Direct summands in the motives of quadrics

Preliminary version.

A. Vishik

1 Introduction

In this paper we will investigate the structure of undecomposable direct summands in the motives of quadrics. We will work mostly in the category of Chow motives $\text{Chow}(k)$ (see, for example, [4], 2.2), but since we rely heavily on the results from [4], we will also be using bigger triangulated category of mixed motives $D^c_{\text{eff}}(k)$ of V.Voevodsky (see [5]) (though, in most cases it will be done just to make terminology compatible with that of [4]). By the latest reason our considerations are restricted to the case of characteristic 0.

Let $Q$ be $n$-dimensional quadric over the field $k$. If $Q$ is hyperbolic, then the Chow motive of $Q$ is very simple - it is isomorphic to the direct sum of Tate motives $\oplus_{i=0}^{n} \mathbb{Z}(i)[2i]$ ($\oplus \mathbb{Z}(n/2)[n]$ if $n$ is even), so, all undecomposable direct summands in this case are given by the Tate motives above. In particular, this happens if $k$ is algebraically closed.

For arbitrary $Q$ and $k$ the situation is more delicate, but it follows from the Rost Nilpotence Theorem (see [3], Proposition 9) (see also [4], Lemma 3.10) that for any direct summand $N$ in the motive of $Q$, $N$ is defined up to isomorphism by it’s restriction to $k$ (see [4], Lemma 3.21), and this restriction is isomorphic to $\oplus_{i \in I(N)} \mathbb{Z}(i)[2i]$, so $N$ is defined up to isomorphism by the set $I(N)$ (plus the $Q$ itself). The natural problem arises - to describe possible sets $I(N)$ for various direct summands $N$.

It appears that if $N$ is undecomposable, then $I(N)$ has a symmetry, coming from the isomorphism $\text{Hom}(N, \mathbb{Z}(\dim(N))[2 \dim(N)]) \simeq N$ (see Corollary 1 ), and, consequently, consists of even number of elements (if $Q$ is anisotropic). Moreover, there is interaction between the splitting pattern of $Q$ (or, the set of higher Witt indices) and the possible decomposition of it’s motive - see the Statement . The Statement can be also used to answer the question: “When subform $p \subset q$ is isotropic over $k(Q)$?”; as one could expect, this happens iff $\text{codim}(p \subset q) < i_1(q)$, where $i_1$ is 1-st higher Witt index - see Corollary 3 . Another interesting question is - to describe minimal elements $i$ of $I(N)$ for all possible $N$ (ans so, describe the number of undecomposable direct summands in the motive of $Q$). Using Statement , we show in Proposition 1 that if there exists such quadric $P$, that $p$ is isotropic.
if and only if \( q \) is \( i + 1 \) times isotropic, then \( i \) is minimal for some set \( I(N) \). We believe, that this condition should be also necessary - see Question 1. Finally, we improve Lemma 4.5 and Proposition 3.4 from [4] - see Corollary 2 and Corollary 4.

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We need to make some brief introduction into the terminology of [4].

In the Voevodsky’s category \( DM_{eff}^-(k) \) we have not only the motives of all smooth projective varieties (as in \( Chow(k) \)), but also of all smooth simplicial schemes. If \( P/k \) is smooth variety (of finite type) over \( k \), then we denote as \( X_P \) the smooth simplicial scheme with \( X^n_P = P \times_k P \times_k \cdots \times_k P - n + 1 \)-times, where the maps of faces and degenerations are given by partial projections and partial diagonals. Also we will denote in the same way the image of \( X_P \) in \( DM_{eff}^-(\cdot) \) (actually, we will denote motives of all smooth varieties and simplicial schemes in the same way as objects themselves, so, omitting \( M(\cdot) \)). As soon as \( P \) has a rational point (or even the 0-cycle of degree 1), the motive \( X_P \) is isomorphic to the trivial Tate motive \( \mathbb{Z} \). In particular, over \( \overline{k} \), the motive \( X_P \) will be isomorphic to \( Z \), so we can say that \( X_P \) is a form of \( Z \). More generally, for two smooth (connected) varieties \( P \) and \( R \) we have: the motives \( X_P \) and \( X_R \) are isomorphic if and only if \( R \) has a 0-cycle of degree 1 over \( k(P) \) and \( P \) has a 0-cycle of degree 1 over \( k(R) \) (see [4], 2.3 for details). The last statement justifies the following notation: we will write \( X_P \cong X_R \) if \( R \) has a 0-cycle of degree 1 over \( k(P) \).

If we have triangulated category \( D \) and \( X,Y,Z \in D \), then we say that \( Z \) is an elementary extension of \( X \) and \( Y \) iff there exists an exact triangle either of the form \( X \rightarrow Z \rightarrow Y \rightarrow X[1] \), or of the form \( Y \rightarrow Z \rightarrow X \rightarrow Y[1] \). If we have objects \( X_1, \ldots, X_m, Z \in D \), then (inductively) \( Z \) is called an extension of \( \{X_j\}_{1 \leq j \leq m} \), if there exist \( 1 \leq i \leq m \) and an exact triangle either of the form \( X_i \rightarrow Z \rightarrow Y \rightarrow X_i[1] \), or of the form \( Y \rightarrow Z \rightarrow X_i \rightarrow Y[1] \), s.t. \( Y \) is an extension of \( \{X_j\}_{j \neq i, 1 \leq j \leq m} \).
If $Q$ is a quadric of dimension $n$, then we can consider the smooth projective varieties $Q^i$ of $i$-dimensional projective planes on $Q$, $i = 0, \ldots, \lfloor n/2 \rfloor$. The Theorem 3.1 from [4] states that the motive of $Q$ is an extension (in the sense specified above, with $\mathcal{D} = DM^{eff}(k)$) of the motives $\mathcal{X}_Q$, $\mathcal{X}_Q(1)[2][\ldots$, $\mathcal{X}_Q(n-1)[2][n-1/2]$, $\mathcal{X}_Q(n-1/2)(n-1)[2n-2][n-1/2]$, $\ldots$, $\mathcal{X}_Q(n-1)[2n-2]$, $\mathcal{X}_Q(n)[2n]$ (plus $k \sqrt{\text{det}(Q)} \times \mathcal{X}_Q(n/2)$, if $n$ is even). Moreover, in the abovementioned Theorem 3.1 it is specified in which order the elementary pieces appear, so (the motive of) $Q$ is a total object in some Postnikov system with graded parts as above. This Postnikov system appears to be compatible with the ring of endomorphisms of the motive $Q$, i.e. any endomorphism of $Q$ extends uniquely to the endomorphism of the whole system (see [4], Theorem 3.7). This shows that for any direct summand $N$ in $Q$, the corresponding projector $p$ in $\text{End}_{DM^{eff}}(Q)$ gives us the decomposition of the Postnikov system into a direct sum of two, and provides us with the projectors $\text{Postnikov system}$ corresponding projector (see [4], Theorem 3.7). This shows that for any direct summand $N$ in $Q$, the corresponding projector $p$ in $\text{End}_{DM^{eff}}(Q)$ gives us the decomposition of the Postnikov system into a direct sum of two, and provides us with the projectors $\text{Postnikov system}$ with

