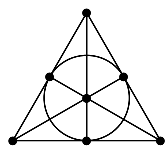
Chapter 3 Types of Matroid

3.1. Representable matroids. A matroid $\mathbf{M} = (S, r)$ is *representable over a field* K if there is a vector space V over K and a map $f : S \to V$ such that $r(X) = \dim f(X)\ \forall X \subseteq S$. Then $X \subseteq S$ is independent in $\mathbf{M}$ iff $\dim f(X) = |X|$, i.e., the multiset $(f(x) : x \in X)$ is linearly independent in V. And $B \subseteq S$ is a base of $\mathbf{M}$ iff $(f(x) : x \in B)$ is a basis for the subspace of V spanned by $f(S)$. $\mathbf{M}$ is *representable* if it is representable over *some* field, *regular* if it is representable over *every* field, and *binary* if it is representable over GF(2). Clearly regular $\Longrightarrow$ binary $\Longrightarrow$ representable. A *minor* of $\mathbf{M}$ is a matroid obtained from $\mathbf{M}$ by restriction and/or contraction.

Theorem 3.1. (W. T. Tutte (1917–2002), 1965.)
(a) A matroid is binary iff it has no minor isomorphic to the 2-uniform matroid on 4 elements.
(b) A binary matroid is regular iff it has no minor isomorphic to $\mathbf{F}$ or $\mathbf{F}^*$, where $\mathbf{F}$, the *Fano matroid*, has as its bases all noncollinear triples of points in the *Fano configuration* or projective plane of order 2. //



Let $\mathbf{M} = (S, \mathbf{I})$ be a matroid that is representable over some field K, and let $f : S \to V$ be a representing map, where V is a vector space over K. Let K^S denote the vector space over K with S as basis. Let $F : K^S \to V$ be the linear transformation induced by f, with null space (kernel) N, say. The *support* $\mathrm{supp}(\mathbf{x})$ of a vector $\mathbf{x}$ in K^S is the subset of S on which the coordinates of $\mathbf{x}$ are nonzero.

Theorem 3.2. The collection $\mathbf{C}$ of circuits of $\mathbf{M}$ is the same as the collection $\mathbf{C}'$ of minimal nonempty supports of vectors in N. (N is called the *circuit subspace* of K^S with respect to the representing map f.)

Proof. Let $X \subseteq S$. Then

$X \in \mathbf{I} \iff (F(x) : x \in X)$ is linearly independent in V

$\iff$ no nonzero linear combination of $(F(x) : x \in X)$ is the zero vector in V

$\iff$ no nonzero linear combination of X is in N

$\iff$ X does not contain the support of any nonzero vector in N

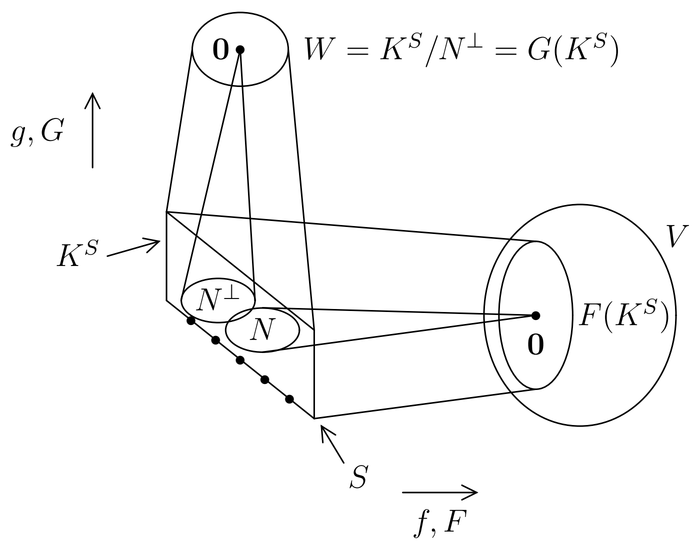
$\iff$ X does not contain any C in $\mathbf{C}'$.

