Matchings and Factors of Graphs

1. **Necessary and sufficient conditions.** Let $G$ be a multigraph and $f : V(G) \rightarrow \mathbb{N} \cup \{0\}$ be a function. An $f$-factor of $G$ is a subgraph $H$ such that $d_H(v) = f(v)$ $\forall v \in V(G)$. A $k$-factor is an $f$-factor s.t. $f(v) = k$ $\forall v \in V(G)$. A matching is a subgraph with maximum degree $\leq 1$; i.e., a set of pairwise nonadjacent edges. A perfect matching is a 1-factor.

**Theorem 1.** Let $G$ be a bipartite multigraph on two sets $S, T$ of vertices. Then $G$ has an $f$-factor if and only if

(i) $\sum_{v \in S} f(v) = \sum_{w \in T} f(w) = \Sigma$, say; and

(ii) $\forall X \subseteq S, Y \subseteq T$, the number $e(X, Y)$ of edges of $G$ between $X$ and $Y$ is at least $\sum_{v \in X} f(v) + \sum_{w \in Y} f(w) - \Sigma$.

**Proof.** Condition (i) is clearly necessary; so suppose it holds. Form $\tilde{G}$ by directing all edges of $G$ from $S$ to $T$, and adding a new vertex $s$ joined to each $v$ in $S$ by $f(v)$ arcs, and a new vertex $t$ with $f(w)$ arcs from each $w$ in $T$ to $t$. Then $G$ has an $f$-factor if and only if $\tilde{G}$ contains $\Sigma$ arc-disjoint $s, t$-paths. The edge-separation analogue of Menger’s theorem says that this happens if and only if every set of arcs separating $s$ from $t$ in $\tilde{G}$ contains at least $\Sigma$ arcs. Suppose $C$ is a separating set of arcs. Let

$$X := \{v \in S : \text{not all } sv\text{-arcs are in } C\},$$

$$Y := \{w \in T : \text{not all } wt\text{-arcs are in } C\}.$$

Then all $XY$-arcs are in $C$, and so

$$|C| \geq \sum_{v \in S \setminus X} f(v) + \sum_{w \in T \setminus Y} f(w) + e(X, Y)$$

$$= 2\Sigma - \sum_{v \in X} f(v) - \sum_{w \in Y} f(w) + e(X, Y).$$
Moreover, $\forall X \subseteq S, Y \subseteq T$, there exists a separating set with this number of arcs. So $G$ has an $f$-factor if and only if, $\forall X \subseteq S, Y \subseteq T$,

$$2\Sigma - \sum_{v \in X} f(v) - \sum_{w \in Y} f(w) + e(X, Y) \geq \Sigma,$$

which is (ii). //

**Corollary 1.1.** $G$ has a 1-factor if and only if $|N(X)| \geq |X| \ \ \forall X \subseteq V(G)$. (Here $N(X) := \bigcup_{x \in X} N(x)$.)

**Proof.** The condition is clearly necessary. To prove that it is sufficient, note first that it implies

$$|S| \geq |N(T)| \geq |T| \ \ \text{and} \ \ |T| \geq |N(S)| \geq |S|,$$

so that $|S| = |T|$, which is (i) in Theorem 1. Also, $\forall X \subseteq S, Y \subseteq T$,

$$e(X, Y) \geq |N(X) \cap Y| = |N(X)| + |Y| - |N(X) \cup Y| \geq |X| + |Y| - |T|,$$

which is (ii). The result follows from Theorem 1. //

**Corollary 1.2.** $G$ has a 1-factor if and only if the following three conditions hold:

(i) $|N(X)| \geq |X| \ \ \forall X \subseteq S,$

(ii) $|N(Y)| \geq |Y| \ \ \forall Y \subseteq T,$

(iii) $|S| = |T|.$

Moreover, any two of these imply the third.

**Proof.** (i) & (ii) $\iff |N(X)| \geq |X| \ \ \forall X \subseteq V(G)$

$\iff G$ has a 1-factor by Corollary 1.1,

$\implies$ (iii).
To see that (i) & (iii) \( \implies \) (ii), let \( Y \subseteq T \), and define \( X := S \setminus N(Y) \). Then \( N(X) \cap Y = \emptyset \) and
\[
|N(Y)| = |S| - |X| = |T| - |X| \quad \text{by (iii)} \\
\geq |T| - |N(X)| \quad \text{by (i)} \\
\geq |Y|.
\]
Similarly (ii) & (iii) \( \implies \) (i). //

If \( S \subseteq V(G) \) (where now \( G \) is not necessarily bipartite), let \( o(S) = o_G(S) \) denote the number of odd components (components with an odd number of vertices) in \( G - S \).

