ON TORSION ELEMENTS IN THE CHOW-GROUPS OF QUADRICS

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0. Introduction

In this paper we present some new example of torsion elements in the Chow-groups of projective quadric. For hyperbolic projective quadric $X$ and any $0 \leq p < \dim(X)/2$, the Chow-group of cycles of codimension $p$ (modulo rational equivalence) is a free abelian group generated by the class $h^p$ of plane section of codimension $p$. By transfer arguments it follows, that for arbitrary (smooth) projective quadric, $CH^p(X) = \mathbb{Z} \cdot h^p \oplus \text{Tors}(CH^p(X))$.

On the other hand, for any projective quadric $X$ of dimension more than 2, the Picard group $CH^1(X)$ is isomorphic to $\mathbb{Z} \cdot h$, in other words, there is no torsion. The natural question arises: can we extend this result to higher Chow-groups.

Basic here is the following conjecture due to N. Karpenko:

Let $X_\varphi$ be a projective quadric, defined by quadratic form $\phi$ of dimension $n$ over a field of characteristic not 2.

Conjecture 1 ([7], Conjecture 0.1).
For any $p$, if $n$ is sufficiently large, then $CH^p(X_\varphi) = \mathbb{Z} \cdot h^p$.

This conjecture was supported by the following computations:

Theorem ([6], Theorem 6.1; [8], Theorem 6.1, Theorem 8.5).
Under above notations,

(a) $CH^2(X_\varphi) = \mathbb{Z}$, for $n > 8$;
(b) $CH^3(X_\varphi) = \mathbb{Z}$, for $n > 12$;
(c) $CH^4(X_\varphi) = \mathbb{Z}$, for $n > 24$;

Moreover, the boundaries in (a) and (b) are exact ones, and more generally:

Theorem ([7], Theorem 2.4).
For any $p > 1$ there exist a $4p$-dimensional quadratic form $\varphi$ (over a suitable field $F$), such that $\text{Tors}(CH^p(X_\varphi)) \neq 0$.

In [7] an attempt was made to make Conjecture 1 more precise:

Conjecture 2 ([7], Conjecture 0.2).
If $n > 4p$ for some $p$, then $CH^p(X_\varphi) = \mathbb{Z} \cdot h$. 

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The principal aim of this note is to disprove this stronger version of the Conjecture - see Corollary 2.1. For any $r \geq 2$ we will construct the quadratic form $\varphi$ of dimension $6 \cdot 2^r$ (over suitable field), such that $\text{Tors.(CH}^{2^r+1}(X_\varphi)) \neq 0$. This clearly will disprove the Conjecture 2.

The main ingredients of the construction are: the special pair of O.Izhboldin (see [3], Section 9), the computations of unramified cohomology by B.Kahn-M.Rost-R.J.Sujatha (including the motivic interpretation of the later)(see [5], and [4], Appendix), and the motivic cohomological operations of V.Voevodsky (see [17]). Although, the original problem is formulated in the classical Chow-motivic language, we have to work in the bigger triangulated category of mixed motives of V.Voevodsky (see [16]). In this category we can use the operations and the decomposition of the motive of a quadric from [14], Theorem 3.1.

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1. Some preliminary computations

Under generalized Albert form we will understand the anisotropic part of the difference of two $n$-fold Pfister forms:

$$(\langle a_1, \ldots, a_n \rangle - \langle b_1, \ldots, b_n \rangle)_{\text{an}}.$$

Such forms were studied extensively in [1].

**Proposition 1.1**

For any $r$, over some field of given characteristic $\neq 2$ there exists anisotropic generalized Albert form of dimension $6 \cdot 2^r$.

**Proof of Proposition 1.1**

**Lemma 1.2**

Let $F$ be a field of characteristic not 2, and $q/F$ be an anisotropic quadratic form. Let $K := F(x_1, \ldots, x_r)$ be a purely transcendental extension, and $p := q \times \langle x_1, \ldots, x_r \rangle$ be a quadratic form over $K$. Then $p$ is anisotropic.

**Proof**
It is evidently sufficient to consider the case $r = 1$.

Suppose that $p := q \times \langle 1, -x \rangle$ is isotropic over $F(x)$. Restricting to $F((x))$, we get a contradiction with the Theorem of Springer (see [10], VI, Proposition 1.9). So, $p$ is anisotropic.

