Eigenvalues of Graphs

0. Preliminaries. Let A be an $n \times n$ matrix. Let

$$\phi(A,x) := \det(xI - A) = (-1)^n \det(A - xI).$$

$\phi(A,x)$ is a monic polynomial of degree n in x, called the *characteristic polynomial* of A. The equation $\phi(A,x) = 0$ is the *characteristic equation* of A and its roots are the *characteristic roots* or *eigenvalues* of A. Their sum is minus the coefficient of x^{n-1} in $\phi(A,x)$, and equals the trace of A, and their product is $(-1)^n$ times the constant term, and equals $\det A$.

Lemma 1.1. λ is an eigenvalue of A iff there exists a nonzero vector $\underset{\sim}{x}$, called an *eigenvector* of A, s.t. $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$. If λ and A are real, then $\underset{\sim}{x}$ can be chosen to be real.

Proof. There exists a nonzero vector $\underset{\sim}{x}$ over the same field as the matrix $A - \lambda I$ s.t. $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$, if and only if $A - \lambda I$ is singular, i.e., $\phi(A,\lambda) = 0$. //

Theorem 1. If A is a real $n \times n$ symmetric matrix then its eigenvalues are all real and it has n orthonormal real eigenvectors.

Proof by induction on n. Let λ_1 be an eigenvalue of A. By Lemma 1.1 there is a corresponding (complex) eigenvector $\underset{\sim}{x}_1 = (x_{11}, \ldots, x_{1n})^\top$. Then

$$A\underset{\sim}{x}_1 = \lambda_1\underset{\sim}{x}_1 \Longrightarrow A\bar{\underset{\sim}{x}}_1 = \bar{A}\bar{\underset{\sim}{x}}_1 = \bar{\lambda}_1\bar{\underset{\sim}{x}}_1 \Longrightarrow \bar{\underset{\sim}{x}}_1^\top A = \bar{\underset{\sim}{x}}_1^\top A^\top = \bar{\lambda}_1\bar{\underset{\sim}{x}}_1^\top.$$

So

$$\bar{\lambda}_1\bar{\underset{\sim}{x}}_1^\top\underset{\sim}{x}_1 = \bar{\underset{\sim}{x}}_1^\top A\underset{\sim}{x}_1 = \lambda_1\bar{\underset{\sim}{x}}_1^\top\underset{\sim}{x}_1.$$

Since $\bar{\underset{\sim}{x}}_1^\top\underset{\sim}{x}_1 = |x_{11}|^2 + \ldots + |x_{1n}|^2 > 0$, it follows that $\bar{\lambda}_1 = \lambda_1$, that is, λ_1 is real. By Lemma 1.1 we can choose $\underset{\sim}{x}_1$ to be real, and of unit norm. If $n = 1$ we are finished. Otherwise, extend $\underset{\sim}{x}_1$ to an orthonormal basis $B = [\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n]$ for $\mathbb{R}^n$. Let P be the $n \times n$ real matrix whose columns are the vectors $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n$. Then $P^\top P = I$, so $P^\top = P^{-1}$. If $\underset{\sim}{y}$ is a vector expressed in terms of the basis B, then $P\underset{\sim}{y}$ is the same vector in standard coordinates. Thus the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ that is represented by A with respect to the standard basis is represented by $C = P^{-1}AP$ with respect to the basis B.

Since $C^\top = (P^{-1}AP)^\top = P^\top A^\top (P^{-1})^\top = P^\top AP = C$, C is symmetric. Since $A\underset{\sim}{x}_1 = \lambda_1\underset{\sim}{x}_1$, $C(1,0,\ldots,0)^\top = \lambda_1(1,0,\ldots,0)^\top$, and so C has the form

$$C = \left[\begin{array}{c|ccc} \lambda_1 & 0 & \ldots & 0 \\ \hline 0 & & & \\ \vdots & & C_0 & \\ 0 & & & \end{array}\right].$$

By the induction hypothesis, the $(n-1) \times (n-1)$ real symmetric matrix C_0 has $n-1$ real eigenvalues $\lambda_2, \ldots, \lambda_n$ and $n-1$ orthonormal real eigenvectors $\underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_n$. Let $\underset{\sim}{z}_i := P\left[\frac{0}{\underset{\sim}{y}_i}\right]$ $(i = 2, \ldots, n)$. Then

$$A\underset{\sim}{z}_i = PP^{-1}AP\left[\frac{0}{\underset{\sim}{y}_i}\right] = PC\left[\frac{0}{\underset{\sim}{y}_i}\right] = P\left[\frac{0}{C_0\underset{\sim}{y}_i}\right] = \lambda_i P\left[\frac{0}{\underset{\sim}{y}_i}\right] = \lambda_i\underset{\sim}{z}_i$$

$(i = 2, \ldots, n)$. Also, if $2 \leqslant i \leqslant j \leqslant n$, then

$$\underset{\sim}{z}_i \cdot \underset{\sim}{z}_j = \underset{\sim}{z}_i^\top\underset{\sim}{z}_j = [\,0\,|\,\underset{\sim}{y}_i^\top]P^\top P\left[\frac{0}{\underset{\sim}{y}_j}\right] = \underset{\sim}{y}_i \cdot \underset{\sim}{y}_j = \delta_{ij}$$

