Jacobi Forms of Lattice Index

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Aim of the talk

1. Definition of Jacobi forms
2. Examples and parallels
3. Some results
I. What are Jacobi forms?

Some notation:

- As usual, $e_m(x) = e^{2\pi i x/m}$ and write $e(x)$ when $m = 1$.
- $\Gamma = \text{SL}_2(\mathbb{Z})$, with elements $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- Upper-half plane: $\mathbb{H} = \{ \tau \in \mathbb{C} : \Im(\tau) > 0 \}$.
- The weight of a Jacobi form will be $k \in \mathbb{Z}_+$.

Apart from the weight, Jacobi forms also have an index. Some prerequisites:

- Denote by $L = (L, \beta)$, where:
  - $L$ is a finite rank $\mathbb{Z}$-module.
  - $\beta : L \times L \to \mathbb{Z}$ is symmetric, positive-definite, even $\mathbb{Z}$-bilinear form.

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Remark

1. **Even** means $\beta(x, x) \in 2\mathbb{Z}$, $\forall x \in L$.
2. We denote $\beta(x) := \frac{1}{2}\beta(x, x)$.

The **rank** of $L$ is $\text{rk}(L)$ (note: $L \simeq \mathbb{Z}^{\text{rk}(L)}$).

The **determinant** of $L$ is $\det(L) := \det(G)$, where

- $G$ is the **gram matrix** of $L$ with respect to $\beta$: pick $\{e_i\}_{i=1, \text{rk}(L)}$ a $\mathbb{Z}$-basis for $L \implies G = (\beta(e_i, e_j))_{i,j}$.
- Note this also gives $\beta(x, y) = x^t G y$.

The **dual** of $L$ is $L^\# := \{y \in L \otimes \mathbb{Q} : \beta(x, y) \in \mathbb{Z}, \forall x \in L\}$.

The **level** of $L$ is $\text{lev}(L) := \min_{\mathbb{N}^+}\{N : N \cdot \beta(y) \in \mathbb{Z}, \forall y \in L^\#\}$
What is the modular group?
The Jacobi Group

More prerequisites: the Heisenberg group associated to \( L \) is
\[
H_L(\mathbb{Z}) := \{ h = (x, y, 1) : x, y \in L \}, \text{ with } hh' = (x + x', y + y', 1).
\]

**Remark**

\( \Gamma \) acts on \( H_L(\mathbb{Z}) \) from the right via \((x, y, 1)^A = ((x, y)A, 1)\).

Combine action of \( \Gamma \) and \( H_L(\mathbb{Z}) \) to get

**Definition (The Jacobi group associated to \( L \))**

We define \( J_L(\mathbb{Z}) \) to be the semi-direct product \( \Gamma \ltimes H_L(\mathbb{Z}) \), with composition law:

\[
(A, h)(A', h') = (AA', h^A h').
\]
More actions

- $J_L(Z)$ acts on $\mathcal{H} \times (L \otimes \mathbb{C})$ via $(A, h)(\tau, z) = \left(A\tau, \frac{z+x\tau+y}{c\tau+d}\right)$. We have a *modular* variable and an *elliptic* variable.

- $J_L(Z)$ acts on $\text{Hol}(\mathcal{H} \times (L \otimes \mathbb{C}))$. If $\phi \in \text{Hol}(\mathcal{H} \times (L \otimes \mathbb{C}))$, then

$$\phi|_{k, L}(A, h) := (\phi|_{k, L} A)|_{k, L} h,$$

where

$$\phi|_{k, L} A(\tau, z) := \phi \left(A\tau, \frac{z}{c\tau+d}\right) (c\tau+d)^{-k} e \left(\frac{-c\beta(z)}{c\tau+d}\right)$$

and

$$\phi|_{k, L} h(\tau, z) := \phi(\tau, z + x\tau + y)e(\tau \beta(x) + \beta(x, z)).$$
Jacobi forms

Definition (Jacobi forms of lattice index)

The space $J_{k,L}$ of Jacobi forms of weight $k$ and index $L$ consists of all $\phi \in \text{Hol}(\mathfrak{H} \times (L \otimes \mathbb{C}))$ that satisfy

1. $\phi|_{k,L}(A, h) = \phi$, $\forall (A, h) \in J_L(\mathbb{Z})$.
2. $\phi$ has a Fourier expansion of the form:

$$\sum_{n \in \mathbb{Z}, r \in L^\# \atop n \geq \beta(r)} c(n, r)e(n\tau + \beta(r, z)).$$
We have a ‘modular interpretation’: elliptic modular forms $f \in M_k(\Gamma)$ are in 1 : 1 correspondence with functions $F(\Lambda_\tau)$ ($\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$) satisfying $F(\lambda \Lambda_\tau) = \lambda^{-k} F(\Lambda_\tau)$, for all $\lambda \in \mathbb{C}^\times$ (Koblitz).

