A Proportionality Problem

(P. C. Fishburn, F. K. Hwang and H. Lee, 1986; D. R. Woodall, 1992.) Let \( \mathcal{R} = (v_1, \ldots, v_n, v_1) \) be a ring of green and blue vertices. If \( 0 \leq l \leq r \), let

\[
N_{l,r}(v_i) := \{ v_{i-l}, \ldots, v_{i-l+1}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i+r} \}
\]

(subscripts reduced modulo \( n \), repetition allowed). For any set (or multiset) \( S \), let \( G(S) \) denote the number of green vertices in \( S \), and \( P(S) := G(S)/|S| \). \( \mathcal{R} \) has property \((l, r, p)\) if \( G(\mathcal{R}) > 0 \) and \( P(N_{l,r}(v)) \geq p \) for each green vertex \( v \) in \( \mathcal{R} \). What does this tell us about \( P(\mathcal{R}) \)?

**Example 1.** If \( \mathcal{R} \) is periodic, \( l \) and \( r \) are multiples of the period, and \( P(\mathcal{R}) = p \), then \( P(N_{l,r}(v)) = p \) for every vertex in \( \mathcal{R} \). Thus property \((l, r, p)\) does not imply \( P(\mathcal{R}) > p \) in general.

**Example 2.** A simple ring \( \mathcal{R}(a, b) \) consists of \( a \) green vertices followed by \( b \) blue vertices. If \( p \leq \frac{l}{r+l} \) then \( \mathcal{R}(l+1, n-l-1) \) has property \((l, r, p)\), but \( P(\mathcal{R}) = \frac{l+1}{n} \) can be arbitrarily small, and so no conclusion can be drawn (unless \( p = 0 \)).

**Example 3.** If \( \frac{2l}{r+l} < p < \frac{1}{2} \) (\( \Leftrightarrow \) \( l < \frac{1}{2}r \)) and \( p(r + l) \in \mathbb{Z} \), define \( a := p(r + l) - l \) and \( b := r - l - a \), so that \( l < a < \frac{1}{2}(r - l) < b \). In \( \mathcal{R}(a, b) \), since \( r - l = a + b \) is the period, \( N_{l,r}(v) = N_{l,l}(v) \cup (1 \text{ period}) \), so that \( P(N_{l,r}(v)) \geq \frac{l+a}{r+l} = p \) if \( v \) is green. Thus \( \mathcal{R}(a, b) \) has property \((l, r, p)\), and

\[
P(\mathcal{R}(a, b)) = \frac{a}{a+b} = \frac{p(r + l) - l}{r - l}.
\]

**Conjectures.** Suppose \( l \leq r \) and \( \mathcal{R} \) has property \((l, r, p)\).
1. $P(\mathcal{R}) \geq p$ if $p > \frac{1}{2}$, or if $p = \frac{1}{2}$ and $l \neq r$.

2. $P(\mathcal{R}) \geq \frac{p(r+l)-l}{r-l}$ if $\frac{l}{r+l} < p < \frac{1}{2}$.

3. $P(\mathcal{R}) \geq \frac{r-\sqrt{r^2-2p(r^2-l^2)}}{2(r-l)}$ if $\frac{l}{r+l} < p < \frac{1}{2}$.

Note that all bounds $= \frac{1}{2}$ when $p = \frac{1}{2}$, and $(2) > (3)$ iff $\frac{2l}{r+l} < p < \frac{1}{2}$.

All three conjectures hold for simple rings.

**Theorem 1.** Conjecture 1 holds for simple rings.

**Proof.** Let $\mathcal{R} = \mathcal{R}(a,b)$ be a simple ring with property $(l,r,p)$, where $l \leq r$, $p \geq \frac{1}{2}$, and $p > \frac{1}{2}$ if $l = r$. Let $v_1, \ldots, v_b$ be blue and $v_{b+1}, \ldots, v_{b+a} = v_0$ be green. Let

$$r = n(a+b) + r', \quad l = m(a+b) + l',$$

where $n \geq 0$, $m \geq 0$, $0 \leq r' < a+b$, $0 \leq l' < a+b$. If $b \leq l' < a+b$ then $v = v_{l'+1}$ is green and $N_{l,r}(v)$ starts with $b$ consecutive blue vertices, so that

$$P(\mathcal{R}) = \frac{a}{a+b} > P(N_{l,r}(v)) \geq p.$$  

Thus we may suppose that $0 \leq l' < b$, and similarly $0 \leq r' < b$.

Now at least one $v \in \{v_0, v_{b+1}\}$ has the property that $G(N_{l',r'}(v)) \leq \min\{r', l'\}$, so that

$$\frac{1}{2}(r + l) \leq p(r + l) \leq G(N_{l,r}(v)) \leq (n + m)a + \min\{r', l'\} \leq (n + m)a + \frac{1}{2}(r' + l').$$

If $n = m = 0$, then $r = r'$, $l = l'$, and we have a contradiction since $\min\{r, l\} < \frac{1}{2}(r + l)$ unless $r = l$, when $\frac{1}{2} < p$. So $n + m > 0$ and

$$P(\mathcal{R}) = \frac{a}{a+b} = \frac{(n + m)a}{n(a+b) + m(a+b)} \geq \frac{p(r+l) - \frac{1}{2}(r'+l')}{(r-r') + (l-l')} \geq \frac{p(r+l-r'-l')}{r+l-r'-l'} = p. \quad //$$
Theorem 2. Conjecture 1 holds in the following cases.