(Kronecker δ) since $P^\top P = I$ and $\underset{\sim}{y}_2, \ldots, \underset{\sim}{y}_n$ are orthogonal. Thus $\underset{\sim}{x}_1, \underset{\sim}{z}_2, \ldots, \underset{\sim}{z}_n$ are n orthonormal real eigenvectors of A with real eigenvalues $\lambda_1, \ldots, \lambda_n$. //

Lemma 2.1. Let $\lambda_1 \geqslant \ldots \geqslant \lambda_k$ be *some* eigenvalues of a real symmetric matrix A with corresponding orthonormal eigenvectors $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_k$, and let $\underset{\sim}{y} = \sum_i a_i\underset{\sim}{x}_i$ be any nonzero vector in $\langle\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_k\rangle$. Then the *Rayleigh quotient* $\dfrac{\underset{\sim}{y}^\top A\underset{\sim}{y}}{\underset{\sim}{y}^\top\underset{\sim}{y}}$ satisfies

$$\lambda_1 \geqslant \frac{\underset{\sim}{y}^\top A\underset{\sim}{y}}{\underset{\sim}{y}^\top\underset{\sim}{y}} \geqslant \lambda_k,$$

with equality if and only if $\underset{\sim}{y}$ is an eigenvector with eigenvalue λ_1 or λ_k as appropriate.

Proof. $\underset{\sim}{y}^\top\underset{\sim}{y} = (\sum_i a_i\underset{\sim}{x}_i) \cdot (\sum_i a_i\underset{\sim}{x}_i) = \sum_i a_i^2$ since $\underset{\sim}{x}_i \cdot \underset{\sim}{x}_j = \delta_{ij}$, and $\underset{\sim}{y}^\top A\underset{\sim}{y} = (\sum_i a_i\underset{\sim}{x}_i) \cdot (\sum_i a_i\lambda_i\underset{\sim}{x}_i) = \sum_i \lambda_i a_i^2$. Thus

$$\lambda_1 = \frac{\sum_i \lambda_1 a_i^2}{\sum_i a_i^2} \geqslant \frac{\sum_i \lambda_i a_i^2}{\sum_i a_i^2} = \frac{\underset{\sim}{y}^\top A\underset{\sim}{y}}{\underset{\sim}{y}^\top\underset{\sim}{y}} \geqslant \frac{\sum_i \lambda_k a_i^2}{\sum_i a_i^2} = \lambda_k,$$

with equality if and only if $a_i = 0$ whenever $\lambda_i \neq \lambda_1$ or λ_k as appropriate. //

Lemma 2.2. Let A be a real $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$. Let X be an $n \times m$ matrix with orthonormal columns (hence, $m \leqslant n$), and let the $m \times m$ matrix $B = X^\top AX$ have eigenvalues $\mu_1 \geqslant \ldots \geqslant \mu_m$. Then $\lambda_i \geqslant \mu_i \geqslant \lambda_{i+n-m}$ $(i = 1, \ldots, m)$.

Proof. Let W be the i-dimensional subspace of $\mathbb{R}^m$ spanned by the eigenvectors of B with eigenvalues $\mu_1, \ldots, \mu_i$. Let $W' := \{X\underset{\sim}{w} : \underset{\sim}{w} \in W\} \subseteq \mathbb{R}^n$; then $\dim W' = i$ since the columns of X are linearly independent. Let U be the $(n - i + 1)$-dimensional subspace of $\mathbb{R}^n$ spanned by the eigenvectors with eigenvalues $\lambda_i, \ldots, \lambda_n$. Since $\dim U + \dim W' = n + 1 > n$, there is a nonzero vector $\underset{\sim}{u} = X\underset{\sim}{w}$ in $U \cap W'$. By Lemma 2.1,

$$\lambda_i \geqslant \frac{\underset{\sim}{u}^\top A\,\underset{\sim}{u}}{\underset{\sim}{u}^\top \underset{\sim}{u}} = \frac{\underset{\sim}{w}^\top X^\top AX\underset{\sim}{w}}{\underset{\sim}{w}^\top X^\top X\underset{\sim}{w}} = \frac{\underset{\sim}{w}^\top B\,\underset{\sim}{w}}{\underset{\sim}{w}^\top \underset{\sim}{w}} \geqslant \mu_i.$$

The other inequality follows by symmetry. //

Theorem 2. *The Interlacing Theorem.* Let A be a real $n \times n$ symmetric matrix, write

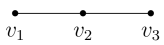
$$A = \left[\begin{array}{c|c} a_{11} & \underset{\sim}{u}^\top \\ \hline \underset{\sim}{u} & A_0 \end{array}\right],$$

and let A and A_0 have eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$ and $\mu_1 \geqslant \ldots \geqslant \mu_{n-1}$ respectively. Then $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \ldots \geqslant \lambda_{n-1} \geqslant \mu_{n-1} \geqslant \lambda_n$.

Proof. Let X consist of the last $n - 1$ columns of the $n \times n$ identity matrix, so that $X^\top AX = A_0$, and apply Lemma 2.2 with $m = n - 1$. //

1. Two graph spectra. Let G be a multigraph (no loops) with vertices $v_1, \ldots, v_n$. Let $D(G)$ denote the diagonal matrix of vertex-degrees $d(v_1), \ldots, d(v_n)$. The *adjacency matrix* $A(G)$ of G is the $n \times n$ matrix in which $[A(G)]_{ij}$ is the number of edges between v_i and v_j. The *Laplacian matrix* $A^{(\mathrm{L})}(G)$ of G is $D(G) - A(G)$. Both are real symmetric matrices and so, by Theorem 1, all their eigenvalues are *real*. The eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$ of $A(G)$ form the (*ordinary*) *spectrum* $\mathrm{Sp}(G) = (\lambda_1, \ldots, \lambda_n)$ of G. The eigenvalues $\lambda_1^{(\mathrm{L})} \geqslant \ldots \geqslant \lambda_n^{(\mathrm{L})}$ of $A^{(\mathrm{L})}(G)$ form the *Laplacian spectrum* $\mathrm{Sp}^{(\mathrm{L})}(G) = (\lambda_1^{(\mathrm{L})}, \ldots, \lambda_n^{(\mathrm{L})})$ of G.