Consider the following:

- $\mathbb{H}_L(\mathbb{Z})$ acts on $\mathfrak{H} \times (L \otimes \mathbb{C})$ via $h(\tau, z) = (\tau, z + x\tau + y)$.
- This is properly discontinuous and fixed point free, so $\mathbb{H}_L(\mathbb{Z}) \setminus (\mathfrak{H} \times (L \otimes \mathbb{C}))$ is an $\text{rk}(L)$—dimensional complex manifold $\mathcal{E}_L$. 
• The projection $\mathcal{H} \times (L \otimes \mathbb{C}) \rightarrow \mathcal{H}$ induces a projection $E_L \rightarrow \mathcal{H}$ whose fiber over $\tau$ is $(L \otimes \mathbb{C})/L_{\tau} \oplus L \cong (\mathbb{C}/\Lambda_{\tau})^{rk(L)} =: T_{\tau,L}$.

• Any $A \in \Gamma$ gives an isomorphism of tori $(T_{\tau,L},0) \cong (T_{A\tau,L},0)$, induced by the map $z \mapsto \frac{z}{c_{\tau}+d}$.

• Consider the action of $\Gamma$ on $\mathcal{H} \times (L \otimes \mathbb{C})$: $(\tau, z) \mapsto \left( A\tau, \frac{z}{c_{\tau}+d} \right)$.

• Combine the actions of $\Gamma$ and $\mathcal{H}_L(\mathbb{Z})$ and set $\mathcal{A}_L = J_L(\mathbb{Z}) \setminus \mathcal{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$. There is a projection $\mathcal{A}_L \rightarrow \Gamma \setminus \mathcal{H}$, whose fiber over $\tau$ is $T_{\tau,L}/\text{Aut}(T_{\tau,L},0)$. 

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Jacobi Forms of Lattice Index
Jacobi forms $\phi \in J_{k,L}$ become functions $\Phi(L_\tau, z)$, with $L_\tau := L_\tau \oplus L$ and $z \in \mathcal{T}_{\tau,L} = (L \otimes \mathbb{C})/L_\tau$, which satisfy:

\[
\Phi(L_\tau, z + \omega) = e(-\tau \beta(x) - \beta(x, z))\Phi(L_\tau, z),
\]

\[
\Phi(\lambda L_\tau, \lambda z) = \lambda^{-k} e(\lambda c \beta(z))\Phi(L_\tau, z),
\]

for $\lambda \in \mathbb{C}^\times$ and $\omega = x\tau + y \in L_\tau$.

Elliptic modular forms can be interpreted as global sections of line bundles on the modular curve $\Gamma \setminus \mathcal{H} \cup \{\text{cusps}\}$. Jacobi forms play a similar role for $J_{L}(\mathbb{Z}) \setminus \mathcal{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \cup \{\text{cusps}\}$.

This also gives:

\[
J_{k,0} \simeq M_k(\Gamma), \quad \text{for } 0 = (L, 0).
\]
Example (Jacobi theta functions associated to $L$)

Fix $x \in L^\#$. Define:

$$\vartheta_{L,x}(\tau, z) := \sum_{\substack{r \in L^\# \
r \equiv x \mod L}} e(\tau \beta(r) + \beta(r, x)).$$

- These transform ‘nicely’ with weight $rk(L)/2$ and index $L$.
- They give isomorphism between spaces of Jacobi forms and spaces of vector-valued Hilbert modular forms (Boylan).
- Every Jacobi form has a theta-expansion (Ajouz). When $rk(L)$ is odd, this gives a connection to half-integral elliptic modular forms.
3) Jacobi forms of scalar index

Example

Fix $L = (\mathbb{Z}, (x, y) \mapsto 2mxy)$, for $m \geq 0$. We get $J_{k,L} = J_{k,m}$, studied extensively by Eichler and Zagier.