$\text{Postnikov system}$ corresponding projector (see [4], Theorem 3.7). This shows that $p_1 \in \text{End}_{DM^{eff}}(\mathcal{X}_Q(i)[2i])$ and $p_j' \in \text{End}_{DM^{eff}}(\mathcal{X}_Q(n-j)[2n-2j])$. But for arbitrary smooth $P/k$, we have $\text{End}_{DM^{eff}}(\mathcal{X}_P) = \mathbb{Z}$ (see [4], Theorem 2.3.2, Theorem 2.3.3 (1)). This shows that $p_i$ and $p_j'$ are either 0-Id, or 1-Id, and $N$ is an extension of some number of “elementary pieces” from the same set: $\mathcal{X}_Q$, $\mathcal{X}_Q(1)[2][\ldots$, $\mathcal{X}_Q(n-1)[2n-2]$, $\mathcal{X}_Q(n)[2n]$ (namely, those ones, for which the corresponding projector ($p_i$ or $p_j'$) is identity). Over $k$, these “elementary pieces” are becoming Tate motives, and their weights give you the set $I(N)$, i.e.: $I(N) = (\cup_{p_i=1}) \cup (\cup_{p_j'=1} n-j) \cup (n/2)$, taken 0, 1 or 2 times, if $n$ is even). This permits one to translate from Chow-motivic terminology to that of [4].

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For a quadric $Q$ over $k$ we will define (following M.Knebusch, see [1], Definition 5.4) it’s higher Witt indices $i_0(q), \ldots, i_s(q)$ inductively in the following way: $q_0 := q$, $k_0 := k$, $i_0(q) := i_W(q_0)$, $k_{i+1} := k_i((q_t)^{\text{anis.}})$, and $q_{i+1} := ((q_t)^{\text{anis.}})_{k_{i+1}}$.

Let $Q$ be a quadric of dimension $n$, which has higher Witt indices $i_1, \ldots, i_s$. Denote $\mathbf{i} := (i_1, \ldots, i_s)$. For $0 \leq i < n/2$, let $1 \leq j(i, \mathbf{i}) \leq s$ be such that $i_1 + \cdots + i_{j(i, \mathbf{i}) - 1} \leq i < i_1 + \cdots + i_{j(i, \mathbf{i})}$, and let $i_1^1 := 2(i_1 + \cdots + i_{j(i, \mathbf{i}) - 1}) + i_{j(i, \mathbf{i})} - i$. Clearly $j(i, \mathbf{i}) = j(i_1^1, \mathbf{i})$.

Let $N$ be a direct summand in the motive of $Q$, and $0 \leq i < n/2$. We say that “$N$ contains $(i)$” iff $N$ contains $\mathcal{X}_Q(i)[2i]$ as an elementary piece in
the sense of [4], Lemma 3.23. Similarly, we say that “$N$ contains $(i)'$” iff it contains $X_{Q'}(n-i)[2n-2i]$ as an elementary piece. Certainly, the above two conditions can be reformulated as: “$I(N)$ contains $i$” and “$I(N)$ contains $n-i'$”, respectively.

The main restriction on the structure of $N$, which we use in this paper, is provided by the following:

**Statement.** Suppose $Q$ is anisotropic.

1. Let $N$ be a direct summand in $Q$. Then $N$ contains $(i)$ iff it contains $(i_1')$.

2. Suppose $N$ is indecomposable. If $N$ contains $(i)$, but does not contain any $(m)$, $0 \leq m < i$, then $N$ contains $(i')$ and does not contain any $(l)'$, $0 \leq l < i_1'$.

**Proof of the Statement.**

1. First of all, we can change $k$ by $k(Q^{i_1+\cdots+i_{(i-1)-1}})$, and assume that $j(i,i') = 1$ (i.e., $X_{Qi} = X_Q$).

   By [4], Lemma 4.5, there are indecomposable direct summands $M(v)[2v]$, $0 \leq v < i_1 - 1$ of $Q$, s.t. $M(v)[2v]$ contains $(v)$. It is enough to prove the statement for $N = M(i)[2i]$, or which is the same, for $M$ and $i = 0$.

   It is clear, that $M$ can’t contain $(m)'$, with $m < i_1 - 1$ (look on $M(i_1 - 1)[2i_1 - 2]$). So, if $M$ does not contain $(i_1 - 1)'$, that means that it does not contain any $X_{Q_m}(n-m)[2n-2m]$ with $X_{Q_m} = X_Q$. Suppose it is the case.