**Theorem 2.** (W. T. Tutte, 1947.) \( G \) has a 1-factor if and only if \( o(S) \leq |S| \quad \forall S \subseteq V(G) \).

**Proof.** (I. Anderson, 1971.) ‘Only if’ holds because in any 1-factor of \( G \), at least one vertex of \( S \) is matched with each odd component of \( G - S \). We prove ‘if’ by induction on \( |V(G)| \). Assume \( o(S) \leq |S| \quad \forall S \subseteq V(G) \). Taking \( S = \emptyset \) we see that \( |V(G)| \) is even. Hence \( o(S) \equiv |S| \pmod{2} \), and
\[
\text{if } o(S) \neq |S| \text{ then } o(S) \leq |S| - 2. \quad (\ast)
\]

If \( |S| = 1 \) then clearly \( o(S) = |S| \). Choose \( S \) maximal such that \( o(S) = |S| \). Then \( G - S \) has \( |S| \) odd components \( C_1, \ldots, C_{|S|} \) — and no even components, otherwise transferring a vertex from one of them to \( S \) would increase \( |S| \) and \( o(S) \) by one and contradict the maximality of \( S \). Let \( T = \{1, 2, \ldots, o(S)\} \), and form a bipartite graph \( H \) on \( S \cup T \) by joining \( s \) to \( t \) iff \( s \in N_G(C_t) \). For each \( Y \subseteq T \),
\[
|N_H(Y)| \geq o_G(N_H(Y)) \geq |Y|
\]
since \( G - N_H(Y) \) has at least the \( |Y| \) odd components \( C_t : t \in Y \). By Corollary 1.2, \( H \) has a 1-factor, and so we can take a vertex \( x_i \)
in each \( C_i \) and match it with a vertex in \( S \). Let \( U \subseteq V(C_i - x_i) \), for some \( i \). Then

\[
o_{C_i - x_i}(U) = o_G(U \cup \{x_i\} \cup S) - (o_G(S) - 1) \\
\leq |U \cup \{x_i\} \cup S| - 2 - (|S| - 1) \\
= |U|.
\]

By the induction hypothesis each set \( C_i - x_i \) has a 1-factor, and hence so does \( G \). \hfill //

**Theorem 3.** (W. T. Tutte, 1952.) \( G \) has an \( f \)-factor if and only if, for every pair of disjoint subsets \( A, B \) of \( V(G) \), the number of components \( C \) of \( G - (A \cup B) \) for which \( e(B, C) + \sum_{v \in C} f(v) \) is odd, is at most \( \sum_{v \in A} f(v) + \sum_{v \in B} [d_{G-A}(v) - f(v)] \).

**Proof.** Both the existence of an \( f \)-factor, and the condition (with \( A = \emptyset, B = \{v\} \)) require \( f(v) \leq d(v) \forall v \in V(G) \); so suppose this is true. We may assume \( d(v) \neq 0, \forall v \in V(G) \). Form \( G' \) from \( G \) by carrying out the following construction at every vertex \( v \):

![Diagram](image)

Then \( G \) has an \( f \)-factor iff \( G' \) has a 1-factor. We must prove that the given condition is equivalent to \( o_{G'}(S) \leq |S|, \forall S \subseteq V(G') \). Choose \( S \) minimal s.t. \( |S| - o_{G'}(S) \) takes its minimum value.

**Claim 1.** \( \forall v \in V(G) , \) either \( \emptyset \neq B(v) \subseteq S \) or \( B(v) \cap S = \emptyset \); otherwise, removing a vertex \( w \) of \( B(v) \) from \( S \) would reduce \( |S| \)
by 1 and change $o_{G'}(S)$ by at most one (since all neighbours of $w$ in $G' - S$ are in the same component), thereby violating the choice of $S$.

**Claim 2.** If $B(v) \subseteq S$ then $A(v) \cap S = \emptyset$ (for the same reason, removing a vertex $w$ of $A(v)$ from $S$).