Example. $G = K_{1,2}$



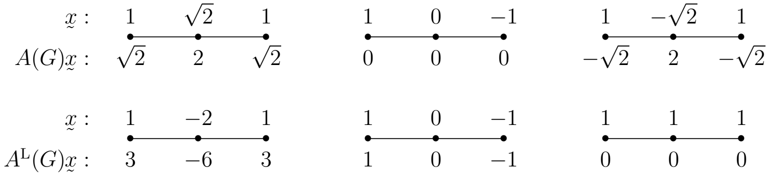
$$A(G) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad A^{(\mathrm{L})}(G) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

$$\phi(A(G), x) = x^3 - 2x, \qquad \phi(A^{(\mathrm{L})}(G), x) = x^3 - 4x^2 + 3x,$$

$$\mathrm{Sp}(G) = (\sqrt{2}, 0, -\sqrt{2}), \qquad \mathrm{Sp}^{(\mathrm{L})}(G) = (3, 1, 0).$$

Eigenvectors (not normalized):

$$\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$



Note that if G is *disconnected*, $G = G_1 \cup G_2$, then

$$\phi(A(G), x) = \phi(A(G_1), x)\phi(A(G_2), x),$$
$$\mathrm{Sp}(G) = \mathrm{Sp}(G_1) \cup \mathrm{Sp}(G_2),$$

and similarly for the Laplacian.

Theorem 3. If G is simple then

$$\phi(A(G),x) = x^n + a_2x^{n-2} + a_3x^{n-3} + \ldots + a_{n-1}x + a_n$$

where a_2 is minus the number of edges of G, a_3 is minus twice the number of triangles, a_4 is the number of pairs of nonadjacent edges minus twice the number of quadrilaterals, and in general

$$a_i = \textstyle\sum_H (-1)^{c(H)} 2^{\gamma(H)},$$

where the sum is over all $H \subseteq G$ s.t. $|V(H)| = i$ and H is a vertex-disjoint union of edges and circuits, where $c(H)$ and $\gamma(H)$ denote the numbers of components and circuits. //

Corollary 3.1. If $a_i = 0$ for all odd $i \leqslant k$, then G has no odd circuits with length $\leqslant k$. //

Theorem 4. (A. K. Kel'mans and V. M. Chelnikov, 1978.) If G is simple then

$$\phi(A^{(\mathrm{L})}(G),x) = x^n - a_1^{(\mathrm{L})}x^{n-1} + a_2^{(\mathrm{L})}x^{n-2} - \ldots + (-1)^{n-1}a_{n-1}^{(\mathrm{L})}x$$

where, for each i, $a_i^{(\mathrm{L})}$ is the sum, over all i-edged spanning subforests F of G, of the product of the numbers of vertices in the $n-i$ components of F. //

2. The ordinary spectrum and the spectral radius.

Assume throughout that G has n vertices $v_1, \ldots, v_n$, eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$, and adjacency matrix $A = A(G)$.

Theorem 5. $\sum_{i=1}^n \lambda_i = 0$; and if $k \geqslant 0$ then $\sum_{i=1}^n \lambda_i^k$ is the number of closed walks of length k in G (counting rotations as distinct).

Proof. Since $A\underset{\sim}{x} = \lambda\underset{\sim}{x} \Longrightarrow A^k\underset{\sim}{x} = \lambda^k\underset{\sim}{x}$, $\sum_{i=1}^n \lambda_i^k$ is the sum of the eigenvalues of A^k, which is the trace of A^k. But it is a well-known (and easy) result that $[A^k]_{ij}$ is the number of walks of length k from v_i to v_j. The result follows. //

Theorem 6. If G has diameter d then G has more than d distinct eigenvalues.

Proof. Suppose G has exactly k distinct eigenvalues $\mu_1, \ldots, \mu_k$, and let X be an $n \times n$ matrix whose columns are orthonormal eigenvectors of $A = A(G)$. Then $AX = XD$, where D is a diagonal matrix with only $\mu_1, \ldots, \mu_k$ on the diagonal. Thus $A = XDX^{-1}$ and

$$\begin{aligned}(A-\mu_1 I)\ldots(A-\mu_k I) &= X(D-\mu_1 I)X^{-1}X\ldots X^{-1}X(D-\mu_k I)X^{-1}\\ &= X(D-\mu_1 I)\ldots(D-\mu_k I)X^{-1}\\ &= Z,\end{aligned}$$

the zero matrix. But if $k \leqslant d$ then G contains vertices v_i and v_j whose distance apart is k. Thus $[A^k]_{ij} \neq 0$, but $[A^l]_{ij} = 0$ if $l < k$, and so $(A-\mu_1 I)\ldots(A-\mu_k I) \neq Z$, a contradiction. //

Theorem 7. (O. Perron, 1907, and G. Frobenius, 1908 & 1912.) Suppose G is *connected*. Then: (a) λ_1 is a simple eigenvalue and $|\lambda_i| \leqslant \lambda_1$, $\forall\, i$. (b) λ_1 has an eigenvector with all its coordinates strictly positive. (c) If $\lambda_n = -\lambda_1$ then $\lambda_i = -\lambda_{n+1-i}$ for each i. //

λ_1 is called the *spectral radius* of G (even if G is not connected), written $\lambda_1(G)$ or $r(G)$.