   Let $S$ be a plane section of $Q$ of codimension $i_1 - 1$. Since $X_S = X_Q = X_{Q_1-1}$, we have a map $\varphi : S(i_1 - 1)[2i_1 - 2] \to Q$, s.t. over $\overline{k}$, $\varphi|_\overline{k}$ maps $(\mathbb{Z})(i_1 - 1)[2i_1 - 2]$ to $\mathbb{Z}(i_1 - 1)[2i_1 - 2]$ isomorphically. Let $\psi : Q \to S(i_1 - 1)[2i_1 - 2]$ is the map given by the cycle “$S$”, embedded “diagonally” to $Q \times S$ (the map, dual to the embedding). Clearly, $\psi|_\overline{k}$ maps $\mathbb{Z}(i_1 - 1)[2i_1 - 2]$ to $(\mathbb{Z})(i_1 - 1)[2i_1 - 2]$ isomorphically. By [4], Lemma 3.26, it follows that $S(i_1 - 1)[2i_1 - 2]$ contains a direct summand, isomorphic to $M(i_1 - 1)[2i_1 - 2]$, i.e., $S$ contains one isomorphic to $M$. And, by our assumption, $M$ does not contain any $X_{S_m}(n'-m)[2n'-2m]$ with $X_{S_m} = X_S$ ($n' = n-i_1 + 1$ is the dimension of $S$). Since, $\dim(S) < \dim(Q)$, repeating this procedure, if necessary, we get a quadric $S'$, which contains a direct summand, isomorphic to $M$, and $i_1(S') = 1$ (in particular, $M$ does not contain $(0)'$).

   So, finally, we can assume, that $i_1(Q) = 1$, $M$ contains $(0)$, but not $(0)'$.

**Lemma 1.**
Let $N$ be a direct summand of $Q$. Then the natural map $Q \times N \to N$ has a splitting $N \to Q \times N$, i.e.: $N$ is a direct summand of $Q \times N$.

**Proof of the Lemma 1**

We have an exact triangle in $DM^{eff}(k)$: $R^1 \to Q \to \mathcal{X}_Q \to R^1[1]$.

From this we get an exact triangle: $R^1 \times N \to Q \times N \to \mathcal{X}_Q \times N \to R^1[1] \times N$. We have: $\mathcal{X}_Q \times N = N$ (since $\mathcal{X}_Q \times Q = Q$, and $N$ is a direct summand in $Q$).

$R^1[1] \times N$ is a direct summand in $R^1[1] \times Q$ and the later is the direct summand in $Q[1] \times Q$ (since the map $Q \times Q \to \mathcal{X}_Q \times Q = Q$ has a splitting - the diagonal).

So, $\text{Hom}(N, R^1[1] \times N)$ is a subgroup in $\text{Hom}(Q, Q[1] \times Q) = \text{Hom}(Q \times Q \times Q, \mathbb{Z}(2n)[4n+1]) = 0$, since $Q \times Q \times Q$ is a smooth projective variety and $4n+1 > 2(2n)$.

Lemma 1 is proven.

Let’s take $N = M^\vee$ - dual to $M$ via duality $\text{Hom}(\cdot, \mathbb{Z}(n)[2n])$.

Since $M$ does not contain $(0)^\vee$, but contains $(0)$, we have that $N$ does not contain $(0)$, but contains $(0)^\vee$. In particular, $N$ is a direct summand in $R^1$ (notations as above) (since for corresponding projector $p_N$ we have $(p_N)_0 = 0$). But $Q \times R^1 = Q^1[1][2] \oplus Q(n)[2n]$ (since we have an exact triangle $R^1 \to Q^1[1][2] \to \mathcal{X}_Q(n)[2n+1] \to R^1[1]$, the composition $Q(n)[2n] = Q \times \mathcal{X}_Q(n)[2n] \to Q \times R^1 \to Q \times Q$ has a splitting - the map dual to the diagonal via duality $\text{Hom}(\cdot, \mathbb{Z}(2n)[4n])$, and $Q \times Q$ is isomorphic to $Q^1$ - see [4], Claim 3.2). So, by Lemma 1, $N$ is a direct summand in $Q^1[1][2] \oplus Q(n)[2n]$. Let $\rho_1 : N \to Q^1[1][2]$, $\rho_2 : N \to Q(n)[2n]$, and $\pi_1 : Q^1[1][2] \to N$, $\pi_2 : Q(n)[2n] \to N$ be corresponding maps. So, $\pi_1 \circ \rho_1 + \pi_2 \circ \rho_2 = \text{id}_N$. As we know, $N$ contains $\mathcal{X}_Q(n)[2n]$. Since $Q$ is anisotropic, over $\bar{k}$, $\rho_2|_{\bar{k}}$ should send $\mathbb{Z}(n)[2n]$ to $\mathbb{Z}(n)[2n]$ via multiplication by an even number (otherwise, the composition $\mathbb{Z}(n)[2n] \to Q \xrightarrow{\text{proj.}} N \xrightarrow{\rho_2} Q(n)[2n]$ would give us a 0-cycle of odd degree on $Q$). So, over $\bar{k}$, $(\pi_1 \circ \rho_1)|_{\bar{k}}$ should act on $\mathbb{Z}(n)[2n]$ via multiplication by an odd number. Let $K = k(Q)$. Then over $K$, $q|_K = \mathbb{H} \perp p$, where $\mathbb{H}$ is hyperbolic plane and $p/K$ is anisotropic (since $i_1(Q) = 1$). Hence, $Q^1|_K = P \oplus P(1)[2] \oplus P(1)[2] \oplus P(1)[2] \oplus P(n-1)[2n-2] \oplus P(n)[2n]$. Also, $N_K = N' \oplus \mathbb{Z}(n)[2n]$, and we get the maps: $\alpha : \mathbb{Z}(n)[2n] \to P \oplus P(1)[2] \oplus P(1)[2] \oplus P(n-1)[2n-2] \oplus P(n)[2n]$ and $\beta : P \oplus P(1)[2] \oplus P(1)[2] \oplus P(n-1)[2n-2] \oplus P(n)[2n] \to \mathbb{Z}(n)[2n]$, s.t. $\beta \circ \alpha : \mathbb{Z}(n)[2n] \to \mathbb{Z}(n)[2n]$ is a multiplication by an odd number.
Lemma 2.

