT}, Q \rangle \in \mathcal{D}(X)$ to seek optimal or ε-optimal Markov control functions $u(t)$ directly as functions $u(t, x)$ of the instantaneous state x.

5. Quantum control with discrete observation

As an example we consider the controlled dynamics of a simple quantum system described between discrete measurement times $T = \{t_k\}$ by the Schrődinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(t) = H(t, u(t)) \psi(t), \qquad t \in T \tag{5.1}$$

Here $H(t,u)$ is the controlled Hamiltonian, i.e., a self-adjoint operator in $\mathcal{H}$ with a dense domain of definition $\mathcal{D} \subseteq \mathcal{H}$, written in the usual form

$$H(t,u) = H_0(t) + \sum_{i=1}^{m} u_i(t) H_i(t),$$

where $u_i(t) \in \mathbb{R}$; $H_i(t)$ are simple[4] functions of t. Under the stated assumption there exists a unique consistent family $\{T_t^\tau\}$ of unitary propagators $T_t^\tau(u_t^\tau)$, representing for any $\psi(t-\tau) = \varphi \in \mathcal{D}$, a solution of (5.1) between adjacent measurement times $t_k < t_{k+1}$ in the form $\psi(t) = T_{t-\tau}^\tau\left(u_{t-\tau}^\tau\right)\psi$, $t \in [t_k, t_{k+1})$, where $\lim_{\tau\searrow 0} \psi(t) = \psi$.

Let E_{vk} demote Hermitian projectors, which determine orthogonal decompositions $I = \sum_{v\in V_k} E_{vk}$ of the unit operator in $\mathcal{H}$ and specify measurements at times t_k of quantum physical quantities described by self-adjoint operators

$$A_k = \sum_{v\in V_k} vE_{vk} \tag{5.2}$$

with discrete spectra $V_k \subseteq \mathbb{R}$.

As a result of measurement of the quantity A_k there occurs a reduction [18] of the quantum state $\varrho \mapsto \varrho\Pi_{vk}$, $v \in V_k$, described by the superoperators $\Pi_{vk}Q = E_{vk}QE_{vk}$, which determines *a priori* direct transfer operators

$$\Pi_k Q = \sum_{v\in V_k} E_{vk}QE_{vk}. \tag{5.3}$$

The states $\varrho_{vk} = \varrho\Pi_{vk}$ to which the system transfers instantaneously depending on the result of this measurement $v \in V_k$ are normalized to the probabilities $\pi_{vk} = \langle \rho, E_{vk}\rangle$ of these transitions, where if ρ is a vector state $\langle \rho_\psi, Q\rangle = \langle \psi|Q\psi\rangle$, the states ρ_{vk} are also vectorial, determined by the projections $\psi_{vk} = E_{vk}\psi$. The product $E_{vk}T_t^\tau(u_t^\tau) = F_{vt}^\tau(u_t^\tau)$ for $\tau = t_k - t$ determines a transformation $\psi(t) \mapsto E_{vk}\psi(t_k)$ corresponding to the evolution (5.1) on the interval $[t, t_k)$ with subsequent measurement of the quantity A_k.

We introduce the notation $F_{vk}(u_k) = F_{vt_k}^{\tau_k}\left(u_{t_k}^{\tau_k}\right)$, where $\tau_k = t_{k+1} - t_k$, and we set $V_t^\tau = \prod_{k\in K_t^\tau} V_k$, where $K_t^\tau = \{k : t_k \in [t, t+\tau)\}$ is the set of all indices of times in the interval $(t, t+\tau)$ (in the case of an empty set $K_t^\tau = \emptyset$ we assume that V_t^τ consists of some single point $\{w\}$).