Lemma 1.3.

Suppose the Pfister form $\langle a, b \rangle$ over the field $F$ is anisotropic. Then the Albert form $\langle a, b, -ab, yz, -y, -z \rangle$ is anisotropic over the field $L := F(y, z)$.

Proof

If our Albert form is isotropic over $L$, then it is isotropic over $F((y))((z))$, and by [10], VI, Proposition 1.9, we have that $\langle a, b, -ab, -y \rangle$ is isotropic over $F((y))$ (since $\langle y, -1 \rangle$ is evidently anisotropic over the last field). Again by the Theorem of Springer we get a contradiction.

Suppose $K = k(a, b, c, d, x_1, \ldots, x_r)$ be purely transcendental extension, generated by specified variables. Consider quadratic form $\varphi := \rho \times \langle x_1, \ldots, x_r \rangle$, where $\rho := \langle a, b, -ab, cd, -c, -d \rangle$.

Lemma 1.4.

The form $\varphi$ is anisotropic.

Proof

By Lemma 1.2, it is sufficient to prove that $\langle a, b, -ab, cd, -c, -d \rangle$ is anisotropic over $L := k(a, b, c, d)$. Take $F := k(a, b)$. Then evidently $\langle a, b \rangle$ is anisotropic over $F$. By Lemma 1.3 we get what we need.

Proposition 1.1 is proven.

2. The Main Theorem

Main Theorem.

Let $\varphi$ is anisotropic generalized Albert form of dimension $6 \cdot 2^r$ over the field $k$ of characteristic 0. Then $\text{Tors. CH}^{2^r+1}(X_{\varphi}) \neq 0$.

Combining the Main Theorem with Proposition 1.1, we get:

Corollary 2.1.

Conjecture 2 is wrong.

Proof of the Main Theorem

Since $\varphi$ is anisotropic generalized Albert form of dimension $6 \cdot 2^r$, we have: $\varphi = \rho \times \langle x_1, \ldots, x_r \rangle$, where $\rho = \langle a, b, -ab, -c, -d, cd \rangle$ is
6-dimensional Albert form, and \( \langle x_1, \ldots, x_r \rangle \) is some Pfister form (see [1]).

Denote \( \psi := \langle a, b, -ab, -c, -d \rangle \times \langle x_1, \ldots, x_r \rangle \). Then \((\varphi, \psi)\) is a special pair of Izhboldin (see [3], Section 9), and by condition, it is an anisotropic special pair.

By [3], Theorem 9.4 (3), we have that \( \psi|_{k(\varphi)} \) is anisotropic Pfister neighbor, i.e., it is proportional to a subform (of dimension more than half) of anisotropic Pfister form \( \pi \). This Pfister form corresponds to a nontrivial pure symbol \( \alpha \) in \( K_{r+3}^M(k(\varphi))/2 \). Under the natural map \( K_{r+3}^M(k(\varphi))/2 \to H^*_e(k(\varphi), \mathbb{Z}/2) \) (which is an isomorphism by “Milnor’s conjecture” - see [17]), \( \alpha \) goes to the nontrivial element \( e^{r+3}(\text{Pf}(\psi|_{k(\varphi)})) \) of \( H^{r+3}_e(k(\varphi), \mathbb{Z}/2) \) (we are following the notations of [3], Lemma 7.4). By [3], Corollary 7.3, \( e^{r+3}(\text{Pf}(\psi|_{k(\varphi)})) \) actually belongs to the unramified part: \( H^{r+3}_{nr}(k(\varphi))/k, \mathbb{Z}/2 \) of \( H^{r+3}_e(k(\varphi), \mathbb{Z}/2) \).

We have natural map: \( \eta_2^* : H^*_e(k, \mathbb{Z}/2) \to H^*_e(k(\varphi), \mathbb{Z}/2) \). Let us denote \( \bar{H}^{r+3}_{nr}(k(\varphi))/k, \mathbb{Z}/2) = H^*_e(k(\varphi), \mathbb{Z}/2)/\text{image}(\eta_2^*). \) For arbitrary element \( x \) from \( H^*_e(k(\varphi), \mathbb{Z}/2) \), we denote it’s image in \( \bar{H}^{r+3}_{nr}(k(\varphi))/k, \mathbb{Z}/2) \) as \( \bar{x} \).