**Claim 3.** $\forall v \in V(G)$, either $A(v) \subseteq S$ or $A(v) \cap S = \emptyset$. This follows from Claim 2 if $B(v) \subseteq S$, and so by Claim 1 we may assume that $B(v) \neq \emptyset$ and $B(v) \cap S = \emptyset$. Now a vertex $w \in A(v) \cap S$ has neighbours in at most two components of $G' - S$, and the result follows by the same reasoning as before.

Let $A := \{v \in V(G) : A(v) \subseteq S\}$

and $B := \{v \in V(G) : B(v) \subseteq S\}$.

Then $A$ and $B$ are disjoint by Claim 2, and, by Claims 1 and 3, $A(v) \cap S = \emptyset$ if $v \notin A$ and $B(v) \cap S = \emptyset$ if $v \notin B$. Each component $C$ of $G - (A \cup B)$ corresponds to a component of $G' - S$ with $\sum_{v \in C} [2d_G(v) - f(v)] + e(B, C)$ vertices, which is odd iff $e(B, C) + \sum_{v \in C} f(v)$ is. The other odd components of $G' - S$ are all isolated vertices: $\sum_{v \in A} [d_G(v) - f(v)]$ of them corresponding to vertices in $A$, and $e(A, B) = \sum_{v \in B} d_A(v)$ of them corresponding to edges between $A$ and $B$. Thus $o_{G'}(S) \leq |S|$ iff the number of components $C$ of $G - (A \cup B)$ for which $e(B, C) + \sum_{v \in C} f(v)$ is odd is at most

$$|S| - \sum_{v \in A} [d_G(v) - f(v)] - \sum_{v \in B} d_A(v). \quad (1)$$

Since

$$|S| = \sum_{v \in A} d_G(v) + \sum_{v \in B} [d_G(v) - f(v)],$$

(1) is equal to

$$\sum_{v \in A} f(v) + \sum_{v \in B} [d_G - A(v) - f(v)],$$

as required.
It follows that if the condition given in the Theorem holds, then \( o_{G'}(S) \leq |S|, \forall S \subseteq V(G') \). To prove the converse, let \( A \) and \( B \) be disjoint subsets of \( V(G) \), and let

\[
S := \bigcup \{A(v) : v \in A\} \cup \bigcup \{B(v) : v \in B\} \subseteq V(G').
\]

Then the results of Claims 1–3 hold by definition, and the above argument shows that if \( o_{G'}(S) \leq |S| \) then the condition in the Theorem holds for \( A \) and \( B \). //

[In fact, the necessity of the condition in Theorem 3 is not difficult to see directly. Suppose \( G \) has an \( f \)-factor \( F \). Then the number of ends of edges of \( F \) in \( B \) is \( \sum_{v \in B} d_F(v) = \sum_{v \in B} f(v) \). Of the other ends of these edges, \( e_F(A, B) \) are in \( A \), and \( \sum_{v \in B} d_{F-A}(v) \) are not in \( A \). However, for each component of \( G - (A \cup B) \) for which \( e(B, C) + \sum_{v \in C} f(v) \) is odd, the number of edges between \( A \) and \( C \) that are in \( F \), plus the number between \( B \) and \( C \) that are not in \( F \), is odd. So if there are \( s \) such components, then \( e_F(A, B) + \sum_{v \in B} d_{F-A}(v) \leq \sum_{v \in A} f(v) + \sum_{v \in B} d_{G-A}(v) - s \). Thus \( \sum_{v \in B} f(v) \leq \sum_{v \in A} f(v) + \sum_{v \in B} d_{G-A}(v) - s \), which is the given condition.]

A multigraph \( G \) has the odd cycle property (OCP) if each two of its odd circuits either overlap (in at least one vertex) or are joined by an edge. Equivalently, if \( S \subseteq V(G) \), then at most one component of \( G - S \) contains an odd circuit.

**Theorem 4.** If \( G \) has the OCP then \( G \) has a 1-factor if and only if \( |V(G)| \) is even and \( |N(X)| \geq |X| \) \( \forall X \subseteq V(G) \) s.t. \( X \) is independent.

