THE STRONG CIRCUIT-DOUBLE-COVER CONJECTURE

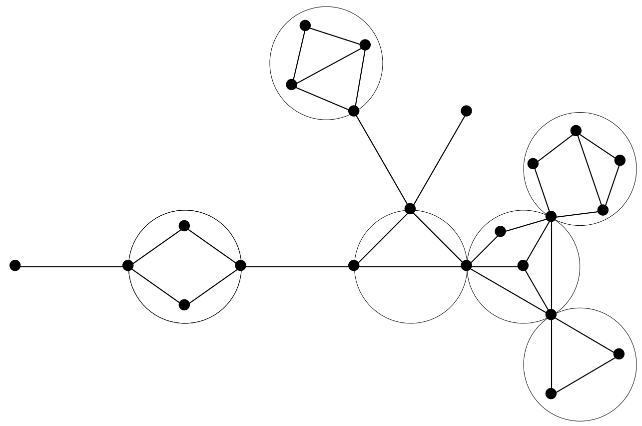
0. Preliminaries. Throughout this survey, graphs will be finite but may contain loops and multiple edges. If G is a graph, the subgraph $\langle X \rangle$ of G **induced** by a subset X of $V(G)$ consists of X together with all edges of G that join pairs of vertices in X, including any loops based at these vertices. When contracting an edge e, we do not delete any loops or parallel edges formed in the process; so $|E(G/e)| = |E(G)| - 1$ always.

If $X \subseteq V(G)$, the set of all edges of G that join a vertex in X to a vertex in $V(G) \backslash X$ is a **coboundary**. The **vertex coboundary** corresponding to a vertex v is the set of all edges of G joining v to other vertices of G (not including any loops based at v).

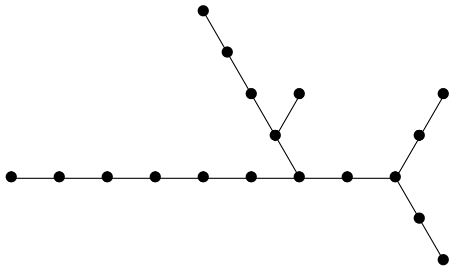
An **edge-cutset** or simply **cutset** in G is a set of edges whose removal increases the number of components in the graph. It is not difficult to see that the minimal non-empty coboundaries are the same as the minimal cutsets.

A connected graph G is ***k*-edge-connected** if it cannot be disconnected by the removal of fewer than k edges, **weakly *k*-edge connected** if the removal of fewer than k edges cannot leave more than one component that contains an edge, and **cyclically *k*-edge-connected** if the removal of fewer than k edges cannot leave more than one component that contains a circuit. Clearly k-edge-connected $\Rightarrow$ weakly k-edge-connected $\Rightarrow$ cyclically k-edge-connected.

A **cut-edge**, **bridge**, **isthmus** or **coloop** in G is an edge whose removal disconnects the component in which it lies; that is, an edge that lies in no circuit or, equivalently, an edge that lies in no 2-connected subgraph of G. A **block** of a simple graph G is a maximal 2-connected subgraph or a subgraph consisting of a cut-edge and its two end vertices. It is easy to see that two distinct blocks can have at most one vertex in common, and that the block structure of a connected graph is *tree-like* in the following sense: if a new graph T is defined to contain a vertex $v(B)$ for each block B of G and a vertex $v(w)$ for each cut-vertex w of G, with $v(B)$ and $v(w)$ adjacent if and only if $w \in B$, then T is a tree. For example, if G is the graph



(where the maximal 2-connected subgraphs are circled) then the block-cutvertex tree of G is



An **end block** is a block that corresponds to an end vertex of the tree.

1. Flows and tensions. Let G be a graph and K an additive abelian group. Let $\vec{G}$ be an **orientation** of G; that is, a digraph obtained from G by assigning a direction (at random) to each edge of G. Given a function $f : E(\vec{G}) \to K$, let $I(f, v)$ be the **net flow into vertex** $\boldsymbol{v}$, defined by

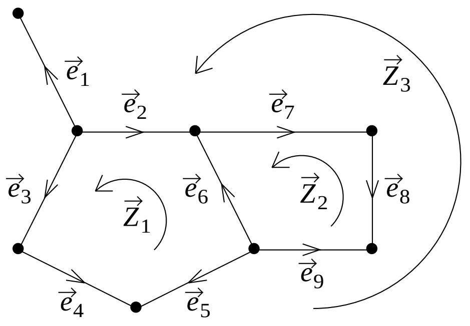
$$I(f, v) := \textstyle\sum_1 f(\vec{e}) - \sum_2 f(\vec{e}),$$

where $\sum_1$ is over all edges $\vec{e}$ of $\vec{G}$ that enter v and $\sum_2$ is over all edges $\vec{e}$ leaving v. A function $f : E(\vec{G}) \to K$ is a ***K*-flow** in $\vec{G}$ if $I(f, v) = 0$ for each vertex v of $\vec{G}$. A function $t : E(\vec{G}) \to K$ is a ***K*-tension** if there is a function $p : V(\vec{G}) \to K$, the **potential function**, such that $t(\vec{ab}) = p(b) - p(a)$ for each edge $\vec{ab}$ of $\vec{G}$.