Proposition 2.2 (cf. [3], Lemma 10.5, Lemma 7.4).

Under the above notations, \( e^{r+3}(\text{Pf}(\psi|_{k(\varphi)})) \) is nonzero.

Proof

Suppose, \( \bar{e}^{r+3}(\text{Pf}(\psi|_{k(\varphi)})) = 0 \). That means that \( e^{r+3}(\text{Pf}(\psi|_{k(\varphi)})) = \eta_2^{r+3}(\lambda) \) for some \( \lambda \in H^{r+3}_e(k, \mathbb{Z}/2). \) By the results of V.Voevodsky, we have the natural identification:

\[
K_{r+3}^M(k)/2 = H^{r+3}_e(k, \mathbb{Z}/2) = I^{r+3}(W(k))/I^{r+4}(W(k)),
\]

where \( I \subseteq W(k) \) is the ideal of even-dimensional form in the Witt-ring \( W(k) \) (see [17] and [11], Section 3.1). So, there exists quadratic form \( q \subseteq I^{r+3}(W(k))/I^{r+4}(W(k)) \), s.t. \( \mathfrak{q} \in I^{r+3}(W(k))/I^{r+4}(W(k)) \) corresponds to \( \lambda \) under the identifications above. By “J-filtration Conjecture” we have \( I^{r+3}(W(k)) = J^{r+3}(W(k)) \) (see [11], Section 3.3, Statement 2). That means that the degree (see [9]) of \( q \) is \( r + 3 \), and there exists a Generalized Splitting Tower of Manfred Knebusch (see [9]) \( k = k_0 \subset k_1 \subset \cdots \subset k_{s-2} \subset k_s \), s.t. for all \( 0 \leq l \leq s-2, k_{l+1} = k_l(Q_l) \), where \( Q_l \) is quadric (over \( k_l \)) of dimension greater than \( 2^{r+3} - 2 \), and \( (q|_{k_{s-1}})_{\text{anis.}} \) is proportional to anisotropic \( r + 3 \)-fold Pfister form over \( k_{s-1} \). That means that \( \mathfrak{q}|_{k_{s-1}} = \lambda|_{k_{s-1}} \in K_{r+3}^M(k_{s-1})/2 \) is a nonzero pure symbol.

Since \( \dim(q_l) > 2^{r+3} > 6 \cdot 2^r = \dim(\varphi) \), by the result of Detlew Hoffmann (see [2], Theorem 1), we have that \( \varphi|_{k_{s-1}} \) is anisotropic.
Let us denote $F := k_{s-1}$. We have: $\varphi|_F$ is anisotropic, and (if you want, again by [3], Theorem 9.4 (3)) $\psi|_{F(X_\varphi)}$ is a neighbor of anisotropic Pfister form $\langle \lambda|_F \rangle|_{F(X_\varphi)}$ (notice that $\lambda|_F$ is a pure symbol). In particular, $\lambda|_{F(X_\varphi)} = 0$. Since $\psi \subset \varphi$, we have $\lambda|_{F(X_\varphi)} = 0$. That means that $\psi|_F$ is a Pfister neighbour of $\langle \lambda|_F \rangle$ By [3], Lemma 9.4 (2), we get: $\varphi|_F$ is isotropic - a contradiction. So, $\hat{\varphi}^{r+3}(\Pf(\psi|_{k(X_\varphi)})) \neq 0$

Let $\gamma := \hat{\varphi}^{r+3}(\Pf(\psi|_{k(X_\varphi)})) \in \text{He}_{nr}^{r+3}(k(X_\varphi)/k; \mathbb{Z}/2)$. By Proposition 2.2 , $\gamma \neq 0$.

Together with the map $\eta^m_2 : H_{et.}^m(k, \mathbb{Z}/2) \rightarrow H_{nr}^m(k(X_\varphi)/k, \mathbb{Z}/2)$ we can consider the map: $\eta^m : H_{et.}^m(k, \mathbb{Q}/\mathbb{Z}(m-1)) \rightarrow H_{nr}^m(k(X_\varphi)/k, \mathbb{Q}/\mathbb{Z}(m-1))$.