Thus the sets in $\mathbf{C}'$ are precisely the minimal dependent sets of $\mathbf{M}$, as required. //

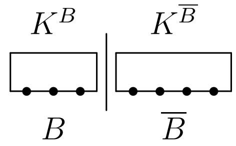
Corollary 3.2.1. If N' is any subspace of K^S, then the minimal nonempty supports of vectors in N' form the circuits of a matroid $\mathbf{M}'$ on S that is representable over K.

Proof. Define $f' : S \to K^S/N'$ by $x \to x + N'$. Then f' is a representing map for $\mathbf{M}'$ with circuit subspace N'. //

Theorem 3.3. Let S, $\mathbf{M}$, K, V, f, F and N be as in Theorem 3.2. Let $N^\perp$ be the orthogonal complement of N w.r.t. the basis S. Let $W := K^S/N^\perp$, define $G : K^S \to W$ by $\mathbf{x} \to \mathbf{x} + N^\perp$, and let $g := G|_S$. Then g is a representing map for the dual matroid $\mathbf{M}^*$. Thus $\mathbf{M}$ is representable over K iff $\mathbf{M}^*$ is, and $\mathbf{M}$ and $\mathbf{M}^*$ correspond to projection from K^S along orthogonal subspaces N and $N^\perp$.



Proof. If B is a subset of S of cardinality $r(S)$, let $\overline{B} := S \setminus B$, and let K^B and $K^{\overline{B}}$ be the subspaces of K^S spanned by B and $\overline{B}$. Note that $|\overline{B}| = r^*(S)$ and $K^{\overline{B}} = (K^B)^\perp$. Also $\dim N = \dim K^S - \dim \operatorname{Im} F = |S| - r(S)$ and $\dim N^\perp = r(S)$. So



$\overline{B}$ is a base of $\mathbf{M}^* \iff B$ is a base of $\mathbf{M}$

$\iff$ no nonzero linear combination of B is in N (cf. Theorem 3.2)

$\iff K^B \cap N = \{\mathbf{0}\}$

$\iff K^B + N = K^S$

(since $\dim K^B + \dim N = |B| + |S| - r(S) = |S| = \dim K^S$)

$\iff (K^B + N)^\perp = \{\mathbf{0}\}$

$\iff K^{\overline{B}} \cap N^\perp = \{\mathbf{0}\}$

$\iff$ no nonzero linear combination of $\overline{B}$ is in $N^\perp$

$\iff (g(x) : x \in \overline{B})$ is a basis for W

(since $\dim W = |S| - r(S) = r^*(S) = |\overline{B}|$).

Thus g is a representing map for $\mathbf{M}^*$. //

3.2. Graphic matroids.

Theorem 3.4. All graphic and cographic matroids are regular.

Proof. Given a field K, and a graph G with m edges, orient each edge, circuit and cocircuit of G arbitrarily, so that each circuit and cocircuit becomes a vector with coordinates 0, 1 and -1. Let V denote the m-dimensional vector space over K with the oriented edges of G as basis, and let W and W^* denote the subspaces spanned by the circuit and cocircuit vectors respectively. Note that every vector in W is orthogonal to every vector in W^*, since every circuit is orthogonal to every cocircuit.

If $\mathbf{0} \neq \mathbf{x} \in W$, let $X := \operatorname{supp}(\mathbf{x})$. If X is circuit-free, choose $e \in X$ and a skeleton $S \supseteq X$. Then the fundamental cutset vector determined by $\{e\} \cup \overline{S}$ is not orthogonal to $\mathbf{x}$, $\Rightarrow\Leftarrow$. So X must contain a circuit. And, by construction, every circuit is the support of some vector in W. Thus the circuits are the minimal nonempty supports of the vectors in W, and $\mathbf{M}(G)$ is representable over K by Corollary 3.2.1. The result for $\mathbf{M}^*(G)$ is similar. //

Theorem 3.5. (W. T. Tutte, 1959 & 1965.) A matroid is graphic (cographic) iff it is regular and has no minor isomorphic to $\mathbf{M}^*(K_5)$ or $\mathbf{M}^*(K_{3,3})$ ($\mathbf{M}(K_5)$ or $\mathbf{M}(K_{3,3})$). Thus it is planar iff it is regular and has no minor isomorphic to $\mathbf{M}(K_5)$, $\mathbf{M}(K_{3,3})$ or their duals. //

3.3. Binary matroids.

Theorem 3.6. Let $\mathbf{M} = (S, \mathbf{I})$ be a matroid. The following statements are equivalent.

