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A coagulation–disintegration model of Oort–Hulst cluster-formation

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Abstract
We propose and analyse a model for the collision of highly energetic particles in which collisions cause the complete fragmentation of the particles into dust and in which particles nucleate and grow through the accretion of dust. This is inspired by the work of Oort and van Hulst. We analyse a variety of aggregation and disintegration kernels, using asymptotic techniques to illustrate the form of solutions, and find steady states and self-similar behaviour.

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1. Introduction
Motivated by the work of Oort and Hulst [22] on the effects of energetic collisions of asteroids in a dusty environment, we consider a coagulation–fragmentation process in which clusters (or particles) grow in size through stepwise coalescence with the smallest particles in the system (the ‘dust’), and where collisions between two larger clusters cause the complete and instantaneous disintegration of both clusters into dust. This dust can then nucleate fresh clusters or coalesce with other clusters, as noted above. Modern astronomical techniques are able to image the presence of these dust clouds around stars, see for example, Rolf Olsen’s photographs of protoplanetary disc around the star β-Pictoris [20].

The general mathematical modelling and analysis of coagulating systems is not a commonly studied physical process; however, it is included in the recent textbook of Krapivsky et al [12]. In chapter 5 they cover general theories of coagulation. The most common models permit aggregation of arbitrary sized particles, and these often give rise to self-similar behaviour. Leyvraz [15] has reviewed this wide-ranging field. A narrower introduction to the field is given in [27] which also covers the special case of Becker–Döring coagulation which only permits aggregation between clusters and the smallest-sized particle in the system. Whilst Krapivsky et al [12] also cover fragmentation, they focus on binary fragmentation, which is quite different from the disintegration we model herein. It should be noted that analysis of
models which include both coagulation and fragmentation is significantly rarer than those which focus on just one of these processes.

A model Oort–Hulst’s proposed mechanisms of coagulation and fragmentation has been suggested earlier by Safranov [24], and this partial differential equation known as the OHS equation has been studied by other authors, most notably, by Lachowicz et al [13] and Bagland and Laurencot [2], who have shown the presence of self-similar solutions. A discrete version of the OHS equation has been proposed by Bagland [1], who showed strong connections between his discrete model and the continuum counterpart. This model allows a cluster to grow by modelling the outcome of a collision to be a disintegration of one cluster, immediately followed by several coagulation processes. A generalisation of this model is proposed by Dubovski [10], who writes the outcome of a collision between \( C_j \) and \( C_i \) as

\[
C_j + C_i + (\Upsilon - 1)C_i \rightarrow C_{j-\Upsilon} + \Upsilon C_1 + C_i + (\Upsilon - 1)C_i \rightarrow C_{j-\Upsilon} + \Upsilon C_{i+1},
\]

(1.1)

with the system passing from the first state to the last state instantaneously. Here, \( \Upsilon \) depends on the cluster size \( j \), and Dubovski shows that taking \( \Upsilon = j \) and then taking a continuum limit leads to the OHS equation. In our model derived below, we explicitly account for the creation of monomers from a collision, and these are then available to coalesce with clusters of any size, not just those sizes involved in the original collision.

However, we return to the original mechanisms proposed by Oort and Hulst, and formulate an alternative simple discrete model which we then show to exhibit a range of interesting behaviour. Our model is based on the Becker–Döring model of cluster growth, combined with a more general form of fragmentation in which collisions result in the fragmentation of both clusters into their many smallest constituent parts. This has some similarities with the modified Becker–Döring [3] model proposed by Dreyer and Duderstadt [9] and Hermann et al [11] and previously studied in [28], the similarities lie in that all terms on the right-hand side of the governing equations are quadratic in the unknown variables, (2.2)–(2.3). The model proposed here shares some behaviour seen in the standard Becker–Döring system with continual input of monomer [26], in that we observe some self-similar behaviour, as has been reported in many coagulation–fragmentation systems. The self-similar growth of large clusters over longer timescales, through weak interactions with small clusters, is most reminiscent of the work of Lifshitz and Slyozov [16] and Wagner [25]. Connections between the Becker–Döring equations and Lifshitz–Slyozov–Wagner theory have been explored extensively by Penrose [23], Laurencot and Mischler [14], and Niethammer [17]. Alongside this, the stability of similarity solutions of the Lifshitz–Slyozov–Wagner system have been investigated by Carr and Penrose [6], Niethammer and Pego [18], Carr [5], Niethammer and Velazquez [19], and Conlon [7].

In section 2 we propose a discrete model of the processes involved, we show that the system possesses a conserved quantity corresponding to the total mass in the system. We consider the large-time asymptotic form of solutions, showing the existence of a steady state. In section 3 a more detailed analysis is given for the solution of certain special parameter values. Our work is summarized and conclusions drawn in section 4.

2. The general model

2.1. Formulation of model

As well as the condensation process by which particles grow, Oort and Hulst [22] identify three mechanisms by which particles reduce in size. These are evaporation, photodissociation and collisions; however, they reject the first two as ineffective, improbable and subdominant.
Hence we are left with just collisions, which may be elastic (leaving clusters unchanged in size), result in coalescence (leading to cluster growth), or lead to the evaporation of the particles concerned.

We propose a model in which particles, or clusters, of a wide range of sizes exist and interact; interactions between large clusters and the smallest particles cause the clusters to grow in size through coagulation/coalescence. In contrast, collisions of larger clusters cause the destruction of both clusters, and all the matter involved is immediately converted to the smallest size, which we refer to as ‘dust’ or ‘monomers’. Furthermore, to allow new clusters to be formed we allow monomer–monomer interactions to result in the nucleation of new clusters which can then grow by coalescence with more dust.

Denoting a cluster of size \( j \) by \( C_j \), with \( j = 1, 2, 3, \ldots \), we have the three processes

\[
C_j + C_1 \rightarrow C_{j+1}, \quad C_j + C_i \rightarrow (j+i)C_1, \quad C_1 + C_1 \rightarrow C_2,
\]

occurring simultaneously, for \( i, j \geq 2 \) at rates \( a_j, k_{j,i} \) (and \( a_1 \)), respectively. Denoting the concentration \( [C_j] \) by \( c_j(t) \) we obtain the general system of evolution equations

\[
\frac{d c_j}{dt} = c_1(a_{j-1}c_{j-1} - a_j c_j) - c_j \sum_{i=2}^{\infty} k_{j,i} c_i,
\]

\[
\frac{d c_1}{dt} = -2a_1 c_1^2 - \sum_{i=2}^{\infty} a_i c_i c_1 + \frac{1}{2} \sum_{j=2}^{\infty} \sum_{i=2}^{\infty} (j + i) k_{j,i} c_j c_i.
\]

Here we have assumed that the result of a collision of a cluster (\( C_j \) with \( j > 1 \)) and a monomer or dust particle (\( C_1 \)) is always coalescence, resulting in the formation of a larger cluster, \( C_{j+1} \). The model could be generalized to allow two possible outcomes: either coalescence (as above), or the disintegration of the cluster, resulting in \( (j+1)C_1 \), each occurring at its own rate.