From [5], Theorem 7.4 and Remark after it (and from “Milnor’s conjecture”, see [17]), we have an exact sequence:

$$0 \rightarrow (\text{Ker}(\eta^m_2))_0 \rightarrow \text{coker}(\eta^m_2) \rightarrow \text{coker}(\eta^m),$$

where $(\text{Ker}(\eta^m_2))_0 = \{ y \in \text{Ker}(\eta^m_2) \mid \{-1\} \cdot y = 0 \}$.

Lemma 2.3 .

The group $\text{Ker}(\eta^{r+3}_2) : H_{et.}^{r+3}(k, \mathbb{Z}/2) \rightarrow H_{nr.}^{r+3}(k(X_\varphi)/k, \mathbb{Z}/2)$ is zero. Consequently, $(\text{Ker}(\eta^{r+3}_2))_0 = 0$.

Proof

Suppose, our Ker is nontrivial. Since $H_{nr}^{r+3}(k(X_\varphi)/k, \mathbb{Z}/2)$ is a subgroup in $H_{et.}^{r+3}(k(X_\varphi), \mathbb{Z}/2)$, in this case we get a nontrivial element $h \in \text{Ker}(H_{et.}^{r+3}(k, \mathbb{Z}/2) \rightarrow H_{et.}^{r+3}(k(X_\varphi), \mathbb{Z}/2)$.

By [11], Lemma from Section 3, there exists field extension $L/k$, s.t. $h|_L$ is nonzero pure symbol $\{a_1, \ldots, a_{r+3}\}$ for some $a_i \in L$. Then $\varphi|_L$ (up to coefficient) is a subform in the anisotropic Pfister form $\langle a_1, \ldots, a_{r+3} \rangle$. Suppose $\rho$ is a complementary form, i.e.: $\varphi|_L \perp \rho = \langle a_1, \ldots, a_{r+3} \rangle$.

Since $\varphi|_L \in I^{r+2}(W(L))$ and $\langle a_1, \ldots, a_{r+3} \rangle \in I^{r+3}(W(L))$, we have that $\rho \in I^{r+2}(W(L))$. But, the dim($\rho$) = $2^{r+3} - 6 \cdot 2^r = 2^{r+1} < 2^{r+2}$ - contradiction.

So, we have an exact sequence:

$$0 \rightarrow \text{coker}(\eta^{r+3}_2) \rightarrow \text{coker}(\eta^{r+3}).$$

Lemma 2.4 .

Let $X$ be arbitrary quadric, then $\text{coker}(\eta^m_{Q(p)/Z(p)}) : H_{et.}^m(k, \mathbb{Q}(p)/\mathbb{Z}(p)) \rightarrow H_{nr.}^m(k(X), \mathbb{Q}(p)/\mathbb{Z}(p))) = 0$ for any $p \neq 2$ and any $m$.

Proof
Let Spec$(L) \subset X$ be arbitrary point on our quadric. Since $X_{\varphi}|L$ is isotropic, it is birationally equivalent to projective space, and hence, $\text{coker}(\eta^m_A : H^m_{\text{et}}(L, A) \to H^m_{\text{nr}}(L(X), A)) = 0$ for any group of coefficients $A$ (see [12], 12.10, 7.3). Since $Q_r/F$ is uniquely 2-divisible, by transfer arguments, we have that $\text{coker}(\eta^m_{Q_r/F} : H^m_{\text{et}}(k, Q_r/F) \to H^m_{\text{nr}}(k(X), Q_r/F)) = 0$.

The above Lemma shows:

\[ \text{coker}(\eta^{r+3}) = \text{coker}(\eta^{r+3}_{Q(2)/Z(2)} : H^{r+3}_{\text{et}}(k, Q(2)/Z(2)) \to H^{r+3}_{\text{nr}}(k(X), Q(2)/Z(2))). \]

In this light, we will denote $\eta^{r+3}_{Q(2)/Z(2)}$ simply as $\eta^{r+3}$.

