Modelling of Homeless Populations

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Abstract

A simple mathematical model is proposed for the changes of homelessness, and of numbers of people housed in the private sector and those resident in council housing. The model, applying to a single local authority, is analysed to see how changing priorities can affect waiting times and the size of the waiting list for council accommodation.

1 Introduction.

In 1996 in England and Wales, the obligations of local authorities towards housing families deemed to be homeless were being revised. In order to see what a change of policy – for instance making a council's system “fairer”, meaning that homeless and other applicants for council housing would be treated in exactly the same way – would have, a simple mathematical model for the numbers of families, or households, is derived. The model is analysed, varying one particular constant which gives a measure of how much priority is given to homeless applicants for council housing.

It is seen that steady states of the model are stable: after a change in populations or operating procedures (say due to a shift in local authority boundaries) population numbers settle down to steady values. For the specific case considered in the present paper, three distinct time scales play a role in this transient behaviour, the longest of which is about fifteen years.

The mathematical model is also solved numerically and exhibits the transient behaviour noted above, settling down to its steady state, which depends upon a constant, here denoted by $k_1$, which determines the rate at which the council houses homeless households. Both the numerical and asymptotic solutions can be used to see how different populations depend upon $k_1$:

The numbers of households satisfactorily housed or on the register (waiting list) for council accommodation are fairly insensitive to changing $k_1$. However, the number of homeless households is approximately inversely proportional to $k_1$.

This indicates that reducing $k_1$ towards corresponding values for applicants in private accommodation and for council-housed applicants wanting transfers, i.e. making the system in some sense fairer, has little advantage in reducing these two waiting lists but is clearly disadvantageous for the homeless.

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It should be noted that there are many assumptions, not all of them totally realistic, made in deriving the present mathematical model. Some of these are mentioned in the following section which derives the model and discusses it in more detail. One of the simplifications is that births and deaths (household creation and loss) can be neglected; this is clearly open to doubt given the importance of the fifteen-year timescale. Also, there is no detailed representation of the actual housing allocation process; rates of rehousing are simply taken to be jointly proportional to the number of “voids” (currently vacant council housing stock) and to “demand” (the current size of the housing registers). Both these issues are tackled in forthcoming work. Variants of the simple law for allocating housing appear in §5.

One more controversial issue not addressed in this article is the reason, or reasons, for homelessness. In particular, the rates at which people become homeless is taken to be governed by a constant of proportionality \( k_5 \). In reality rates of homelessness will be affected by varying circumstances such as the economic situation, including house prices. It has to be emphasized that these rates are here supposed to be independent of the current homelessness level and of perceived waits on the housing lists.

Although we generally find it convenient to work with populations in terms of households (families, couples and single people), to get a correspondence between houses and households, when we carry out the numerical solutions in §3 we measure populations in terms of people. We then assume that on average one household equals three people.

More details of the present work can be found in [2].

2 The Model

The population is not considered in detail but rather three main classes of household are considered: the homeless (registered as such and in temporary accommodation, e.g. temporary occupants of council housing not presently permanently occupied or in hostels), permanent occupants of council housing stock, and the remainder (the “general” population). (The term “household” is used as we do not distinguish between single people, couples, and families. Other demographic features, such as age, are also not taken into account into this preliminary study.) The numbers of households of these categories in some borough (the district or town corresponding to a single local authority) are denoted by \( T \), \( P \) and \( G \) respectively. The homeless are taken to be on the housing register for consideration towards permanent council housing. Numbers of families \( P_R \) from those already in council houses and \( G_R \) from the general population are also on the register (for a change of council house – e.g. because of need for a larger home due to increased family size – and to move out of the private sector, respectively; the former is referred to as a “transfer”). The remaining numbers of council-housed and general households are \( P_N \) and \( G_N \):

\[
T = \text{no. of households in temporary accommodation (the homeless)}
\]

\[
P_R = \text{no. of households in council stock and seeking transfer}
\]

\[
P_N = \text{no. of households in council stock and not seeking transfer}
\]

\[
G_R = \text{no. of households in private sector and seeking council accommodation}
\]

\[
G_N = \text{no. of households in private sector and not seeking council accommodation}
\]

\[
P = \text{no. of households in council stock} = P_R + P_N \tag{1}
\]

\[
G = \text{no. of households in private sector} = G_R + G_N \tag{2}
\]

\[
R = \text{no. of households on register} = T + P_R + G_R \tag{3}
\]
The movement between the different population types is illustrated in Fig. 1.