(a) $\mathbf{M}$ is binary.
(b) The circuits of $\mathbf{M}$ are the minimal nonempty sets in the collection that they generate under $+_2$.
(c) Every Boolean sum of circuits is a union of disjoint circuits.
(d) No nonempty independent set of $\mathbf{M}$ is a Boolean sum of circuits.
(e) $|C \cap C^*|$ is even, for each circuit C and cocircuit C^*.
(f) (P. D. Seymour, 1975.) $|C \cap C^*| \neq 3$, for each circuit C and cocircuit C^*.
(g) (J.-C. Fournier, 1981.) Whenever $C_1 \neq C_2$ are circuits and $x, y \in C_1 \cap C_2$, then $(C_1 \cup C_2) \setminus \{x, y\}$ contains a circuit.

Proof that (e) $\Longrightarrow$ (d). (The rest of the proof of equivalence of (a)–(e) is an exercise.) Suppose (e) holds and some nonempty Boolean sum of circuits, X, is independent. Choose $x \in X$ and a cobase $\overline{B} \subseteq S \setminus X$. Then $\overline{B} \cup \{x\}$ contains a cocircuit C^* containing x, so $|X \cap C^*| = 1$. But (e) $\Longrightarrow$ $|X \cap C^*|$ is even, $\Rightarrow\Leftarrow$. //

Let 2^S denote the usual vector space over GF(2) consisting of all subsets of S. If U is a subspace of 2^S, let $U^\perp := \{Y \subseteq S : |X \cap Y|$ is even, $\forall X \in U\}$. Since the parity of $|X \cap Y|$ is $X \cdot Y$ with the 'obvious' dot product, it follows by standard linear algebra that $(U^\perp)^\perp = U$ and $(U \cap W)^\perp = U^\perp +_2 W^\perp$.

Theorem 3.7. If $\mathbf{M} = (S, \mathbf{I})$ is a binary matroid, then S is a disjoint union of circuits and cocircuits.

Proof. Let N be the subspace consisting of all Boolean sums of circuits, so that $N^\perp$ is the set of Boolean sums of cocircuits, by Theorems 3.2 and 3.3. If $X \in N^\perp \cap N$, then $|S \cap X| = |X \cap X|$ is even by Theorem 3.6 (e), and so $S \cdot X = 0$. Thus $S \in (N^\perp \cap N)^\perp = N +_2 N^\perp$. That is, $S = C +_2 D = C \cup D$ where $C \in N$ and $D \in N^\perp$. The result follows by Theorem 3.6 (c). //

Corollary 3.7.1. (J. Edmonds ? 1971) Let $G = (V, E)$ be a graph. Then

(a) E is the union of a cycle C and a coboundary D. (Possibly C or $D = \emptyset$.)

(b) E is a disjoint union of circuits and cocircuits.

(c) V is a disjoint union $V = X \cup Y$ (possibly X or $Y = \emptyset$) such that the induced subgraphs $\langle X \rangle$ and $\langle Y \rangle$ have all degrees even. //

Theorem 3.8. (T. A. McKee, 1984.) Let $\mathbf{M} = (S, \mathbf{N})$ be a binary matroid, where $\mathbf{N}$ is the set of Boolean sums of circuits. Then $S \in \mathbf{N}$ iff every element lies in an odd number of circuits.

Corollary 3.8.1. A connected graph is Eulerian iff each edge lies in an odd number of circuits.

Corollary 3.8.2. A graph is bipartite iff each edge lies in an odd number of cocircuits.

Proof of Theorem 3.8. (D. R. Woodall, 1990.) 'If' is obvious. We prove 'only if' by induction on $|S|$, noting that it holds if $|S| = 1$.