In order to consider more concrete systems, we postulate rate kernels of the form \( a_j = a j^\alpha \), \((\alpha \leq 1), \ k_{j,i} = k(j^\beta i^\gamma + j^\beta i^\delta)\), \((\beta, \gamma \leq 1)\), which have been discussed in more detail by da Costa [8]. The system of equations can be written more compactly by introducing moments of the cluster distribution \([c_j]_{j = 2}^{\infty}\) defined by

\[
M_\beta = \sum_{j=2}^{\infty} j^\beta c_j,
\]

note that this definition does not include the dust, \( j = 1 \); then

\[
\frac{d c_j}{dt} = a c_1 [(j-1)^\alpha c_{j-1} - j^\alpha c_j] - k j^\beta c_j M_\beta - k j^\delta c_j M_\gamma,
\]

\[
\frac{d c_1}{dt} = -2a c_1^2 - ac_1 M_\alpha + k (M_{\beta+1} M_\gamma + M_{\delta} M_{\gamma+1}).
\]

The problem (2.5)-(2.6) has a set of identities given by

\[
\sum_{j=1}^{\infty} g_j \frac{d c_j}{dt} = \sum_{j=1}^{\infty} (g_{j+1} - g_j - g_1) a_j c_j c_j + \frac{1}{2} \sum_{j=2}^{\infty} \sum_{i=2}^{\infty} (i + j) g_1 - g_j) k_{i,j} c_i c_j,
\]

the example \( g_j = j \) demonstrates density conservation, \( d q / dt = 0 \), where \( q = c_1 + M_1 \). Since the reactions (2.1) are not reversible, the system does not satisfy detailed balancing. So, whilst there may be a steady-state solution to the kinetic equations (2.5), it will not be a thermodynamic equilibrium solution.
2.2. The steady-state solution

A natural question to ask of the problem (2.5)–(2.6) is whether it possesses steady-state solutions, and if so what form such a solution takes. In this case it is remarkably easy to describe the steady-state solution. Setting the time derivatives to zero, and rearranging (2.5), we obtain

\[
\frac{(j-1)^a c_{j-1}}{\beta^a c_j} = 1 + \frac{k}{ac_1} \left( j^{\gamma-a} M_\beta + j^{\beta-a} M_\gamma \right). \tag{2.8}
\]

Taking logs and summing, we obtain

\[
c_j = c_1 j^{-a} \exp \left( - \sum_{i=2}^{j} \log \left( 1 + \frac{k}{ac_1} \left( i^{\gamma-a} M_\beta + i^{\beta-a} M_\gamma \right) \right) \right). \tag{2.9}
\]

All that remains to be determined is: (i) the ranges of \( \alpha, \beta, \gamma \) over which this formula is physically relevant; (ii) the value of \( c_1 \), which determines the total mass in the system, \( \rho = c_1 + M_1 \); and (iii) the appropriate values for \( M_\beta, M_\gamma \) which make these moments consistent. It is this last part of the analysis which is the most difficult, as illustrated by the calculations given in section 3.3.1, where the calculation for general parameter values can only be solved numerically, and we consider two special asymptotic limits. We consider various ranges below; without loss of generality we assume \( \beta \geq \gamma \). Physically, it is reasonable to assume that both coalescence and fragmentation rates grow with cluster size; however, it is also useful to consider simple models in which they are size-independent; thus in the examples we consider later we restrict our attention to cases where \( \alpha, \beta, \gamma \geq 0 \).

2.2.1. The case \( \alpha > \beta + 1 \). If we consider the limit \( j \to \infty \) in (2.9) in this case we find \( c_j \sim C j^{-a} \) for some \( C \). This is because the terms involving \( j \) in the logarithm are small so \( \log(1+h) \sim h \) for \( h \ll 1 \) implies

\[
c_j \sim c_1 j^{-a} \exp \left( K + \frac{k}{ac_1} \left( \frac{M_\beta j^{\gamma-a+1}}{\alpha - \gamma - 1} + \frac{M_\gamma j^{\beta-a+1}}{\alpha - \beta - 1} \right) \right). \tag{2.10}
\]

for some \( K \). Whilst this expression decays to zero at large \( j \), as the exponential term asymptotes to \( \exp(K) \), the expression as a whole does not decay fast enough for \( M_1 \) to be well-defined (since \( \alpha \leq 1 \)). Hence, in this case, there is no physically-relevant steady-state solution.

2.2.2. The case \( \max(\beta, \gamma + 1) < \alpha < \beta + 1 \). In this case we once again obtain (2.10); however, now the second term in the exponent grows in magnitude with increasing \( j \), but since it has a negative sign, the expression

\[
c_j \sim c_1 j^{-a} \exp \left( K - \frac{kM_\gamma j^{\beta+1-a}}{ac_1(\beta+1-a)} \right), \tag{2.11}
\]

as a whole decreases to zero fast enough at large \( j \) for \( M_1, M_\beta \) and \( M_\gamma \) to be well-defined.

2.2.3. The case \( \alpha < \min(\beta, \gamma + 1) \). In this case the large-\( j \) asymptotic limit of (2.9) is

\[
c_j \sim c_1 j^{-a} \exp \left( K - j \left( (\beta - \alpha)(\log j - 1) + \log \frac{kM_\gamma}{ac_1} \right) \right), \tag{2.12}
\]

which again decays rapidly enough for all required moments to be well-defined.

2.2.4. The case \( \beta < \alpha < \gamma + 1 \). This case reduces to

\[
c_j \sim c_1 j^{-a} \exp \left( K - \frac{kM_\gamma j^{\beta+1-a}}{ac_1(\beta+1-a)} - \frac{kM_\beta j^{\gamma+1-a}}{ac_1(\gamma+1-a)} \right), \tag{2.13}
\]

hence all moments exist.
2.2.5. The case $\gamma + 1 < \alpha < \beta$. In this case the large-$j$ limit of (2.10) is as equation (2.12), which again decays sufficiently rapidly with increasing $j$ that all moments exist.

2.2.6. The case $\alpha - 1 = \beta = \gamma$. In this case (2.9) can be simplified to

$$c_j \sim A j^{-\alpha - 2kM_{\beta}/\alpha c_1},$$  \hspace{1cm} (2.14)

for some $A$, and where $M_{\beta}$ has to be determined in a self-consistent manner. Later we consider a specific example of this form, and find that equation (3.39) has this simple large-$j$ asymptotic form. Due to this algebraic decay, it is possible that some of the moments $M_\beta$, $M_\gamma$, $M_1$ may not be defined.

2.2.7. The case $\alpha = \beta > \gamma$. Here, simplifying equation (2.9) we obtain

$$c_j \sim c_1 j^{-\alpha} \exp \left( C - j \log \left( 1 + \frac{kM_{\gamma}}{ac_1} \right) - \frac{kM_{\beta} j^{1+\gamma-\alpha}}{(1 + \gamma - \alpha)(ac_1 + kM_{\gamma})} \right),$$  \hspace{1cm} (2.15)

for some constant $C$. At large $j$, although at moderately large $j$ we may observe algebraic decay due to the term $j^{-\alpha}$, the dominant term at large $j$ is the $j \log(j)$ which means that any steady-state solution decays exponentially. Thus all required moments, $M_1$, $M_\beta$, $M_\gamma$ are well-defined. In section 3.3 we consider an example of this case, and (3.64) clearly shows both algebraic decay for moderately large $j$ and exponential decay at larger $j$. The same behaviour can be seen in (3.59), once one notes $\Gamma(j+p)/\Gamma(j) \sim j^p$ for large $j$ and $p = \mathcal{O}(1)$.