In [4] B.Kahn interpreted $\text{coker}(\eta^{r+3})$ in terms of motivic cohomology, see [4], Theorem A.1: there is an exact sequence:

\[ 0 \to \text{coker}(\eta^{r+3}) \to H^{r+5}_{B}(X_{\varphi}, Z(2)(r + 2)), \]

where $X_{\varphi}$ is a standard simplicial scheme, corresponding to the morphism $X_{\varphi} \to \text{Spec}(k)$ (see, for example, [14], Definition 2.3.1), and $H^{r+5}_{B}(X_{\varphi}, Z(2)(r + 2)) = \text{Hom}_{DM_{\text{eff}}}(M(X_{\varphi}), Z(2)(r + 2)[r + 5])$ - the Hom-group in the triangulated category of mixed motives of V.Voevodsky, where $Z$ by definition is the motive of a point ($M(\text{Spec}(k))$).

Remark Notice the discrepancy in the notations for $\eta^{m}$ in [4] in comparison to [5] (where it coincides with our notation). $\eta^{m}$ in [4] corresponds to $\eta^{m+1}$ in [5].

This shows, that we have an exact sequence:

\[ 0 \to \text{coker}(\eta^{r+3}) \to \text{Hom}_{DM_{\text{eff}}}(M(X_{\varphi}), Z(2)(r + 2)[r + 5]). \]

Since we have a nontrivial element $\gamma$ in $\text{coker}(\eta^{r+3})$, its image $\gamma' \in \text{Hom}_{DM_{\text{eff}}}(M(X_{\varphi}), Z(2)(r + 2)[r + 5])$ will also be nontrivial.

Using the fact that $\text{Hom}_{DM_{\text{eff}}}(M(\text{Spec}(L)), Z(i)[j]) = 0$ for $j > i$, where $L$ is a field (see [17], Corollary 2.3), and the fact that $M(X_{\varphi})|L = M(\text{Spec}(L))$ for any point $\text{Spec}(L) \subset X_{\varphi}$ on a quadric (see [17], Lemma 3.8, or [14], Theorem 2.3.4), by usual transfer arguments we have that the groups $\text{Hom}_{DM_{\text{eff}}}(M(X_{\varphi}), Z(2)(i)[j])$ have exponent 2 for $j > i$. That means, that we have an embedding: $\text{Hom}_{DM_{\text{eff}}}(M(X_{\varphi}), Z(2)(r + 2)[r + 5]) \to \text{Hom}_{DM_{\text{eff}}}(M(X_{\varphi}), Z/2(r + 2)[r + 5])$. Let $\gamma''$ be the image of $\gamma'$ under this map. We have: $\gamma'' \neq 0$.

From this point we will drop subscript $DM_{\text{eff}}$ from Hom, and $M(-)$.

Together with $X_{\varphi}$ we can consider $\tilde{X}_{\varphi} := \text{Cone}[1](pr : X_{\varphi} \to Z)$, where $pr$ is induced by the natural projection $X_{\varphi} \to \text{Spec}(k)$.
Now, we can use Voevodsky’s cohomological operations.

We have cohomological operations $Q_i$ of bidegree $(2^i - 1)[2^{i+1} - 1]$, s.t. $Q_i$ is a differential (it’s square is zero), and for quadric $X$ of dimension greater or equal $2^m - 1$, $Q_m$ acts without cohomology on $\text{Hom}(\mathcal{X}_X, \mathbb{Z}/(2(*)[*]))$ (see [17], Theorem 3.17, Theorem 3.25 and Lemma 4.11 (notice that in the proof of the later no specific of Pfister case is used))

In particular, in our case, $Q_1, \ldots, Q_{r+2}$ will act without cohomology on $\text{Hom}(\mathcal{X}_X, \mathbb{Z}/(2(*)[*]))$.

Since $\text{Hom}(\mathbb{Z}, \mathbb{Z}/(2(i)[j]) = 0$ for $j > i$ (see [17], Corollary 2.3), we can identify $\text{Hom}(\mathcal{X}_X, \mathbb{Z}/(2(i)[j])$ with $\text{Hom}(\mathcal{X}_X, \mathbb{Z}/(2(i)[j])$ if $j > i$.