If $d, e \in S$, choose $f \notin S$ and define $\mathbf{M}(de \to f) = (S_1, \mathbf{N}_1)$ by: $S_1 := (S \setminus \{d, e\}) \cup \{f\}$, and, if $C \subseteq S \cap S_1$, then

$$C \in \mathbf{N}_1 \iff C \in \mathbf{N},$$
$$C \cup \{f\} \in \mathbf{N}_1 \iff C \cup \{d, e\} \in \mathbf{N}.$$

It is easy to see that $\mathbf{N}_1$ is a subspace of 2^{S_1}, and $S_1 \in \mathbf{N}_1 \iff S \in \mathbf{N}$.

Assume $S \in \mathbf{N}$ and let $e \in S$. We shall prove that e lies in an odd number of circuits. Choose a cocircuit D containing e. (If there isn't one, then e is a loop and the result is obvious.) For each $d \in D \setminus \{e\}$, let $c(d, e)$ denote the number of circuits containing f in the matroid $\mathbf{M}(de \to f)$. By induction, $c(d, e)$ is odd. But $|D|$ is even since $S \in \mathbf{N}$, and so $\Sigma := \sum_{d \in D \setminus \{e\}} c(d, e)$ is odd.

Now, C is a circuit containing f in $\mathbf{M}(de \to f)$ iff $C' := (C \setminus \{f\}) \cup \{d, e\}$ is either a circuit in $\mathbf{M}$ containing d and e, or a disjoint union $C_1 \cup C_2$ of two circuits where $d \in C_1$, $e \in C_2$ and $C' \setminus \{d, e\}$ contains no circuits. The result will follow if we can prove that every circuit C' containing e makes an odd contribution to Σ, and every set of the form $C' = C_1 \cup C_2$ makes an even contribution. The first holds because C' contributes 1 to $c(d, e)$ for an odd number of d's (since $|C' \cap D|$ is even). And if $C' = C_1 \cup C_2$ as above, then this is the only representation of C' as a disjoint union of two circuits, since if $C' = C_1' \cup C_2'$ where $e \in C_2'$ (w.l.o.g.) and $C_1' \neq C_1$, then either C_1' or $C_1 +_2 C_1'$ contains a circuit in $C' \setminus \{d, e\}$, $\Rightarrow\Leftarrow$. But $|C_1 \cap D|$ is even, and so C' contributes 1 to $c(d, e)$ for an even number of d's.

Thus the number of circuits of $\mathbf{M}$ containing e has the same parity as Σ, which is odd. //

3.4. Matroids and transversals. Let $I := \{1, \ldots, n\}$ and let $\mathcal{A}(I) = (A_1, \ldots, A_n)$ be a family of subsets of a finite set S. If $K \subseteq I$ let

$$\mathcal{A}(K) := (A_i : i \in K) \quad \text{and} \quad A(K) := \bigcup_{i \in K} A_i.$$

A *transversal* or *system of distinct representatives* (*SDR*) of $\mathcal{A}(I)$ is a set $\{x_1, \ldots, x_n\}$ of distinct elements such that $x_i \in A_i$ for each i in I. A subset X of S is a *partial transversal* (*PT*) of $\mathcal{A}(I)$, of *length* $|X|$ and *defect* $n - |X|$, if X is a transversal of some subfamily $\mathcal{A}(J)$ of $\mathcal{A}(I)$ ($J \subseteq I$, $|J| = |X|$).

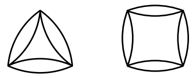
Theorem 3.9. (J. Edmonds and D. R. Fulkerson, 1965; L. Mirsky and H. Perfect, 1967.) If $\mathcal{A}(I)$ is a family of subsets of a set S, then the PTs of $\mathcal{A}(I)$ are the independent sets of a matroid on S, called a *transversal matroid*, with rank function r given by

$$r(X) = \min_{K \subseteq I} (|A(K) \cap X| + |I| - |K|) \quad \forall\, X \subseteq S.$$