2.2.8. Summary. If $\alpha < 1$ and $\alpha < \beta + 1$, there may be a steady state. Both (2.11) and (2.13) have the property that in the case $k \ll a$ the distribution has a region of algebraic decay, namely $c_j \sim j^{-\alpha}$ for $j = \mathcal{O}(1)$, followed by a region of more rapid decay, of the form $c_j \sim \exp(-C j^{\beta+1-\alpha})$ for some positive constant $C$, which occurs when $j = \mathcal{O}((a/k)^{1/(\beta+1-\alpha)})$. If $\alpha > \beta + 1$ then there is no steady-state solution.

In section 3 we consider various specific choices of parameter values $\alpha$, $\beta$, $\gamma$, the steady-state solutions are described in more detail.

3. Example asymptotic solutions

In this section we consider in detail the behaviour exhibited by a few special cases of $\alpha$, $\beta$, $\gamma$. We start with the simplest case where all reactions are size-independent, that is, $\alpha = \beta = \gamma = 0$ (section 3.1). Later we consider the case $\alpha = 1$, $\beta = \gamma = 0$ (section 3.2), and finally the case $\alpha = \beta = 1$, $\gamma = 0$ (section 3.3).

In each case we seek steady-state solutions or use asymptotic techniques to determine the large-time kinetics of the system. We then seek to illustrate the behaviour of the system in more detail by constructing the solution of the system starting from a state in which all matter is in the form of dust. If we were to choose $k \gg a$, where disintegration dominates coagulation then the system would stay in a state where almost all the mass was dust ($C_1$). Instead we choose to consider the case where coagulation dominates fragmentation, that is, $a \gg k$, and use asymptotic techniques to construct an approximate solution through a sequence of increasingly long timescales.

3.1. The case $\alpha = \beta = \gamma = 0$

Here we study the case where both aggregation rates and disintegration rates are independent of particle size. In this case the governing equations can be written as

$$\frac{dc_j}{dt} = ac_1(c_{j-1} - c_j) - kc_j M_0,$$  \hspace{1cm} (3.1)
\[
\frac{dc_1}{dt} = -2ac_1^2 - ac_1M_0 + kM_0(q - c_1),
\]
\[
\frac{dM_0}{dt} = ac_1^2 - kM_0^2,
\]
(3.2)

Note that the last two equations decouple from the rest of the system. In section 3.1.1 we describe the steady-state solution and, in the remainder of section 3.1, we analyse the special case where \( k \ll a \), taking \( a = O(1) \), and \( k \ll 1 \), and assume initial conditions of the form
\[
c_1(0) = \varrho, \quad c_j(0) = 0, \quad (j \geq 2),
\]
(3.4)

that is, initially all mass is in the form of dust.

3.1.1. Steady-state solution. The steady solution of (3.1) is \( c_j = c_1 e^{-\lambda(j-1)} \), where \( e^k = 1 + kM_0/ac_1 \). The consistency condition that \( kM_0^2 = ac_1^2 \), implies \( e^k = 1 + \sqrt{k/a} \); and density conservation implies \( c_1 = \varrho k/(\sqrt{a} + \sqrt{k})^2 \). Hence the steady state is given by
\[
c_j^{\text{ss}} = \frac{\varrho k}{(\sqrt{a} + \sqrt{k})^2} \left( 1 + \sqrt{\frac{k}{a}} \right)^{-(j-1)}, \quad M_0^{\text{ss}} = \frac{\varrho \sqrt{ak}}{(\sqrt{a} + \sqrt{k})^2}.
\]
(3.5)

For small \( k/a \) this can be approximated by
\[c_j^{\text{ss}} \sim \varrho k/a, \quad M_0^{\text{ss}} \sim \varrho (k/a)^{3/2}, \quad c_1^{\text{ss}} \sim (\varrho k/a) \exp(-(j-1)\sqrt{k/a}).\]
(3.6)

In this case we observe a distribution spread broadly with the range of cluster sizes being typically \( j \sim \sqrt{a/k} \gg 1 \).

3.1.2. Timescale 1: \( t = O(1) \). We now consider only the case \( a = O(1), k \ll 1 \) and construct a solution through a series of timescales. Ignoring the \( O(k) \) terms, the system (3.2)–(3.3) simplifies to
\[
\frac{dc_1}{dt} = -2ac_1^2 - ac_1M_0, \quad \frac{dM_0}{dt} = ac_1^2.
\]
(3.7)

which can be solved by using \( M_0 \) as a surrogate time variable, so that (3.7) becomes
\[
\frac{dc_1}{dM_0} = -2 - \frac{M_0}{c_1}.
\]
(3.8)

Noting that this equation is homogeneous, we solve by writing \( c_1 = yM_0 \), whence
\[
\log K_1 - \log M_0 = \frac{1}{1+y} + \log(1+y).
\]
(3.9)

The initial data (3.4) implies \( K_1 = \varrho \), hence
\[
\varrho = (M_0 + c_1) \exp \left( \frac{M_0}{M_0 + c_1} \right),
\]
(3.10)

since we have \( c_1 = \varrho \) and \( M_0 = 0 \) at \( t = 0 \) and, as \( t \) increases, \( c_1 \) decreases and \( M_0 \) increases. Towards the end of this timescale, but before we start the next, that is, in the range \( 1 \ll t \ll O(-\log k) \), we have \( M_0 \sim \varrho/e + O(1) \), and so \( c_1 \) decays according to \( c_1 \sim \exp(-aq/\varrho) \). As \( t \to +\infty \) this timescale will cease to be valid due to the exhaustion of monomers, hence in the next timescale, \( c_1 \ll 1 \).

We define the time at which this exhaustion of monomers occurs by \(-t_{1e} \), \( \log k \), this value can be found using the calculation given in section 3.1.6. We find that \( c_1 = O(k) \) when \( \tau = 1 - O(k) \) and
\[
t = \int \frac{dt}{ac_1} = \frac{1}{aq} \int \frac{e^\varrho dt}{1-\tau} = e^\varrho \left[ E_1(1-\tau) - E_1(1) \right] \sim -\frac{e^\varrho}{aq} \log(1-\tau),
\]
(3.11)

where the last asymptotic approximation is for \( \tau = 1^- \). Hence \( t_{1e} = e/aq \).
3.1.3. Timescale 2: \( t = -t_1 \log k + \mathcal{O}(1) \). Balancing terms, we deduce that new scalings must be introduced when \( c_1 = \mathcal{O}(k) \). To determine the evolution on the current timescale, we introduce \( t = t_2 - t_1 \log k \), and \( c_1 = \tilde{c} \), with \( t_2, \tilde{c}, M_0 = \mathcal{O}(1) \). At leading-order, the governing equations are then

\[
\frac{dM_0}{dt_2} = 0, \quad \frac{d\tilde{c}}{dt_2} = M_0\tilde{c} - aM_0\tilde{c}, \tag{3.12}
\]

thus on this timescale, we simply observe a slowing down in the rate of reduction of the monomer concentration \( c_1 \) as it becomes small. The solution of (3.12) is

\[
M_0(t_2) = \frac{\theta}{e}, \quad \tilde{c}(t_2) = -\frac{\theta}{a} + K_2 e^{-a\tilde{c}/e}, \tag{3.13}
\]

for some constant \( K_2 \), obtained by matching back into timescale 1.