Proposition 1. *Let the set K_t^{s-t} be finite for any $t < s$. Then the chronological product*

$$F_{vt}^{s-t}\left(u_t^{s-t}\right) = T_{t_1}^{s-t_1}\left(u_{t_1}^{s-t_1}\right) F_{v_1 l-1}(u_{l-1}) \ldots F_{v_{k+1}k}(u_k) F_{v_k t}^{t_k - t}\left(u_t^{t_k - t}\right), \tag{5.4}$$

where $k = \min K_t^{s-t}$, $l = \max K_t^{s-t}$, and $v = (v_k, \ldots, v_l) = v_t^{s-t}$, determines controllable quantum dynamical system described by superoperators $\{\Pi_t^\tau\}$ of the form (2.9), (3.10):

$$\Pi_{vt}^\tau(u_t^\tau) Q = F_{vt}^\tau(u_t^\tau)^\dagger QF_{vt}^\tau(u_t^\tau), \tag{5.5}$$

under the counting measure $\mu_t^\tau = 1$ on $V_t^\tau \ni v$.

[4]In other words, having one-sided limits; for unbounded self-adjoint operators $H_i(t)$, $i = 0, \ldots, m$, this means that $H(t, u(t))\psi$ is a simple function for any $\psi \in \mathcal{D}$.

The proof is reducible to the verification of conditions (3.11) and (3.12), which take the form

$$\sum_{v\in V_t^\tau} F_{vt}^\tau\left(u_t^\tau\right)^\dagger F_{vt}^\tau\left(u_t^\tau\right)=I \qquad \forall u_t^\tau\in U_t^\tau, \tag{5.6}$$

$$F_{v't+\tau}^{\tau'}\left(u_{t+\tau}^{\tau'}\right)F_{vt}^\tau\left(u_t^\tau\right)=F_{(v',v)t}^{\tau'+\tau}\left(u_t^{\tau'+\tau}\right) \tag{5.7}$$

(where $v'\in V_{t+\tau}^{\tau'}$, $v\in V_t^\tau$), which is easily verified by induction, owing to the finiteness of the product (4.4).

Because of the spatial form (5.5) of the consistent family $\{\Pi_{vt}\}$ on the basis of Corollary 1, we infer that the space X of normalized vectors $\psi\in\mathcal{H}$, $\|\psi\|=1$ forms a space of sufficient coordinates, the *a posteriori* evolution $\psi\mapsto T_{vt}^\tau\left(u_t^\tau,v_t^\tau\right)\psi$ of which is described by the nonlinear propagators (3.13): $T_{vt}^\tau\left(u_t^\tau\right)/\left\|T_{vt}^\tau\left(u_t^\tau\right)\psi\right\|$, and the *a priori* evolution by transfer operators of the form (3.15):

$$\Pi_t^\tau\left(u_t^\tau\right)Q=\sum_{v\in V_t^\tau}F_{vt}^\tau\left(u_t^\tau\right)^\dagger QF_{vt}^\tau\left(u_t^\tau\right).$$

We give special consideration to the case of complete measurements described by the operators A_k with a non-degenerate spectrum.

Proposition 2. *Let $\{\psi_{vk}\}_{v\in V_k}$ denote the complete orthonormal systems of eigenvectors of the operators A_k, and let E_{vk} be the corresponding one-dimensional projectors onto ψ_{vk}. Then the a posteriori states at the times $\{t_k\}$ are vector states, which are completely determined by the last result of measurement $v_k\in V_k$:*

$$\left\langle\rho_t^{t_k-t},Q\right\rangle=\left\langle Q\psi_{v_kk},\psi_{v_kk}\right\rangle \qquad \forall t<t_k, \qquad \varrho=\varrho_t, \tag{5.8}$$

and the measurement process $\{v_k\}$ is a Markov process, which is described by the controllable transition probabilities

$$\pi_{vk}\left(u_k,v_k\right)=\left|\left(T_k\left(u_k\right)\psi_{v_kk},\psi_{v_kk+1}\right)\right|^2, \tag{5.9}$$

where $T_k\left(u_k\right)=T_{t_k}^{\tau_k}\left(u_{t_k}^{\tau_k}\right)$, $\tau_k=t_{k+1}-t_k$.