A function $c : E(\vec{G}) \to K$ is called a ***K*-chain** of $\vec{G}$. If the edges of $\vec{G}$ are ordered as $\vec{e}_1, \ldots, \vec{e}_m$, then the K-chain c can be written as an m-tuple $(c(\vec{e}_1), \ldots, c(\vec{e}_m))$. The set of all K-chains forms an additive abelian group, called the ***K*-chain group** $C(\vec{G}, K)$ of $\vec{G}$. It is an m-dimensional vector space over K in the event that K is a field.

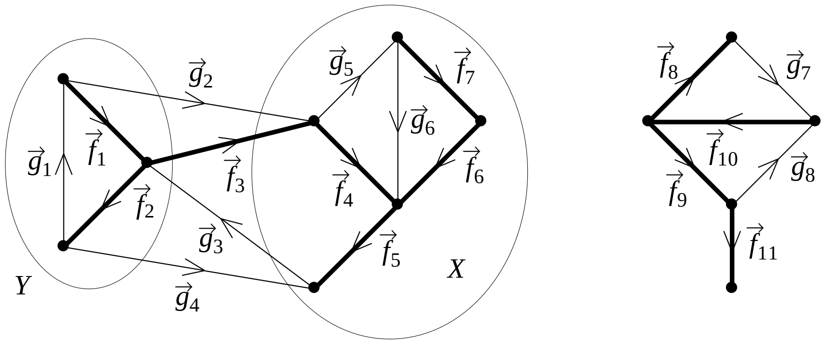
Let us orient each circuit Z of G in one of the two possible directions (chosen at random) to form a **circuit vector** $\vec{Z}$, which can be represented in the form $\vec{Z} = \sum a_i \vec{e}_i$ where each a_i is (the integer) 0, 1 or -1. (Here $-\vec{e}_i$ denotes $\vec{e}_i$ traversed in the opposite direction to the arrow.) The subgroup $Z(\vec{G}, K)$ consisting of all linear combinations $\sum x_i \vec{Z}_i$ ($x_i \in K$, $\vec{Z}_i$ a circuit vector) is called the ***K*-cycle group** of $\vec{G}$, and its elements are ***K*-cycles**; these definitions are clearly independent of which orientation we choose for each circuit. (Of course, $x_i \times 0$ (the integer) $= 0$ (the zero element of K), $x_i \times 1 = x_i$ and $x_i \times -1 = -x_i$.) If K is a field then $Z(\vec{G}, K)$ is a subspace of $C(\vec{G}, K)$, called the **circuit subspace** or **current subspace**, and in this case we can think of the circuit vectors as being K-cycles rather than $\mathbb{Z}$-cycles. For example, the circuit subspace of the graph below is a 2-dimensional subspace, spanned by any two of the vectors $\vec{Z}_1$, $\vec{Z}_2$ and $\vec{Z}_3$. A typical $\mathbb{Z}$-cycle might be $(0, -3, 3, 3, -3, 8, 5, 5, -5) = 3\vec{Z}_1 - 5\vec{Z}_2$.

$$\begin{aligned}
\vec{Z}_1 &= -\vec{e}_2 + \vec{e}_3 + \vec{e}_4 - \vec{e}_5 + \vec{e}_6 \\
&= (0, -1, 1, 1, -1, \;\; 1, \;\; 0, \;\; 0, 0), \\
\vec{Z}_2 &= (0, \;\; 0, 0, 0, \;\; 0, -1, -1, -1, 1), \\
\vec{Z}_3 &= (0, -1, 1, 1, -1, \;\; 0, -1, -1, 1), \\
&= \vec{Z}_1 + \vec{Z}_2.
\end{aligned}$$



If $X \subseteq V(\vec{G})$, let $\delta(X) := \sum_1 \vec{e} - \sum_2 \vec{e}$, where $\sum_1$ is over all edges from $V(\vec{G}) \setminus X$ to X and $\sum_2$ is over all edges from X to $V(\vec{G}) \setminus X$; $\delta(X)$ is a **coboundary vector**, which is a **vertex coboundary vector** if X consists of a single vertex. The subgroup $B(\vec{G}, K)$ consisting of all linear combinations $\sum x_i \vec{B}_i$ ($x_i \in K$, $\vec{B}_i$ a coboundary vector) is called the ***K*-coboundary group** of $\vec{G}$, and its elements are ***K*-coboundaries**. If K is a field then $B(\vec{G}, K)$ is a subspace of $C(\vec{G}, K)$, called the **cutset subspace** or **voltage subspace**.