3.1.4. Timescale 3: \( t = \mathcal{O}(k^{-1}) \). There being no abrupt singularity at the end of timescale 2, the next timescale has the same scalings for \( M_0 \) and \( c_1 \), but evolves on a longer timescale, thus we introduce \( t_3 = \tau t_2 \), alongside \( c_1 = \tilde{c} \). The leading order terms in the governing equations are all \( \mathcal{O}(k) \) and are

\[
\frac{dM_0}{dt_3} = -M_0^2, \quad 0 = M_0(-a\tilde{c} + \tilde{c}). \tag{3.14}
\]

With initial data of \( \tilde{c} = \theta/a, M_0 = \theta/e \), we have the solution

\[
\tilde{c} = \frac{\theta}{a}, \quad M_0 = \frac{\theta}{\theta t_3 + e}. \tag{3.15}
\]

This solution will cease to be valid when the number of clusters, \( M_0 \), becomes small.

3.1.5. Timescale 4: \( t = \mathcal{O}(k^{-3/2}) \). In this region we retain \( c_1 = \mathcal{O}(k) \) but introduce a scaling for \( M_0 \) smaller than \( \mathcal{O}(1) \), namely \( M_0 = k^{1/2} m \), introduce \( t_4 = k^{3/2} t \), defining \( c_1 = \tilde{c} \), with \( m, t_4, \tilde{c} = \mathcal{O}(1) \). Hence we find

\[
\frac{dm}{dt_4} = ac\tilde{c}^2 - m^2, \quad 0 = m(\theta - ac\tilde{c}), \tag{3.16}
\]

which has the solution

\[
\tilde{c} = \frac{\theta}{a}, \quad m = \frac{\theta}{\sqrt{a}} \coth \left( \frac{\theta(t_4 - K_4)}{\sqrt{a}} \right). \tag{3.17}
\]

The constant \( K_4 \) is obtained by matching back into the previous timescale, which yields \( K_4 = 0 \). At large \( t_4 \), the above solution asymptotes to \( M_0 \sim \theta \sqrt{k/a}, c_1 \sim \theta k/a \). This is consistent with the leading order expression for the steady-state solution (3.5). Hence this is the final timescale of interest in this case.

3.1.6. The distribution in timescales 1 and 2. In timescale 1, the solution for the shape of the distribution at leading order can be found from the leading order governing equation

\[
\frac{1}{ac_1} \frac{dc_1}{d\tau} = c_{j-1} - c_j. \tag{3.18}
\]

Defining \( \tau = \int ac \, dt, \) and \( y_j = e^\tau c_j \) yields

\[
\frac{dy_j}{d\tau} = y_{j-1}, \quad \frac{dc_1}{d\tau} = -2c_1 - M_0, \quad \frac{dM_0}{d\tau} = c_1. \tag{3.19}
\]

The latter two equations separate and are solved by

\[
c_1 = \theta(1 - \tau) e^{-\tau}, \quad M_0 = \theta \tau e^{-\tau}, \tag{3.20}
\]
and timescale 1 ends as $t \to 1$ which corresponds to $c_1 \to 0$ and $1 \ll t_1 \ll -t_1, \log k$. Using the initial data $y_j(0) = 0$ for $j \geq 2$ and boundary data of $y_1(\tau) = \varrho(1 - \tau)$, we obtain the solution

$$y_j(\tau) = \frac{\varrho \tau^{j-1}}{j!} (j - \tau).$$  

(3.21)

Hence at the end of T1, and all through T2, the distribution is described by

$$c_j = \frac{\varrho (j - 1)}{j!}.\quad (3.22)$$

In timescale 2 the leading order equations are $\frac{dc_j}{dt_2} = 0$ for all $j \geq 2$. Hence the shape of the distribution at the end of T2 is the same as that at the end of T1, and this, namely (3.22), forms the initial conditions for the start of T3.

3.1.7. Self-similar distribution at end of timescale 3. Towards the end of timescale 3 (and start of timescale 4) we note that the number of clusters, $M_0$, decays as an inverse power of time, $M_0 \sim 1/t_3 = 1/kt$ (3.15), and the monomer concentration is constant, $c_1 = \varrho k/a$. Since the typical cluster size in the system is given by $M_1/M_0$, which is large and growing, we expect the distribution to be spread over a wide range of cluster sizes. Thus we take $j$ to be continuous and seek a similarity solution, of the form $c_j = \varrho t_3^p f(\eta)$ with $\eta = j/t_3^q$ for some exponents $p, q$. Since $M_1/M_0$ is growing linearly with time, we take $p = 1$ and in order to conserve $M_1 \sim \int \varrho t_3^{q+1} f(\eta) d\eta$ we find $q = -2$. The governing equations for $f(\eta)$ can then be derived from

$$\frac{dc_j}{dt_3} = \varrho (c_{j-1} - c_j) = \frac{c_j}{t_3},$$

(3.23)

and recalling $t_3 = kt$.

Over this period, the distribution is attracted to a self-similar shape, given by

$$c_j(t) = \frac{1}{k t_3^q} f(\eta) = \frac{1}{t_3^q} f(\eta), \quad \eta = \frac{j}{kt} = \frac{j}{t_3},$$

(3.24)

where the shape of the distribution is given by $2f + \eta f' = \varrho f' + f$, hence

$$f(\eta) = \frac{K}{\varrho - \eta}.$$  

(3.25)

Unfortunately, for this solution, the quantities $\int f(\eta) d\eta$ and $\int \eta f(\eta) d\eta$ are divergent, whatever value is chosen for $K$. Thus we have to investigate the form of the solution in more detail.

3.1.8. Intermediate asymptotics. The problem with the above asymptotic solution (3.24)–(3.25) is the area of interest is around the divergence at $\eta = \varrho$, which corresponds to the time-dependent aggregation size of $j = kqt$. It may seem counterintuitive that there would be a cluster size in the problem (2.5) which occurs in significantly larger concentrations than other sizes, and that this particular size of interest increases in size over time. Hence in this section we examine this region of the solution in more detail.