Theorem 8. If G is connected and simple, then the following statements are equivalent; if G is not connected, delete (c); if G is not simple, perhaps delete (d) (?).

(a) G is bipartite.
(b) $\lambda_i = -\lambda_{n+1-i}\ \forall\, i$.
(c) $\lambda_n = -\lambda_1$.
(d) $a_i = 0$ whenever i is odd. (Recall $\phi(A(G),x) = \sum_{i=0}^n a_i x^{n-i}$.)
(e) $\sum_{j=1}^n \lambda_j^i = 0$ whenever i is odd.
(f) $\sum_{j=1}^n \lambda_j^k = 0$, where $k = n$ (n odd) or $n-1$ (n even).

Proof. If G is bipartite with bipartition $V(G) = S \cup T$, and if $\underset{\sim}{x}_i$ is an eigenvector with eigenvalue λ_i, then $\underset{\sim}{x}_i'$, obtained by negating $\underset{\sim}{x}_i$ on the vertices of T, is an eigenvector with eigenvalue $-\lambda_i$; thus (a) $\Longrightarrow$ (b) $\Longrightarrow$ ((c) and (d)). But (a) $\Longleftrightarrow$ (d) if G is simple, by Theorem 3 and Corollary 3.1. By Theorem 7, (c) $\Longrightarrow$ (b) if G is connected. Finally, (e) $\Longleftrightarrow$ (a) $\Longleftrightarrow$ (f) by Theorem 5. //

Theorem 9. (A. J. Hoffman, 1970, and H. S. Wilf, 1967.) The chromatic number $\chi(G)$ of G satisfies

$$\left\lceil 1 - \tfrac{\lambda_1}{\lambda_n} \right\rceil \leqslant \chi(G) \leqslant \lfloor 1 + \lambda_1 \rfloor .$$

Proof of the upper bound. We may suppose that G is connected. Let $\underset{\sim}{x}_1 = (x_{11}, \ldots, x_{1n})^\top$ be a positive eigenvector corresponding to λ_1, and suppose w.l.o.g. $x_{11} \geqslant \ldots \geqslant x_{1n}$. For each i,

$$\lambda_1 x_{1i} = [A\underset{\sim}{x}_1]_i = \textstyle\sum_{v_j \in N(v_i)} x_{1j} \geqslant \displaystyle\sum_{\substack{v_j \in N(v_i)\\ j<i}} x_{1i} = s_i x_{1i},$$

where $s_i = \#\{v_j \in N(v_i) : j < i\}$; so $s_i \leqslant \lambda_1$. Suppose $\lfloor 1+\lambda_1 \rfloor > \lambda_1$ colours are available, and colour $v_1, \ldots, v_n$ in sequence. When v_i is reached, at most $s_i < \lfloor 1 + \lambda_1 \rfloor$ of its neighbours have been coloured, and so there must be a colour available for v_i. [We have proved that every graph G is $\lfloor \lambda_1(G) \rfloor$-*degenerate*.] //

Lemma 10.1. If $\underset{\sim}{y}$ is any positive vector then

$$\frac{\underset{\sim}{y}^\top A \underset{\sim}{y}}{\underset{\sim}{y}^\top \underset{\sim}{y}} \leqslant \lambda_1 \leqslant \max_{1 \leqslant i \leqslant n} \frac{[A\underset{\sim}{y}]_i}{[\underset{\sim}{y}]_i},$$

with equality in the lower bound, and (if G is connected) in the upper bound, iff $A\underset{\sim}{y} = \lambda_1 \underset{\sim}{y}$.

Proof. The lower bound follows from Lemma 2.1 (even if $\underset{\sim}{y}$ is not positive). For the upper bound, let $\underset{\sim}{x}_1 = (x_{11}, \ldots, x_{1n})^\top$ be

a nonnegative eigenvector with eigenvalue λ_1, so that

$$\lambda_1 = \frac{\lambda_1 \underset{\sim}{x}_1^\top \underset{\sim}{y}}{\underset{\sim}{x}_1^\top \underset{\sim}{y}} = \frac{\underset{\sim}{x}_1^\top A \underset{\sim}{y}}{\underset{\sim}{x}_1^\top \underset{\sim}{y}} = \frac{\sum_i x_{1i}[A\underset{\sim}{y}]_i}{\sum_i x_{1i}[\underset{\sim}{y}]_i} \leqslant \max_{1 \leqslant i \leqslant n} \frac{[A\underset{\sim}{y}]_i}{[\underset{\sim}{y}]_i},$$

with equality iff $[A\underset{\sim}{y}]_i = \lambda_1 [\underset{\sim}{y}]_i$ whenever $x_{1i} \neq 0$. If G is connected then $x_{1i} \neq 0 \; \forall\, i$, and so equality holds iff $A\underset{\sim}{y} = \lambda_1 \underset{\sim}{y}$. //

Theorem 10. If G has n vertices, m edges and maximum degree Δ, then

(a) $\frac{2m}{n} \leqslant \lambda_1 \leqslant \Delta$, and

(b) $$\frac{1}{m} \sum_{v_i v_j \in E(G)} \sqrt{d(v_i) d(v_j)} \leqslant \lambda_1 \leqslant \max_{1 \leqslant i \leqslant n} \frac{1}{\sqrt{d(v_i)}} \sum_{v_j \in N(v_i)} \sqrt{d(v_j)} \leqslant \max_{1 \leqslant i \leqslant n} \sqrt{\sum_{v_j \in N(v_i)} d(v_j)},$$

with equality in the lower bound of (a), and (if G is connected) in the upper bound of (a), iff G is regular.