Suppose that G has n vertices, m edges and c components. Let S be the set of edges of $\vec{G}$ corresponding to a skeleton of G. Let $\vec{f}_1, \ldots, \vec{f}_{n-c}$ be the $n - c$ edges of S and let $\vec{g}_1, \ldots, \vec{g}_{m-n+c}$ be the $m - n + c$ edges of $\bar{S} := E(\vec{G}) \setminus S$. For each j, $S \cup \{\vec{g}_j\}$ contains a unique undirected circuit Z_j; let $\vec{Z}_j$ be the corresponding circuit vector with its orientation determined by the direction of the edge $\vec{g}_j$, so that it has coordinate 1 on $\vec{g}_j$. The set of such circuits Z_j is called the **system of fundamental circuits** associated with S. In a similar way, each edge $\vec{f}_i$



in S determines a unique partition of the vertices of the component of $\vec{G}$ in which it lies into two disjoint non-empty sets X and Y, with $\vec{f}_i$ directed from Y to X; let $\vec{B}_i$ be the corresponding coboundary vector $\delta(X)$, with coordinate 1 on the edge $\vec{f}_i$. For example, in the graph above, $\vec{B}_3 = \delta(X) = \vec{f}_3 + \vec{g}_2 - \vec{g}_3 + \vec{g}_4$. The set of such coboundaries is called the **system of fundamental cutsets** associated with S.

If $c = \sum c_i \vec{e}_i$ is a K-chain and $a = \sum a_i \vec{e}_i$ where each a_i is 0, 1 or -1 (or, if K is a field, a is a K-chain), then their dot product $a \cdot c = c \cdot a$ is defined to be $\sum a_i c_i$, and they are **orthogonal** if $a \cdot c = 0$.

Theorem 1. (a) The K-cycles are precisely the K-chains that are orthogonal to every coboundary vector; if K is a field, they are precisely the K-chains that are orthogonal to every K-coboundary.
(b) The K-coboundaries are precisely the K-chains that are orthogonal to every circuit vector; if K is a field, they are precisely the K-chains that are orthogonal to every K-cycle.
(c) The K-flows are precisely the same as the K-cycles.
(d) The K-tensions are precisely the same as the K-coboundaries.

2. Nowhere-zero flows and tensions. A K-chain c of $\vec{G}$ is **nowhere zero** if $c(\vec{e}) \neq 0$ for each edge $\vec{e}$ of $\vec{G}$. Let $k \geqslant 2$ be an integer and K an arbitrary additive abelian group with k elements.

Theorem 2. The following three statements are equivalent.
(a) G is (properly vertex) k-colourable.
(b) $\vec{G}$ has a nowhere-zero $\mathbb{Z}_k$-tension.
(c) $\vec{G}$ has a nowhere-zero K-tension.

(b) and (c) are clearly independent of the particular orientation $\vec{G}$ of G. A nowhere-zero $\mathbb{Z}_k$-tension will be referred to as a ***k*-tension**.

Theorem 3. Suppose that G is a plane graph. Then the following three statements are equivalent.
(a) G is face-k-colourable (that is, the map formed from G is properly k-colourable).
(b) $\vec{G}$ has a nowhere-zero $\mathbb{Z}_k$-flow.
(c) $\vec{G}$ has a nowhere-zero K-flow.

Theorem 4. The following three statements are equivalent.

(a) $\vec{G}$ has a nowhere-zero $\mathbb{Z}$-flow f with $1 \leqslant |f(\vec{e})| \leqslant k-1$ for each edge $\vec{e}$.

(b) $\vec{G}$ has a nowhere-zero $\mathbb{Z}_k$-flow.

(c) $\vec{G}$ has a nowhere-zero K-flow.

Again, these are independent of the particular orientation $\vec{G}$ of G. A nowhere-zero $\mathbb{Z}_k$-flow will be referred to as a ***k*-flow**. The following conjectures are all due to W. T. Tutte (1954).

Conjecture A. (*The* 5-*flow conjecture.*) Every 2-edge-connected graph has a 5-flow.

Conjecture B. If G is a 2-edge-connected graph that does not have the Petersen graph as a minor (subcontraction), then G has a 4-flow.

Conjecture C. If G is a 2-edge-connected graph with no minimal cutsets of 3 edges (in particular, if G is 4-edge-connected), then G has a 3-flow.

F. Jaeger (1979) proved that every 2-edge-connected graph has an 8-flow and every 2-edge-connected graph without minimal cutsets of 3 edges has a 4-flow. P. D. Seymour (1981) proved that every 2-edge-connected graph has a 6-flow (Theorem 10 below). M. Kochol (1999) proved that every 2-edge-connected graph without a Petersen minor has a 5-flow.

By Theorem 3, the 5-colour theorem shows that the 5-flow conjecture holds for (2-edge-connected) planar graphs (and the 4-colour theorem shows that the 4-flow conjecture also holds for planar graphs). R. Steinberg (1984) proved that the 5-flow conjecture holds for graphs embeddable in the projective plane. M. Möller, H. G. Carstens and G. Brinkmann (1988) improved this by showing that the 5-flow conjecture holds for all graphs with up to 28 vertices and (hence) for graphs of orientable genus $\leqslant 2$ or nonorientable genus $\leqslant 4$; E. Steffan (1996) showed that the 5-flow conjecture holds for graphs with up to 43 vertices and (hence) for graphs of nonorientable genus 5.