The governing equation is

$$\frac{dc_j}{dt} = ac_1(c_{j-1} - c_j) - kc_jM_0(t),$$

(3.26)

with $M_0(t) \sim 1/kt$ as $t \to \infty$ and $c_1 = \varrho k/a$. Let us write

$$j = kqt + \xi \sqrt{kt}, \quad \xi = \frac{j - kqt}{\sqrt{kt}} = O(1), \quad c_j(t) = \frac{g(\xi)}{(kt)^{3/2}},$$

(3.27)

J A D Wattis
then we find that \( g(\xi) \) satisfies
\[
0 = \varrho g''(\xi) + \xi g'(\xi) + g(\xi).
\]

The general solution of this is
\[
g(\xi) = K_1 e^{-\xi^2/2\varrho} + K_2 e^{-\xi^2/2\varrho}.
\]

We now substitute the intermediate ansatz (3.27) into the outer solution (3.24)–(3.25), obtaining
\[
c_j = \frac{1}{k^2 t^2} \int \left( \frac{k \varrho t + \xi \sqrt{k t}}{k t} \right) = \frac{1}{k^2 t^2} \frac{K}{\varrho - (\varrho + \xi/\sqrt{k t})} = -\frac{K}{\xi (k t)^{3/2}}.
\]

The time-dependence of this is the same as in the last equation in (3.27). The solution (3.25) is valid for \( \eta = O(1) \) where
\[
\eta = \frac{j}{k t} = \varrho + \frac{\xi}{\sqrt{k t}},
\]
hence the two formulae (3.25) and (3.29) should agree when \( \eta \to \varrho^- \) and \(-\xi \gg 1\).

We are interested in the solution which satisfies \( g(\xi) > 0 \) and the matching condition
\[
g(\xi) \sim \frac{K}{-\xi} \quad \text{as} \quad \xi \to -\infty,
\]
which determined \( K_1 \) in terms of \( K \). The constant \( K \) is then determined by requiring the total mass in the system to be \( \varrho \). However, given the complexity of the function (3.29), it is not possible to determine \( K \) explicitly in terms of \( \varrho, a, k \). We illustrate both the ‘outer’ solution given by (3.24)–(3.25) and the ‘inner’ solution (3.27), (3.29) in figure 1.

It is now possible to show that the asymptotic form for the total number of clusters, \( M_0 \), derived earlier, (3.15), is consistent with the solution given by (3.24)–(3.25), (3.27), and (3.29).

We split the sum \( M_0(t) = \sum_{j=2}^{\infty} c_j(t) \) into two parts, above and below a value of \( j \) in the matching region, \( j_c = k \varrho t - B(k t)^{3/4} \), for some \( B = O(1) \). Each sum is approximated by an integral, using either (3.24)–(3.25), or (3.27) and (3.29). Hence
\[
M_0 \sim \frac{K}{k t} \int_{\eta=0}^{\varrho-B(k t)^{3/4}} \frac{d\eta}{\varrho - \eta} - \frac{i K}{2 \varrho} \int_{\xi=-B(k t)^{3/4}}^{0} e^{-\xi^2/2\varrho} \text{erf} \left( \frac{i \xi}{\sqrt{2 \varrho}} \right) d\xi.
\]

Figure 1. Plots of \( c_j(t) \) against \( j \) for fixed time. The dotted line shows the solution (3.24)–(3.25) which has a singularity at \( j = 100 \) and is zero for \( j \geq 100 \). The solid line the intermediate asymptotic approximation (3.27), (3.29) shows how this singularity is removed.
The first integral can be found explicitly, it is \((K/kt) \log(\varrho(kt)^{1/4}/B)\); the second integral can only be found numerically or asymptotically, the latter calculation giving \((K/kt)[O(1) - \log(B(kt)^{1/4})]\). Combining the two results, the terms involving \(\log(kt)\) cancel leaving \(M_0 \sim 1/kt\) for \(kt \gg 1\).

3.1.9. Summary. Whilst we cannot describe the shape of the distribution in timescale 4, the typical sizes of cluster present in each timescale can be determined by calculating \(\langle j \rangle = M_1/M_0\). These are \(O(1)\) in timescales 1 and 2, growing as \(kt\) in timescale 3, and \(O(k^{-1/2})\) in timescale 4. The form of the kinetics in timescale 4 suggest that the steady-state solution is stable, and acts as an attractor for arbitrary initial data. However, the most noteworthy feature of the solution is that in the relatively long timescale 3, the self-similar distribution has a peak, showing considerable numbers of clusters of a large and increasing size.

3.2. The case \(\alpha = 1, \beta = \gamma = 0\)

In this case, the aggregation rates increase with particle size in a linear fashion, whilst the fragmentation rates remain size-independent. Hence the governing equations are

\[
\begin{align*}
\frac{dc_j}{dt} &= ac_1[(j-1)c_{j-1} - wc_j] - 2kc_jM_0, \\
\frac{dc_1}{dt} &= -ac_1(2c_1 + M_1) + 2kc_1M_1, \\
\frac{dM_0}{dt} &= ac_1^2 - 2kc_1^2,
\end{align*}
\]

the last being a consequence of the first, but being useful in our analysis of the system below. Note that we can replace the moment \(M_1\) by \(\varrho - c_1\) since, total mass is conserved in the case where aggregation rate coefficients are given by \(a_j = aj\), as noted by Brilliantov and Krapivsky [4]. This can be seen by calculating the rate of change of the second moment of the system, which grows exponentially, so can have no finite time singularity. Before considering the kinetics of the system we seek steady states.

3.2.1. Steady-state solution. Setting the time-derivatives in (3.34)–(3.36) to zero, and solving for \(c_1, M_0, c_j\) we find \(M_0 = c_1 \sqrt{a/2k}\) and hence

\[
2ac_1^2 + ac_1M_1 = 2kc_1c_1 \sqrt{a/2k}.
\]

This equation has the trivial solution \(c_1 = 0\) and a nontrivial solution which leads to the steady state

\[
c_1 = \varrho \left( \frac{\sqrt{2k} - \sqrt{a}}{\sqrt{2k} + \sqrt{a}} \right), \quad M_0 = c_1 \sqrt{\frac{a}{2k}}; \quad M_1 = \varrho - c_1,
\]

\[
c_j = c_1(j-1)!! \Gamma \left( 2 + \sqrt{\frac{2k}{a}} \right) / \Gamma \left( j + 1 + \sqrt{\frac{2k}{a}} \right).
\]

The large-\(j\) asymptotic form of this solution is \(c_j \sim c_1 \Gamma(2 + \sqrt{2k/a}) j^{-1/2} e^{-\sqrt{2k/a}},\) which has the form noted in (2.14).

