Colourfully Panconnected Subgraphs

A set \( X \subseteq V(G) \) is \( \text{distance-}k\text{-connected} \) in \( G \) if, \( \forall u, w \in X \), there is a sequence \( u = x_0, x_1, \ldots, x_l = w \) of vertices of \( X \) such that the distance \( d_G(x_{i-1}, x_i) \leq k \) for each \( i \) (1 \( \leq i \leq l \)).

**Theorem 1.** (J. Ouyang (unpublished manuscript, 1994), J. Griggs and E. Czabarka.) If \( G \) is a connected \( k\)-chromatic graph, then \( G \) has a (proper vertex) \( k\)-colouring such that each colour class is distance-\( k\)-connected in \( G \).

This was conjectured by Chen, Schelp and Shreve (DM 170 (1997) 231–236), who note that ‘distance-\( k\)-connected’ is best possible—consider two copies of \( K_k \) connected by a long path.

If \( v \in V(G) \), a \( k\)-colouring of \( G \) is:
- **variegated at \( v \)** if the \( k \) colours can be ordered as \( c_0, c_1, \ldots, c_{k-1} \) so that, \( \forall i \), colour \( c_i \) occurs within distance \( i \) of \( v \) (so \( v \) has colour \( c_0 \));
- **variegated** if it is variegated at every vertex;
- **panconnected** if, for each \( i \) (1 \( \leq i \leq k \)), the union of any \( i \) colour classes is distance-(\( k+1-i \))-connected in \( G \).

Ouyang’s proof of Theorem 1 can be modified to prove:

**Theorem 2.** (O. V. Borodin and D. R. Woodall, 1996.) Let \( G \) be a connected \( k\)-colourable graph with at least \( k \) vertices. Then \( G \) has a variegated panconnected \( k\)-colouring.

A subgraph \( H \) of \( G \) is **\( k\)-colourfully panconnected (\( k\)-cp)** in \( G \) if \( G \) has a \( k\)-colouring that induces a variegated panconnected \( k\)-colouring of \( H \); clearly then \( H \) is connected and \( |V(H)| \geq k \). If \( G \) is a connected \( k\)-colourable graph with \( n \geq k \) vertices, let

\[
s_k(G) := \min\{|E(H)| : H \text{ is a spanning } k\text{-cp subgraph of } G\};
\]
this makes sense because, by Theorem 2, $G$ is a spanning $k$-cp subgraph of itself. If $G$ is bipartite then clearly $s_2(G) = n - 1$: any spanning tree will do for $H$. So suppose $k \geq 3$.

**Conjecture.** (Borodin and Woodall, 1996.) If $G$ is a connected $k$-colourable graph with $n \geq k \geq 4$ vertices, then $s_k(G) \leq n$; that is, $G$ has an at-most-unicyclic spanning $k$-cp subgraph. [Moreover, if $G$ has an edge that is contained in no circuit with length $\equiv 1 \pmod{k}$, then $s_k(G) = n - 1$.]

**Theorem 3.** Assume $n \geq k \geq 3$.

(a) $s_k(C_n) = \begin{cases} n & \text{if } n \equiv 1 \pmod{k}, \\ n - 1 & \text{otherwise.} \end{cases}$

(b) $s_k(K_{r,s}) = \begin{cases} n + \delta - 2 & \text{if } k = 3, \\ n - 1 & \text{if } k \geq 4, \end{cases}$

where $n = r + s$, $\delta = \min\{r, s\}$.

**Proof.** (a) Clearly $s_k(C_n) = n - 1$ or $n$. Let $P : v_0v_1\ldots v_{n-1}$ be a spanning tree (path) of $C_n$. It is easy to see that $c : V(P) \rightarrow \{0, 1, \ldots, k - 1\}$ is a variegated panconnected $k$-colouring of $P$ if and only if (after possibly relabelling the colours) $c(v_i) \equiv i \pmod{k}$, $\forall i$. This gives a proper $k$-colouring of $C_n$ iff $n \not\equiv 1 \pmod{k}$. If $n \equiv 1 \pmod{k}$, recolour $v_{n-1}$ with colour 1 to give a variegated panconnected $k$-colouring of $C_n$ itself.

(b) Let $G = K_{r,s}$. If $\delta = 1$, then $s_k(G) = n - 1$ for all $k$. (Any colouring using all $k$ colours will do.) If $\delta \geq 2$ and $k \geq 4$, then colourings and spanning trees as in Fig. 1(a) will work. So suppose $\delta \geq 2$ and $k = 3$. It follows from Theorem 5 or Fig. 1(b) that $s_3(G) \leq n + \delta - 2$. We must prove that $s_3(G) \geq n + \delta - 2$.