Let $P/K$ be anisotropic quadric, and $0 \leq m \leq \dim(P)/2$. Suppose for some $l$ we have maps: $\alpha : \mathbb{Z}(l)[2l] \to P^m$, $\beta : P^m \to \mathbb{Z}(l)[2l]$. Then the composition $\beta \circ \alpha : \mathbb{Z}(l)[2l] \to \mathbb{Z}(l)[2l]$ is a multiplication by an even number.

Proof of the Lemma 2.

We have the natural identification: $\text{Hom}(\mathbb{Z}(l)[2l], P^m) = CH_1(P^m)$, and $\text{Hom}(P^m, \mathbb{Z}(l)[l]) = CH^1(P^m)$, and if $\alpha$ is represented by a cycle $A$, and $\beta$ by cycle $B$, then the composition $\beta \circ \alpha : \mathbb{Z}(l)[2l] \to \mathbb{Z}(l)[2l]$ is a multiplication by the degree of the intersection $A \cap B \in CH_0(P^m)$. If this number would be odd, then we would have a point of odd degree on $P^m$, and, because of the natural projection $P^m \to P$, also on $P$. By Springer’s theorem, we then would have a rational point on $P$ - contradiction. So, the degree of $A \cap B$ is even.

Lemma 2 is proven. \hfill \Box

Using Lemma 2, we get a contradiction. So, (1) is proven.

(2) follows from (1) applied to $N$ and $N^\vee$ (the dual to $N$ via duality $\text{Hom}(-, \mathbb{Z}(n)[2n])$, $n = \dim(Q)$). Really, clearly, by (1), $N$ will contain $(i_1^\perp)'$.

Let $N$ contains $l)'$, where $l < i_1^\perp$. Let $i_1 = l_1^\perp$. We have: $(i_1^\perp)^\perp = i$, and if $j(i, \vec{i}) = j(i_1^\perp, \vec{i}) > j(l, \vec{i})$, then $i > i_1$, and so, $N^\vee$, containing $(l)$, by (1), should contain also $(i_1)^\perp$ (i.e., $N$ contains $(i_1)$), which is not the case by the condition. So, $j(i, \vec{i}) = j(i_1^\perp, \vec{i}) = j(l, \vec{i})$. Now, we can change $k$ by $K = k(Q^{h_1 + \cdots + h_d, 0, -1})$, and assume that $j(i, \vec{i}) = 1$. Then we have by [4], Lemma 4.5, that for each $0 \leq u \leq i_1 - 1$, $(u)$ is contained in an undecomposable direct summand isomorphic to $N(u - i)[2u - 2i]$. Taking $u = i_1 - 1$, and applying (1), we get that $l$ can’t be $i$.

Statement is proven. \hfill \Box

From the result above we immediately get that undecomposable direct summands should have some kind of symmetry (the simplest consequence of which is: if $Q$ is anisotropic, and $N$ undecomposable, then $N|_F$ consists of even number of Tate motives).

Corollary 1.

Let $N$ be an undecomposable direct summand in the motive of anisotropic quadric $Q$. Let lowest term of $N$ is $(i)$, and highest - $(j)'$. Let $M = N(-i)[-2i]$. Define the dimension of $M$ as $d = n - i - j$, where $n = \dim(Q)$. Then $\text{Hom}(M, \mathbb{Z}(d)[2d])$ is isomorphic to $M$.  

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Proof of Corollary 1

Clearly, to prove this statement for $N$ is the same as to prove it for $N^\vee = \text{Hom}(N, \mathbb{Z}(n)[2n])$. Evidently, $j = i_{l,t}^+$, by Statement (2). Changing $N$ by $N^\vee$, if necessary, we can assume that $i \leq i_{l,t}^+$. Take $S$ to be a plane section of codimension $l = i_{l,t}^+ - i$. We have: $X_{Q^\vee} = X_{Q^\vee + l} = X_{Q^\vee l}$. By [4], Lemma 4.5, $Q$ contains direct summand isomorphic to $N(l)[2l]$. Let $\varphi : S \to Q$ be map, given by the embedding $S \subset S \times Q$, and $\psi : Q \to S(l)[2l]$ be a map dual to it. By duality $\text{Hom}(\mathbb{Z}(n)[2n])$, let $\rho_N : N \to Q$, $\rho_{N(l)[2l]} : N(l)[2l] \to Q$, and $\pi_N : Q \to N$, $\pi_{N(l)[2l]} : Q \to N(l)[2l]$ be maps defining $N$ and $N(l)[2l]$ as direct summands in $Q$. Consider $\varepsilon := \psi(-l)[-2l] \circ \rho_{N(l)[2l]}(-l)[-2l] \circ \pi_N : Q \to S$. It is easy to see, that over $\mathbb{K}$, $\varphi \circ \varepsilon : Q \to Q$ maps $Z(i)[2i]$ to itself isomorphically. Hence, by [4], Lemma 3.26, $S$ contains a direct summand $N_1$ isomorphic to $N$. The lowest term of $N_1$ is $X_{S}(i)[2i]$, and the highest term should be $X_{S}(n_1 - i)[2n_1 - i]$, where $n_1 = n - l = \dim(S)$ (really, $i = i_{l,t}^+ - l$). Since $N_1$ contains $(i)$ and $(i)'$, $N_1^\vee$ (dual to $N_1$ via duality $\text{Hom}(\mathbb{Z}(n_1)[2n_1])$) should also contain $(i)$ and $(i)'$. By [4], Lemma 3.21, that means that $N_1^\vee$ is isomorphic to $N_1$, i.e.: $\text{Hom}(M, \mathbb{Z}(d)[2d])$ is isomorphic to $M$.

\[ \square \]

Also, we can improve a bit the Lemma 4.5 from [4].