Proof. If $\underset{\sim}{y} = (y_1, \ldots, y_n)^\top$, then

$$\underset{\sim}{y}^\top A \underset{\sim}{y} = \sum_{i,j} y_i [A]_{ij} y_j = 2 \sum_{v_i v_j \in E(G)} y_i y_j .$$

So for (a), apply Lemma 10.1 with $\underset{\sim}{y} = (1, 1, \ldots, 1)^\top$, when

$$\frac{\underset{\sim}{y}^\top A \underset{\sim}{y}}{\underset{\sim}{y}^\top \underset{\sim}{y}} = \frac{2m}{n}, \text{ the average degree of } G, \text{ and } \frac{[A\underset{\sim}{y}]_i}{[\underset{\sim}{y}]_i} = \frac{d(v_i)}{1}.$$

Clearly $A\underset{\sim}{y} = \lambda_1 \underset{\sim}{y}$ iff G is λ_1-regular.

For (b), put $\underset{\sim}{y} = \left(\sqrt{d(v_1)}, \ldots, \sqrt{d(v_n)} \right)^\top$ in Lemma 10.1, when

$$\frac{\underset{\sim}{y}^\top A \underset{\sim}{y}}{\underset{\sim}{y}^\top \underset{\sim}{y}} = \frac{2}{2m} \sum_{v_i v_j \in E(G)} \sqrt{d(v_i) d(v_j)}$$

and

$$\frac{[A\underset{\sim}{y}]_i}{[\underset{\sim}{y}]_i} = \frac{1}{\sqrt{d(v_i)}} \sum_{v_j \in N(v_i)} \sqrt{d(v_j)} \leqslant \sqrt{\sum_{v_j \in N(v_i)} d(v_j)}$$

since, by Cauchy–Schwarz,

$$\Big(\sum_{v_j \in N(v_i)} \Big(1 \cdot \sqrt{d(v_j)}\Big)\Big)^2 \leqslant \Big(\sum_{v_j \in N(v_i)} 1^2 \Big)\Big(\sum_{v_j \in N(v_i)} \sqrt{d(v_j)}^{\,2} \Big)$$

$$= d(v_i) \sum_{v_j \in N(v_i)} d(v_j). \quad //$$

Theorem 11. Suppose G has eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$.

(a) If $v \in V(G)$ and $G - v$ has eigenvalues $\mu_1 \geqslant \ldots \geqslant \mu_{n-1}$, then $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \ldots \geqslant \lambda_{n-1} \geqslant \mu_{n-1} \geqslant \lambda_n$.

(b) If $H \subsetneqq G$ then $\lambda_1(H) \leqslant \lambda_1(G)$, with strict inequality if G is connected.

Proof. (a) follows immediately from Theorem 2. To prove (b), let B be the adjacency matrix of $(V(G), E(H))$, and let $\underset{\sim}{y}$ be a nonnegative eigenvector of B with eigenvalue $\lambda_1(H)$. Then

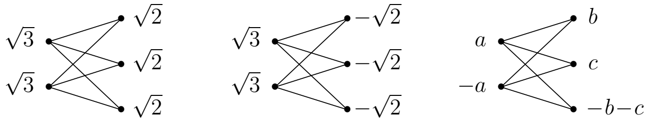
$$\lambda_1(H) = \frac{\underset{\sim}{y}^\top B\, \underset{\sim}{y}}{\underset{\sim}{y}^\top \underset{\sim}{y}} \leqslant \frac{\underset{\sim}{y}^\top A\, \underset{\sim}{y}}{\underset{\sim}{y}^\top \underset{\sim}{y}}\,.$$

But $\dfrac{\underset{\sim}{y}^\top A\, \underset{\sim}{y}}{\underset{\sim}{y}^\top \underset{\sim}{y}} \leqslant \lambda_1(G)$ by Lemma 2.1, with strict inequality unless $A\underset{\sim}{y} = \lambda_1(G)\underset{\sim}{y}$, in which case, if G is connected, $\underset{\sim}{y}$ is a positive eigenvector and so $\underset{\sim}{y}^\top B \underset{\sim}{y} < \underset{\sim}{y}^\top A \underset{\sim}{y}$. //

Theorem 12. (J. H. Smith, 1970.) A graph has exactly one positive eigenvalue if and only if it is a complete multipartite graph plus (possibly) some isolated vertices. //

Further examples.

Eigenvalues: 3   −1 (multiplicity 3)



Eigenvalues: $\sqrt{6}$ $-\sqrt{6}$ 0 (multiplicity 3)

Eigenvalues: τ $-\tau$ $-\tau^{-1}$ τ^{-1}

$(\tau^2 = \tau + 1, \quad 1 - \tau^{-1} = \tau^{-2}(\tau^2 - \tau) = \tau^{-2})$

3. Line graphs and leapfrogs.

Lemma 13.1. Let A be a real $n \times n$ symmetric matrix.