EXERCISES

1 Prove Theorem 2. (Hint: Use potentials.)

2 Let $t(G,K)$ be the number of different nowhere-zero K-tensions on $\vec{G}$. (Clearly this is independent of the orientation $\vec{G}$.) Express $t(G,K)$ in terms of a value of the chromatic polynomial of G.

3 Prove directly, without using the previous exercise, that $t(G,K) = t(G-e,K) - t(G/e,K)$ for each edge e of G that is not a cut-edge, and that $t(G-e,K) = t(G/e,K)$ if e is a cut-edge.

4 Let $f(G,K)$ be the number of different nowhere-zero K-flows on $\vec{G}$. Prove that $f(G,K) = f(G/e,K) - f(G-e,K)$ for each non-loop edge e of G.

5 Use the previous exercise to prove that (b) $\Leftrightarrow$ (c) in Theorem 4. (In fact, $f(G,K)$ depends only on G and $|K|$.)

6 Show, further, that there are a **flow polynomial** $F(G,k)$ and a **tension polynomial** $T(G,k)$ such that $f(G,K) = F(G,|K|)$ and $t(G,K) = T(G,|K|)$ for every G and K, and find the degrees of these polynomials.

7 Suppose that $\vec{G}$ has a nowhere-zero $\mathbb{Z}$-chain f such that $1 \leqslant |f(\vec{e})| \leqslant k-1$ for each edge $\vec{e}$ and the net flow into each vertex v is congruent to 0 (mod k). Show how it can be modified (if necessary) to satisfy the condition in Theorem 4(a), and deduce that (a) $\Leftrightarrow$ (b) in Theorem 4. (Hint: If the net flow into v is negative, prove that there is a vertex w such that the net flow into w is positive and there is a path P from v to w such that the flow along each edge of P is in the direction from v towards w.)

8 Prove that G has a 2-tension if and only if G is bipartite and a 2-flow if and only if each component of G is Eulerian.

9 Let G be a plane graph and G^* a geometric dual of G. Let $\vec{G}$ be an orientation of G and $\vec{G}^*$ the orientation of G^* such that if we traverse any edge $\vec{e}$ of $\vec{G}$ in the direction of the arrow, then the corresponding edge $\vec{e}^*$ of $\vec{G}^*$ crosses from left to right. If $c = \sum c_i \vec{e}_i$ is any K-chain of $\vec{G}$, let $c^* = \sum c_i \vec{e}_i^*$ be the corresponding K-chain of $\vec{G}^*$. Prove that c is a K-coboundary in $\vec{G}$ if and only if c^* is a K-cycle in $\vec{G}^*$ and *vice versa*.

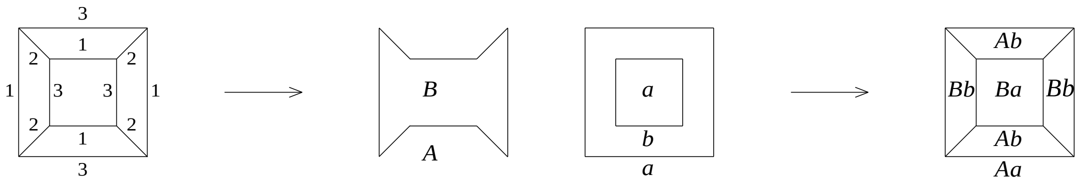
10 Use Theorem 2 and the previous exercise to prove Theorem 3.

11 Use Theorem 3 to prove that a 3-regular plane graph G is face-4-colourable if and only if G is edge-3-colourable.

12 Here is a direct proof of the result in the previous exercise, which does not appear to use groups. Reinterpret it using an appropriate group.

‘If G is face-4-coloured with colours a, b, c, d, colour each edge with colour 1 if it lies between faces of colours a and b or c and d, 2 if it lies between a and c or b and d and 3 if it lies between a and d or b and c. It is easy to see that this edge-3-colouring is proper.

‘Suppose conversely that G is edge-3-coloured. The edges coloured 1 and 2 form disjoint simple closed curves in the plane: colour each resulting region A or B according as it lies inside an even or an odd number of these curves. The edges coloured 1 and 3 similarly give regions which we can colour a and b. Now each face of the map has been assigned one of the four combinations (“colours”) Aa, Ab, Ba and Bb (see the diagram below), and this colouring is clearly proper.’



3. Circuit double covers. A **circuit double cover (CDC)**, or simply **double cover**, of a graph G is a set $\mathscr{C}$ of circuits in G such that every edge of G is contained in exactly two of them. $\mathscr{C}$ is **orientable** if it is possible to orient every circuit in $\mathscr{C}$ in such a way that each edge of G is traced in opposite directions in the two circuits of $\mathscr{C}$ containing it. $\mathscr{C}$ is ***k*-colourable** if it is possible, using only k colours, to assign a colour to every circuit in $\mathscr{C}$ in such a way that, for each edge e of G, the two circuits of $\mathscr{C}$ containing e have different colours.

Example 1. If G is a 2-connected plane graph, then the circuits bounding the faces form an orientable double cover, which is 4-colourable by the four-colour theorem.

Example 2. If G is a graph embedded in a surface S in such a way that each face is homeomorphic to a disc, then the circuits bounding the faces form a double cover, which is orientable if and only if the surface is.