Corollary 2.

Let $N$ be an undecomposable direct summand in $Q$, with the lowest term $(i)$. Then for any $m$, s.t. $X_{Q^m} = X_{Q^r}$, there exist an undecomposable direct summand of $Q$, isomorphic to $N(m - i)[2m - 2i]$.

Proof of Corollary 2 By Statement (2), the highest term of $N$ will be $(i_{l,t}^+)'. \text{X}_{Q^r} = X_{Q^r l}$, it is equivalent to prove the statement for $N$ or for $N^\vee = \text{Hom}(N, \mathbb{Z}(n)[2n])$. Changing $N$ by $N^\vee$, if necessary, we can assume that $i \leq i_{l,t}^+$. Then $Q(i)'$ contains $N$, and from [4], Lemma 4.5 follows the statement for required $m \geq i$. Moreover, if $r = i_1 + \cdots + i_{l(t, t) - 1}$, then $N(r - i)[2r - 2i]$ contains $(r)$ and $(i_1 + \cdots + i_{l(t, t) - 1})'$. Again, applying [4], Lemma 4.5 to $(N(r - i)[2r - 2i])^\vee$ (and then, dualizing back), we get the statement for required $m < i$.

Corollary 2 is proven.

\[ \square \]

One more application of the Statement explains, in which cases the sub-form of $q$ will be isotropic over the generic point of $Q$, and computes the 1-st
higher Witt index for such subforms in terms of that for $q$.

Corollary 3.

(1) Let $P$ and $Q$ be such anisotropic quadrics that $X_P = X_Q$ (in other words, $P$ has a rational point over $k(Q)$, and $Q$ has a rational point over $k(P)$. Then $\dim(P) - i_1(P) = \dim(Q) - i_1(Q)$.

(2) Let $Q$ be anisotropic quadric, and $P \subseteq Q$ - a subquadric of codimension $i$. Then the following conditions are equivalent:

(a) $X_P = X_Q$.
(b) $0 \leq i < i_1(Q)$.

Moreover, if these conditions are satisfied, then $i_1(P) = i_1(Q) - i$.

Proof of Corollary 3 (1) Since $X_P = X_Q$, we have direct summands $N$ of $Q$, and $M$ of $P$, s.t. $N$ contains $X_Q$, $M$ contains $X_P$, and $M \cong N$. To construct such summands, consider rational maps: $f : Q \to P$, $g : P \to Q$. Closure of their graphs in $Q \times P$ and $P \times Q$, respectively, gives us motivic maps $\phi : Q \to P$ and $\psi : P \to Q$, s.t. $\phi$ and $\psi$, restricted to $\bar{k}$ map $\mathbb{Z}$ to $\mathbb{Z}$ isomorphically. By [4], Lemma 3.26, this implies the existence of specified motives $M$ and $N$. From Statement it follows, that the dimension (see Corollary 1 ) of $N$ is $\dim(Q) - i_1(Q) + 1$, and the dimension of $M$ is $\dim(P) - i_1(Q) + 1$. The isomorphism $M \cong N$ completes the proof.

(2) Clearly, from the existence of rational point on $P$ follows the existence of such on $Q$. Hence, $X_P \geq X_Q$. Also, from the existence of $i$-dimensional projective subspace on $Q$ follows the existence of rational point on $P$. Hence, $X_Q \geq X_P \geq X_Q$.

If $0 \leq i < i_1(Q)$, then $X_{Q'} = X_Q$, and from the above inequality we get: $X_Q = X_P$. So, $(b) \Rightarrow (a)$.

In the other direction: if $X_Q = X_P$, then by (1), $\dim(P) - i_1(P) = \dim(Q) - i_1(Q)$. Since $i_1(P) \geq 1$, we get: $i \leq i_1(Q)$.

The last statement is evident in the light of (1).

The following interesting question in the study of direct summands of $Q$ arises: for which $i$ we have a direct summand $N$, “starting” from $(i)$ (that is: $N|_\pi$ contains $\mathbb{Z}(i)[2i]$, but does not contain any $\mathbb{Z}(j)[2j]$ with $j < i$)? We can give here some sufficient condition (see Proposition 1 below), which, we
believe, should be also necessary one (see Question 1). Our Statement is very useful here.

Lemma 3.
Let $Q/k$ be some quadric, and $K/k$ be some field, such that $K$ has a smooth point over $k(Q)$. Then $\mathcal{I}(Q/K) = \mathcal{I}(Q/k)$.

Proof of Lemma 3
We just need to check, that over $K(Q^{1+\cdots+t-1})$, $Q^{1+\cdots+t}$ has no smooth point for any $1 \leq t \leq s$. But $Q$ has a smooth point over $k(Q^{1+\cdots+t-1})$, and $K$ has a smooth point over $k(Q)$. By transitivity, from the existence of a $K(Q^{1+\cdots+t-1})$-point on $Q^{1+\cdots+t}$ would follow the existence of $k(Q^{1+\cdots+t-1})$-point there, which is not the case.

Lemma 3 is proven.

Lemma 4.
Let $Q$ be a quadric, and $i_1, \ldots, i_t$ be its higher Witt indices. Let $1 \leq t \leq s$, and $S$ be a plane section of $Q$ of codimension $i_t(Q) - 1$. Let $i = i_1 + \cdots + i_t - 1$. Then $Q$ contains an undecomposable direct summand with “lowest” term $(i)$ iff $S$ does.

Proof of Lemma 4
Let $\dim(Q) = n$.