**Proof.** The conditions are clearly necessary. To prove they are sufficient, we use Theorem 2. Suppose \( \exists S \subseteq V(G) \) s.t. \( o(S) > |S| \). Since \( |V(G)| \) is even, \( o(S) \geq |S| + 2 \). Since \( G \) has the OCP, at
most one component of \( G - S \) contains an odd circuit. Let \( C_i \)
\((i = 1, 2, \ldots, |S| + 1)\) be bipartite odd components of \( G - S \). Let
\( X_i \) be the larger partite set of \( C_i \), so that \(|N_{C_i}(X_i)| \leq |X_i| - 1\), and
let \( X := \bigcup_{i=1}^{|S|+1} X_i \). Then \( X \) is independent and
\[
|N(X)| \leq |S| + \sum_{i=1}^{|S|+1} |N_{C_i}(X_i)| \\
\leq |S| + \sum_{i=1}^{|S|+1} (|X_i| - 1) \\
= |S| + |X| - |S| - 1 < |X|.
\]
This contradiction shows that \( o(S) \leq |S|, \forall S \subseteq V(G) \), so that \( G \)
has a 1-factor by Theorem 2. //

**Theorem 5.** (D. R. Fulkerson, A. J. Hoffman and M. H. McAndrew, 1965; Canad. J. Math. 17, 166–177.) If \( G \) has the OCP then
\( G \) has an \( f \)-factor if and only if \( \sum_{v \in V(G)} f(v) \) is even and
\[
t(A, B) := \sum_{v \in A} f(v) + \sum_{v \in B} \left[ d_{G-A}(v) - f(v) \right] \geq 0
\]
for every pair of disjoint subsets \( A, B \) of \( V(G) \).

**Proof.** Let \( s(A, B) \) denote the number of odd components \( C \) of
\( G - (A \cup B) \), where \( odd \) means now that \( e(B, C) + \sum_{v \in C} f(v) \) is
odd. By Tutte’s \( f \)-factor theorem (Theorem 3),
\[
G \text{ has an } f \text{-factor } \iff s(A, B) \leq t(A, B) \ \forall A, B \\
\implies \sum_{v \in V(G)} f(v) \text{ is even and } \\
t(A, B) \geq 0 \ \forall A, B.
\]
So suppose \( \sum_{v \in V(G)} f(v) \) is even and \( t(A, B) \geq 0 \ \forall A, B \), and
suppose \( s(A, B) > t(A, B) \) for some \( A, B \). Choose \( A, B \) such that
\( A \cup B \) is maximal with this property. Note that \( s(A, B) + t(A, B) \)
has the same parity as
\[
r(A, B) := \sum_{C \text{ a cpt of } G = (A \cup B)} \left[ e(B, C) + \sum_{v \in C} f(v) \right] + t(A, B).
\]
But
\[ \sum_{v \in B} d_{G-A}(v) = 2e(B, B) + e(B, V(G) - (A \cup B)), \]
and so \( r(A, B) \) has the same parity as
\[ 2e(B, V(G) - (A \cup B)) + \sum_{v \in V(G)} f(v), \]
which is even. Thus \( s(A, B) \) has the same parity as \( t(A, B) \), and \( s(A, B) \geq t(A, B) + 2 \geq 2 \). At most one odd component of \( G - (A \cup B) \) contains an odd circuit, since \( G \) has the OCP; so let \( C \) be one that is bipartite, with bipartition \((X, Y)\), say. The two quantities
\[ r_X := e(B, X) - \sum_{v \in X} f(v) \]
and
\[ r_Y := e(B, Y) - \sum_{v \in Y} f(v) \]
are not equal, since their sum has the same parity as \( e(B, C) + \sum_{v \in C} f(v) \), which is odd. Suppose w.l.o.g. \( r_X > r_Y \). Let \( A' := A \cup X, B' := B \cup Y \). Then \( s(A', B') = s(A, B) - 1 \), and, since \( N(Y) \subseteq X \cup A \cup B = A' \cup B \),
\[ t(A', B') = t(A, B) + \sum_{v \in X} f(v) - \sum_{v \in Y} f(v) - e(B, X) + e(B, Y) = t(A, B) - r_X + r_Y < t(A, B). \]
So \( s(A', B') > t(A', B') \). This contradicts the maximality of \( A \cup B \), so we conclude that \( s(A, B) \leq t(A, B) \ \forall A, B \), whence \( G \) has an \( f \)-factor by Theorem 3. //