We get a connection to Siegel modular forms, because:

- Let $\Gamma^J$ denote $J_L(\mathbb{Z})$ for this particular choice of $L$.

\[ \Gamma \hookrightarrow \Gamma^J \hookrightarrow \text{Sp}_2(\mathbb{Z}). \]

- Every SMF of degree 2 has a Jacobi-Fourier expansion (Piatetski–Shapiro). This also holds for degree $g > 2$, where now the expansion is in terms of Jacobi forms of matrix index $(g - 1)$ (Bringmann).
From the work of Gritsenko:

- Modular forms of *orthogonal type* can be obtained as liftings of Jacobi forms. The former determine Lorentzian Kac–Moody Lie (super) algebra of Borcherds type.
- Jacobi forms are solutions to the *mirror symmetry* problem for $K3$ surfaces.
- For a compact complex manifold, one defines its elliptic genus, which can be a weak Jacobi form ($n \geq 0$).
- And much more...
III. Work done
1) Poincaré and Eisenstein series

**Definition**

Let \( r \in L^\# / L \) and \( D \in \mathbb{Q}_{\leq 0} \) be such that \( \beta(r) \equiv D \mod \mathbb{Z} \). We define

\[
g_{L,r,D} := e(\tau(\beta(r) - D) + \beta(r, z)).
\]

When \( D < 0 \), we define

\[
P_{k,L,r,D} := \sum_{\gamma \in J_L(\mathbb{Z})_\infty \setminus J_L(\mathbb{Z})} g_{L,r,D} |_{k,L} \gamma
\]

and, when \( \beta(r) \in \mathbb{Z} \), let

\[
E_{k,L,r} := \frac{1}{2} \sum_{\gamma \in J_L(\mathbb{Z})_\infty \setminus J_L(\mathbb{Z})} g_{L,r,0} |_{k,L} \gamma.
\]
Why the interest?

- Both are elements of $J_{k,L}$.
- Eisenstein series:
  - Perpendicular to *Jacobi cusp forms* ($n > \beta(r)$) with respect to a suitably defined *Petersson scalar product*.
  - We get a decomposition:
    $$J_{k,L} = S_{k,L} \oplus J_{k,L}^{Eis}.$$
  - Their *twists* by Dirichlet characters modulo $N_x$ (level of $x$) form a *basis of eigenforms* of $J_{k,L}^{Eis}$ with respect to (again) suitably defined *Hecke operators*. 
Poincaré series (previously undefined in this setting):

- They are cusp forms.
- They reproduce Fourier coefficients of other cusp forms via the Petersson scalar product.
- Furthermore, they generate $S_k(\Gamma)$.
- Our main interest is in reproducing kernels of linear operators defined between spaces of Jacobi cusp forms and elliptic modular forms.
Proposition

For any $\phi \in S_{k,L}$,

$$\langle \phi, P_{k,L,r,D} \rangle = \lambda_{k,L,D} c(n, r),$$

where

$$\lambda_{k,L,D} := 2^{-2k + \frac{rk(L)}{2} + 2} \Gamma \left( k - \frac{rk(L)}{2} - 1 \right) \det(L)^{-\frac{1}{2}} (\pi |D|)^{-k + \frac{rk(L)}{2} + 1}$$

and $c(n, r)$ is the Fourier coefficient of $\phi$ corresponding to $e(\tau(\beta(r) - D) + \beta(r, z))$. 
Theorem

For $k > \text{rk}(L) + 2$, $P_{k,L,r,D}$ is a cusp form. It has the following Fourier expansion:

$$P_{k,L,r,D}(\tau, z) = \sum_{n' \in \mathbb{Z}, r' \in L^\# \atop n' > \beta(r')} G_{k,L,D,r}(n', r') e \left( n' \tau + \beta(r', z) \right),$$

where

$$G_{k,L,D,r}(n', r') := \delta_L(D, r, D', r') + (-1)^k \delta_L(D, r, D', -r') + 2\pi i^k$$

$$\times \det(L)^{-\frac{1}{2}} \left( \frac{D'}{D} \right)^{\frac{k - \text{rk}(L)}{4} - \frac{1}{2}} \cdot \sum_{c \geq 1} (H_{L,c}(n, r, n', r'))$$

$$+ (-1)^k H_{L,c}(n, r, n', -r')) \cdot J_{k - \frac{\text{rk}(L)}{2} - 1} \left( \frac{4\pi (DD')^{\frac{1}{2}}}{c} \right),$$

where $D' = \beta(r') - n'$, we use $J_{k - \frac{\text{rk}(L)}{2} - 1} \left( \cdot \right)$ for the Bessel function and

$$H_{L,c}(n, r, n', r') := c^{-\frac{\text{rk}(L)}{2} - 1} \sum_{\lambda(c)} e_c(\beta(r', \lambda + r)) K(n', \beta(\lambda) + \beta(r + \lambda) + n; c).$$

In the last equation, $\lambda$ runs through a complete set of representatives of $L/cL$ and $K(n', \beta(\lambda) + \beta(r + \lambda + n); c)$ is a Kloosterman sum.
2) Operators on the spaces of Jacobi forms

- Operators give *structure* to the space.
- They facilitate *equivariant lifts* between different types of modular forms.
- They have algebraic interpretations in terms of the surfaces that our modular forms underlie.