$(\Rightarrow)$ If $Q$ contains such a summand $N$, then by [4], Lemma 4.5, it contains also one isomorphic to $N(i_t(Q) - 1)[2i_t(Q) - 2]$, with lowest term $(i + i_t - 1)$ (since $\mathcal{X}_{Q^{1+\cdots+t-1}} \neq \mathcal{X}_{Q^{1+\cdots+t-1}} = \cdots = \mathcal{X}_{Q^{1+\cdots+t-1}}$). Let $\varphi : S \rightarrow Q$ be a map, corresponding to the inclusion $S \subset Q$, and $\varphi^\vee : Q(-i_t(Q) + 1)[2i_t(Q) + 2] \rightarrow S$ be the dual map via duality $\text{Hom}(-, \mathbb{Z}(n - i_t + 1)[2n - 2i_t + 2])$. Let $j_N : N \rightarrow Q$, $j_{N(i_t-1)[2i_t-2]} : N(i_t-1)[2i_t-2] \rightarrow Q$, and $\pi_N : Q \rightarrow N$, and $\pi_{N(i_t-1)[2i_t-2]} : Q \rightarrow N(i_t-1)[2i_t-2]$ be maps, realizing $N$ and $N(i_t-1)[2i_t-2]$ as direct summands in $Q$. Then we have the pair of maps: $\varphi : S \rightarrow Q$, and $\psi := \varphi^\vee \circ j_{N(i_t-1)[2i_t-2]}(1 - i_t)[2 - 2i_t] \circ \pi_N : Q \rightarrow S$. It is easy to see, that for $\psi \circ \varphi : S \rightarrow S$ we have: $(\psi \circ \varphi)_i = 1$ (see [4], Theorem 3.7 for notations). That means (see [4], Lemma 3.26) that $Q$ and $S$ contain isomorphic direct summands, containing $(i)$. Such summands should be isomorphic to $N$, so the $(i)$ is the “lowest” term in them.

$(\Leftarrow)$ If $S$ contains such a summand $M$, then $M(-i)[-2i]$ is a direct summand in $S'$ (see [4], Claim 3.2 and Lemma 4.6). Really, we need only to check, that $M$ “is contained” in $S(i)'$, i.e.: $i \leq i_1(i_S)$ (by the Statement). Let
$K = k(Q^{i-1})$. Suppose $M$ is not contained in $S(i)'$, then $i > i^\perp_{(1)(S)}$, and by Lemma 3, $i > i^\perp_{(1)(S)}$.

But over $K$, $Q|_K$ is $(i-1)$-times isotropic. Then $Q$ contains undecomposable direct summand $N$, whose “lowest” term is $(i)$, then, by the Statement (2), the “highest” term of $N$ will be $(i+i_i-1)'$. By $(\rightarrow)$, $S|_K$ also contains direct summand $M'$ isomorphic to $N$, and $M'$ contains $(i)$ and $(i)'$ as it’s “lowest” and “highest” terms. By the statement, we get contradiction with the assumption that $i > i^\perp_{(1)(S)}$. So, $M(−i)[−2i]$ is a direct summand in $S_i$.

Since $\mathcal{X}_{Z^i} = \mathcal{X}_{S_i} = \mathcal{X}_{Q^i} = \mathcal{X}_{Q^i+i_i-1}$, we have the map $\varepsilon': S^i(i+2i_i-1)[2i + 2i_i - 2] \rightarrow Q$, which over $\mathbb{F}$ maps $(\mathbb{Z})(i + i_i - 1)[2i + 2i_i - 2]$ to $\mathbb{Z}(i + i_i - 1)[2i + 2i_i - 2]$ isomorphically. Since $M(−i)[−2i]$ is a direct summand in $S_i$, we get a map $\varepsilon'': M(i_i - 1)[2i_i - 2] \rightarrow Q$ with the same property. Let $\varepsilon := \varepsilon'' \circ \pi_M(i_i - 1)[2i_i - 2] : S(i_i - 1)[2i_i - 2] \rightarrow Q$, where $\pi_M : S \rightarrow M$ is the natural projection. We can see, that $\varepsilon|_\mathbb{F}$ still maps $(\mathbb{Z})(i + i_i - 1)[2i + 2i_i - 2]$ to $\mathbb{Z}(i + i_i - 1)[2i + 2i_i - 2]$ isomorphically.

Let $\varphi : S \rightarrow Q$ be natural map (corresponding to the embedding $S \subset Q$), and $\varphi' : Q(1 - i_i)[2 - 2i_i] \rightarrow S$ be map dual to $\varphi$ via duality $\text{Hom}(−, \mathbb{Z}(n - i_i + 1)[2n - 2i_i + 2])$. Then $\varphi' \circ \varepsilon(1 - i_i)[2 - 2i_i] : S \rightarrow S$ over $\mathbb{F}$ maps $\mathbb{Z}(i)[2i]$ to $\mathbb{Z}(i)[2i]$ isomorphically. By [4], Lemma 3.26 that means that $Q(1 - i_i)[2 - 2i_i]$ and $S$ contain isomorphic undecomposable direct summands $N_1$ and $N_2$, containing $\mathcal{X}_{Q^{i_i-1}}(i)[2i]$ and $\mathcal{X}_{S_i}$, respectively. So, they should be isomorphic to $M$ (see [4], Lemma 3.21). In particular, $\mathcal{X}_{Q^{i_i-1}}(i)[2i]$ is the “lowest” term in $N_1$, and $\mathcal{X}_{Q^{i_i-1}}(i_i - 1)[2i + 2i_i - 2]$ is the “lowest” term in $N_1(i_i - 1)[2i_i - 2]$. By the Statement (2), $\mathcal{X}_{Q^n}(n - i)[2n - 2i]$ is the “highest” term in it. If $(N_1(i_i - 1)[2i_i - 2])'$ is the summand dual to $N_1(i_i - 1)[2i_i - 2]$ via duality $\text{Hom}(−, \mathbb{Z}(n)[2n])$, then it is evidently undecomposable and it’s “lowest” term is $(i)$.

Lemma 4 is proven.

\[\square\]

Proposition 1.