(a) If A is *nonnegative semidefinite*, i.e., $\underset{\sim}{x}^\top A \underset{\sim}{x} \geqslant 0 \;\forall\, \underset{\sim}{x} \in \mathbb{R}^n$, then A has no negative eigenvalues.

(b) If $R := \max\limits_{1 \leqslant i \leqslant n} \sum_{j=1}^n |[A]_{ij}|$, then no eigenvalue of A exceeds R.

Proof. (a) If $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ then $\lambda \underset{\sim}{x}^\top \underset{\sim}{x} = \underset{\sim}{x}^\top A \underset{\sim}{x} \geqslant 0$. If $\underset{\sim}{x} \neq \underset{\sim}{0}$ then $\underset{\sim}{x}^\top \underset{\sim}{x} > 0$ and so $\lambda \geqslant 0$.
(b) Let $\underset{\sim}{x} = (x_1, \ldots, x_n)^\top$ be an eigenvector of A with eigenvalue $\lambda \geqslant 0$, and choose i s.t. $|x_i|$ is maximal. Then

$$\lambda |x_i| = |[A\underset{\sim}{x}]_i| = \Big|\sum_{j=1}^n [A]_{ij} x_j\Big| \leqslant \sum_{j=1}^n |[A]_{ij}||x_j| \leqslant R|x_i|,$$

and so $\lambda \leqslant R$. //

Lemma 13.2. Let A be an $m \times n$ matrix and B an $n \times m$ matrix. Then AB and BA have the same nonzero eigenvalues; in fact,

$$x^n \phi(AB, x) = x^m \phi(BA, x).$$

Proof. Note that

$$\begin{array}{cc} & \\ m & \\ n & \\ & \end{array}\!\!\begin{array}{c} \left[\begin{array}{c|c} AB & -A \\ \hline Z & Z \end{array}\right] \\ \begin{array}{cc} m & n \end{array} \end{array} \begin{array}{c} \left[\begin{array}{c|c} I & Z \\ \hline B & I \end{array}\right] \\ \begin{array}{cc} m & n \end{array} \end{array} = \begin{array}{c} \left[\begin{array}{c|c} Z & -A \\ \hline Z & Z \end{array}\right] \\ \begin{array}{cc} m & n \end{array} \end{array} = \begin{array}{c} \left[\begin{array}{c|c} I & Z \\ \hline B & I \end{array}\right] \\ \begin{array}{cc} m & n \end{array} \end{array} \begin{array}{c} \left[\begin{array}{c|c} Z & -A \\ \hline Z & BA \end{array}\right] \\ \begin{array}{cc} m & n \end{array} \end{array},$$

so that, if $P = \left[\begin{array}{c|c} I & Z \\ \hline B & I \end{array}\right]$, $Q = \left[\begin{array}{c|c} AB & -A \\ \hline Z & Z \end{array}\right]$ and $R = \left[\begin{array}{c|c} Z & -A \\ \hline Z & BA \end{array}\right]$, then $QP = PR$ and $Q = PRP^{-1}$. Thus

$$\begin{aligned} \det(Q - xI) &= \det[P(R - xI)P^{-1}] \\ &= \det P \det(R - xI) \det P^{-1} \\ &= \det(R - xI), \end{aligned}$$

so that $\phi(Q, x) = \phi(R, x)$. But $\phi(Q, x) = x^n \phi(AB, x)$ and $\phi(R, x) = x^m \phi(BA, x)$. The result follows. //

The *line graph* $L(G)$ of G is the (multi)graph whose vertex-adjacency matrix is the edge-adjacency matrix of G.

G $\qquad$ $L(G)$

Theorem 13. All eigenvalues of the matrices $D(G) \pm A(G)$ lie between 0 and $M := \max\limits_{v_iv_j \in E(G)} [d(v_i) + d(v_j)]$. All eigenvalues of $A(L(G))$ lie between -2 and $M - 2$.

Proof. If $\underset{\sim}{x} = (x_1, \ldots, x_n)^\top$, then

$$\begin{aligned} \underset{\sim}{x}^\top A^{(\mathrm{L})}(G)\underset{\sim}{x} = \underset{\sim}{x}^\top (D(G) - A(G))\underset{\sim}{x} &= \sum_{i=1}^{n} d(v_i)x_i^2 - 2 \sum_{v_iv_j \in E(G)} x_ix_j \\ &= \sum_{v_iv_j \in E(G)} (x_i - x_j)^2 \geqslant 0. \end{aligned}$$

Similarly, $\underset{\sim}{x}^\top (D(G) + A(G))\underset{\sim}{x} = \sum_{v_iv_j \in E(G)} (x_i + x_j)^2 \geqslant 0$. Thus the matrices $D(G) \pm A(G)$ are nonnegative semidefinite, and so have no negative eigenvalues by Lemma 13.1.

Let C denote the $n \times m$ *incidence matrix* of G, in which $[C]_{ij}$ is 1 if v_i is incident with e_j, and 0 otherwise. Then

$$CC^\top = D(G) + A(G) \quad \text{and} \quad C^\top C = 2I + A(L(G)).$$

Now, an edge v_iv_j is adjacent to $d(v_i) + d(v_j) - 2$ other edges, and so, by Lemma 13.1(b), no eigenvalue of $A(L(G))$ can exceed $\max\limits_{1 \leqslant r \leqslant m} \sum_{s=1}^{m} |[A(L(G))]_{rs}| = M - 2$. Since $CC^\top$ and $C^\top C$ have the same nonzero eigenvalues by Lemma 13.2, it follows that all eigenvalues of both lie between 0 and M, and all eigenvalues of $A(L(G))$ lie between -2 and $M - 2$.