Below we will deal only with such cohomology ($j > i$), so, we will use $\mathcal{X}_X$, instead of $\mathcal{X}_X$.

Let us consider $Q_{r-1} \circ Q_{r-2} \circ \cdots \circ Q_1(\gamma'')$.

**Lemma 2.5.**

$Q_{r-1} \circ Q_{r-2} \circ \cdots \circ Q_1(\gamma'') \neq 0$.

**Proof**

Suppose that $Q_{r-1} \circ Q_{r-2} \circ \cdots \circ Q_1(\gamma'') = 0$. Then there exists such $0 \leq i \leq r - 2$, that $Q_i \circ \ldots \circ Q_1(\gamma'') = 0$, but $Q_{i+1} \circ \ldots \circ Q_1(\gamma'') = 0$.

Since $Q_j$ has bidegree $(2^j - 1)[2^{j+1} - 1]$ (see [17], Theorem 3.17), we have: $y := Q_i \circ \ldots \circ Q_1(\gamma'') \in \text{Hom}(\mathcal{X}_X, \mathbb{Z}/(2(r - i + 2^{i+1})[r - i + 2^{i+2} + 1])$. Since $Q_{i+1}(y) = 0$ and $Q_{i+1}$ acts without cohomology on $\text{Hom}(\mathcal{X}_X, \mathbb{Z}/(2(*)[*]))$, we have that there exists $z \in \text{Hom}(\mathcal{X}_X, \mathbb{Z}/(2(r - i + 1)[r - i + 2]))$, s.t. $y = Q_{i+1}(z)$.

By [16], Lemma 6.4, [17], Theorem 4.1 (see also [4], Theorem A.1), we have an identification of $\text{Hom}(\mathcal{X}_X, \mathbb{Z}/(2(r - i + 1)[r - i + 2])$ with the $\text{Ker}(\mathbb{K}_r \rightarrow \mathbb{K}_r)$.

Since $\dim(\mathcal{X}_X) = 6 \cdot 2^r > 2^{r+2}$ we have by “Kahn-Rost-Sujatha Conjecture” (see [11], Statement 1 from Section 3.2) that $\text{Ker}(\mathbb{K}_r \rightarrow \mathbb{K}_r(k(\mathcal{X}_X))/2) = 0$. We get a contradiction. So, $Q_{r-1} \circ Q_{r-2} \circ \cdots \circ Q_1(\gamma'') \neq 0$.

So, we get a nontrivial element $u := Q_{r-1} \circ Q_{r-2} \circ \cdots \circ Q_1(\gamma'') \in \text{Hom}(\mathcal{X}_X, \mathbb{Z}/(2(r + 1)[r + 2])$.

Since $\gamma''$ came from cohomology group with $\mathbb{Z}_2$ coefficients, and $Q_i = [\beta, q_i]$ for some operation $q_i$ ($\beta$ here is a Bockstein morphism) (see [17], Theorem 3.17(3)), we have that there exists $v \in \text{Hom}(\mathcal{X}_X, \mathbb{Z}(2^r + 1)[2^{r+1} + 2])$ s.t. $v = u$.

The main conclusion is:

**Lemma 2.6.**
The group $\text{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}(2^r + 1)[2^{r+1} + 2])$ is nontrivial.

Now, we need to relate the motivic cohomology of $\mathcal{X}_{X_\varphi}$ and those of $X_\varphi$. Let $X_\varphi^j$ be (as usually) the variety of $j$-dimensional projective spaces on $X_\varphi$. By [14], Theorem 4.1, the motive $M(X_\varphi)$ of $X_\varphi$ decomposes as: $M(X_\varphi) = \bigoplus_{i=0, \ldots, 2^r + 1} F_\alpha(X_\varphi)(i)[2i]$, where $\alpha := \{x_1, \ldots, x_r\} \in K^M_r(k)/2$, and $F_\alpha(X_\varphi)$ is so-called higher form of 4-dimensional quadric $X_\varphi$, and $F_\alpha(X_\varphi)$ is an extension (in $\text{DM}^{-}_{\text{eff}}(k)$) of $M(X_\varphi)$, $M(\mathcal{X}_{X_\varphi}^r)(2^r)[2^{r+1}]$, $M(\mathcal{X}_{X_\varphi}^{r+1})(2^r)[2^{r+1}]$, $M(\mathcal{X}_{X_\varphi}^r)(3 \cdot 2^r - 1)[3 \cdot 2^{r+1} - 2]$, $M(\mathcal{X}_{X_\varphi}^r)(4 \cdot 2^r - 1)[4 \cdot 2^{r+1} - 2]$ and $M(\mathcal{X}_{X_\varphi}^r)(5 \cdot 2^r - 1)[5 \cdot 2^{r+1} - 2]$.