Proof. The first result follows from the result of A. J. Hoffman and H. W. Kuhn (1956) that a PT is maximal (by inclusion) iff it is maximum (in length). (In particular, if $\mathcal{A}(I)$ has a transversal, then its PTs are the subsets of its transversals.) The rank function follows from the result of O. Ore (1955) that $\mathcal{A}(I)$ has a PT of defect d iff $|A(K)| \geqslant |K| - d \;\forall\, K \subseteq I$, so that X has a subset of t elements that is a PT iff $|A(K) \cap X| + |I| - |K| \geqslant t$, $\forall\, K \subseteq I$. //

Theorem 3.10. (a) (M. J. Piff and D. J. A. Welsh, 1970.) A transversal matroid is representable over every infinite field and every sufficiently large finite field.
(b) (J. De Sousa and D. J. A. Welsh, 1972.) A transversal matroid is graphic (and hence regular) iff it is binary.
(c) (J. A. Bondy, 1972.) A graphic matroid is transversal iff it is

the circuit matroid of a graph that has no subgraph homeomorphic to K_4 or a 'doubled up' circuit on at least 3 vertices:

 $\dots$ //

Theorem 3.11. *The Rado–Hall theorem* (R. Rado (1906–1989), 1942). If (S, r) is a matroid then $\mathcal{A}(I)$ has an independent transversal iff $r(A(K)) \geqslant |K|\ \forall\, K \subseteq I$. (Hall's theorem is the special case where (S, r) is the free matroid.)

Proof. (R. Rado, 1967.) *Only if* is obvious. *If*: We prove first that if $\mathcal{A}(I) = (A_1, A_2, \dots, A_n)$ satisfies the hypothesis and $|A_1| \geqslant 2$, then $\exists\, a \in A_1$ s.t. $(A_1 \setminus \{a\}, A_2, \dots, A_n)$ satisfies the hypothesis. The result will immediately follow, since we can reduce each set A_i to a singleton in this way, and the result is then obvious.

So suppose there is no $a \in A_1$ as required. Let $b, c \in A_1$. Then $\exists\, J, K \subseteq \{2, 3, \dots, n\}$ s.t.

$$r(A(J) \cup (A_1 \setminus \{b\})) < |J| + 1, \quad \text{i.e.,} \leqslant |J|,$$

$$\text{and} \quad r(A(K) \cup (A_1 \setminus \{c\})) \leqslant |K|.$$

But then

$$\begin{aligned} |J| + |K| &\geqslant r(A(J) \cup (A_1 \setminus \{b\})) + r(A(K) \cup (A_1 \setminus \{c\})) \\ &\geqslant r(A(J \cup K \cup \{1\})) + r(A(J \cap K)) && \text{by submodularity} \\ &\geqslant |J \cup K| + 1 + |J \cap K| && \text{by hypothesis} \\ &= |J| + |K| + 1, \ \Rightarrow\Leftarrow \quad // \end{aligned}$$

Theorem 3.12. *The matroid intersection theorem* (J. Edmonds, 1969). Two matroids $\mathbf{M}_1 = (S, r_1)$ and $\mathbf{M}_2 = (S, r_2)$ share an independent set X of rank r iff $r_1(X_1) + r_2(X_2) \geqslant r$ whenever $S = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$.

Proof. 'Only if' is obvious. We prove 'if' by induction on $|S|$, noting that it is obvious if $r = 0$ or $|S| = 1$. So suppose $r > 0$, $|S| \geqslant 2$ and the condition holds. Choose $x \in S$ s.t. $r_1(\{x\}) = r_2(\{x\}) = 1$ (which is clearly possible).

There is a set X as required with $x \notin X$ iff $\mathbf{M}_1|(S \setminus \{x\})$ and $\mathbf{M}_2|(S \setminus \{x\})$ share an independent set of rank r, i.e. (by induction) iff $r_1(X_1) + r_2(X_2) \geqslant r$ whenever $S \setminus \{x\} = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$.

There is a set X as required with $x \in X$ iff $\mathbf{M}_1 \cdot (S \setminus \{x\})$ and $\mathbf{M}_2 \cdot (S \setminus \{x\})$ share an independent set of rank $r - 1$, i.e. (by induction) iff

$$r_1(Y_1 \cup \{x\}) - 1 + r_2(Y_2 \cup \{x\}) - 1 \geqslant r - 1,$$

that is,

$$r_1(Y_1 \cup \{x\}) + r_2(Y_2 \cup \{x\}) \geqslant r + 1,$$

whenever $S \setminus \{x\} = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \emptyset$.