Clearly this steady-state solution (3.39) is only valid for \(a < 2k\), whilst for \(a \gg 2k\), we expect the system to approach some other solution. The simplest is a similarity solution in which the large-time asymptotics are governed by \(c_1 \to 0\). The structure of the \((c_1, M_0)\) phase plane is illustrated in figure 2.
3.2.2. Similarity solution. As noted above and illustrated in figure 2, for $2k < a$, we have no steady state; and we now seek the similarity solution pertaining as $c_1, M_0 \to 0$. Near the point $(0, 0)$, the equation (3.36) is quadratically small in the unknowns $c_1, M_0$, whilst we have $c_1$ being determined by the linear equation
\[
\frac{dc_1}{dt} = M_1 (2kM_0 - ac_1),
\] (3.40)
so trajectories are strongly attracted to the centre manifold which is locally approximated by $c_1 = 2kM_0/a$. This behaviour occurs for both $a < 2k$ and $a > 2k$. Motion on the centre manifold is approximated by
\[
\frac{dM_0}{dt} = 2kM_0 \left( \frac{2k}{a} - 1 \right),
\] (3.41)
so, for $2k > a$, we have repulsion away from $(0, 0)$, and towards the equilibrium point described above; and for $2k < a$, we have convergence towards $(0, 0)$, given by
\[
M_0 \sim \frac{a}{2k(a - 2k)}, \quad c_1 \sim \frac{1}{(a - 2k)t}, \quad \text{as } t \to \infty.
\] (3.42)
For the remainder of this subsection we focus exclusively on the case $2k < a$. From (3.34), using the relations (3.42), the distribution at large times is governed by
\[
\left( 1 - \frac{2k}{a} \right) t \frac{dc_j}{dt} = (j - 1)c_{j-1} - jc_j - c_j.
\] (3.43)
Formally, this is solved by
\[
c_j(t) = \frac{(j - 1)!}{(a - 2k)t} \Gamma \left( 2 + \frac{2k}{a} \right) / \Gamma \left( j + 1 + \frac{2k}{a} \right),
\] (3.44)
however, this has a divergent mass $(c_1 + M_1)$, since $c_j$ decays slowly with increasing $j$, namely, $c_j \sim j^{-1-2k/a}$. Hence we introduce a transition region, $j \sim s(t)$, such that for $j \ll s(t)$ the solution above (3.44) holds, and for $j \gg s(t)$, we have $c_j(t) = 0$, and there is a transition for $j \sim s(t)$, where $s(t) \to \infty$ as time $t$ increases. Thus we write the large-time asymptotic solution for this equation in the form
\[
c_j(t) = \frac{(j - 1)! \Psi(j/s(t))}{(a - 2k)t} \Gamma \left( 2 + \frac{2k}{a} \right) / \Gamma \left( j + 1 + \frac{2k}{a} \right),
\] (3.45)
where the function $\Psi(\eta)$ with $\eta = j/s(t)$ describes the shape of the transition from unity to zero, satisfying $\Psi(0) = 1$ and $\Psi \to 0$ as $\eta \to \infty$. Note that for any given cluster size $j$, $\Psi(j/s(t)) \to 1$ as $t \to \infty$. 

Figure 2. Schematic illustration of $(c_1, M_0)$ phase space for (3.35)-(3.36), for the cases of $a < 2k$, where a steady state exists (left) and $2k < a$ (right), which has a similarity solution. In both figures the open circle shows the location of the stable node (on the left, in physically realisable space; on the right, not physically relevant); the solid circle shows the degenerate stationary point. The diagonal line represents the centre manifold. Double arrows show the general direction of trajectories, single arrows the direction of motion on the centre manifold.
3.2.4 Timescale 2: \( t = O(1) \). In this timescale, we treat \( c_j, t, a = O(1) \), and ignore terms involving \( k \ll 1 \). We take initial conditions in which only dust is present, namely \( c_j(0) = 0, \) \( c_j(0) = 0 \) for all \( j \geq 2 \). The governing equation for the dust is then \( dc_j/d\tau = -a c_1 (c_1 + \varrho) \), hence

\[
\frac{dc_1}{d\tau} = \frac{\varrho}{2e^{\varrho \tau} - 1}.
\]  

(3.47)

Defining a new time variable in which \( \hat{c}_i = ac_1 \hat{a}_i \), we note that this implies \( c_1 = \varrho (2e^{-\tau} - 1) \). Hence \( e^\tau - 1 = 1 - e^{-\varrho \tau} \) and \( \tau \to \log 2 \) as \( t \to \infty \). The evolution of the rest of the distribution can be found by introducing a generating function, as in Brilliantov and Krapivsky \[4\], which yields

\[
c_j = \frac{\varrho}{j} (1 - e^{-\tau})^{j-1} (j e^{-\tau} + e^{-\tau} - 1),
\]

(3.48)

thus at the end of this timescale the distribution is given by

\[
c_j = \frac{2^{-j} \varrho (j - 1)}{j}, \quad M_0 = \varrho (1 - \log 2).
\]

(3.49)

There follows a timescale over which \( c_1 \) makes the transition from \( O(1) \) and decaying, to \( O(k/\alpha) \) and steady.

3.2.4 Timescale 2: \( t = -t_c \log k + O(1) \). Timescale 1 ends due to the concentration \( c_1 \) becoming small, hence in this timescale we assume \( c_1 = O(k) \); the quantity \( t_c \) defines the time at which this occurs, and its value will be determined later. Putting \( c_1 = k \tilde{c}_1 \) and \( t_2 = t + t_c \log k \) we have

\[
\frac{dc_1}{dt_2} = 2 \varrho M_0 - a \varrho \tilde{c}_1,
\]

(3.50)

hence \( \tilde{c}_1 = 2 M_0 / a + K e^{-\varrho \tau_c} \), where \( K \) is a constant determined by matching back into timescale 1. This leads to

\[
c_1 = \frac{\varrho}{2 e^{\varrho \tau} - 1} = k K e^{-a - \varrho \tau_c} \log k + \frac{2k M_0}{a},
\]

(3.51)

which can be solved by \( K = \frac{1}{2} \varrho \) and \( t_c = 1/\varrho \). In this timescale the concentrations \( c_j = O(1) \) for \( j \geq 2 \) and are determined by \( dc_j/dt_2 = 0 \), and so retain their values from the end of timescale 1.
3.2.5. Timescale 3: $t = O(k^{-1})$. Since, in the last timescale, the system approached a steady state, we retain the scalings from that regime, namely $c_1 = k\tilde{c}_1$, and $M_0 = O(1)$, and postulate a longer timescale $t_3 = kt = O(1)$ to introduce new terms into the leading order balance. Hence we obtain

$$\frac{dM_0}{dt_3} = -2M_0^2, \quad \tilde{c}_1 = \frac{2M_0}{a},$$  \hspace{1cm} (3.52)

Using the conditions from the end of timescale 1 (3.49), we obtain the solution

$$M_0(t_3) = \frac{\rho(1 - \log 2)}{1 + 2\rho t_3(1 - \log 2)}, \quad c_1 = \frac{2k\rho(1 - \log 2)}{a + 2\rho t_3(1 - \log 2)}.$$  \hspace{1cm} (3.53)

This differs from the third timescale in the previous example, where the dust concentration $c_1$ remained constant whilst $M_0$ decreased, see (3.15).

Here, the distribution is governed by

$$t_3 \frac{dc_j}{dr_3} = (j - 1)c_{j-1} - (j + 1)c_j,$$  \hspace{1cm} (3.54)

which has the solution $c_j(t_3) = \psi(j/t_3)/J_j$, where $\psi(\eta)$ describes the shape of the transition region in terms of $\eta = j/t_3$. Were $\psi$ to be equal one for all $j$, the mass flux $J_j = ac_j c_j$ would be given by $J_j = 2kM_0/t_3$, which does not converge to zero as $j \to \infty$. Hence there would be a flux of mass out of the system.

The transition region is located at $j = s_0 t_3$ for some $s_0$. This corresponds to the asymptotic limit $k \to 0$ of the solution of (3.46). Thus, here $s(t)$, which is the maximum size of clusters present in significant numbers in the system, grows only linearly in time. Since the solution $c_j = 1/jt_3$ has a self-similar form, timescale 3 is the final timescale for the parameter regime $\alpha = 1$, $\beta = \gamma = 0$, $2k < a$, and is approached by $c_j = (1/jt_3)\psi(j/t_3)$.