This proposition follows from the property

$$E_{vk}QE_{vk}=\left\langle\psi_{vk}|Q\psi_{vk}\right\rangle E_{vk}$$

of the one-dimensional orthogonal projection operators E_{vk} corresponding to the eigenvectors ψ_{vk}, so that the application of any state ϱ to (5.5) at $t=t_k-\tau$ leads to (5.8), up to normalization. Inasmuch as the *a posteriori* state (5.8) does not depend on the previous measurements, the conditional probability given by expression (5.9) for the even $v_{k+1}=v$ for fixed preceding results is Markovian.

In the proposition proved above, the controllable sufficient coordinate $x_k=v_k$ is indicated, provided only that the quantum system is analyzed at discrete measurement times $\{t_k\}$.

We now consider the optimal control problem for a discretely observed quantum system. Let the control performance, as a function of the initial t, be described by the operator (4.1), which is determined by the integral (4.2) of some operator-valued function $S(t,u):\mathcal{H}\mapsto\mathcal{H}$.

Proposition 3. *Under the conditions of Theorem 3, for a vector initial state* ϱ_ψ *the minimum risk*

$$q(\psi,t)=\inf_{u_t(\cdot)\in U_t(\cdot)}\int\limits_{v_t}\langle\psi|Q_t(u_t(v_t),dv_t)\psi\rangle \tag{5.10}$$

in the intervals (t_k,t_{k+1}) *between measurements satisfies the functional equation in variational derivatives*

$$-\frac{\partial}{\partial t}q(\psi,t)=\inf_{u\in U(t)}\left(\|\psi\|^2_{S(t,u)}+2\hbar^{-1}\operatorname{Im}\langle\delta q(\psi,t)/\delta\psi|H(t,u)\psi\rangle\right), \tag{5.11}$$

where $\|\psi\|^2_S=\langle\psi|S\psi\rangle$*, and at the measurement times* $\{t_k\}$ *it satisfies the recursive equation*

$$q_k(\psi)=\inf_{u_k\in U_k}\left(\|\psi\|^2_{S_k(u_k)}+\sum_{\vee\in V_{k+1}}\pi^u_{vk}(\psi)\,q_{k+1}\left(\widehat{\psi}\right)\right), \tag{5.12}$$

which determines the boundary values $q(t_k-0,\psi)=q_k(\psi)$ *for (5.11). Here* $\pi^u_{vk}(\psi)=\|T_{vk}(u_k)\psi\|^2$, $\psi=T_{vk}(u_k)\psi/\sqrt{\pi^u_{vk}(\psi)}$*, and*

$$S_k(u_k)=\int\limits_0^\tau T^{t-t_k}_{t_k}\left(u^{t-t_k}_{t_k}\right)^\dagger S(t,u(t))\,T^{t-t_k}_{t_k}\left(u^{t-t_k}_{t_k}\right)dt. \tag{5.13}$$

Equation (5.11) is readily proved on the assumption of analyticity of the function $\psi\mapsto q(\psi,t)$, as is natural for a quadratic boundary condition $q(\psi,t)=\|\psi\|^2_Q$ at some final time T. Here (5.11) represents a functional version of the Bellman equation corresponding to the Schrődinger equation (5.1) and a quadratic transition objective function $S(t,u,\psi)=\|\psi\|^2_{S(t,u)}$. Equation (5.12) follows directly from (4.7) for $t=t_k$, $\tau=t_{k+1}-t_k$ and $\varrho=\varrho_\psi$ if it is taken into account that the integral (4.2) now has the form (5.13).

