Edge-Colourings of Simple Graphs

If a graph $G$ is properly edge-coloured, let $C(v)$ denote the set of colours present on edges at vertex $v$. An \((a, b)\)-chain is a component (a maximal path or a circuit) of the subgraph of edges coloured $a$ or $b$.

**Theorem 1.** (D. König, 1916.) If $G$ is a bipartite multigraph then $\chi'(G) = \Delta(G)$.

**Proof.** Suppose $\chi'(G) \geq \Delta(G) + 1$ and let $t := \Delta(G)$. Let $H$ be a minimal subgraph of $G$ that is not edge-$t$-colourable, and let $e = uw \in E(H)$. Choose an edge-$t$-colouring of $H - e$. Every colour $c$ is used at $u$ or $w$, otherwise we could colour $e$ with $c$. But there are colours $c_u \notin C(u)$ and $c_w \notin C(w)$, since $d_{H-e}(u) < t$ and $d_{H-e}(w) < t$. Let $P$ be the \((c_w, c_u)\)-chain starting from $u$. Since $H$ is bipartite, $P$ could only reach $u$ or $w$ along an edge coloured $c_u$ or $c_w$ respectively, which is impossible. So interchange the colours $c_u$ and $c_w$ along $P$ and colour $e$ with $c_w$. This shows that $\chi'(H) \leq t$, a contradiction. //

Vizing proved a general upper bound for $\chi'(G)$, which we will prove only for simple graphs. From now on, assume $G$ is simple. A \textit{fan sequence} at a vertex $w$ is a sequence of distinct edges of the form $wv_1, \ldots, wv_s$ such that, for each $i \geq 2$, $c(wv_i) \notin C(v_{i-1})$.

**Lemma 2.1.** Suppose $t \geq \Delta(G)$ and $G$ is not edge-$t$-colourable but $G - e$ is, for some edge $e = uw$ of $G$. For some edge-$t$-colouring $c$ of $G - e$, suppose there is an edge $wv_1$ s.t. $c(wv_1) \notin C(u)$, and let $wv_1, \ldots, wv_s$ be a longest fan sequence starting with $wv_1$. Then

(i) $\forall i, C(u) \setminus C(w) \subseteq C(v_i)$;
(ii) if $j < i$ then $c(wv_j) \in C(v_i)$;
(iii) $d(v_s) = \Delta(G) = t$. 

Proof. If \( a \in C(u) \setminus C(w) \) and \( a \notin C(v_i) \) then we can (re)colour
\[
e, \quad wv_1, \ldots, wv_{i-1}, wv_i
\]
with \( c(wv_1), c(wv_2), \ldots, c(wv_i), a \). (*)

This contradiction proves (i). To prove (ii), note that \( C(u) \setminus C(w) \neq \emptyset \), since clearly \( |C(u) \cup C(w)| = t \) and so \( C(u) \subseteq C(w) \) would imply \( d(w) \geq 1 + t > \Delta(G) \). Fix \( a \in C(u) \setminus C(w) \), and suppose \( \exists j < i \) s.t. \( c(wv_j) \notin C(v_i) \). Let \( P \) be the \((a, c(wv_j))\)-chain starting from \( v_i \).

**Case 1:** \( j = 1 \). If \( P \) does not end at \( u \) (even if \( P \) ends along \( v_1w \)) interchange colours along \( P \) and then use (*). If \( P \) does end at \( u \) then interchange colours along the \((a, c(wv_1))\)-chain starting along \( wv_1 \) (which does not end at \( u \)) and colour \( e \) with \( c(wv_1) \). Either way we have a contradiction.

**Case 2:** \( j \geq 2 \). If \( P \) does not end at \( v_{j-1} \), interchange colours along \( P \) and then use (*). Otherwise, interchange colours along \( P \) and then (re)colour
\[
e, \quad wv_1, \ldots, wv_{j-2}, wv_{j-1}
\]
with \( c(wv_1), c(wv_2), \ldots, c(wv_{j-1}), a \).

This contradiction proves (ii). Now (iii) follows because if any colour \( b \) were missing from \( v_s \), then by (i) and (ii), and since \( |C(u) \cup C(w)| = t \), there would be an edge at \( w \) of colour \( b \notin \{c(wv_1), \ldots, c(wv_s)\} \), and so there would be a longer fan sequence. Thus all \( t \) colours are present at \( v_s \). //

**Theorem 2.** Vizing’s Theorem (V. G. Vizing, 1964; R. P. Gupta, 1966.) For every simple graph \( G \) with maximum degree \( \Delta \), \( \chi'(G) \leq \Delta + 1 \).