Let $Q$ and $P$ be such quadrics, that for some $i$, $\mathcal{X}_{Q^i} = \mathcal{X}_{P^i}$. Then there exists a direct summand $N$ in $Q$, starting from $\mathcal{X}_{Q^i}(i)[2i]$ (i.e. $N$ contains no $\mathcal{X}_{Q^i}(l)[2i]$ with $l < i$, but contains $\mathcal{X}_{Q^i}(i)[2i]$).

Proof

Changing $P$, if necessary, by it’s plane section, we can assume, that $i_1(P) = 1$. By Lemma 4, we can also assume, that $i_{j_1(i)(Q^i)} = 1$, i.e.
$X_{Q^i-1} \neq X_{Q^i} \neq X_{Q^{i+1}}$. Really, first of all we can assume, by [4], Lemma 4.5 and our Statement, that $i = i_1 + \cdots + i_{t-1}$ for some $t$. If the corresponding higher Witt index $i_t(Q)$ is $> 1$, change $Q$ by its plane section $S$ of codimension $i_t - 1$. Since $\dim(S) < \dim(Q)$, after few such steps we should get the quadric $S'$ with $i_j(Q_{S'}) = 1$. And the existence of the required direct summand for $Q$ follows from that for $S'$ (by Lemma 4).

Let $\dim(Q) = n$, $\dim(P) = m$.

Since $X_{Q^i} = X_{P^i}$, we have a motivic map $\varphi : P(i)[2i] \to Q$, s.t. $\varphi|_{\overline{k}}$ sends $(\mathbb{Z})(i)[2i]$ to $\mathbb{Z}(i)[2i]$ isomorphically. Let $\varphi' : Q(m + 2i - n)[2(m + 2i - n)] \to P(i)[2i]$ will be map dual to $\varphi$ with respect to duality $\text{Hom}(\cdot, \mathbb{Z}(m + 2i)[2m + 4i])$.

Let $K = k(Q^{i-1})$. Then over $K$, $q$ is (precisely) $i$-times isotropic (since $X_{Q^{i-1}} \neq X_{Q^i}$); let $R/K$ be anisotropic part of $Q|_K$.

Then $Q_K = \oplus_{l=0,\ldots,i-1} (\mathbb{Z}(l)[2i] \oplus \mathbb{Z}(n-l)[2n-2l]) \oplus R(i)[2i]$, and $\varphi|_K(-i)[-2i]$ gives a map $\psi : P|_K \to R$.

But $X_{P/K} = X_{Q'K} = X_{P/K}$, that means that there exist a map: $\rho : R \to P$, s.t. for the composition $\alpha = \rho \circ \psi : P \to P$, we have $\alpha_0 = 1$ (in the sense of [4], Theorem 3.7, and the text after the Corollary 3.9)(i.e., $\alpha|_{\overline{k}}$ sends $\mathbb{Z}$ to $\mathbb{Z}$ isomorphically). By [4], Lemma 3.26, that means, that $R$ and $P|_K$ contain isomorphic direct summands $N$ and $M$, containing $(0)$ and $(0)$, respectively. But by the Statement, $M$ should contain $(0)'$ (since, by the Lemma 3, $i_1(P/K) = i_1(P) = 1$), and it will be the “highest” elementary piece of $M$, and in the same way, the “highest” elementary piece of $N$ will be $(0)'$ (again, by the Lemma 3, $i_1(R) = i_1(Q) = 1$). Since $M$ is isomorphic to $N$, and $\dim(R) = n - 2i$, $\dim(P) = m$, we get that: $m = n - 2i$, and $\psi|_{\overline{k}}$ sends $\mathbb{Z}(m)[2m]$ to itself via multiplication by an odd number (hence, $\varphi(-i)[-2i]$ does the same). Then $\varphi'(n - m - 3i)[2n - 2m - 6i]|_{\overline{k}}$ maps $\mathbb{Z}$ to $\mathbb{Z}$ via multiplication by an odd number. By [4], Lemma 3.20 we can find a map $\rho : Q(-i)[-2i] \to P$, which, over $\overline{k}$, will map $\mathbb{Z}$ to $\mathbb{Z}$ isomorphically. Hence, for $\varepsilon := \rho \circ \varphi(-i)[-2i] : P \to P$ we have $\varepsilon_0 = 1$, and by [4], Lemma 3.26, $P$ and $Q(-i)[-2i]$ have isomorphic direct summands, containing $X_P$ and $X_{Q^i}$, respectively. That means that $Q$ contains a direct summand starting from $X_{Q^i}(i)[2i]$.

Proposition 1 is proven.

In connection with Proposition 1 it is natural to ask the following:

Question 1.
Are the following conditions equivalent?

1) \( Q \) contains a direct summand with the lowest term \((i)\).

2) There exists quadric \( P/k \), s.t. \( X_P = X_Q \).

In a meantime, we can characterize those \( i \), for which there exists a direct summand \( N \) starting from \((i)\), in the following way:

**Proposition 2 .**

Let \( Q \) be a quadric, and \( 0 \leq i < n/2 \), where \( n = \dim(Q) \). Then the following conditions are equivalent:

1) There exists undecomposable direct summand \( N \) in \( Q \), with the lowest term \((i)\).

2) The natural map \( \alpha_i : Q \to \mathbb{Z}(i)[2i] \) (corresponding to a plane section of codimension \( i \)) is a composition \( Q \overset{u}{\to} X_Q(i)[2i] \to \mathbb{Z}(i)[2i] \), for some \( u (X_Q(i)[2i] \to \mathbb{Z}(i)[2i] \) here is a natural projection).

3) The map \( \alpha_i : Q \to \mathbb{Z}(i)[2i] \) is a composition \( Q \overset{v}{\to} Q(i)[2i] \to \mathbb{Z}(i)[2i] \), for some \( v \), where \( Q(i)[2i] \to \mathbb{Z}(i)[2i] \) is again a natural projection.

4) There exists a subvariety \( T \) of \( Q \) of codimension \( i \) and of degree not divisible by \( 4 \), s.t. \( Q^i \) has a rational point over \( k(T) \).