In conclusion we consider the optimal control problem described above in the complete measurement case. Making use of the fact that the process of complete measurement at discrete times $\{t_k\}$ induces a Markov sufficient coordinate $x_k=v_k$, from (5.12) we deduce the customary equation

$$q_k(v_k)=\inf_{u_k\in U_k}\left(s_k(u_k,v_k)+\sum_{v\in V_{k+1}}\pi_{vk}(u_k,v_k)\,q_{k+1}(v)\right), \tag{5.14}$$

which describes the optimum risk for the control of a discrete Markov process $\{v_k\}$ with the transition probabilities (5.9), an objective function $s_k(u_k,v_k)=\left\|\psi_{v_k}\right\|^2_{S_k(u_k)}$ and a boundary condition of the form $q_k(v)=\left\|\psi_{v_k}\right\|^2_Q$. The solution of the given Bellman equation (5.14) is easily implemented on a computer by standard dynamic programming methods in the case of piecewise-constant admissible strategies, for which $U_k=U(t_k)\subseteq\mathbb{R}^m$.

Appendix A. Notations, Definitions and Facts

(1) Let $\{Q_i\}_{i\in\mathrm{I}}$ be a family of self-adjoint operators acting in a complex Hilbert space $\mathcal{H}$. The von Neumann algebra generated by the family $\{Q_i\}$ is defined as the minimal weakly closed self-adjoint sub-algebra $\mathfrak{A}$ of bounded operators in $\mathcal{H}$ containing the spectal projectors of this operators, along

with the unit operator I. It consists of all bounded operators that commute with the commutant $\{Q_i\}' = \{Q : QQ_i = Q_iQ \quad \forall i \in \mathrm{I}\}$, i.e., it is the seconed commutant $\mathfrak{A} = \{Q_i\}''$ of the family $\{Q_i\}$. The latter can be taken as the definition of the von Neumann algebra generated by the family $\{Q_i\}$ in the case of unbounded self-adjoint operators Q_i densely defined on a domain $\mathcal{D} \subseteq \mathcal{H}$. The simplest example of von Neumann algebra is the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators acting in $\mathcal{H}$ [11].

(2) A (normal) state on a von Neumann algebra $\mathfrak{A}$ is defined as a linear ultraweakly continuous functional $\varrho : \mathfrak{A} \to \mathbb{C}$ (which will be denoted as $\varrho(Q) = \langle \rho, Q \rangle$) satisfying the positivity and normalization conditions

$$\langle \rho, Q \rangle \geq 0, \qquad \forall Q \geqq 0, \qquad \langle \rho, I \rangle = 1 \tag{A.1}$$

[$Q \geqq 0$ signifies the nonnegative definiteness $\langle \psi | Q\psi \rangle \geqq 0 \ \forall \psi \in \mathcal{H}$ called Hermitian positivity of Q]. They are usually identified with the density operators ρ as the elements of the opposite, or transposed algebra $\mathfrak{A}^\intercal = \{A^\intercal : A \in \mathfrak{A}\}$ with respect to thre pairing $\langle \rho, Q \rangle$. The linear span of all *normal* states is a Banach subspace $\mathfrak{A}_\star$ of the dual space $\mathfrak{A}^\star$, called predual to $\mathfrak{A}$ as $\mathfrak{A}_\star^\star = \mathfrak{A}$. A state ρ is called vector state if $\langle \rho, Q \rangle = \langle \psi | Q\psi \rangle \ \rho = \rho_\psi)$ for some $\psi \in \mathcal{H}$. Any state is a closed convex hull of vector states ρ_ψ, $\|\psi\| = 1$. If on an algebra $\mathfrak{A}$ there exists a normal semi-finite trace $Q \mapsto \mathrm{tr}\{Q\}$, then the states on $\mathfrak{A}$ can be described by the transposed density operators $\rho^\intercal = \bar{\rho} \in \mathfrak{A}$ (or affilaited to $\mathfrak{A}$, if they are unbounded), determining ρ by means of the bilinear form $\langle \rho, Q \rangle = \mathrm{tr}\{\rho^\intercal Q\}$. For the case $\mathfrak{A} = \mathcal{B}(\mathcal{H})$ the density operator ρ is any nuclear positive operator with unit trace [11].