**Proof.** Suppose \( \chi'(G) \geq \Delta + 2 \). Set \( t = \Delta + 1 \) and let \( H \) be a minimal subgraph of \( G \) that is not edge-\( t \)-colourable. Then \( H - e \)
is edge-$t$-colourable, for each edge $e = uw$ of $H$, and so Lemma 2.1 implies that $t = \Delta(H) \leq \Delta$, a contradiction.  

It follows from Vizing’s theorem that $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$ for every simple graph $G$. $G$ is said to be of class one if $\chi'(G) = \Delta(G)$ and of class two if $\chi'(G) = \Delta(G) + 1$. Theorem 1 shows that every bipartite graph is of class one. So is $K_n$ ($n$ even):

$$\begin{array}{c}
\text{times } n-1 \text{ rotations}
\end{array}$$

But $K_n$ ($n$ odd) is of class two, since each colour can be used on at most $\lceil \frac{n}{2} \rceil = \frac{n}{2}(n-1)$ edges and so $\chi'(K_n) \geq \frac{n(n-1)}{2} = n = \Delta(K_n)+1$. A graph $G$ is called overfull if $|E(G)| > \Delta(G)\lceil \frac{1}{2}|V(G)| \rceil$. If $G$ is overfull then $|V(G)|$ is odd and $G$ is of class two.

**Conjecture.** (A. G. Chetwynd and A. J. W. Hilton, 1986.) If $\Delta(G) > \frac{1}{3}|V(G)|$, then $G$ is of class two if and only if $G$ contains an overfull subgraph $H$ with $\Delta(H) = \Delta(G)$.

The Petersen graph minus 1 vertex is of class two, and has no such overfull subgraph, and has $\Delta = \frac{1}{3}|V|$.

**Lemma 3.1.** Let $G$ and $c$ be as in Lemma 2.1 with $t = \Delta(G)$. Let $wv_1, \ldots, wv_s$ and $wx_1, \ldots, wx_r$ be fan sequences at $w$ with $v_1 \neq x_1$ and $c(wv_1), c(wx_1) \notin C(u)$. Then $v_i \neq x_j$, $\forall i, j$.

**Proof.** Suppose $v_i = x_j$ and (w.l.o.g.) $i \geq 2$ and the vertices $v_1, \ldots, v_i, x_1, \ldots, x_j$ are otherwise distinct. Let $a \in C(u) \setminus C(w)$ as before, and interchange colours along the $(c(wv_i), a)$-chain $P$ starting along $wv_i = wx_j$. If $P$ does not end at $v_i-1$ we can use (*). So suppose $P$ ends at $v_i-1$. If $j = 1$, we can simply colour $e$ with $c(wv_i)$, since $c(wv_i) = c(wx_1) \notin C(u)$ and $P$ does not end at $u$. If $j \geq 2$, we can reverse the roles of the two fans and then use (*) (since $P$ does not end at $x_{j-1}$).  

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Theorem 3. Vizing’s Adjacency Lemma (1965). Let $G$ be a minimal graph that is not edge-$\Delta$-colourable, where $\Delta = \Delta(G)$, and let $u, w$ be adjacent vertices of $G$, where $d(u) = k$. Then $w$ is adjacent to at least $\Delta - k + 1$ vertices of degree $\Delta$ different from $u$.

Proof. Consider an edge-$\Delta$-colouring of $G - uw$. For each of the $\Delta - k + 1$ colours appearing at $w$ but not at $u$, there is a longest fan sequence at $w$ starting with an edge of that colour. By Lemma 3.1 these fan sequences are all disjoint, and by Lemma 2.1 each ends with an edge $wv_s$ with $d(v_s) = \Delta$. //

Corollary 3.1. Each vertex of $G$ has at least two neighbours of degree $\Delta$, and $G$ contains at least three $\Delta$-vertices. //

Theorem 4. (V. G. Vizing, 1968.) If $G$ is a simple planar graph with maximum degree $\Delta \geq 8$, then $\chi'(G) = \Delta$.

