Restricted Colourings of Graphs

1. **Vertex-colourings.** If \( c : V \to \{1, \ldots, k\} \) is an improper vertex-\( k \)-colouring of a finite graph \( G = (V, E) \), let \( d_i(v) \) denote the number of neighbours of vertex \( v \) with colour \( i \), and let \( d(v) \) denote the degree of \( v \). Call an edge *bad* if its endvertices have the same colour.

**Theorem 1.1.** (First version, O. V. Borodin and A. V. Kostochka, 1977.) Let \( f_1, f_2 : V \to \mathbb{Z} \) satisfy \( f_1(v) + f_2(v) \geq d(v) - 1 \), \( \forall v \in V \). Then there exists a 2-colouring \( c : V \to \{1, 2\} \) such that \( d_{c(v)}(v) \leq f_{c(v)}(v), \forall v \in V \).

**Exercise.** Show that \( f_1(v) + f_2(v) \geq d(v) - 2 \) is not enough.

**Proof.** Choose a 2-colouring \( c \) that minimizes

\[
N_c := B_c - \frac{1}{2} \sum_{v \in V} (f_{c(v)}(v) - f_{3-c(v)}(v)),
\]

where \( B_c \) is the number of bad edges. If \( d_{c(v)}(v) > f_{c(v)}(v) \) for some \( v \in V \), change the colour of \( v \) from \( c(v) \) to \( 3 - c(v) \). Then \( B_c \) increases by \( d(v) - 2d_{c(v)}(v) \), and so \( N_c \) increases by

\[
\begin{align*}
d(v) - 2d_{c(v)}(v) + f_{c(v)}(v) - f_{3-c(v)}(v) \\
\leq d(v) - 2d_{c(v)}(v) + 2f_{c(v)}(v) - d(v) + 1 \\
= 2[f_{c(v)}(v) - d_{c(v)}(v)] + 1 \\
< 0, \quad \Rightarrow \Leftarrow.
\end{align*}
\]

Therefore \( d_{c(v)}(v) \leq f_{c(v)}(v) \) for all \( v \in V \).

**Corollary 1.1.1.** Let \( f_1, \ldots, f_k : V \to \mathbb{Z} \) satisfy \( \sum_i f_i(v) \geq d(v) - k + 1, \forall v \in V \), where \( k \geq 1 \). Then there exists a \( k \)-colouring \( c : V \to \{1, \ldots, k\} \) such that \( d_{c(v)}(v) \leq f_{c(v)}(v), \forall v \in V \).

**Proof.** Exercise. (Induction on \( k \).)
Theorem 1.1. (Second version, C. Bernardi, DM 64 (1987) 95–96.) Let $f_1, f_2 : V \to \mathbb{R}$ satisfy $f_1(v) + f_2(v) > d(v), \forall v \in V$. Then there exists a 2-colouring $c : V \to \{1, 2\}$ such that $d_{c(v)}(v) < f_{c(v)}(v), \forall v \in V$.

Proof. W.l.o.g. $f_1(v)$ and $f_2(v)$ are integers whose sum is $d(v) + 1$. Let $m(v) := \max\{f_1(v), f_2(v)\}$. If $f_1(v) > f_2(v)$, join $v$ to $f_1(v) - f_2(v) = m(v) - f_2(v)$ new independent vertices permanently coloured 2; and analogously if $f_2(v) > f_1(v)$. Do this for all $v \in V$ to form a new graph $G'$ in which each (old) $v$ has degree $2m(v) - 1$, and choose a 2-colouring $c : V \to \{1, 2\}$. Now, $v$ is adjacent to < $f_{c(v)}(v)$ vertices of colour $c(v)$ in $G$ if and only if $v$ is adjacent to < $m(v)$ vertices of colour $c(v)$ in $G'$.

If this is not so, change the colour of $v$ and it will be so. Since this change reduces the number of bad edges in $G'$, a finite number of repetitions will achieve the desired result. 

Exercise. Prove that the two versions of Theorem 1.1 are equivalent.

Exercise. State and prove the second version of Corollary 1.1.1.

Theorem 1.2. (R. H. Cowen and W. Emerson, unpublished?) Let $p_1, \ldots, p_k$ be such that $0 \leq p_i \leq 1, \forall i$, and $\sum_{i=1}^{k} p_i \geq 1$. Then there is a $k$-colouring $c : V \to \{1, \ldots, k\}$ s.t. $d_{c(v)}(v) \leq p_{c(v)} d(v)$ for each $v \in V$.

Exercise. Show that this does not follow if $\sum_{i=1}^{k} p_i < 1$.

Exercise. Derive Theorem 1.2 from Theorem 1.1.

Proof. Choose a $k$-colouring $c$ that minimizes $N_c := \sum_i \frac{b_i}{p_i}$, where $b_i$ is the number of edges with both ends coloured $i$. If $v$ has colour $i$ and $d_i(v) > p_i d(v)$, then $\exists j$ s.t. $d_j(v) < p_j d(v)$. Change the colour of $v$ from $i$ to $j$. $N_c$ goes down by $\frac{d_i(v)}{p_i} - \frac{d_j(v)}{p_j} > d(v) - d(v) = 0$, $\Rightarrow \Leftarrow$. 