Conjecture D. (*The circuit-double-cover conjecture*, G. Szekeres (1973) and P. D. Seymour (1979).) Every 2-edge-connected graph has a circuit double cover.

Conjecture E. (*The strong circuit-double-cover conjecture*, M. Preissmann (1981) and U. Celmins (1984).) Every 2-edge-connected graph has a 5-colourable orientable circuit double cover.

The following theorem shows that the strong circuit-double-cover conjecture would imply Tutte's 5-flow conjecture.

Theorem 5. If a graph G has a k-colourable orientable double cover $\mathscr{C}$, then it has a k-flow. (The converse is true for $k = 2$, 3 or 4—see Theorems 6, 7 and 8—and is strongly conjectured to hold for all k.)

A **cycle** in a graph G is a set of edges forming a subgraph in which every vertex has even degree; equivalently, it is a union of edge-disjoint circuits. (So a cycle is essentially the same as a $\mathbb{Z}_2$-cycle.) Note that G has a k-colourable double cover if and only if it contains k cycles such that every edge is in exactly two of them.

Theorem 6. For a graph G, the following statements are equivalent.
(a) G has a 2-flow.
(b) $E(G)$ is a cycle.
(c) G has a 2-colourable double cover.
(d) G has a 2-colourable orientable double cover.

Theorem 7. For a graph G, the following statements are equivalent.
(a) G has a 3-flow.
(b) G has a 3-colourable orientable double cover.

Theorem 8. For a graph G, the following statements are equivalent.
(a) G has a 4-flow.
(b) $E(G)$ is the union of two cycles.
(c) G has a 3-colourable double cover.
(d) G has a 4-colourable double cover.
(e) G has a 4-colourable orientable double cover.

As part of the proof of Theorem 8 we need the following Lemma of W. T. Tutte (1950).

Lemma 8.1. If a graph G has a 3-colourable double cover $\mathscr{C}$ then it has a 4-colourable orientable double cover.

B. Alspach, L. Goddyn and C. Q. Zhang (1994) proved that a 2-connected graph with no Petersen minor has a circuit double cover, and C. Q. Zhang (1996a) proved that a 2-connected graph with no K_5 minor also has a circuit double cover.

F. Jaeger (1988) defined a **nearly F_4-graph** to be a graph $H = G - e$ where G has a 4-flow, and he showed that every 2-edge-connected nearly F_4-graph has a 5-flow. He also showed that a 2-edge-connected 3-regular graph with a 2-factor having at most 2 odd components is a nearly F_4-graph, and so has a 5-flow. A. Huck and M. Kochol (1995) proved that every graph of this last type has a 5-colourable double cover, and C.-Q. Zhang (1996b) extended this to nearly F_4-graphs, but as far as I know it is not known in either case whether every such graph has a 5-colourable *orientable* double cover.

EXERCISES

13 Prove Theorem 6. (Note that the circuits in a double cover do not need to be distinct: we can take the same circuit twice.)

14 Prove that a 3-regular graph has a 3-colourable double cover if and only if it is edge-3-colourable.

15 Prove Theorem 7 for 3-regular graphs by proving that, in this case, (a) $\Rightarrow$ bipartite $\Rightarrow$ (b) and then using Theorem 5. (For the first implication, note that, in $\mathbb{Z}_3$, $2 = -1$, and so by reorienting the edges of G if necessary we can make the flow +1 on every edge. For the second implication use Hall's theorem to prove that a 3-regular bipartite graph has a perfect matching and so is edge-3-colourable. The fact that Theorem 7 holds in general if it holds for 3-regular graphs is proved in the next section.)

16 Prove that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) in Theorem 8. (For the first equivalence, consider $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$-flows. For the second, note that if C_1 and C_2 are cycles then so is $C_1 +_2 C_2$, the **Boolean sum** or **symmetric difference** of C_1 and C_2, which is the set of edges that belong to exactly one of C_1 and C_2.)

17 Prove that (c) $\Leftrightarrow$ (d) in Theorem 8. (Hint: If C_1, C_2, C_3, C_4 are cycles, then so are $C_1 +_2 C_2$, $C_1 +_2 C_3$ and $C_1 +_2 C_4$.)

18 Let K be the additive abelian group $\mathbb{Z}_2^k$, each of whose elements can be written as a k-tuple $(a_1, \ldots, a_k)$ of 0's and 1's, and let D_k be the subset of K consisting of all elements made up from two 1's and $k-2$ 0's. Prove that a graph G has a k-colourable double cover if and only if it has a D_k-flow (that is, a K-flow using only values in D_k).

19 Let K be the additive abelian group $\mathbb{Z}^k$, each of whose elements can be written as a k-tuple $(a_1, \ldots, a_k)$ of integers, and let $\vec{D}_k$ be the subset of K consisting of all elements made up from one 1, one -1 and $k-2$ 0's. Prove that a graph G has a k-colourable orientable double cover if and only if it has a $\vec{D}_k$-flow (that is, a K-flow using only values in $\vec{D}_k$).