**Exercises.** Theorem 5 \( \implies \) Theorem 1,
Theorem 3 \( \implies \) Theorem 1 (directly).

**Problem.** Is there any other class of multigraphs, other than bipartite and those with the OCP, for which there is a n.s.c. for the existence of an \( f \)-factor that is different from Theorem 3?
2. **Sufficient conditions.** I. Anderson (1971) proved that if $|V(G)|$ is even and $|N(X)| \geq \frac{4}{3}|X| \forall X \subseteq V(G)$ s.t. $|X| \leq \frac{3}{4}|V(G)|$, then $G$ has a 1-factor. D. R. Woodall (1973) defined the *binding number* of $G$ to be

$$
\text{bind}(G) := \min \left\{ \frac{|N(X)|}{|X|} : \emptyset \not\subseteq X \subseteq V(G) \text{ and } N(X) \neq V(G) \right\}
$$

$$
= \max\{c : |N(X)| \geq c|X| \forall X \subseteq V(G) \text{ s.t. } N(X) \neq V(G)\}. 
$$

So Anderson’s result says that if $|V(G)|$ is even and $\text{bind}(G) \geq \frac{4}{3}$ then $G$ has a 1-factor. The figure $\frac{4}{3}$ here is best possible in view of the graph $(t + 2)K_3 + tK_1$ (where + denotes ‘join’), which has no 1-factor and has binding number

$$
\frac{3(t + 1) + t}{3(t + 1)} = \frac{4t + 3}{3t + 3} \to \frac{4}{3} \quad \text{as } t \to \infty.
$$

However, the result can be improved in other ways.

**Theorem 6.** *(Fundamental Lemma.)* Let $G$ be a multigraph and $a, b, c, d$ real numbers such that $a, b \geq 0$. Then

$$
a|N(X)| \geq b|X| + c|V(G)| + d
$$

whenever $\emptyset \not\subseteq X \subseteq V(G)$ \hfill (2)

if and only if

$$
b|N(X)| \geq a|X| + (c + b - a)|V(G)| + d
$$

whenever $X \subseteq V(G), N(X) \neq V(G)$. \hfill (3)

**Proof.** Suppose (2) holds and let $Y \subseteq V(G), N(Y) \neq V(G)$. Put $X := V(G) \setminus N(Y)$, so that $X \neq \emptyset$ and $N(X) \subseteq V(G) \setminus Y$. Then

$$
b|N(Y)| = b|V(G)| - b|X|
$$

$$
\geq b|V(G)| - a|N(X)| + c|V(G)| + d \quad \text{by (2)}
$$

$$
\geq b|V(G)| - a|V(G)| + a|Y| + c|V(G)| + d
$$

$$
= a|Y| + (c + b - a)|V(G)| + d.
$$
Thus (2) implies (3). The converse is proved similarly. //

\(G\) is *sesquiconnected* if \(G\) is connected and, \(\forall v \in V(G), G - v\) has at most two components. Let \(\delta(G) := \min\{|N(v)| : v \in V(G)\}\). (If \(G\) is simple, this is the minimum degree.)

**Theorem 7.** If \(G\) is sesquiconnected, \(n = |V(G)|\) is even, \(\delta(G) \geq \frac{1}{4}(n + 3)\) and
\[
|N(X)| \geq \frac{1}{4}(2|X| + n - 5) \tag{4}
\]
for every nonempty *independent* subset \(X\) of \(V(G)\), then \(G\) has a 1-factor.

**Proof.** Suppose not. Then \(\exists S \subseteq V(G)\) s.t. \(o(S) \geq |S| + 2\). Since \(\delta(G) \geq \frac{1}{4}(n + 3)\), each of the odd components of \(G - S\) has at least \(\frac{1}{4}(n + 3) + 1 - |S|\) vertices, and so
\[
n \geq |S| + (|S| + 2)
\left[
\frac{1}{4}(n + 3) + 1 - |S|
\right],
\]
whence
\[
|S|^2 - \frac{1}{4}(n + 3)|S| + \frac{1}{2}(n - 7) \geq 0,
\]
i.e.,
\[
(|S| - 2)(|S| - \frac{1}{4}(n - 5)) \geq 1.
\]
This is impossible if \(2 \leq |S| \leq \frac{1}{4}(n - 5)\). Since \(|S| \geq 2\) by the sesquiconnectedness of \(G\), \(|S| > \frac{1}{4}(n - 5)\). Let \(x\) of the odd components consist of a single vertex each, and let \(X\) be the set of these. If \(x = 0\) then \(n \geq |S| + 3(|S| + 2)\) and \(|S| \leq \frac{1}{4}(n - 6)\), which is impossible. So \(X \neq \emptyset\) and, by (4),
\[
|S| \geq |N(X)| \geq \frac{1}{4}(2x + n - 5). \tag{5}
\]
Also,
\[
n \geq |S| + x + 3(|S| + 2 - x). \tag{6}
\]
Adding 4 times (5) to (6) gives \(0 \geq 6 - 5 = 1\), a contradiction. //
Corollary 7.1. If \( n = |V(G)| \) is even and
\[ |N(X)| \geq \frac{1}{4}(2|X| + n + 1) \]
whenever \( \emptyset \not\subseteq X \subseteq V(G) \) (7)
then \( G \) has a 1-factor.