Proof. (H. Hind and Y. Zhao, DM 190 (1998) 107–114.) Let $G = (V, E, F)$ be a minimal counterexample. Then $d(u) \geq 2$ and $d(u) + d(w) \geq \Delta + 2 \geq 10$ for each edge $uw$. Let $d(f)$ denote the number of edges bounding face $f$. Euler’s formula $|V| - |E| + |F| \geq 2$ gives

$$\sum_{v \in V}(4 - d(v)) + \sum_{f \in F}(4 - d(f)) \geq 8.$$ 

(1)

Assign ‘charge’ $M(x) := 4 - d(x)$ to each $x \in V \cup F$. Note that only 2-vertices, 3-vertices and 3-faces have positive charge, but the total charge is positive by (1). Now redistribute the charge as follows.

For each edge $uw$,

(R1) if $d(u) = 2$ then $d(w) = \Delta$: send 1 from $u$ to $w$;

(R2) if $d(u) = 3$ then $d(w) \geq \Delta - 1$: send $\frac{1}{3}$ from $u$ to $w$.

For each 3-face $f = uvw$,

(R3) if $d(u) \leq 4$ and $d(v), d(w) \geq 7$, $f$ sends $\frac{1}{2}$ to each of $v, w$: $\frac{1}{3}$ directly and $\frac{1}{6}$ via $u$;
(R4) if \( d(u) = 4 \) and \( d(v) = 6 \), then \( d(w) = \Delta = 8 \), since \( d(u) + d(v) \geq \Delta + 2 \), and if \( d(u), d(w) < \Delta \) then \( d(v) \geq 2 + (\Delta - d(u) + 1) \geq 7 \) by Theorem 3; and \( f \) sends \( \frac{1}{3} \) to \( v \) and \( \frac{2}{3} \) to \( w \): \( \frac{1}{3} \) directly and \( \frac{1}{6} \) via each of \( u, v \);

(R5) if \( d(u), d(v), d(w) \geq 5 \) then \( f \) sends \( \frac{1}{3} \) to each of \( u, v, w \).

Finally,

(R6) each 5-vertex \( v \) sends \( \frac{2}{(3d_{\geq 7}(v))} \) to each vertex in \( N_{\geq 7}(v) \), where \( d_i(v) \) and \( N_i(v) \) denote the number and set of neighbours of \( v \) with degree \( i \), etc. Note that \( d_{\geq 7}(v) \geq 2 \) by Corollary 3.1, and \( v \) loses exactly \( \frac{2}{3} \) by (R6).

**Claim.** If \( d(w) \geq 7 \), then the charge that \( w \) receives by (R6):

(a) from each neighbouring 5-vertex is \( \leq \frac{1}{6} \) if \( d(w) = 7 \), \( \leq \frac{2}{9} \) otherwise;

(b) in total, is \( \leq \frac{1}{2} \) if \( d(w) = 7 \), \( \leq \frac{8}{9} \) otherwise.

**Proof of Claim.** Suppose \( v \in N_5(w) \). If \( d_{\leq 6}(v) \geq 1 \), then \( d_\Delta(v) \geq \Delta - 6 + 1 \geq 3 \) by Theorem 3. Thus \( d_{\geq 7}(v) \geq 3 \), and \( \geq 4 \) if \( d(w) < \Delta \). This proves (a). (b) follows because if \( d_5(w) > 0 \) then \( d_\Delta(w) \geq \Delta - 4 \) by Theorem 3, so that \( d_5(w) \leq 4 \), and \( \leq 3 \) if \( d(w) < \Delta \). This proves the Claim. \(/\)

If \( x \in V \cup F \), let the charge on \( x \) after the redistribution be \( M'(x) = M(x) + R(x) \), where \( R(x) \) is the net change. If \( x \) is a 3-face then \( M(x) = 1, R(x) = -1 \) by (R3)–(R5), and so \( M'(x) = 0 \). If \( x \) is any other face then \( M'(x) = M(x) \leq 0 \). So suppose \( x = v \), a vertex.