20 Prove that every Hamiltonian graph has a 3-colourable double cover. (Use the ideas of Lemmas 9.2–9.4 below to show that it suffices to prove this for every 3-regular Hamiltonian graph, and then use the result of Exercise 14. M. Tarsi (1986) and L. Goddyn (1989) both proved that every 2-edge-connected graph with a Hamiltonian path has a 6-colourable double cover; A. Huck and M. Kochol (1995) have proved that every such graph has a 5-colourable double cover.)

4. A reduction procedure. Note that a connected graph is 2-edge-connected if and only if every edge in it belongs to a circuit.

Lemma 9.1. Suppose that G is a 2-edge-connected graph with a cut-vertex v, so that $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \{v\}$ (and $G_1 \setminus \{v\}$, $G_2 \setminus \{v\}$ are non-empty). Then G_1 and G_2 are 2-edge-connected graphs with fewer edges than G, and G has a k-flow/double cover/k-colourable double cover/orientable double cover/k-colourable orientable double cover if and only if G_1 and G_2 both do.

Proof. For k-flows, this follows from the fact that if $I(f, v) = 0$ for all but one of the vertices of a graph (G_1 or G_2) then it must hold at the remaining vertex also. For double covers the result follows from the fact that every circuit in G is contained in G_1 or G_2. $\square$

In view of Lemma 9.1 and similar reasoning for a graph with a loop, it often suffices to consider loopless 2-connected graphs.

Lemma 9.2. Suppose that G is a 2-edge-connected graph with a cutset of two edges, e and e'. Then G/e is a 2-edge-connected graph with fewer edges than G, and G has a k-flow/double cover/k-colourable double cover/orientable double cover/k-colourable orientable double cover if and only if G/e does.

Proof. 'Only if' is easy to see, so we consider 'if'. Given a k-flow in G/e, restore e and give it the same flow as in e' but in the opposite direction. And given a double cover of G/e, form a double cover of G by including e in the two circuits that contain e'. □

Thus it often suffices to consider loopless 2-connected 3-edge-connected graphs.

Lemma 9.3. Suppose that G is a 2-edge-connected graph with a minimal cutset of three edges e_1, e_2, e_3 such that each component C_1, C_2 of $G \setminus \{e_1, e_2, e_3\}$ contains an edge. Let G_1 and G_2 be the results of contracting C_1 and C_2, respectively, into a single vertex. Then G_1 and G_2 are 2-edge-connected graphs with fewer edges than G. If G has a k-flow then so do G_1 and G_2. And G has a double cover/k-colourable double cover/orientable double cover/k-colourable orientable double cover if and only if G_1 and G_2 both do.

Proof. A double cover in G_1, G_2 or G must contain exactly three circuits that use edges from among e_1, e_2, e_3, each of which uses exactly two of these edges (in 'opposite directions'). □

Thus it often suffices to consider graphs that are loopless, 2-connected, 3-edge-connected and weakly 4-edge-connected (which implies cyclically 4-edge-connected).

Lemma 9.4. Let G be a loopless 3-edge-connected graph with a vertex v of degree 4 or more. If e_1, e_2 are two edges incident with v, let G' be obtained from G by deleting e_1 and e_2 and adding a new edge joining the end vertices of e_1 and e_2 distinct from v. Then G' has fewer edges than G, and if G' has a k-flow/double cover/k-colourable double cover/orientable double cover/k-colourable orientable double cover then so does G. Moreover, for *some* choice of e_1, e_2, G' is 2-edge-connected.

Proof. This is all straighforward except for the final sentence, which was proved by H. Fleischner (1976). □

Thus it often suffices to consider 2-edge-connected graphs that are 3-regular. Note that 'weakly 4-edge-connected' and 'cyclically 4-edge-connected' are the same for such graphs, since after the deletion of a minimal cutset of three or fewer edges, any component containing an edge must contain a circuit. A 3-regular cyclically 3-edge-connected graph with more than two vertices is necessarily simple and 3-connected. A 3-regular cyclically 4-edge-connected graph that is not edge-3-colourable is called a **snark**.

Theorem 9. Consider the conjecture that every 2-edge-connected graph has a double cover/5-colourable double cover/orientable double cover/5-colourable orientable double cover. A minimum counterexample to any of these conjectures ('minimum' in the sense of having fewest edges) is a snark. Also, a minimum counterexample to the 5-flow conjecture is a 3-regular 3-edge-connected (hence, simple and 3-connected) graph that is not edge-3-colourable.

Proof. Exercise 14 and Theorem 8 show that all the conjectures hold for 3-regular edge-3-colourable graphs. The result for double covers now follows from Lemmas 9.1–9.4, and the result for flows from Lemmas 9.1, 9.2 and 9.4. □

U. Celmins (1984) proved that a minimum counterexample to the 5-flow conjecture is cyclically 5-edge-connected (hence, it is a snark) and has girth (length of shortest circuit) at least 7, and M. Kochol (2004) proved that it must be cyclically 6-edge-connected. L. Goddyn (1985) proved that a minimum counterexample to the double cover conjecture also has girth at least 7. This was interesting since, up to 1996, no snark with girth at least 7 was known, and it was conjectured that none exist. However, M. Kochol (1996) showed how to construct cyclically 5-edge-connected snarks with arbitrarily large girth.