More precisely, there exists the following diagram in $\text{DM}^{-}_{\text{eff}}(k)$ (we drop $M(\cdot)$ from the notations):

\[
\begin{array}{ccc}
\mathcal{X}_{X_\varphi} & \mathcal{X}_{X_\varphi}^r(2^r)[2^{r+1}] & \mathcal{X}_{X_\varphi}(5 \cdot 2^r - 1)[5 \cdot 2^{r+1} - 2] \\
\nearrow \bullet[1] & \nearrow \star[1] & \nearrow \star[1] & \nearrow \star[1] & \nearrow \star[1] & \nearrow \star[1] & \nearrow \star[1] \\
Y_1 & Y_2 & \ldots & Y_5 & 0 \\
F_\alpha(X_\varphi) & Y_1 & Y_2 & \ldots & Y_5 & Z_1 & \ldots & Z_5
\end{array}
\]

(see [14], Lemma 3.23, Theorem 3.1, and use octahedron axiom).

In particular, we have an exact sequences:

$$
\text{Hom}(F_\alpha(X_\varphi), \mathbb{Z}(2^r + 1)[2^{r+1} + 2]) \leftarrow \text{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}(2^r + 1)[2^{r+1} + 2]) \leftarrow \text{Hom}(Y_1, \mathbb{Z}(2^r + 1)[2^{r+1} + 1]).
$$

and

$$
\text{Hom}(Y_2, \mathbb{Z}(2^r + 1)[2^{r+1} + 1]) \leftarrow \text{Hom}(Y_1, \mathbb{Z}(2^r + 1)[2^{r+1} + 1]) \leftarrow \text{Hom}(\mathcal{X}_{X_\varphi}^r(2^r)[2^{r+1}], \mathbb{Z}(2^r + 1)[2^{r+1} + 1]) \leftarrow \text{Hom}(Y_2, \mathbb{Z}(2^r + 1)[2^{r+1} + 1]).
$$

Notice, that $Y_2$ is an extension of some $\mathcal{X}_P(j)[2j]$, where $P$ are smooth projective varieties and $j \geq 2^{r+1} > 2^r + 1$. By [17], Corollary 2.2(1), we have: $\text{Hom}(Y_2, \mathbb{Z}(2^r + 1)[*]) = 0$ and, consequently, $\text{Hom}(Y_1, \mathbb{Z}(2^r + 1)[2^{r+1} + 1]) = \text{Hom}(\mathcal{X}_{X_\varphi}^r(2^r)[2^{r+1}], \mathbb{Z}(2^r + 1)[2^{r+1} + 1])$, and the later group is equal to: $\text{Hom}(\mathcal{X}_{X_\varphi}^r, \mathbb{Z}(1)[1])$.

So, we have an exact sequence:

$$
\text{Hom}(F_\alpha(X_\varphi), \mathbb{Z}(2^r + 1)[2^{r+1} + 2]) \leftarrow \text{Hom}(\mathcal{X}_{X_\varphi}, \mathbb{Z}(2^r + 1)[2^{r+1} + 2]) \leftarrow \text{Hom}(\mathcal{X}_{X_\varphi}^r, \mathbb{Z}(1)[1]),
$$

where the last map is induced by the map $f : \mathcal{X}_{X_\varphi} \to \mathcal{X}_{X_\varphi}^r(2^r)[2^{r+1} + 1]$, which is the composition: $\mathcal{X}_{X_\varphi} \to Y_1[1] \to \mathcal{X}_{X_\varphi}^r(2^r)[2^{r+1} + 1]$ in the diagram above.
By [13], Corollary 3.2.1, Hom($\mathcal{X}_{\mathcal{X}_X^{2r}}$, $Z(1)[1]$) = $H^0(\mathcal{X}_{\mathcal{X}_X^{2r}}, \mathcal{O}^*) = H^0(\text{Spec}(k), \mathcal{O}^*) = \text{Hom}(\mathbb{Z}, \mathbb{Z}(1)[1])$, and this isomorphism is induced by projection $pr: \mathcal{X}_{\mathcal{X}_X^{2r}} \rightarrow \mathbb{Z}$ (as usually, I'm dropping $M(-)$).