\[
\begin{align*}
d(v) = 2: & \quad M(v) = 2, R(v) = -2 \text{ by (R1), } M'(v) = 0. \\
d(v) = 3: & \quad M(v) = 1, R(v) = -1 \text{ by (R2), } M'(v) = 0. \\
d(v) = 4: & \quad M(v) = 0, R(v) = 0, M'(v) = 0. \\
d(v) = 5: & \quad M(v) = -1, R(v) \leq \frac{5}{3} - \frac{2}{3} \text{ by (R5) & (R6), } M'(v) \leq 0. \\
d(v) = 6: & \quad M(v) = -2, R(v) \leq \frac{5}{3} \text{ by (R4) & (R5), } M'(v) \leq 0. \\
d(v) = 7: & \quad M(v) = -3, \text{ so we must prove } R(v) \leq 3. \text{ Now,}
\end{align*}
\]
\[ R(v) \leq \frac{7}{3} + \frac{2d_3(v)}{3} + \frac{d_4(v)}{3} + \frac{d_5(v)}{6}. \]

If \( d_3(v) > 0 \) then \( d_\Delta(v) = 6 \) by Theorem 3, so \( R(v) \leq \frac{7}{3} + \frac{2}{3} = 3 \).
If \( d_3(v) = 0 \) but \( d_4(v) > 0 \) then \( d_\Delta(v) \geq 5 \) by Theorem 3, so \( d_4(v) + d_5(v) \leq 2 \); again \( R(v) \leq \frac{7}{3} + \frac{2}{3} = 3 \).
If \( d_\leq 4(v) = 0 \) then \( R(v) \leq \frac{7}{3} + \frac{1}{2} < 3 \) by (b) of the Claim.

If \( d(v) = k \geq 8 \): \( M(v) = 4 - k \) and
\[
R(v) \leq \frac{k}{3} + \frac{5d_2(v)}{6} + \frac{2d_3(v)}{3} + \frac{d_4(v)}{3} + \frac{2d_5(v)}{9} + \min\{d_4(v), d_6(v)\}\
\]

since (i) if \( w \in N_2(v) \) then, since \( G \) is simple, \( vw \) lies in the boundary of at most one 3-face, and so \( v \) receives 1 from \( w \) by (R1) and \( \leq \frac{1}{2} \) from the two faces either side of edge \( vw \) by (R3), instead of the \( \frac{2}{3} \) allowed for in the first term; and (ii) a 4-vertex is adjacent to at most one 6-vertex, and a 6-vertex to at most one 4-vertex, by Theorem 3. It suffices to prove that \( R(v) \leq \frac{k}{3} + \frac{4}{3} \), since \( 4 - k + \frac{k}{3} + \frac{4}{3} = \frac{2}{3}(8 - k) \leq 0 \).
If \( d_2(v) > 0 \) then \( d_\Delta(v) = \Delta - 1 \) and \( R(v) \leq \frac{k}{3} + \frac{5}{6} < \frac{k}{3} + \frac{4}{3} \).
If \( d_2(v) = 0 \) but \( d_3(v) > 0 \) then \( d_\Delta(v) \geq \Delta - 2 \) and \( R(v) \leq \frac{k}{3} + \frac{4}{3} \).
If \( d_\leq 3(v) = 0 \) but \( d_4(v) > 0 \) then \( d_\Delta(v) \geq \Delta - 3 \) and
\[
R(v) \leq \frac{k}{3} + \frac{3}{3} < \frac{k}{3} + \frac{4}{3}. 
\]
If \( d_\leq 4(v) = 0 \) then \( d_\Delta(v) \geq \Delta - 4 \) and \( R(v) \leq \frac{k}{3} + \frac{8}{9} < \frac{k}{3} + \frac{4}{3} \).
Thus \( M'(x) \leq 0 \ ) \forall x \in V \cup F \), and this contradicts (1). //

Vizing conjectured the result of Theorem 4 if \( \Delta \geq 6 \). (It is false if \( 2 \leq \Delta \leq 5 \): subdivide an edge of \( C_4 \), \( K_4 \), octahedron, icosahedron.) L. Zhang (Graphs Combin. 16 (2000) 467–495) and D. P. Sanders and Y. Zhao (JCT(B) 83 (2001) 201–212) proved it if \( \Delta = 7 \). Both this and the proof of Theorem 4 work also in the projective plane. For the torus and Klein bottle, the result is false if \( \Delta = 6 \) (consider \( K_7 - e \)); it was proved by Hind and Zhao (loc. cit.) for \( \Delta \geq 8 \), and by Sanders and Zhao (JCT(B) 87 (2003) 254–263) for \( \Delta = 7 \).