Proof of Theorem 7. Theorem 5 proves one implication. To prove the other, suppose that there is a graph with a 3-flow but no 3-colourable orientable double cover, and let G be such a connected graph with as few edges as possible and (subject to this condition) as many vertices as possible. Clearly G is 2-edge-connected. By Lemmas 9.1 and 9.2, G is loopless, 2-connected and 3-edge-connected. Choose a 3-flow in G, noting that the flow in every edge is ± 1 (mod 3): thus, after reorienting the edges if necessary, the flow is +1 in every edge. If G is 3-regular then the result follows by Exercise 15; so assume that G has a vertex v with degree 4 or more.

If there are two edges e_1, e_2 at v that carry flow in opposite directions, then the graph G' constructed in Lemma 9.4 has a 3-flow but no 3-colourable orientable double cover (else G would have one), and this contradicts the minimality of G.

So we may suppose that all edges at v carry flow in the same direction (i.e., all into v or all out of v). This requires that the degree of v is a multiple of 3. Split v into nonadjacent vertices v_1 of degree 3 and v_2 of degree at least 3 to form a graph G'' with the same number of edges as G but more vertices. The 3-flow in G induces a 3-flow in G'', which does not have a 3-colourable orientable double cover or else G would have one. Thus G'' contradicts the choice of G. □

5. The 6-flow theorem. If X is a subgraph of G, we form the **2-closure** $\langle X \rangle_2$ of X as follows. Form a maximal sequence of subgraphs $X = X_0 \subset X_1 \subset \ldots \subset X_k$ where, for each $i \geqslant 1$, X_i is obtained from X_{i-1} by taking its union with a circuit that contains exactly one or two edges not in X_{i-1}. The resulting graph X_k (which cannot be extended in this way) is $\langle X \rangle_2$. It is clearly independent of the precise sequence of subgraphs chosen.

Lemma 10.1. Every 3-connected simple graph G contains a cycle Z (in fact, a union of *vertex*-disjoint circuits) such that $\langle Z \rangle_2 = G$.

Lemma 10.2. If $X \subseteq G$ is such that $\langle X \rangle_2 = G$, then there is a $\mathbb{Z}_3$-flow ϕ in G that is nonzero on $E(G) \setminus E(X)$.

Theorem 10. (P. D. Seymour (1981).) Every 2-edge-connected graph has a 6-flow.

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THE STRONG CIRCUIT-DOUBLE-COVER CONJECTURE

0. Preliminaries.

1. Flows and tensions.

Proof of Theorem 1. Note first that if $\vec{Z}$ is a circuit vector and $\vec{B} = \delta(X)$ a coboundary vector, then $\vec{Z} \cdot \vec{B}$ is the number of times that the directed circuit corresponding to $\vec{Z}$ crosses from $V(\vec{G}) \setminus X$ to X minus the number of times it crosses from X to $V(\vec{G}) \setminus X$, which is zero; so $\vec{Z} \cdot \vec{B} = 0$. It follows by linearity that every K-cycle is orthogonal to every coboundary vector, every K-coboundary is orthogonal to every circuit vector and, if K is a field, every K-cycle is orthogonal to every K-coboundary.

Now let $c = \sum a_i \vec{f}_i + \sum b_j \vec{g}_j$ be a K-chain that is orthogonal to every coboundary vector (or, if K is a field, to every K-coboundary). Then so is $c' := c - \sum b_j \vec{Z}_j$, which is of the form $c' = \sum a_i' \vec{f}_i$. Thus $a_i' = c' \cdot \vec{B}_i = 0$ for each i, so $c' = 0$ (the zero chain) and $c = \sum b_j \vec{Z}_j$, a K-cycle. This completes the proof of (a).

The proof of (b) is exactly similar.

The K-flows are by definition the K-chains that are orthogonal to every vertex coboundary vector. But every coboundary vector is a sum of vertex coboundary vectors since, for each $X \subseteq V(\vec{G})$, $\delta(X) = \sum_{v \in X} \delta(\{v\})$. So, by (a), the K-flows are precisely the same as the K-cycles.

A K-tension is clearly orthogonal to every circuit vector (since the sum of the tensions around any circuit must be 0), and so it is a K-coboundary. Finally, let $c = \sum a_i \delta(X_i)$ be a K-coboundary, and consider the potential function p on $V(\vec{G})$ determined by the K-chain $\sum a_i X_i$ on $V(\vec{G})$; that is, $p(v) := \sum \{a_i : v \in X_i\}$ for each v in $V(\vec{G})$. Then c is the K-tension determined by p. This proves (d) and so completes the proof of the theorem. □