3.2.6. Summary. For the case $\alpha = 1$, $\beta = \gamma = 0$ we observe the system approaches the steady state (3.39) in the case $a < 2k$, this corresponds to a coexistence of dust and larger clusters, whose typical size is given by

$$\frac{M_1}{M_0} = \sqrt{\frac{2k}{a} \left(\frac{\sqrt{2k} + \sqrt{a}}{\sqrt{2k} - \sqrt{a}}\right)},$$  \hspace{1cm} (3.55)

which diverges as $a \to 2k$. For $2k < a$, the system approaches a similarity solution, in which the typical cluster size grows algebraically in time.

3.3. The case $\alpha = \beta = 1$, $\gamma = 0$

We now consider the case where both aggregation and fragmentation rates depend on particle size in a linear fashion. The system is governed by

$$\frac{dc_j}{dr} = ac_1[(j - 1)c_{j-1} - jc_j] - kc_j(jM_0 + M_1),$$  \hspace{1cm} (3.56)

$$\frac{dc_1}{dr} = -2ac_1^2 - ac_1M_1 + k(M_0M_2 + M_1^2),$$  \hspace{1cm} (3.57)

$$\frac{dM_0}{dr} = ac_1^2 - 2kM_0M_1,$$  \hspace{1cm} (3.58)

where the last equation is a consequence of the first, and we note that $M_1 = \rho - c_1$. 


3.3.1. Steady state. This is determined by setting the left-hand side of (3.56) to zero and solving to find a general solution, for \( c_j \) with \( j \geq 2 \) in terms of \( c_1 \), which in order to simplify the calculations, we write as \( c_1 = \rho \delta \). Once the arbitrary constant has been determined, by requiring the formulate to hold for \( c_1 \) as well as \( j \geq 2 \), we obtain

\[
c_j = \rho \delta \left( \frac{2 - 2\delta}{2 - \delta} \right)^{(j-1)} \frac{(j-1)!\Gamma(2 + Q)}{\Gamma(j + 1 + Q)}, \quad Q = \frac{2k(1 - \delta)^2}{a\delta(2 - \delta)}.
\] (3.59)

There remains one constraint to satisfy, that of density conservation \( (\rho = \sum_{j=1}^\infty j c_j) \), which yields an equation for \( \delta \), namely

\[
\frac{1}{\delta} = \sum_{j=1}^\infty \left( \frac{2 - 2\delta}{2 - \delta} \right)^{(j-1)} \frac{j!\Gamma(2 + Q)}{\Gamma(j + 1 + Q)}.
\] (3.60)

Whilst this equation cannot be solved in general, there are a couple of cases which can be approximated using asymptotic techniques. In the case \( k \gg a \), we obtain a solution in which \( \delta = 1 + o(1) \). Writing \( \delta = 1 - \varepsilon \) with \( 0 < \varepsilon \ll 1 \) we find each term in the sum is smaller than the previous, and so (3.60) reduces to

\[
1 + \varepsilon + O(\varepsilon^2) = 1 + 4\varepsilon/(Q + 2) + O(\varepsilon^2),
\] (3.61)

hence \( Q = 2 \) and \( k/a = \varepsilon^{-2} \), and given \( k, a \) the parameter \( \delta \) is given by

\[
\delta \sim 1 - \sqrt{a/k}, \quad a \ll k.
\] (3.62)

In this case disintegration dominates aggregation, and as expected, the system’s steady state is one in which almost all the mass is in the form of dust.

More interesting is the case where aggregation dominates disintegration, and so we expect most matter to end up in clusters. In the limit \( k \ll a \), we expect to find the system has little dust and clusters have considerable size, hence \( \delta \ll 1 \), and the sum in (3.60) is dominated by terms of size \( 1/\delta \). Putting \( j = 1 + \sqrt{x/\delta} \) into (3.60), we find

\[
\frac{1}{\delta} \sim \Gamma(2 + Q)\delta^{Q-1} \int_0^\infty x^{-Q}e^{-x/2}dx = \frac{(Q + 1)Q_{21}^{1-Q}Q_{Q-1}}{\sin(\pi Q)},
\] (3.63)

since the integral evaluates to \( 2^{1-Q}\Gamma(1 - Q) \) and using (5.5.1) and (5.5.3) of Olver et al [21]. Whilst it appears that in the above equation the left-hand side is larger than the right, the two sides balance when \( Q = 1 - 2\delta \). This relationship implies \( \delta \sim k/a \). Hence for the case \( k \ll a \), we have the solution \( \delta = k/a \). In this case, the distribution \( c_j \) decays algebraically for \( j = O(1) \) and exponentially when \( j = O(\delta^{-1}) \), being given by

\[
c_j \sim \frac{2\rho\delta e^{-3j/2}}{j(j+1)},
\] (3.64)

This large-\( j \) asymptotic form is consistent with the calculation (2.15) carried out in section 2.2.7. Finally, we note that at steady state \( M_0 = O(\delta) \), and \( M_2 = 4\rho \).

We now consider the kinetics by which the system evolves to this steady state starting from the all-dust initial conditions \( c_j(0) = 0, \ c_1(0) = \rho \). Hereon, we focus exclusively on the case \( k \ll 1, a = O(1) \).

3.3.2. Timescale 1: \( t = O(1) \). This timescale is identical to that of section 3.2.3. This is due to both cases having \( \alpha = 1 \) and the differences in \( \beta \) values only affect the fragmentation terms, which are irrelevant for the kinetics in timescale 1.
3.3.3. Timescale 2: $t = t_c + O(1)$. Here all the scalings are as in timescale 1, with the exception of

$$c_1 = \frac{k}{a} \tilde{c}_1, \quad t = t_2 + t_c,$$

where $t_c = O(-\log k) \gg 1$ is some timeshift with respect to $t = 0$. Hence the leading order governing equations are

$$\frac{dc_j}{dt_2} = 0, \quad \frac{dM_0}{dt_2} = 0, \quad \frac{dM_2}{dt_2} = 0, \quad \frac{d\tilde{c}_1}{dt_2} = aq^2 + aqM_0M_2 - aq\tilde{c}_1.$$

Since at the end of timescale 1, we have $M_0 = \varrho (1 - \log 2)$ and $M_2 = 4\varrho$, the solution for $\tilde{c}_1$ is

$$\tilde{c}_1 = \varrho (5 - 4 \log 2 + e^{-aq(t_2+B)}),$$

for some constant $B$, which is obtained by matching the solution back to the end of timescale 1, namely $c_1 \sim \frac{1}{2} \varrho e^{-aq\tilde{t}}$. This yields $(k/a)e^{aq(t_2+B)} = \frac{1}{2}$, hence we choose $B = 0$ and $t_c = (1/a\varrho) \log (a/2k)$.

Thus over this timescale, all quantities remain at their values from the end of timescale 1, and all that happens is that the dust concentration stops decreasing, and saturates at a small value. The next timescale will be considerably longer.