**Proof of the Proposition 2**

(1 \( \to \) 2) Follows from [4], Lemma 3.23.

(2 \( \to \) 3) In DM\textsuperscript{eff}(\( k \)) we have the following exact triangle: \( Q^i \to X_Q \to Y \to Q^i[1] \), where \( Y \) is an “extension” of \( Q^i \times Q^i[1] \), \( Q^i \times Q^i \times Q^i[2] \), etc ...

Since \( \text{Hom}_{\text{DM}\textsuperscript{eff}}(Q, Q^i \times \cdots \times Q^i(i)[2i + p]) = 0 \), for any positive \( p \), we have that any map \( u : Q \to X_Q(i)[2i] \) can be lifted to the map \( v : Q \to Q^i(i)[2i] \).

(3 \( \to \) 4) The map \( v : Q \to Q^i(i)[2i] \) is given by some cycle \( V \subset Q \times Q^i \) of dimension \( n - i \), and the composition \( Q \overset{v}{\to} Q^i(i)[2i] \to \mathbb{Z}(i)[2i] \) is given by the cycle \( W = (\pi_1)_*(V) \subset Q \), where \( \pi_1 : Q \times Q^i \to Q \) is the projection on the first factor. Since the composition coincides with \( \alpha_i \) (given by plane section of codimension \( i \)), we have that the degree of \( W \) is 1. Hence \( W \) should contain irreducible component \( T \) of odd degree in odd multiplicity. That means that over \( k(T) \), \( Q^i \) has a point of odd degree, and by Springer theorem it has a rational point over that field.
(4 → 1) The rational map \( T \rightarrow Q^i \) give us cycle \( V \subset Q \times Q^i \) of dimension \( n - i \), and so, a map \( v : Q \rightarrow Q^i(2i) \). Consider the standard map \( \varphi : Q^i(2i) \rightarrow Q \) (given by the cycle \( \Phi \subset Q^i \times Q, \Phi = \{(l, x) : x \in l\} \)). The composition \( Q^i(2i) \xrightarrow{v} Q \xrightarrow{\rho} Z(i)[2i] \) coincides with the natural projection \( Q^i(2i) \rightarrow Z(i)[2i] \). That means that the composition \( Q \rightarrow Q^i(2i) \xrightarrow{v} Q \xrightarrow{\rho} Z(i)[2i] \) is given by the cycle \( T \subset Q \). Consider the map \( \rho := \varphi \circ v : Q \rightarrow Q \). Since \( \text{Hom}(Z(j)[2j], Q^i(2i)) = 0 \) for any \( j < i \), we have that over \( \overline{k} \), \( \rho|_\overline{k} \) maps \( Z(j)[2j] \) to 0 for such \( j \). On the other hand \( \rho|_\overline{k} : Z(i)[2i] \rightarrow Z(i)[2i] \) is a multiplication by the degree of \( T \) divided by 2, which is odd. By [4], Lemma 3.12 and Lemma 3.25, there is a direct summand of \( Q \), starting from \( i \).

The Proposition 2 permits us to clarify the picture in the Proposition 3.4 from [4].

We remind, that \( \beta_i : Z(n - i)[2n - 2i] \rightarrow Q \) is a natural map given by the plane section of codimension \( i \) in \( Q \). Then we can define the natural map \( \oplus \beta'_i : \oplus_{0 \leq i < n/2} \mathcal{X}_{Q^i}(n - i)[2n - 2i] \rightarrow Q \), where \( \beta'_i \) is a composition of natural projection \( \mathcal{X}_{Q^i}(n - i)[2n - 2i] \rightarrow Z(n - i)[2n - 2i] \) and \( \beta_i \). Let \( P' = \text{Cone}(\oplus \beta'_i) \).

By [4], Proposition 3.4, \( P' \) is an extension of \( \mathcal{X}_{Q^i}(i)[2i], 0 \leq i < n/2 \), and also \( k(\sqrt{\text{det}(Q)}) \times \mathcal{X}_{Q^{n/2}}(n/2)[n] \) (if \( n \) is even).

**Corollary 4.**

Let \( Q \) be anisotropic quadric. The following conditions are equivalent:

1. \( P' \) is isomorphic to a direct sum \( \oplus_{0 \leq i < n/2} \mathcal{X}_{Q^i}(i)[2i] \) \( (\oplus k(\sqrt{\text{det}(Q)}) \times \mathcal{X}_{Q^{n/2}}(n/2)[n], \text{if } n \text{ is even}). \)

2. \( Q \) consists of “binary motives” (i.e., motives, consisting of just two elementary pieces).

**Proof** (1 → 2) Since \( P' \) is a direct sum we have that the map \( \alpha_i : Q \rightarrow Z(i)[2i] \) can be lifted to the map \( \alpha'_i : Q \rightarrow \mathcal{X}_{Q^i}(i)[2i] \). By Proposition 2, that means that there is an undecomposable direct summand of \( Q \), starting from \( i \). Since it is true for all \( 0 \leq i \leq n/2 \), all undecomposable direct summands of \( Q \) are binary (they contain only one “simple piece” from the lower half of \( Q \) ⇒ only one from the upper half as well).

(2 → 1) The decomposition of \( Q \) into the binary motives gives the decomposition of the map \( \oplus \beta'_i : \oplus_{0 \leq i < n/2} \mathcal{X}_{Q^i}(n - i)[2n - 2i] \rightarrow Q \), which
gives a decomposition of $P'$ into the direct sum of elementary components of $P'$ (i.e. $X_Q, X_{Q^2}(1)[2]$, etc. ...).

**Remark** (1) and (2) in the Corollary 4 should be equivalent to (3): $Q$ is *Excellent* quadric. In the one direction it is a result of M.Rost (see [2], Proposition 4). In the other: we know only (from the proof of the Statement 6.1 from [4]) that $Q$ should have *excellent splitting pattern*, and our binary motives are “of the Rost-motive size”.

**References**