In [Ajouz, 2015], we are given

- Hecke operators:

\[
T_0(l)\phi := l^{k-2-rk(L)} \sum_{\gamma \in J_L(\mathbb{Z}) \setminus J_L(\mathbb{Z})} \phi |_{k,L} \gamma,
\]

- Action of the *orthogonal group* of \( L \):

\[
W(\alpha)\phi(\tau, z) = \sum_{n \in \mathbb{Z}, r \in L^\#} c(n, \alpha(r)) e(n\tau + \beta(r, z)).
\]
We want a theory of *newforms*. For that, we need:

**Definition**

We define the operator $U(l)$ on the space $J_{k,L}$ by:

$$U(l)\phi(\tau, z) := \phi(\tau, lz).$$

**Remark**

1. The operator $U(l)$ corresponds to the endomorphism “multiplication by $l$” on $\mathcal{T}_{\tau,L} = (L \otimes \mathbb{C})/(L\tau \oplus L)$.

2. Think of $U(l) : M_k(N) \rightarrow M_k(lN)$,

$$U(l)f(\tau) = \sum a(ln)q^n.$$
Theorem

The operator $U(l)$ maps $J_{k,L}$ to $J_{k,L'}$, where $L' = (L, \beta')$, where $\beta' = l^2 \beta$. Moreover, if $\phi \in J_{k,L}$ has the Fourier expansion

$$\phi(\tau, z) = \sum_{n \in \mathbb{Z}, r \in L^\# \atop n \geq \beta(r)} c(n, r) e(n\tau + \beta(r, z)),$$

then $U(l)\phi$ has the following Fourier expansion:

$$U(l)\phi(\tau, z) = \sum_{n \in \mathbb{Z}, r' \in L'^\# \atop n \geq \beta'(r')} c(n, lr') e(n\tau + \beta'(r', z)),$$

with the convention $c(n, lr') = 0$ unless $r'$ is an $l$–th multiple of another element of $L'^\#$.

- Note that the level of $L'$ is $\text{lev}(L') = l^2 \cdot \text{lev}(L)$.
**Definition**

We define the operator $V(I)$ on the space $J_{k,L}$ by:

$$V(I)\phi(\tau, z) = I^{k-1} \sum_{\substack{M \in \Gamma \backslash \mathcal{M}_2(\mathbb{Z}) \\det(M) = I}} U(\sqrt{I}) \left( \phi_{k,L,M} \right) (\tau, z).$$

**Remark**

1. Assume that $L_\tau$ is contained in $L'$ with index $I$. If $\{\omega_1, \omega_2\}$ is a basis for $L'$, then there exists $M \in \mathcal{M}_2(\mathbb{Z})$ with determinant $I$, such that $(\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}) = M (\begin{smallmatrix} \omega_1 \\ \omega_2 \end{smallmatrix})$.

2. If $M = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$, then $U(\sqrt{I}) \left( \phi_{k,L,M} \right) (\tau, z)$ contains a factor of $\phi(M\tau, \frac{Iz}{c\tau + d})$. 
Theorem

The operator $V(l)$ maps $J_{k,L}$ to $J_{k,L''}$, where $L'' = (L, \beta'')$, where $\beta'' = l \beta$. Moreover, if $\phi \in J_{k,L}$ has the Fourier expansion

$$\phi(\tau, z) = \sum_{n \in \mathbb{Z}, r \in L^\#} c(n, r) e(n \tau + \beta(r, z)),$$

then $V(l)\phi$ has the following Fourier expansion:

$$V(l)\phi(\tau, z) = \sum_{n, r''} \sum_{\substack{n \geq \beta''(r''), \quad a| (n,l) \quad \frac{l'r''}{a} \in L'^{**}} \quad a^{k-1} \quad c \left( \frac{n}{a^2}, \frac{l'r''}{a} \right) \quad e(n \tau + \beta''(r'', z)).$$
Goal

- Find isomorphisms of the type
  
  \[ J_{k,m} \cong \mathfrak{m}_{2k-2}(m), \]

  like in [Skoruppa & Zagier, 1988].

- Find decomposition of the type
  
  \[ S_{k,m} \bigoplus V(l') U(l) S_{k,m/l^2 l'}, \]

  like in [Eichler & Zagier, 1985].
Thank you!