Finally, change *one* 1 to -1 in each column of C to form $\vec{C}$. (This is equivalent to orienting each edge of C.) Then

$$\vec{C}\vec{C}^\top = D(G) - A(G) = A^{(\mathrm{L})}(G) \quad \text{and} \quad \vec{C}^\top\vec{C} = 2I + A',$$

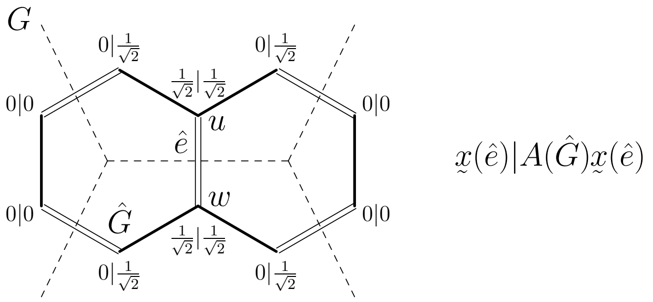
where A' is formed by replacing some 1's by -1's in $A(L(G))$. No eigenvalue of A' can exceed $M - 2$, by the same argument as for $A(L(G))$, and so no eigenvalue of $\vec{C}^\top\vec{C}$ or $\vec{C}\vec{C}^\top$ can exceed M, as before. //

Now let G be a 3-regular plane graph. Its *leapfrog* $\hat{G}$ is obtained by 'omnicapping' G (i.e., triangulating each face of G by inserting one new vertex) and then taking the dual (see handout). If

G has $n = 2k$ vertices, $3k$ edges and r faces, then $\hat{G}$ has $6k$ vertices, $9k$ edges and $n + r$ faces.

Theorem 14. At least half the (ordinary) eigenvalues of $\hat{G}$ are $\geqslant 0$ and at least half are $\leqslant 0$.

Proof. $\hat{G}$ has $3k$ nonadjacent edges that cross the edges of G and form a 1-factor in $\hat{G}$. For each of them, say $\hat{e} = uw$, form the vector $\underset{\sim}{x}(\hat{e})$ on $V(\hat{G})$ whose coordinate is $\frac{1}{\sqrt{2}}$ on u and w and 0 elsewhere. Let X be the $6k \times 3k$ matrix with these $3k$ vectors $\underset{\sim}{x}(\hat{e})$ as columns. Then the columns of X are orthonormal, and $X^\top A(\hat{G})X = I + \frac{1}{2}A(L(G))$.



Since all eigenvalues of $L(G)$ are $\geqslant -2$ by Theorem 13, it follows that all eigenvalues of $X^\top A(\hat{G})X$ are $\geqslant 0$, whence the largest $3k$ eigenvalues of $A(\hat{G})$ are $\geqslant 0$ by Lemma 2.2.

Finally, change *one* $\frac{1}{\sqrt{2}}$ to $-\frac{1}{\sqrt{2}}$ in each column of X to form $\vec{X}$. Then $\vec{X}^\top A(\hat{G})\vec{X} = -(I + \frac{1}{2}A')$, where A' is formed by replacing some 1's by -1's in $A(L(G))$. As in the proof of Theorem 13, the nonzero eigenvalues of $2I + A'$ are the same as those of $A^{(\mathrm{L})}(G)$, which are all $\geqslant 0$. It follows that all eigenvalues of $\vec{X}^\top A(\hat{G})\vec{X}$ are $\leqslant 0$, whence the smallest $3k$ eigenvalues of $A(\hat{G})$ are $\leqslant 0$ by Lemma 2.2. //

The first half of the above proof, that at least half the eigenvalues are $\geqslant 0$, will work for any graph having a 1-factor whose contraction leaves a line graph — e.g., the fullerene with 14 faces and

24 vertices has a 1-factor whose contraction leaves the line graph of the cube. But the second half of the proof does not extend, because the matrix A' obtained has the wrong signs, and is not of the form $\vec{C}^\top\vec{C} - 2I$ for any $\vec{C}$.

4. The Laplacian spectrum and the algebraic connectivity.

Assume the vertices of G are labelled so that $\Delta = d(v_1) \geqslant d(v_2) \geqslant \ldots \geqslant d(v_n)$.

Theorem 15. $\lambda_1^{(\mathrm{L})} \leqslant d(v_1) + d(v_2)$ and $\lambda_n^{(\mathrm{L})} = 0$.