Proof. Clearly (7) implies (4), and also implies \( \delta(G) \geq \frac{1}{4}(n + 3) \)
take \( |X| = 1 \). If \( G \) is disconnected, let \( X \) be the vertex-set of a
smallest component, for which \( |N(X)| \leq |X| \leq \frac{1}{2}n \). Then (7) gives
\[ |X| \geq \frac{1}{4}(2|X| + n + 1) \implies \frac{1}{2}|X| \geq \frac{1}{4}(n + 1) \implies |X| \geq \frac{1}{2}(n + 1), \]
a contradiction. If \( G - v \) has more than two components, let \( X \) be
the vertex-set of a smallest one, for which \( |N(X)| \leq |X| + 1 \) and
\( |X| \leq \frac{1}{3}(n - 1) \). Then (7) gives
\[ |X| + 1 \geq \frac{1}{4}(2|X| + n + 1) \implies |X| \geq \frac{1}{2}(n - 3) > \frac{1}{3}(n - 1) \]
if \( n > 7 \). But if \( n \leq 7 \) then \( n \leq 6 \) (because \( n \) is even), so \( |X| \leq \frac{5}{3} \),
i.e., \( |X| = 1 \). But then \( |N(X)| = |X| \), and we have already seen
that it is impossible that \( |N(X)| \leq |X| \leq \frac{1}{2}n \). So (7) implies that
\( G \) is sesquiconnected, and so \( G \) has a 1-factor by Theorem 7. //

Corollary 7.2. If \( n = |V(G)| \) is even then \( G \) has a 1-factor if:

(a) (I. Anderson, 1971) \( \operatorname{bind}(G) \geq \frac{4}{3} \);
(b) (I. Anderson, 1972) \( |N(X)| \geq \frac{1}{2}(4|X| - n + 1) \) whenever
\( X \subseteq V(G), N(X) \neq V(G) \).

Proof. (a) \( \iff \frac{3}{4}|N(X)| \geq |X| \)
whenever
\( X \subseteq V(G), N(X) \neq V(G) \)
\( \iff |N(X)| \geq \frac{3}{4}|X| + \frac{1}{4}n \)
whenever \( \emptyset \not\subseteq X \subseteq V(G) \) by Thm 6
\( \iff |N(X)| \geq \frac{1}{2}|X| + \frac{1}{4}n + \frac{1}{4} \)
whenever \( \emptyset \not\subseteq X \subseteq V(G) \iff (7) \)
\( \iff \frac{1}{2}|N(X)| \geq |X| - \frac{1}{4}n + \frac{1}{4} \)
whenever \( X \subseteq V(G), N(X) \neq V(G) \)
by Thm 6
\(\iff (b)\).
The result follows from Corollary 7.1.  //

There could be 12 theorems analogous to Theorem 7: for 1-factors, \(k\)-factors, \([a, b]\)-factors and \(f\)-factors, and in arbitrary graphs, bipartite graphs and graphs with the OCP. The 1-factor theorems are given above, A. M. Robertshaw did \(k\)-factors in bipartite graphs, and T. R. Poole did \([a, b]\)-factors in bipartite and arbitrary graphs. For \(k\)-factors in arbitrary graphs we have the following.

