On the kernels in Milnor's K-theory under function field extensions

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March 30, 1998

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0 Introduction

In this paper we will consider the following question:

Question 1.

Let k be a field of characteristic different from 2, and Q/k be some quadric.

Is it true, that the ideal $\mathrm{Ker}_Q := \mathrm{Ker}(\mathrm{K}_*^M(k)/2 \rightarrow \mathrm{K}_*^M(k(Q))/2)$ *in* $\mathrm{K}_*^M(k)/2$ *is generated by pure symbols ?*

The answer to our question is known to be positive in the case Q - *Pfister neigbour* (i.e.: a subquadric of dimension more or equal than half in a *big Pfister quadric*) by [6], and for 2-dimensional *non-Pfister* quadric by [8].

The positive answer to our question would imply the following:

1) Conjecture of B.Kahn, M.Rost and R.J.Sujatha (see Conjecture 1 of the original version of [3], where authors asked basically the same question about the low degree part of the kernel).

2) The fact that any *conservative* quadric (i.e. such quadric, for which the

kernel above is nontrivial) is a subquadric in a *big Pfister quadric.*
3) The fact, that if some quadric Q is a *normvariety* for some $h \in \mathrm{K}_m^M(k)/2$, then h is a *pure symbol.* From this, on it's part, follows nice motivic description of *excellent* quadrics.

We will show, that our question is equivalent to the fact that Ker_Q, considered as a *cyclic module* of M.Rost (see [7]) is generated by one precisely described *pure* symbol. So, in the case of positive answer to the question we would have explicit description of the Ker_Q; and we have such a description for the *pure* part of the kernel unconditionally.

I would like to thank Yuri Ivanovich Manin for helpful remarks to the original version of the paper. Also I'm grateful to Max-Planck Institut für Mathematik for supporting me during the writing of this work.

1 Universal symbol

In this section we will show, that for arbitrary quadric Q/k there exists some purely transcendental extension $k(A_Q)/k$, and some *pure* symbol $\alpha_Q \in \mathrm{K}_*^M(k(A_Q))/2$, s.t. for arbitrary field extension E/k, our quadric Q has E-point if and only if $\alpha_Q|_{E(A_Q)} = 0$.

Statement 2.

Suppose we have complete flag F of quadrics $Q_0 \subset Q_1 \subset \cdots \subset Q_n = Q$, where $n = \dim(Q)$.

Then there exists purely transcendental field extension $k(A_F)/k$, and a pure symbol $\{y_0, \ldots, y_n\} \in \mathrm{K}_{n+1}^M(k(A_F))/2$, s.t. for arbitrary field extension E/k, Q_i has E-point if and only if $\{y_0, \ldots, y_i\}|_{E(A_F)} = 0$.

Proof

Induction on the dimension of Q.

1) $q = \langle 1, -a_0 \rangle$. Evidently, $\alpha_{k\sqrt{a_0}} = \{a_0\} \in \mathrm{K}_1^M(k)/2$ will have desired property.

2) Let now $q_i = \langle 1, -a_0, \ldots, -a_i \rangle$, and F' is a flag: $Q_0 \subset \cdots \subset Q_{n-1}$. By inductive assumption: we have $\alpha_{F'} = \{y_0, \ldots, y_{n-1}\} \in \mathrm{K}_n^M(k(A_{F'}))/2$, which has desired property with respect to F'.

In particular $\alpha_{F'}|_{k(Q_{n-1})(A_{F'})} = 0$ (Since over $k(Q_{n-1})$ Q_{n-1} has a point). By the "Third representation theorem" (see [5], IX, Theorem 2.8) this means, that $q_{n-1}|_{k(A_{F'})}$ is similar to a subform of $\langle\langle y_0, \ldots, y_{n-1} \rangle\rangle$, and since q_{n-1} represents 1, it is a subform itself. We have: $q_{n-1}|_{k(A_{F'})} \perp q'' = \langle\langle y_0, \ldots, y_{n-1} \rangle\rangle$, where q'' is some form, defined over $k(A_{F'})$.

From this: if for some field extension E/k, $\{y_0, \ldots, y_{n-1}\} \neq 0$ (as an element of $\mathrm{K}_n^M(E(A_{F'}))/2$), then the following four conditions are equivalent:

a) $q|_E$ - is isotropic.

b) $q|_{E(A_{F'})}$ - is isotropic.

c) $(\langle\langle y_0, \ldots, y_{n-1} \rangle\rangle \perp -(q'' \perp \langle a_n \rangle))|_{E(A_{F'})}$ is $dim(q'') + 1$-times isotropic.

d) $(q'' \perp \langle a_n \rangle)|_{E(A_{F'})}$ - is a subform in $\langle\langle y_0, \ldots, y_{n-1} \rangle\rangle|_{E(A_{Q'})}$.