If *both* conditions fail, for some X_1, X_2, Y_1, Y_2, then

$$r_1(X_1) + r_2(X_2) \leqslant r - 1,$$

$$r_1(Y_1 \cup \{x\}) + r_2(Y_2 \cup \{x\}) \leqslant r,$$

and so

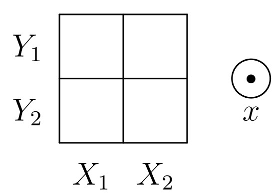
$$r_1(X_1 \cup Y_1 \cup \{x\}) + r_1(X_1 \cap Y_1) + r_2(X_2 \cup Y_2 \cup \{x\}) + r_2(X_2 \cap Y_2)$$
$$\leqslant 2r - 1 \text{ by submodularity.}$$

So either the representation

$$S = (X_1 \cup Y_1 \cup \{x\}) \cup (X_2 \cap Y_2)$$

or

$$S = (X_2 \cup Y_2 \cup \{x\}) \cup (X_1 \cap Y_1)$$



violates the hypothesis of the theorem, $\Rightarrow\Leftarrow$. //

3.5. Matroids induced by graphs. Let G be a graph or digraph. If $X, Y \subseteq V(G)$, X is *linked onto* Y by Π if $|X| = |Y|$ and Π is a set of $|X|$ disjoint XY-paths in G.

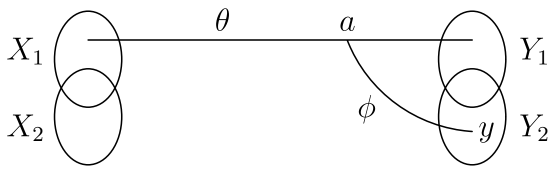
Theorem 3.13. (R. A. Brualdi, 1971; J. H. Mason, 1972.) If $(V(G), \mathbf{I})$ is a matroid, let $\mathbf{I}_G := \{X \subseteq V(G) : X \text{ is linked onto some set in } \mathbf{I}\}$. Then $(V(G), \mathbf{I}_G)$ is a matroid with the same rank as $(V(G), \mathbf{I})$, called the matroid *induced* from $\mathbf{I}$ by G.

Proof. (I1) and (I2) are straightforward, as is the rank. To prove (I3), let Θ and Φ be sets of disjoint paths linking sets X_1, X_2 onto sets Y_1, Y_2 in $\mathbf{I}$, where $|\Theta| = |X_1| = |Y_1| < |\Phi| = |X_2| = |Y_2|$. We must find $x \in X_2 \setminus X_1$ such that $X_1 \cup \{x\}$ is linked onto some set in $\mathbf{I}$.

If $\theta \in \Theta$, $\phi \in \Phi$ and $a \in V(\theta) \cap V(\phi)$, let $\theta a \phi$ denote the walk that follows θ as far as a and ϕ thereafter. Let

$$\Pi = \Pi(\Theta, \Phi) := \Theta \cup \Phi \cup \{\theta a \phi : \theta \in \Theta, \phi \in \Phi, a \in V(\theta) \cap V(\phi)\}.$$

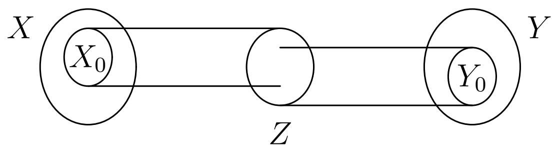
We prove the result by induction on $|\Pi|$.