![Diagram of population movement]

Figure 1: Transfer between different types of household. The k’s denote constants of proportionality governing the rates. The top populations (G) are resident in private accommodation, the bottom (P) in council houses, and the middle (T) in temporary accommodation. The populations on the right (R) are on the housing register and are waiting for a (new) council house, those on the left (N) are not.

It is assumed that the homeless families only come from those in the private sector (councils will not evict tenants except for some misdemeanour, in which case the expelled parties are not considered homeless) and that the rates are proportional to those populations, with the same rate constant $k_5$. The rates of moving from $P$ to $G_N$, from $G_N$ to $G_R$, and from $P_N$ to $P_R$ (perhaps some deterioration of conditions) are similarly taken to be proportional to the relevant populations (with constants of proportionality $k_6$, $k_3$, and $k_7$ respectively). The rates of rehousing are considered to be jointly proportional to the demand ($G_R, T, P_R$) and availability of council housing, $P_O - P$, if $P_O =$ total housing stock ($P =$ number presently occupied), with respective constants $k_4$, $k_1$, $k_8$. (This law allows arbitrarily fast rates of rehousing. The inclusion of a factor inhibiting the speed could be considered.)

Movement from $G_R$ back to $G_N$ and from $T$ directly into $G$ are neglected. (The validity of this assumption is questionable.)

With these laws the variation of the five populations over time can be given by:

rate of change of $G_N = -k_5G_N - k_3G_N + k_6P$; \hspace{1cm} (4)

rate of change of $G_R = -k_5G_R + k_3G_N - k_4(P_O - P)G_R$; \hspace{1cm} (5)

rate of change of $T = k_5G - k_1(P_O - P)T$; \hspace{1cm} (6)

rate of change of $P_N = -k_6P_N - k_7P_N + (P_O - P)(k_4G_R + k_1T + k_8P_R)$; \hspace{1cm} (7)
rate of change of \( P_R = -k_6 P_R + k_7 P_N - k_8 (P_O - P) P_R \). \( (8) \)

In writing down these it has been implicitly assumed that:

(i) birth and death rates can be neglected (these should probably be allowed for if time scales of the model are similar to, or longer than one generation);

(ii) there is no migration between boroughs;

(iii) the numbers of households are large enough for them to be considered as continuum variables; the values of \( T \) are certainly not particularly big in the example discussed below (§3 and §4) in which case some discrete variant of the model (possibly stochastic) ought to be used;

(iv) rates depend only upon present circumstances and any delays (due, say to administrative procedures) are of little importance;

(v) movement from \( T \) to \( G \) is negligible.

We shall make some further remarks concerning (i), (iv) and (v) later.

For this study seasonal variation has been ignored. This would cause cyclic variation of the populations but it might be expected to otherwise give rise to similar overall behaviour.

3 Numerical Solution

Quantitative consideration was based upon information for a “typical” English metropolitan borough, assumed to have approximately steady populations and rates of households coming onto the register and being rehoused.

The population is \( G_O \approx 2 \times 10^5 \) people and the number of council houses is \( 1.8 \times 10^4 \); taking an average of 3 people/household, this gives \( P_O \approx 6 \times 10^4 \) (total capacity for individuals).

Should we put in the calculation of the \( k \)'s, possibly in an appendix?