3.3.4. Timescale 3: $t = O(a/k)$. In this timescale, quantities evolve on the slower timescale where $t \sim a/k$, hence we put $t_3 = kt/a = O(1)$. The leading order governing equations for $c_1$ and $M_0$ are now

$$0 = \varrho^2 + M_0M_2 - \varrho \tilde{c}_1, \quad \frac{dM_0}{dt_3} = -2aqM_0,$$

hence we have

$$M_0 = \varrho (1 - \log 2) e^{-2aq\tilde{t}}, \quad \tilde{c}_1 = \varrho + \frac{M_0M_2}{\varrho}.$$

However, since the equation for $\tilde{c}_1$ depends on $M_2$ and the equation for $M_2$ depends on $M_3$, we cannot solve the rest of the system, due to a lack of closure.

In the next subsection we show that, in a similar fashion to the previous case where $\alpha = \beta = \gamma = 0$, there is a similarity solution which illustrates the behaviour of the system at the end of timescale 3 and start of timescale 4. The similarity solution ceases to be valid, since the concentrations do not decay to zero, rather they saturate, as the system approaches a steady-state solution, as described in section 3.3.1.

3.3.5. Similarity solution. We retain the scalings $t_3 = kt/a$, $c_1 = (k\varrho/a)(1 + M_0M_2/\varrho^2)$ and hence aim to solve the evolution equations

$$\frac{1}{aq} \frac{dc_j}{dt_3} = \left(1 + \frac{M_0M_2}{\varrho^2}\right) \left[(j-1)c_{j-1} - jc_j\right] - c_j \left(1 + \frac{jM_0}{\varrho}\right).$$

At the start of timescale 3, corresponding to $t_3 = 0$, the quantity $c_j(t_3) = O(1)$, is given by (3.49); in timescale 1, the fragmentation does not influence the kinetics, so the solution at the end of timescale 1 is the same for this choice of $\alpha = 1 = \beta, \gamma = 0$ as for the case of $\alpha = 1, \beta = \gamma = 0$ studied in section 3.2.3. The distribution (3.49) is not slowly-varying in $j$, and cannot be solved for explicitly in terms of elementary functions. However, through timescale 3, we expect the concentrations of clusters of size $j = O(1)$ to reduce, as the distribution spreads to larger sizes, in a fashion qualitatively similar to that which occurs in section 3.1.
Later in timescale 3, the concentrations reduce in magnitude, the distribution will become slowly-varying in $j$, and then the differences in $j$ can be approximated by derivatives. Using these properties, the resulting equation has a similarity solutions of the form $c_j = e^{-2\theta t} f(\eta)$, where $\eta = je^{-\theta t}$. Noting that $M_0$ is solved by (3.69) fixes $\theta = 2a\varrho$. Substituting the similarity ansatz into (3.70) leads to

$$4f + 2\eta \frac{df}{d\eta} = \left(1 + \frac{\Phi_0\Phi_2}{\varrho^2} \right) \left( \eta \frac{df}{d\eta} + f \right) + f + \frac{\Phi_0\eta f}{\theta},$$

(3.71)

where the zeroth and second moments are given by $M_0 = \Phi_0 e^{-2a\varrho t}$ and $M_2 = \Phi_2 e^{2a\varrho t}$ respectively, and $\Phi = \int \eta^q f(\eta) d\eta$. These calculations confirm that $M_2/M_0/\varrho^2$ is time-independent. Solving (3.71), we obtain

$$f(\eta) = \frac{\varrho \eta^{\frac{\varrho^2 - a\varrho^2}{\Phi_2^2 - \varrho^2}}}{\Gamma\left(\frac{\varrho^2 - a\varrho^2}{\Phi_2^2 - \varrho^2}\right)} \left( \frac{\Phi_0\varrho}{\Phi_0\Phi_2 - \varrho^2} \right)^{\frac{a\varrho^2}{\Phi_2^2 - \varrho^2}} \exp \left( -\frac{\Phi_0\eta}{\Phi_0\Phi_2 - \varrho^2} \right),$$

(3.72)

where, by Cauchy–Schwarz, we note that $\Phi_0\Phi_2 > \varrho^2$. The arbitrary constant from solving the ODE is determined by requiring the first moment to be $\varrho$; this fixes the amplitude, and then the expressions for the zeroth and second moments are automatically satisfied. Thus we have a two-parameter family of similarity solutions, the free parameters being $\Phi_0$ and $\Phi_2$. Some of these solutions are illustrated in figure 3.

We note that at the start of timescale 3, from (3.49) we have $M_0 = \varrho(1 - \log 2)$ and $M_2 = 4\varrho$, whilst in the steady state (3.59) we have $M_0 = \varrho\delta$ and $M_2 = 4\varrho$. Thus over timescales 3 and 4, $M_0$ only decreases slightly, and $M_2$ returns to its original value, and possibly does not change at all at leading-order. Thus we again note that there is a significant period of self-similar behaviour during of the aggregation-disintegration process, but that the distribution eventually approaches the steady form given by (3.59).

4. Conclusions

We have proposed a family of models of the Oort–Hulst processes by which gas and smoke in interstellar space aggregate to form larger clusters. By choosing the exponents $\alpha, \beta, \gamma$ and the rates $a, k$, a wide range of steady states and dynamical behaviour may be observed. For a few choices of these parameters we have determined steady-state solutions for the system.
It should be noted that these are not equilibrium states, since there is a continual growth in particle size through accretion of dust and disintegration of particles due to collisions. We have shown that the models exhibit a range of behaviours, depending on the relative frequencies of aggregation and collision-induced fragmentation. The size-dependencies of these processes also affect the relative amount of mass found in each state and the distribution of particle sizes.

For a few special choices of the aggregation and disintegration parameters \( a, k, \alpha, \beta, \gamma \) we have determined the typical particle size, and the dynamics by which a steady state is approached, or the large-time asymptotic behaviour where no steady state exists. In solving models asymptotically, our focus has been on the cases where coalescence dominates disintegration, since this leads to the more-interesting scenarios in which clusters form, and we wish to understand how the size-distribution of clusters evolves.

Features which are particularly noteworthy are: (i) the peak in the distribution at a time-varying cluster size, as illustrated in figure 1 for the case of size-independent aggregation and disintegration rates; also (ii) the presence of both self-similar and steady-state behaviour in the large-time dynamics of the case of size-dependent aggregation rate \( a_j = aj \) and size-independent disintegration rate \( k_{ij} = k \). We observe a steady state when \( a < 2k \) and self-similar convergence to vanishingly small amounts of dust, and increasing cluster sizes when \( a > 2k \). Finally, the case where both aggregation and disintegration rates are size-dependent, via \( a_j = aj \) and \( k_{ij} = k(i + j) \) exhibits both self-similar behaviour in the convergence towards a steady state, and a self-similar distribution which gives rise to a peak of clusters of one particular cluster size, as illustrated in figure 3.

Although the models considered here are simple, we have shown that they exhibit many features that are desired in a model of particle formation. Future work could include generalisations to include other processes, to make it more accurate, for example, including the possibility of particle–particle collisions resulting in coagulation, making the stepwise growth reversible, and allowing cluster–cluster collisions to result in the destruction of only one of the particles involved.

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