(a) \( l = 0 \) (even if \( p < \frac{1}{2} \)).
(b) \( l = r \) (and \( p > \frac{1}{2} \)).
(c) \( l = r - 1 \) (and \( p \geq \frac{1}{2} \)).

Proof. (a) is obvious if \( p = 0 \). In all other cases, \( p > \frac{l}{r+l} \), so \( G(N_{0,r}(v)) > 0 \) for every green vertex \( v \). Let \( g_1 \) be any green vertex. Given \( g_i \), let \( g_{i+1} \) be the green vertex in \( N_{0,r}(g_i) \) furthest from \( g_i \). Continue until the first repetition: \( g_{k} = g_{j} \) for some \( j < k \).

(a) The intervals \((g_i, g_{i+1}) \) \((j \leq i \leq k - 1)\) cover \( \mathcal{R} \) uniformly, and \( P((g_i, g_{i+1})) \geq P(N_{0,r}(g_i)) \geq p \) for each \( i \), so that \( P(\mathcal{R}) \geq p \).

(b) and (c): Define multisets

\[
A_i := (g_i, g_{i+1}) + [g_i, g_{i+1}), \quad B_i := N_{0,r}(g_i) + N_{l,0}(g_{i+1}),
\]

\[
A := \sum_{i=j}^{k-1} A_i, \quad B := \sum_{i=j}^{k-1} B_i = \sum_{i=j}^{k-1} N_{l,r}(g_i)
\]

since \( g_k = g_j \). \( A \) covers \( \mathcal{R} \) uniformly, so that \( P(A) = P(\mathcal{R}) \), and \( P(B) \geq p \) by hypothesis. It remains to prove that \( P(A) \geq P(B) \). Let \( v_i^{+t} := v_{i+t} \).

(b) Here \( l = r \). For any \( i \) \((j \leq i \leq k - 1)\), suppose that \( g_{i+1} = g_i^{+t} \) and let \( s := r - t \geq 0 \), so that \( g_i^{+s} = g_i^{+r} \). \( A_i \) is formed from \( B_i \) by deleting \( s \) blue vertices (in \([g_i^{+1}, g_i^{+s}]\)) and \( s \) vertices of unknown colour (in \([g_i^{-s}, g_i^{-1}]\)). This holds for every \( i \). Since at most half the deleted vertices are green, and since \( P(B) \geq p > \frac{1}{2} \), it follows that \( P(A) \geq P(B) \) as required.

(c) Now \( l = r - 1 \). If \( s \) (as in (b)) = 0, then \( A_i \) is formed from \( B_i \) by adding one green vertex \((g_i)\). Otherwise, \( A_i \) is formed from \( B_i \) by deleting \( s \) blue vertices and \( s - 1 \) vertices of unknown colour. Since \( s - 1 < s \), the required conclusion follows as before. //
**Exercise.** \( P(\mathcal{R}) \geq p \) if and only if, for each green vertex \( v \) in \( \mathcal{R} \), \( \exists r(v) > 0 \) s.t. \( P(N_{0,r(v)}(v)) \geq p \).

**Theorem 3.** (B. J. Tarlow, 1998.) Conjecture 1 holds if \( l \leq 3 \).

**Proof.** W.l.o.g. \( p = \frac{g}{r+l} \) for some \( g \in \mathbb{N}, g > l \). It suffices to prove that, for each green vertex \( v \) in \( \mathcal{R} \), \( \exists l(v) > 0 \) s.t. \( P(N_{l(v),0}(v)) \geq p \). Given \( v \), relabel \( \mathcal{R} \) so that \( v = v_{r+1} \).

If \( v_0 \) is green, then \( P([v_l, v_r]) > p \); so assume \( v_0 \) is blue.

If \( v_1 \) is green, then \( P([v_{l+1}, v_r]) \geq p \); so assume \( v_1 \) is blue.

Choose \( s, t > 0 \) minimal such that \( v_s, v_{-t} \) are green. Let there be \( n_G \) green vertices in \([v_2, v_r]\) (or, equivalently, in \([v_s, v_r]\)). Then \( g \leq G(N_{l,r}(v_{-l})) \leq l + n_G \), and so \( n_G \geq g - l \).

Suppose \( n_G = g - l \). If \( s \leq l \) then \( N_{l,r}(v_s) \) contains \( s \) blue vertices \( v_0, \ldots, v_{s-1} \) and the \( s \) vertices \( v_{r+1}, \ldots, v_{r+s} \), and so \( G(N_{l,r}(v_s)) \leq (l - s) + (g - l - 1) + s = g - 1 \), a contradiction. Thus \( s \geq l + 1 \), and so \( P([v_{l+1}, v_r]) = \frac{g-l}{r-l} \geq \frac{g}{r+l} = p \) since \( \frac{1}{2} \leq p \).

Thus we may assume that \( n_G \geq g - l + 1 \geq g - \frac{1}{2}l - \frac{1}{2} \), since \( l \leq 3 \). But then \( P([v_2, v_r]) \geq \frac{g-\frac{1}{2}l-\frac{1}{2}}{r-1} \geq \frac{g}{r+l} = p \). //

Conjecture 1 has been proved also when \( l = 4 \) and \( l = r - 2 \), and (by Ben Tarlow) for ‘rings of order 4’.

**Exercise.** If \( p > \frac{l}{r+l} \) and \( \mathcal{R} \) has property \((l, r, p)\), then \( P(\mathcal{R}) \geq \frac{p(r+l) - l}{r} \).