Let $G$ have partite sets $X, Y$. Let $H$ be a spanning 3-cp subgraph of $G$, and let $c : V(G) \rightarrow \{0, 1, 2\}$ be a 3-colouring that induces a variegated panconnected 3-colouring of $H$. Let $X_i$
be the set of vertices with colour $i$ ($i = 0, 1, 2$). W.l.o.g. $X = X_0$, $Y = X_1 \cup X_2$. Each vertex $v$ of $X$ is adjacent in $H$ to vertices in $X_1$ and $X_2$ since $c|_H$ is variegated at $v$. So, for each such $v$, choose edges $vv_1, vv_2$ of $H$ with $v_1 \in X_1$, $v_2 \in X_2$ and let $H'$ be the subgraph of $H$ comprising the $2|X|$ edges so chosen. Note that, in $H'$, $d(u, w) \geq 4$ if $u, w \in X_1$ or $u, w \in X_2$. Since $X_1$ and $X_2$ are distance-3-connected in $H$, $E(H) \setminus E(H')$ must contain at least $|X_i| - 1$ edges incident with $X_i$ ($i = 1, 2$). Hence

$$|E(H)| \geq 2|X_0| + (|X_1| - 1) + (|X_2| - 1) = n + |X_0| - 2 \geq n + \delta - 2.$$ 

It follows that $s_3(G) \geq n + \delta - 2$, as required. 

**Theorem 4.** Let $v_0v_1$ be an edge in a connected $k$-colourable graph $G$. Then there is a $k$-colouring $c : V(G) \to \{0, 1, \ldots, k - 1\}$, and a spanning tree $T$ of $G$ containing edge $v_0v_1$, such that $c(v) \equiv d_T(v_0, v) \pmod{k}$, $\forall v \in V(G)$.

**Proof.** Choose a $k$-colouring $c$ of $G$ and permute colours if necessary so that $c(v_0) = 0$, $c(v_1) = 1$. Set $T = \{v_0, v_0v_1, v_1\}$.

**Construction 1.** (Growing the tree.) While $\exists uw \in E(G)$ s.t. $u \in V(T)$, $w \in V(G) \setminus V(T)$ and $c(w) \equiv c(u) + 1 \pmod{k}$, add $w$ and $uw$ to $T$. If now (when $\not\exists$ such an edge $uw$) $V(T) = V(G)$, then $T$ is the required spanning tree, so stop; otherwise, invoke Construction 2.

**Construction 2.** (Changing the colouring.) Choose $uw \in E(G)$ s.t. $u \in V(T)$, $w \in V(G) \setminus V(T)$ and $c(w) \equiv c(u) + r \pmod{k}$, where $r \geq 2$ is as small as possible. Set $T' = \{w\}$, and while $\exists w' \in V(T')$, $u' \in V(G) \setminus V(T')$ s.t. $u'w' \in E(G)$ and $c(u') \equiv c(w') - 1 \pmod{k}$, add $u'$ and $u'w'$ to $T'$. Note that $T' \cap T = \emptyset$, since if $c(u') \equiv c(w') - 1$ then $c(w') \equiv c(u') + 1$ and so if $u' \in T$ then $w' \in T$, $\Rightarrow \Leftarrow$. Now reduce $c(v)$ by 1 (mod $k$), $\forall v \in V(T')$. The new colouring $c$ is still
proper, by the definition of $T'$, and now $c(w) \equiv c(u) + r - 1 \pmod{k}$. If $r - 1 > 1$, repeat Construction 2 a further $r - 2$ times. Eventually $c(w) \equiv c(u) + 1 \pmod{k}$, and $w$ and $uw$ can be added to $T$. So return to Construction 1, and iterate until $V(T) = V(G)$. 

**Corollary 4.1.** Let $G$ be a connected $k$-chromatic graph that contains a vertex $v_0$ that is adjacent to vertices of all other colours in every $k$-colouring of $G$. (E.g., $k = 3$ and $G$ contains a triangle, or $k = 4$ and $G$ contains a wheel with odd circumference.) Then $s_k(G) \leq n$.

**Proof.** Choose $v_1$ arbitrarily in $N(v_0)$, and form $T$ and a $k$-colouring of $G$ as in Theorem 4. Let $e$ be any edge of $G$ joining $v_0$ to a vertex of colour $k - 1$, and let $H := T \cup \{e\}$. Then $H$ is a $k$-cp subgraph of $G$ with $n$ edges. 

A graph is $r$-degenerate if every subgraph of it contains a vertex with degree $\leq r$, or, equivalently, if its vertices can be ordered so that each is adjacent to at most $r$ earlier vertices.

**Theorem 5.** Let $G$ be a connected $k$-colourable graph with $n \geq k$ vertices, where $k = 3$ or 4. Then $G$ has a 2-degenerate spanning $k$-cp subgraph with at most $n + \delta - 2$ edges; hence $s_k(G) \leq n + \delta - 2$. Moreover, if $G$ has an edge $e$ that is contained in no circuit with length $\equiv 1 \pmod{k}$, then $s_k(G) = n - 1$.

**Proof.** Form $T$ and a $k$-colouring of $G$ as in Theorem 4, where $v_0v_1$ is the edge $e$ if it exists and is otherwise arbitrary subject to $d(v_0) = \delta$.