Choose $y \in Y_2 \setminus Y_1$ s.t. $Y_1 \cup \{y\} \in \mathbf{I}$, and let $\phi \in \Phi$ be the path from some $x \in X_2$ to y. If ϕ does not meet any path in Θ, then $X_1 \cup \{x\}$ is linked onto $Y_1 \cup \{y\}$ by $\Theta \cup \{\phi\}$, and so $X_1 \cup \{x\} \in \mathbf{I}_G$ as required. Otherwise, follow ϕ backwards from y until it first hits a path $\theta \in \Theta$, at a, say. Let $\Theta' := \Theta \cup \{\theta a \phi\} \setminus \{\theta\}$. Θ' is a set of disjoint paths linking X_1 onto a set $Y_1' \subset Y_1 \cup \{y\} \in \mathbf{I}$. And $|\Pi(\Theta', \Phi)| < |\Pi(\Theta, \Phi)|$ since $\theta \notin \Pi(\Theta', \Phi)$. So the result follows by the induction hypothesis. //

Corollary 3.13.1. *The linkage theorem* (*finite case*) (J. S. Pym, 1969.) Let $X_0 \subseteq X \subseteq V(G)$ and $Y_0 \subseteq Y \subseteq V(G)$. Suppose that

X_0 is linked onto some subset of Y, and some subset of X is linked onto Y_0. Then $\exists\, X', Y'$ s.t. $X_0 \subseteq X' \subseteq X$, $Y_0 \subseteq Y' \subseteq Y$ and X' is linked onto Y'.



Proof. Let Z be a smallest set of vertices separating Y from X. By Menger's theorem, G contains $|Z|$ disjoint XY-paths, and hence $|Z|$ disjoint XZ-paths and $|Z|$ disjoint ZY-paths. Let $\mathbf{I}$ be the free matroid on Z. Then $X_0 \in \mathbf{I}_G$, and X contains a base of $\mathbf{I}_G$, and so X_0 can be extended to a base $X' \subseteq X$ for $\mathbf{I}_G$, which is linked onto Z. Similarly, Y_0 can be extended to a set $Y' \subseteq Y$ such that Z is linked onto Y'. Now clearly X' is linked onto Y'. //

Corollary 3.13.2. (H. Perfect, 1969.) Let G be a bipartite graph on two sets S, T. Let $(T, \mathbf{I})$ be a matroid and define $\hat{\mathbf{I}} := \{X \subseteq S : X$ is linked onto a set in $\mathbf{I}$ by disjoint edges$\}$. Then $(S, \hat{\mathbf{I}})$ is a matroid. //

Corollary 3.13.3. (C. St J. A. Nash-Williams (1932–2001), 1967.) Let $(T, \mathbf{I})$ be a matroid, $f : S \to T$ and $g : T \to S$ functions,

$$\mathbf{I}_1 := \{X \subseteq S : f|_X \text{ is injective and } f(X) \in \mathbf{I}\},$$
$$\mathbf{I}_2 := \{g(Y) : Y \in \mathbf{I}\}.$$

Then $(S, \mathbf{I}_1)$ and $(S, \mathbf{I}_2)$ are matroids. //

Corollary 3.13.4. (J. Edmonds, 1969.) Let $\mathbf{I}_1, \ldots, \mathbf{I}_k$ be k independence structures on S, and let $\mathbf{I} := \{X_1 \cup \ldots \cup X_k : X_j \in \mathbf{I}_j$ for each $j\}$. Then $(S, \mathbf{I})$ is a matroid. $\mathbf{I}$ is called the *union*, *sum* or *join* of $\mathbf{I}_1, \ldots, \mathbf{I}_k$, denoted by $\mathbf{I}_1 \vee \ldots \vee \mathbf{I}_k$ or $\bigvee_{j=1}^{k} \mathbf{I}_j$.

Proof. Let $T_1, \ldots, T_k$ be k disjoint copies of S, and $\mathbf{I}'_j$ an independence structure on T_j isomorphic to $\mathbf{I}_j$ on S $(j = 1, \ldots, k)$. Let $T := \bigcup_{j=1}^{k} T_j$ and $\mathbf{I}' := \{X'_1 \cup \ldots \cup X'_k : X'_j \in \mathbf{I}'_j \text{ for each } j\}$. Since the sets T_j are disjoint, it is easy to see that $\mathbf{I}'$ is an independence structure on T. Now join each $s \in S$ to its copies in $T_1, \ldots, T_k$, and $(S, \mathbf{I})$ is the matroid induced by $(T, \mathbf{I}')$ as in Corollary 3.13.2. //