So, we get an exact sequence:

$$\text{Hom}(F_\alpha(X_\rho), \mathbb{Z}(2^r + 1)[2^{r+1} + 2]) \leftarrow \text{Hom}(\mathcal{X}_{\mathcal{X}_X^{2r}}, \mathbb{Z}(2^r + 1)[2^{r+1} + 2]) \leftarrow \text{Hom}(\mathbb{Z}, \mathbb{Z}(1)[1]),$$

where the last map is induced by the composition

$$g: \mathcal{X}_{\mathcal{X}_X^{2r}} \rightarrow \mathcal{X}_{\mathcal{X}_X^{2r}}(2^r)[2^{r+1} + 1] \xrightarrow{pr(2^r)[2^{r+1} + 1]} \mathbb{Z}(2^r + 1)[2^{r+1} + 1].$$

Lemma 2.7.

\[ g = 0 \]

\[ \text{Proof} \]

By the very construction of the diagram above - see [14], the proof of Theorem 3.1, the composition $Y_1[1] \rightarrow \mathcal{X}_{\mathcal{X}_X^{2r}}(2^r)[2^{r+1} + 1] \rightarrow \mathbb{Z}(2^r)[2^{r+1} + 1]$ equals the composition: $Y_1[1] \rightarrow F_\alpha(X_\rho)[1] \rightarrow X_\varphi[1] \xrightarrow{\alpha_{2r}[1]} \mathbb{Z}(2^r)[2^{r+1} + 1]$, where $\alpha_{2r} \in \text{Hom}(X_\varphi, \mathbb{Z}(2^r)[2^{r+1}]) = CH^{2^r}(X_\varphi)$ is given by the class of a plane section of codimension $2^r$. Hence, $g = 0$.

Now, we can finish the proof of the Main Theorem.

From Lemma 2.7 it follows that the map: Hom($\mathcal{X}_{\mathcal{X}_X^{2r}}, \mathbb{Z}(2^r + 1)[2^{r+1} + 2]$) $\leftarrow$ Hom($\mathbb{Z}, \mathbb{Z}(1)[1]$) is trivial. And we have an exact sequence:

$$\text{Hom}(F_\alpha(X_\rho), \mathbb{Z}(2^r + 1)[2^{r+1} + 2]) \leftarrow \text{Hom}(\mathcal{X}_{\mathcal{X}_X^{2r}}, \mathbb{Z}(2^r + 1)[2^{r+1} + 2]) \leftarrow 0$$

Since $F_\alpha(X_\rho)$ is a direct summand in the motive of $X_\varphi$, we have (by above) an exact sequence:

$$\text{Hom}(X_\varphi, \mathbb{Z}(2^r + 1)[2^{r+1} + 2]) \leftarrow \text{Hom}(\mathcal{X}_{\mathcal{X}_X^{2r}}, \mathbb{Z}(2^r + 1)[2^{r+1} + 2]) \leftarrow 0$$

By Lemma 2.6, the group Hom($\mathcal{X}_{\mathcal{X}_X^{2r}}, \mathbb{Z}(2^r + 1)[2^{r+1} + 2]$) is nontrivial, on the other hand it has exponent 2 (since $2^{r+1} + 2 > 2^r + 1$). That means that the group Hom($X_\varphi, \mathbb{Z}(2^r + 1)[2^{r+1} + 2]$) contains a torsion element of exponent 2.

By [15], Theorem 4.2.9, we have: Hom($X_\varphi, \mathbb{Z}(2^r + 1)[2^{r+1} + 2]$) = $CH^{2^r+1}(X_\varphi)$. The Main Theorem is proven.
References