**Theorem 8.** (Woodall, 1990; Egawa and Enomoto, 1989.) Let \(G\) be a graph with \(n\) vertices, and let \(k \geq 2\) be an integer. If \(k\) is odd, suppose \(n\) is even and \(G\) is connected. Suppose that
\[
|N(X)| \geq \frac{1}{2k-1}(|X| + (k-1)n - 1)
\]
for every nonempty independent subset \(X\) of \(V(G)\), and
\[
\delta(G) \geq \frac{(k-1)(n+2)}{2k-1}.
\]
Suppose further that, if \(n \leq 4k+1-4\sqrt{k+2}\), then
\[
\delta(G) > n+2k-2\sqrt{kn}+2.
\]
Then \(G\) has a \(k\)-factor.  //

**Theorem 9.** (C. Chen, 1995; DM 146, 303–306.) If \(n = |V(G)|\) is even, \(l \geq 1\) and
\[
\text{bind}(G) = b > \max\{l, \frac{7l+13}{12}\} = \begin{cases} 
\frac{5}{3} & \text{if } l = 1, \\
\frac{9}{4} & \text{if } l = 2, \\
l & \text{if } l \geq 3,
\end{cases}
\]
then every matching of \(l\) edges in \(G\) can be extended to a 1-factor.
Proof. Suppose not. Let $L$ be the vertex-set of a matching that is not extendable, so that $|L| = 2l$. Then $\exists T \subseteq V(G - L)$ s.t. $o_{G-L}(T) \geq |T| + 2$; that is, writing $S := T \cup L$,

$$o_G(S) \geq |S| - 2l + 2.$$ (8)

Let $i(G - S)$ denote the number of isolated vertices in $G - S$. There are two cases.

Case 1: $i(G - S) = 0$. Let $X$ be the vertex-set of any $|S| - 2l + 1$ odd components of $G - S$. Since $N(X) \neq V(G)$,

$$|X| + |S| \geq |N(X)| \geq b|X|,$$

and so $|X| \leq \frac{|S|}{b - 1}$. But $|X| \geq 3(|S| - 2l + 1)$, and so

$$3(|S| - 2l + 1) \leq \frac{|S|}{b - 1}.$$ (9)

Since $3(b - 1) \geq 3\left(\frac{5}{3} - 1\right) > 1$ and $|S| \geq 2l$, (9) remains true with $|S|$ replaced by $2l$. Hence $3(b - 1) \leq 2l$, and so $b \leq \frac{3}{2}l + 1$, a contradiction unless $l = 2$. If $l = 2$ and some odd component of $G - S$ has at least 5 vertices, then the LHS of (9) is increased by 2 and we get $5(b - 1) \leq 2l = 4$ or $b \leq \frac{4}{5}$, a contradiction. So we may assume that every odd component of $G - S$ has exactly 3 vertices. Add one vertex from another odd component to $X$, so that

$$|X| + |S| + 1 \geq |N(X)| \geq b|X|$$

and

$$3(|S| - 2l + 1) + 1 \leq |X| \leq \frac{|S| + 1}{b - 1}.$$

This gives $4(b - 1) \leq 2l + 1 = 5$, and so $b \leq \frac{5}{4}$, a contradiction.

Case 2: $i(G - S) > 0$. Let $X := V(G) - S$. Then

$$n - i(G - S) \geq |N(X)| \geq b|X| = bn - b|S|,$$

and so

$$b|S| - i(G - S) \geq (b - 1)n.$$

From this and (8),

$$o_G(S) - i(G - S) \geq |S| - 2l + 2 + (b - 1)n - b|S|$$

$$= (b - 1)(n - |S|) - 2l + 2.$$

But

$$n - |S| \geq 3|o_G(S) - i(G - S)| + i(G - S)$$

$$\geq (3b - 3)(n - |S|) - 6l + 6 + i(G - S).$$

Rearranging,

$$(3b - 4)(n - |S|) \leq 6l - 6 - i(G - S).$$ (10)

Now, $n - |S| \geq o_G(S) \geq 2$, by (8), and so if $i(G - S) \geq 2$ then (10) gives $6b - 8 \leq 6l - 8$, whence $b \leq l$, a contradiction. But if $i(G - S) = 1$ then $n - |S| \geq 4$ (since the second odd component of $G - S$ has at least 3 vertices), and so $12b - 16 \leq 6l - 7$, giving the contradiction $b \leq \frac{6l + 7}{12} < \frac{7l + 13}{12}$. In either case, the theorem is proved. //
Robertshaw and Woodall replaced the binding-number condition in Theorem 9 by the correct (best possible) condition on $|N(X)|$ for $X$ independent as in Theorem 7. Poole extended this to $k$-factors in bipartite graphs, and Philpotts and Woodall extended it to $k$-factors in arbitrary graphs. Nothing is known about $f$-factors or graphs with the OCP.