(3) Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be von Neumann algebras in respective Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and let $\mathrm{M} : \mathfrak{A}_2 \to \mathfrak{A}_1$ be a linear operator that transforms the operators $Q_2 \in \mathfrak{A}_2$ into operators $Q_1 \in \mathfrak{A}_1$ (superoperator, in the terminology of [13]). The operator M is called a transfer operator if it is ultraweakly continuous, completely positive in the sense

$$\sum_{i,k=1} \left\langle \psi_i | \mathrm{M}\left(Q_i^\dagger Q_k\right) \psi_k \right\rangle \geqq 0, \qquad \forall Q_i \in \mathfrak{A}_2, \qquad \psi_i \in \mathcal{H}_1, \tag{A.2}$$

$(i = 1, \ldots, n < \infty)$, and unity-preserving: $\mathrm{M}I_2 = I_1$. A composition $\varrho_1 \circ \mathrm{M}$ with any state $\varrho_1 : \mathfrak{A}_1 \to \mathbb{C}$ is the state $\mathfrak{A}_2 \to \mathbb{C}$ described by the predual action of the superoperator M on ρ_1:

$$\langle \rho_1, \mathrm{M}Q_2 \rangle = \langle \mathrm{M}_\star \rho_1, Q_2 \rangle, \forall Q_2 \in \mathfrak{A}_2, \rho_1 \in \mathfrak{A}_1^\intercal.$$

A transfer operator M is called spatial if

$$\mathrm{M}Q_2 = T^\dagger Q_2 T \quad \text{ or } \quad \mathrm{M}_\star \rho_1 = T_\star^\dagger \rho_1 T_\star, \tag{A.3}$$

where $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a linear isometric operator, $T^\dagger T = I$, called the propagator, and $T_\star$ is the transposed operator such that $T_\star^\intercal = T$. Every transfer operator on $\mathfrak{A}_2 = \mathcal{B}(\mathcal{H}_2)$ is a closed convex hull of spatial transfer operators.

(4) Let V be a measurable space, and $\mathcal{B}$ its Borel σ-algebra. A mapping $\Pi : dv \in \mathcal{B} \mapsto \Pi(dv)$ with values $\Pi(dv)$ in ultraweakly continuous, completely positive superoperators $\mathfrak{A}_2 \to \mathfrak{A}_1$ is called a transfer-operator measure if for any $\rho_1 \in \mathfrak{A}_1^\intercal$, $Q_2 \in \mathfrak{A}_2$ the numerical function

$$\langle \Pi(dv)_\star \rho_1, Q_2 \rangle = \langle \rho_1, \Pi(dv) Q_2 \rangle$$

of the set $dv \subseteq V$ is a countably additive measure normalized to unity for $Q_2 = I$. In other words, $\Pi(dv)$ is an operator-valued measure that is σ-additive in the weak (strong) operator sense and for $dv = V$ is equal to some transfer operator M. The quantum-state transformations $\rho_1 \mapsto \rho_2$ corresponding to ideal measurements are described by transfer-operator measures of the form

$$\Pi(dv) Q = F(v)^\dagger Q F(v) \mu(dv), \tag{A.4}$$

where $F(v)$ denotes linear operators $\mathcal{H}_1 \to \mathcal{H}_2$, the integral under a positive numerical measure μ on V is interpreted in strong operator topology, and $\int F^\dagger(v) F(v) \mu(dv) = I_1$. Every transfer-operator $\mathrm{M} : \mathfrak{A}_2 \to \mathfrak{A}_1$ for $\mathfrak{A}_2 = \mathcal{B}(\mathcal{H})$ can be represented by the integral (A.4) with respect to $dv \subseteq V$ of some ideal measure $\Pi(dv)$.

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