2. Nowhere-zero flows and tensions.

3. Circuit double covers.

Proof of Theorem 5. Let $\vec{G}$ be an orientation of G. If $\vec{e} \in E(\vec{G})$ let $C_1(\vec{e})$ and $C_2(\vec{e})$ be the circuits of $\mathscr{C}$ containing $\vec{e}$, where $C_1(\vec{e})$ is the one whose orientation agrees with that of $\vec{e}$. Choose a k-colouring $c : \mathscr{C} \to \mathbb{Z}_k$ and define $f(\vec{e}) := c(C_1(\vec{e})) - c(C_2(\vec{e})) \neq 0$. To see that f is a flow, note that if u, v, w are three successive vertices encountered round a circuit $C \in \mathscr{C}$ in the direction of its orientation, then C contributes $\pm c(C)$ to $f(uv)$ according as uv is oriented into or out of v, and $\mp c(C)$ to $f(vw)$ according as vw is oriented into or out of v, so that the contribution of C towards the net flow into v is zero. This is true for every $C \in \mathscr{C}$ and every vertex v. □

Proof of Lemma 8.1. Let $\vec{G}$ be an orientation of G, orient each circuit in $\mathscr{C}$ arbitrarily and let C_1, C_2, C_3 be the resulting $\mathbb{Z}$-cycles (with all coordinates 0, 1 or −1) representing the three coloured cycles of $\mathscr{C}$. Every edge has coordinate 0, ±1 and ±1 in C_1, C_2 and C_3 (not respectively). Now define

$$\begin{aligned} B_0 &:= C_1 + C_2 + C_3, & A_0 &:= \tfrac{1}{2}B_0, \\ B_1 &:= C_1 - C_2 - C_3, & A_1 &:= \tfrac{1}{2}B_1, \\ B_2 &:= -C_1 + C_2 - C_3, & A_2 &:= \tfrac{1}{2}B_2, \\ B_3 &:= -C_1 - C_2 + C_3, & A_3 &:= \tfrac{1}{2}B_3. \end{aligned}$$

Each edge has coordinate 0, 0, 2 and −2 in $B_0, \ldots, B_3$ and so has coordinate 0, 0, 1 and −1 in $A_0, \ldots, A_3$ (not respectively). Thus $A_0, \ldots, A_3$ are $\mathbb{Z}$-cycles which represent edge-disjoint unions of directed circuits. Moreover, each edge is in exactly two of them and is oppositely oriented in these two. Thus $A_0, \ldots, A_3$ yield a 4-colourable orientable CDC, as required. □

4. A reduction procedure.

5. The 6-flow theorem.

Proof of Lemma 10.1. Choose r maximal such that there exist disjoint circuits $C_1, \ldots, C_r$ such that $\langle C_1 \cup \ldots \cup C_r \rangle_2$ is connected; this makes sense because clearly $\langle C_1 \rangle_2$ is connected. Let $Z := C_1 \cup \ldots \cup C_r$ and $Y := \langle Z \rangle_2$, so that Y is connected. If $V(Y) = V(G)$ then evidently $Y = G$; so suppose $V(Y) \neq V(G)$ and let H be the induced subgraph $\langle V(G) \setminus V(Y) \rangle$. No vertex of H is adjacent to more than one vertex of Y, so every vertex of H has degree at least 2 in H. Define B to be a 2-connected component of H if H has one, and an end block of a component of H otherwise; then B is 2-connected and has at least three vertices. Since G is 3-connected, there are three pairwise nonadjacent edges b_1v_1, b_2v_2, b_3v_3 leaving B ($b_i \in B$, $v_i \notin B$, $i = 1, 2, 3$). At least two v_i's, say v_1 and v_2, are in Y, since $v_i \notin Y$ implies $v_i \in H$ and b_i is the cutvertex of B. Since B is 2-connected, it contains a circuit C_{r+1} through b_1 and b_2, and now $\langle C_1 \cup \ldots \cup C_{r+1} \rangle_2$ is connected. This contradicts the maximality of r. □

Proof of Lemma 10.2. We use induction on $|E(G) \setminus E(X)|$. The result is trivial if $X = G$, so suppose $X \neq G$ and let $X_1 := X \cup C$ where the circuit C has exactly one or two edges not in X; say $E(C) \setminus E(X) = E_0$. Let $\vec{C}$ be an orientation of C, regarded as a $\mathbb{Z}_3$-flow. Since $|E(G) \setminus E(X_1)| < |E(G) \setminus E(X)|$, we may suppose inductively that there is a $\mathbb{Z}_3$-flow ϕ_1 that is nonzero on $E(G) \setminus E(X_1)$. It is now easy to see that one of ϕ_1, $\phi_1 + \vec{C}$ and $\phi_1 - \vec{C}$ is nonzero on E_0 and hence on $E(G) \setminus E(X)$. □

Proof of Theorem 10. By Lemmas 9.1, 9.2 and 9.4, it suffices to prove the result for 3-regular 3-edge-connected graphs, which are necessarily simple and 3-connected (with one exception: the graph with two vertices and three parallel edges). By Lemmas 10.1 and 10.2, such a graph G contains a cycle Z and a $\mathbb{Z}_3$-flow ϕ that is nonzero on $E(G) \setminus Z$. Then 2ϕ is a $\mathbb{Z}_6$-flow that is nonzero on $E(G) \setminus Z$ and is even on every edge of G. So $2\phi + \vec{Z}$ is a 6-flow, for any orientation $\vec{Z}$ of Z. □