Proof. It follows from Theorem 13 that all eigenvalues of $A^{(\mathrm{L})}(G) = D(G) - A(G)$ lie between 0 and

$$\max_{v_iv_j \in E(G)} d(v_i) + d(v_j) \leqslant d(v_1) + d(v_2).$$

If $\underset{\sim}{x} = (1, 1, \ldots, 1)^\top$ then $A^{(\mathrm{L})}(G)\underset{\sim}{x} = \underset{\sim}{0}$, and so $\lambda_n^{(\mathrm{L})} = 0$. //

Theorem 16. (a) $\sum_{i=1}^n \lambda_i^{(\mathrm{L})} = \operatorname{trace} A^{(\mathrm{L})}(G) = \sum_{i=1}^n d(v_i)$.
Also, if G is connected and $n \geqslant 4$, then (R. Grone and R. Merris, 1994):
(b) $\lambda_1^{(\mathrm{L})} \geqslant d(v_1) + 1$,
(c) $\lambda_1^{(\mathrm{L})} + \lambda_2^{(\mathrm{L})} \geqslant d(v_1) + d(v_2) + 1$,
(d) $\lambda_1^{(\mathrm{L})} + \lambda_2^{(\mathrm{L})} + \lambda_3^{(\mathrm{L})} \geqslant d(v_1) + d(v_2) + d(v_3) + 1$. //

Conjecture. If G is connected then, for each $j \leqslant n - 1$,

$$\textstyle\sum_{i=1}^j \lambda_i^{(\mathrm{L})} \geqslant 1 + \sum_{i=1}^j d(v_i).$$

Theorem 17. If G^+ is formed from G by adding an edge between two nonadjacent vertices, then the $n-1$ largest Laplacian eigenvalues of G interlace the Laplacian eigenvalues of G^+. //

Theorem 18. G is connected if and only if $\lambda_{n-1}^{(\mathrm{L})} > 0$.

Proof. The matrix-tree theorem, attributed to Kirchhoff (1847), is that the number of spanning trees of G is equal to the determinant of every $(n-1)\times(n-1)$ principal submatrix of $A^{(\mathrm{L})}(G)$. The sum of these determinants is $(-1)^{n-1}$ times $a_{n-1}^{(\mathrm{L})}$, the coefficient of x in $\phi(A^{(\mathrm{L})}(G), x)$, which is therefore nonzero iff G has a spanning tree, i.e., is connected. (In fact, since $\phi(A^{(\mathrm{L})}(G), x)$ is multiplicative over components, the number of zero Laplacian eigenvalues of G is equal to its number of components.) //

$\lambda_{n-1}^{(\mathrm{L})}$ is the *algebraic connectivity* of G, written $\lambda_{n-1}^{(\mathrm{L})}(G)$ or $a(G)$.

Theorem 19. (a) (M. Fiedler, 1973.) If G has connectivity κ and edge-connectivity λ, then $2\lambda(1-\cos\frac{\pi}{n}) \leqslant \lambda_{n-1}^{(\mathrm{L})}$ and, if $G \neq K_n$, then $\lambda_{n-1}^{(\mathrm{L})} \leqslant \kappa$.
(b) (B. Mohar, 1988.) If $X \subseteq V(G)$ and $|X| \leqslant \frac{1}{2}n$, then the number ∂X of edges with exactly one end in X is at least $\frac{1}{2}\lambda_{n-1}^{(\mathrm{L})}|X|$.
(c) (B. Mohar and S. Poljak, 1990.) If $V(G) = X \,\dot\cup\, Z \,\dot\cup\, Y$ and $X, Y \neq \emptyset$ and Z separates X from Y in G, then

$$|Z| \geqslant \frac{4|X||Y|\lambda_{n-1}^{(\mathrm{L})}}{n\Delta - \lambda_{n-1}^{(\mathrm{L})}(|X|+|Y|)} \geqslant \frac{4(n-1-|Z|)\lambda_{n-1}^{(\mathrm{L})}}{n\Delta - \lambda_{n-1}^{(\mathrm{L})}(n-|Z|)}.$$

Proof of (b). We may assume $X \neq \emptyset$. Let $\overline{X} := V(G) \setminus X$. Define $\underset{\sim}{x} = (x_1, \ldots, x_n)^\top$ by

$$x_i := \begin{cases} \frac{1}{|X|} & \text{if } v_i \in X, \\ -\frac{1}{|\overline{X}|} & \text{if } v_i \in \overline{X}, \end{cases}$$

so that $\sum_{i=1}^n x_i = 0$—i.e., $\underset{\sim}{x}$ is orthgonal to the eigenvector with eigenvalue $\lambda_n^{(\mathrm{L})}$. Therefore, since $\underset{\sim}{x}^\top A^{(\mathrm{L})}(G)\,\underset{\sim}{x}$ can be expressed as

the sum over all edges v_iv_j of

$$\begin{bmatrix} x_i & x_j \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} = (x_i - x_j)^2,$$

and $x_i - x_j = 0$ if $v_i, v_j \in X$ or $v_i, v_j \in \overline{X}$, it follows from Lemma 2.1 that

$$\lambda_{n-1}^{(\mathrm{L})} \leqslant \frac{\underset{\sim}{x}^\top A^{(\mathrm{L})}(G)\,\underset{\sim}{x}}{\underset{\sim}{x}^\top \underset{\sim}{x}} = \frac{\left(\frac{1}{|X|}+\frac{1}{|\overline{X}|}\right)^2 \partial X}{|X|\frac{1}{|X|^2} + |\overline{X}|\frac{1}{|\overline{X}|^2}}$$

$$= \left(\frac{1}{|X|}+\frac{1}{|\overline{X}|}\right)\partial X \leqslant \frac{2\partial X}{|X|}. \quad //$$

Corollary 19.1. By choosing X s.t. $\partial X = \lambda$, we see that $\lambda_{n-1}^{(\mathrm{L})} \leqslant \left(\frac{1}{|X|}+\frac{1}{|\overline{X}|}\right)\lambda \leqslant (1+\frac{1}{n-1})\lambda$. //