The following highly approximate values were found from data on annual lets etc. and are used here:

\[
\begin{align*}
  k_1 &= 10^{-2} \text{ people}^{-1} \text{ yr}^{-1} \\
  k_3 &= 1.5 \times 10^{-2} \text{ yr}^{-1} \\
  k_4 &= 10^{-4} \text{ people}^{-1} \text{ yr}^{-1} \\
  k_5 &= 5 \times 10^{-3} \text{ yr}^{-1} \\
  k_6 &= 5 \times 10^{-2} \text{ yr}^{-1} \\
  k_7 &= 3 \times 10^{-2} \text{ yr}^{-1} \\
  k_8 &= 10^{-4} \text{ people}^{-1} \text{ yr}^{-1}
\end{align*}
\]

\( (9) \)

(Note that \( k_4 = k_8 \), indicating that transfers and applicants from the private sector are treated equally.)

These values correspond to steady values of

\[
\begin{align*}
  G_N &\approx 1.4 \times 10^5 \text{ people}, \quad G_R \approx 4.8 \times 10^3 \text{ people}, \quad T \approx 17 \text{ people} \\
  P_N &\approx 5.25 \times 10^4 \text{ people}, \quad P_R \approx 3.5 \times 10^3 \text{ people}, \quad R \approx 8.1 \times 10^3 \text{ people} \\
  (P_O - P)/P_O &\approx 0.07 \,(7\%) 
\end{align*}
\]

\( (10) \)
(These figures differ somewhat from the original data. The difficulty appears to lie with the equation for the steady population level. It has three solutions lying quite close together, one just less than $P_0$ and the others just above. The consequent sensitivity on the data means that the determination of the constants should be carried out with care.)

A numerical simulation of the differential equations (4) - (8), see Fig.2, confirms that equilibria are stable.

![Figure 2: Population changes over short and long times for the correct value of $k_1$ with non-equilibrium populations at $t = 0$.](image)

It should be observed that there is a very rapid initial change but the populations only settle down to their steady values over a time scale of 30 years. This suggests that the dynamics are quite important (policies may change a few times in such a period) and that births, deaths, new families etc. should also be taken into account.

Computations can also be done, starting at the above steady state, but with constant $k_1$ reduced by a factor of ten ($k_1 = 10k_4$) so much less priority is given to the homeless, see Fig.3.

The graphs indicate very little change, except that the number of homeless rapidly rise to ten times the previous value.
Figure 3: Population changes over time for $k_1$ reduced to a tenth of its proper value (still ten times greater than $k_4$ and $k_8$).

The final numerical solution, Fig.4, is for the “fair” case of $k_1 = k_4 = k_8 = 10^{-4}$ people$^{-1}$ yr$^{-1}$.

Figure 4: Population changes over time for the “fair” case with $k_1$ made equal to $k_4$ and $k_8$.

Again the populations change little, except that there was another eventual ten-fold increase in the number temporarily housed, $T$. The amount of “vacant” council property, $P_o - P$, which might be used to accommodate the homeless temporarily, was seen to increase a little, but not sufficiently to cope with the change in homeless families.
### 4 Approximate Solution

The constants used (and the numerical solution) indicate some substantially different sizes of populations and time scales. To try and exploit these variations to achieve approximate solutions the equations can be scaled to identify useful large or small parameters (a twenty-year time scale is used).

The most extreme parameter can be exploited to simplify the problem. This indicates a rapid change (over a time scale of a day or so) to give

\[ P \simeq G_O - G + k_4 G_R / k_1. \]

Using the next largest parameter then gives a change of \( G_R \), over a few months, to

\[ G_R \simeq k_3 G / k_4 (G + P_O - G_O). \]

Finally, \( G \) approaches its equilibrium value,

\[ G \simeq k_6 G_O / (k_3 + k_5 + k_6), \]

over a period of about fifteen years.