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2. Edge-colourings. Let $G = (V,E)$ as before. If $c : E \to \{1, \ldots, k\}$ is an improper edge-$k$-colouring of $G$, let $d_i(v)$ now denote the number of edges at $v$ with colour $i$ ($i = 1, \ldots, k; v \in V$).

Conjecture. (D. R. Woodall, 1985/1990.) Let $p_1, \ldots, p_k$ be such that $0 \leq p_i \leq 1$, $\forall i$, and $\sum_{i=1}^{k} p_i \geq 1$. Then there is a $k$-colouring $c : E \to \{1, \ldots, k\}$ s.t.

$$d_i(v) \leq p_i d(v) + 1 \quad (*)$$

$(i = 1, \ldots, k; v \in V)$.

Exercise. Show that this would imply Vizing’s theorem, that $\chi'(G) \leq \Delta + 1$.

Exercise. Show that the 1 in $(*)$ cannot be replaced by any constant less than 1. (Hint: Not every graph is edge-$\Delta$-colourable.)

Theorem 2.1. The conjecture is true if $k = 2$.

Proof. Let $G$ be a minimal counterexample, and $e = uw \in E$, so that $G - e$ has such a 2-colouring. Give $e$ colour 1. If now $d_1(w) > p_1 d(w) + 1$ then

$$d_2(w) = d(w) - d_1(w) < (1 - p_1) d(w) - 1 \leq p_2 d(w) - 1,$$

so give $e$ colour 2 instead. Then $u$ is the only vertex where $(*)$ fails.

Let $V_i := \{v \in V : d_i(v) > p_i d(v)\}$ ($i = 1, 2$). Clearly $V_1 \cap V_2 = \emptyset$. W.l.o.g. $u \in V_1$, so that $p_1 d(u) + 1 < d_1(u) \leq p_1 d(u) + 2$.

Let $H$ be the union of all trails

$$u = v_1, e_1, v_2, e_2, \ldots$$

in which edge $e_j$ has colour 1 (2) if $j$ is odd (even). In each such trail $T$, $v_j \in V_1$ ($V_2$) if $j$ is odd (even), since if $v_j$ were the first vertex along $T$ for which this failed, then changing the colours of $e_1, \ldots, e_{j-1}$
would satisfy (*)& everywhere, \(\Rightarrow \Leftarrow\). Thus \(H\) is bipartite on two sets \(V_1 \cap H\) and \(V_2 \cap H\).

Note that \(H\) contains every edge of \(G\) that has an end in \(V_i \cap H\) and is coloured \(i\) \((i = 1, 2)\). Thus the proportion of edges of \(H\) that have colour \(i\) is \(\geq p_i\). But this cannot hold for \(i = 1, 2\) simultaneously, \(\Rightarrow \Leftarrow\). //

**Exercise.** Use this theorem and induction on \(k\) to prove the weaker version of the conjecture with (*)& replaced by \(d_i(v) \leq p_i d(v) + \lceil \log_2 k \rceil\).

**Theorem 2.2.** (R. O. Davies, unpublished?) Let \(G = (U, W, E)\) be a bipartite graph and let \(f : U \rightarrow \mathbb{Z}\) and \(g : W \rightarrow \mathbb{Z}\) be functions with nonnegative integer values. Then there exists an edge-2-colouring \(c : E \rightarrow \{\text{blue}, \text{red}\}\) such that \(d_{\text{blue}}(u) \leq f(u) \forall u \in U\) and \(d_{\text{red}}(w) \leq g(w) \forall w \in W\), if and only if, for each pair of subsets \(I \subseteq U\) and \(J \subseteq W\), the number of edges of \(G\) between \(I\) and \(J\) is at most \(\sum_{u \in I} f(u) + \sum_{w \in J} g(w)\).

**Proof.** ‘Only if’ is clear, and so we prove ‘if’. For each \(u \in U, w \in W\), let \(u_1, \ldots, u_{f(u)}\) be copies of \(u\) and \(w_1, \ldots, w_{g(w)}\) be copies of \(w\). For each edge \(e = uw \in E\), let \(X_e := \{u_1, \ldots, u_{f(u)}, w_1, \ldots, w_{g(w)}\}\). For each subset \(E'\) of \(E\), let \(I(E')\) and \(J(E')\) be the sets of endvertices of edges of \(E'\) in \(U\) and \(W\) respectively. Then

\[
\left| \bigcup_{e \in E'} X_e \right| = \sum_{u \in I(E')} f(u) + \sum_{w \in J(E')} g(w) \geq |E'|.
\]

It follows from Hall’s theorem that \((X_e : e \in E)\) has a system of distinct representatives. For each edge \(e = uw\), colour \(e\) blue if the representative \(x_e\) of \(X_e\) is \(u_k\) for some \(k\), and red if \(x_e\) is \(w_k\) for some \(k\). Then at most \(f(u)\) edges at \(u\) are blue and at most \(g(w)\) edges at \(w\) are red. //