If $T$ has no vertex of colour 2, then $G$ is a star and the result holds (cf. Theorem 3(b)). If $k = 4$ and $T$ has no vertex of colour 3, then recolour an arbitrary $T$-neighbour of $v_0$ with colour 3 unless $d_T(v_0) = 1$, in which case recolour an arbitrary $T$-neighbour of $v_1$
(other than $v_0$) with colour 3. In either case, $T$ is a $k$-cp subgraph of $G$ and $s_k(G) = n - 1$; and this also holds whenever $d_T(v_0) = 1$. So we may suppose that all colours do occur in $T$ and that $d_T(v_0) \geq 2$.

Let the components of $T - v_0$ be $T_1, \ldots, T_r$, where $v_1 \in T_1$. Let $X_i$ be the set of vertices with colour $i$. $T$ may fail to be a $k$-cp subgraph for three reasons:

(i) if there is ‘short branch’ ($T_3$ or $T_5$ in Fig. 2) then the colouring of $T$ is not variegated at vertices in it;

(ii) $X_{k-1}$ may not be distance $k$-connected in $T$ (although $X_{k-1} \cap T_i$ is, $\forall i$);

(iii) if $k = 4$ then $X_2 \cup X_3$ may not be distance-3-connected in $T$ (although, again, $(X_2 \cup X_3) \cap T_i$ is, $\forall i$).

If $v_0v_1$ is contained in no circuit of $G$ with length $\equiv 1 \pmod{k}$, then we can interchange colours 1 and $k - 1$ throughout $T_1$ to obtain a proper $k$-colouring of $G$ that is a variegated panconnected $k$-colouring of $T$. So $s_k(G) = n - 1$.

The same conclusion will follow if we can interchange colours 1 and $k - 1$ throughout any proper nonempty subset of $T_1, \ldots, T_r$; so suppose we can’t. Then we can reorder $T_2, \ldots, T_r$ if necessary so that, for each $i$ ($1 \leq i \leq r - 1$), there is an edge $e_i$ of $G$ joining a vertex of colour 1 or $k - 1$ in $T_1 \cup \ldots \cup T_i$ to a vertex of colour $k - 1$ or 1 in $T_{i+1}$ (Fig. 3). Adding these $r - 1$ edges to $T$ will create a 2-degenerate spanning $k$-cp subgraph $H$ of $G$ with $n - 1 + r - 1 \leq n + \delta - 2$ edges. \hfill //

**Proof of Theorem 1.** The result is obvious if $k \leq 2$, and it follows from Theorem 5 if $k = 3$ or 4. The following argument works for all $k \geq 4$.

Let $c$ be a $k$-colouring of a connected $k$-chromatic graph $G$. Let $v$ be a vertex adjacent to vertices of all other colours (which exists since $G$ is $k$-chromatic), and let $H := \langle \{v\} \cup N(v) \rangle_G$. Let
$\partial(H) := V(H) \cap N(V(G) \setminus V(H))$. We say that $c$ is strongly variegated at a vertex $x$ if, for each $i$ ($2 \leq i \leq k - 2$), within distance $i$ of $x$ there are vertices with at least $i + 1$ colours different from that of $x$. Then the induced colouring $c_H$ of $H$ has the following properties:

P1. each $c_H$-colour-class is distance-$k$-connected in $H$;
P2. $c_H$ is variegated;
P3. $c_H$ is strongly variegated at all but at most one vertex in $\partial(H)$.

(In fact, P3 holds with no such exceptional vertex.) We now modify $c$ on vertices outside $H$, and add vertices and edges to $H$, while preserving P1–P3, until $\partial(H) = \emptyset$, at which point the result will be proved.

So suppose $\partial(H) \neq \emptyset$ and choose $x \in \partial(H)$ such that $c_H$ is strongly variegated at all vertices in $\partial(H) \setminus \{x\}$. Let $u \in N(x) \setminus V(H)$. Choose a colour $c_x$ whose closest occurrence to $x$ in $H$ is as far from $x$ as possible (at distance $\leq k - 1$). There are three cases.

Case 1: $c(u) = c_x$. Then add $u$ and the edge $ux$ to $H$. It is easy to see that P1–P3 still hold, with $u$ being the exceptional vertex in P3 if there is one. (This is possible even if $x$ is not exceptional.)

Case 2: $u$ is adjacent to a vertex $y$ of colour $c_x$ in $H$. Add $u$ and the edges $ux, uy$ to $H$. Then P1–P3 hold, with $u$ again being the exceptional vertex if there is one.

Case 3: neither of the previous cases arises. For each vertex of colour $c(u)$ or $c_x$ in $V(G) \setminus (V(H) \cup \{u\})$ in turn, change its colour to be different from both $c(u)$ and $c_x$ if possible. Let $C$ be the Kempe chain in $G$ with colours $c(u)$ and $c_x$ containing $u$. If $C \cap H = \emptyset$ then interchange colours in $C$ and we are back in Case 1. Otherwise, $\exists w \in V(G) \setminus (V(H) \cup \{u\})$ and $z \in N(w) \cap V(H)$ s.t. $w, z \in C$. Note that $w$ is adjacent to vertices of all other colours in $G$, since otherwise its colour would have been changed at the start of Case 3. Add $w$ and all its neighbours and neighbouring edges to $H$. Since the old $c_H$ was strongly variegated at $z$, P1–P3 still hold, with $x$ remaining the exceptional vertex if there is one. //