The steady \( G \) can be used to determine the other static values:

\[ G_R \simeq \frac{k_3^2 G_O}{k_4 (k_6 P_O - (G_O - P_O) (k_3 + k_5))}, \quad P \simeq \frac{(k_3 + k_5) G_O}{k_3 + k_5 + k_6}. \]

Note that \( k_1 \) does not affect any of these approximate equilibria, although it can influence the transient values of \( G, G_R \) and \( P \). (The same is true, in this case, for \( P_O \).)

Turning our attention to the homeless,

\[ T \simeq \frac{k_5 k_6 G_O}{k_1 [k_6 P_O - (k_3 + k_5) (G_O - P_O)]}, \]

and is seen to be inversely proportional to \( k_1 \) (approximately).

The insensitivity of all the steady values, other than of \( T \), to the crucial homeless-housing-rate constant \( k_1 \) is associated with the relative smallness of numbers of homeless. Should \( k_1 \) be reduced to such an extent that \( T \) becomes a significant part of the total population then it will have a more noticeable effect on the other category sizes.

### 5 Other Models

(i) Councils may wish to put extra effort into housing homeless families if \( T \) gets above a certain size. Ideally this would mean constraining \( T \) but the procedure could be represented by replacing \( k_1 T \) by \( k_1 T + k_1 T^2 \).

(ii) Administrative delays could possibly be important. These might be represented by replacing terms in flow rates such as \( P(t) \) by \( P(t - t_D) \) with \( t_D = \) delay time.

This variation would change the dynamics, at least over the shorter time scales, but leave the steady states unaltered.

(iii) One possible reading of the new legislation was that after providing one year of temporary housing a council will have discharged its duty to a homeless family. If the council gives automatic reassessment as homeless to such families the model considered here can still apply. However, if such regulations were to be applied very stringently, with further temporary accommodation being denied, some of the equations, particularly that concerning the homeless population, would need substantial modification.
(iv) Councils may aim to relet houses a set time $t_D$ after they become vacant (typically a few weeks). They may be let to households in different categories according to some weighting (assigned priority).

If $V(t) =$ rate of houses becoming vacant at time $t$, and hence ready for reletting at time $t + t_D$,

$$V = k_6 P + w_8 P_R V(t - t_D)/S,$$

rate of change of $G_R = -k_3 G_R + k_3 G_N - w_4 G_R V(t - t_D)/S$

rate of change of $T = k_5 G_R - w_1 T V(t - t_D)/S + k_5 G_N$

rate of change of $P_N = -k_6 P_N - k_7 P_N + V(t - t_D)$,

and rate of change of $P_R = -k_6 P_N + k_7 P_N - w_8 P_R V(t - t_D)/S$.

where $S = w_4 G_R + w_1 T + w_8 P_R$. The $w$‘s are (constant) weights and their relative sizes indicate the priority given to housing the three categories on the register.

(v) Discrete, rather than continuous, time may be used. This could again change the dynamics but leave steady states unaltered. Delays might be more easily incorporated using this approach (taking step length equal to the delay).

(vi) Allowance for difference types (sizes, ages . . . ) of households might be made. Although this would make models more accurate there would be considerable complication.

To date, these different models have not been taken further.

6 Conclusions

The main outcome of this brief study is the identification of the key role played by the constant $k_1$ – the constant which fixes the speed at which the homeless are rehoused in permanent council property. Reducing this constant, i.e. making the system “fairer” with less priority to accommodating homeless families, appears to have little effect on the sizes of other categories on the waiting list but there is a marked increase in the number of households in temporary accommodation. It also increases the typical time during which a household remains homeless.

The model, indicated by the size of its longest time-scale, should be modified to allow for births etc.

It could be varied by including flows from $G_R$ to $G_N$ (people removing themselves from the register) or by allowing the rates at which registered and unregistered people become homeless to differ, but these modifications are unlikely to substantially change the main result.

The inclusion of movement from the homeless to the general population (from $T$ to $G$) would have the effect of limiting the numbers in temporary accommodation. If this effect is small a great reduction in $k_1$ (and vast swelling of $T$) would be needed for this flow to become significant.

References