By the "Third representation theorem" (see [5], IX, Theorem 2.8) the last condition is equivalent to the fact: $\langle\langle y_0, \ldots, y_{n-1} \rangle\rangle|_{E(A_{F'})(V_{q'' \perp \langle a_n \rangle})}$ represents the "generic value" $(q'' \perp \langle a_n \rangle)(\mathfrak{v})$, where $V_{q'' \perp \langle a_n \rangle}$ is the underlying $k(A_{F'})$-vector space of quadratic form $q'' \perp \langle a_n \rangle$ (defined over that field), and $\mathfrak{v}$ - coordinates in $V_{q'' \perp \langle a_n \rangle}$. And this is equivalent to: $\{y_0, \ldots, y_{n-1}, (q'' \perp \langle a_n \rangle)(\mathfrak{v})\} = 0 \in \mathrm{K}_n^M(E(A_{F'})(V_{q'' \perp \langle a_n \rangle}))/2$.

Put: $\alpha_F = \{y_0, \ldots, y_{n-1}, (q'' \perp \langle a_n \rangle)(\mathfrak{v})\}$, and $k(A_F) = k(A_{F'})(V_{q'' \perp \langle a_n \rangle})$.

If $\{y_0, \ldots, y_{n-1}\}|_{E(A_{F'})} \neq 0$, then we already proved that Q_E has a point if and only if $\alpha_F|_{E(A_F)} = 0$.

If $\{y_0, \ldots, y_{n-1}\}|_{E(A_{F'})} = 0$, then $Q_{n-1} \subset Q$ has a rational point over E, and, on the other hand, trivially, $\alpha_F|_{E(A_F)} = 0$.

□

Corollary 3. *All anisotropic quadrics are stably embedable, i.e., for each such quadric Q/k there exists purely transcendental extension $k(A_Q)$, s.t. $Q|_{k(A_Q)} \subset Q_\alpha$, where $Q_\alpha/k(A_Q)$ is Pfister quadric.*

Proof

In the previous statement take: $E = k(Q)$, $k(A_Q) = k(A_F)$ and $\alpha = \alpha_F$ for some full flag F of subquadrics in Q. Then since $Q|_{k(Q)}$ is isotropic, we have: $\alpha_Q|_{k(A_Q)(Q)} = 0$. So, $\alpha \in \mathrm{K}_{n+1}^M(k(A_Q))$ is a nontrivial representative of the kernel. And $Q|_{k(A_Q)}$ is a subquadric in Q_{α_Q}.

□

Corollary 4.

For anisotropic quadric Q, and some quadric R we have, that $Q|_{k(R)}$ is isotropic if and only if $R|_{k(A_Q)}$ is a subquadric in a Pfister quadric Q_{α_Q}.

Remark By the result of D.W.Hoffmann, $Q|_{k(R)}$ is anisotropic if $\dim(R) \geq 2^k - 1 > \dim(Q)$ (for some k) (see [2], Theorem 1).

This shows, that Q_{α_Q} does not contain subquadrics of big dimension defined over k.

Corollary 5.

Let Q, R be quadrics over k, and $\dim(Q) = \dim(R) = n$. Let Q^i (resp. R^i) be variety of i-dimensional projective planes on Q (resp R).

Then the following is equivalent:

(1) $M(Q) = M(R)$, where $M(X)$ is a Chow motive of X.

(2) $\forall 0 \leq i \leq [n/2]$, $\forall E/k$ *- field extension,* $\mathrm{Ker}(\mathrm{K}_*^M(E)/2 \to \mathrm{K}_*^M(E(Q^i))/2) = \mathrm{Ker}(\mathrm{K}_*^M(E)/2 \to \mathrm{K}_*^M(E(R^i))/2)$.

Proof

By [8], Proposition 5.1 we have: (1)$\Leftrightarrow$ *Universal splitting towers of M.Knebusch* for Q and R are equivalent (see [4], or [8], Definition 2.4.19).

That is: $M(Q) = M(R) \ \Leftrightarrow\ \forall 0 \leq i \leq [n/2]$, Q^i has a rational point over $k(R^i)$, and R^i has a rational point over $k(Q^i)$.

So, evidently, (1)$\Rightarrow$(2). And to prove the opposite conclusion, it is enough to show, that if $\mathrm{Ker}(\mathrm{K}_*^M(E)/2 \to \mathrm{K}_*^M(E(Q))/2) = \mathrm{Ker}(\mathrm{K}_*^M(E)/2 \to \mathrm{K}_*^M(E(R))/2)$, $\forall E/k$, then Q has a rational point over $k(R)$ and R has a rational point over $k(Q)$ (the case of arbitrary $0 \leq i \leq [n/2]$ can be easily obtained from this).

Take $E = k(Q)(A_R)$ (notations as above).
Then $\mathrm{Ker}(\mathrm{K}_*^M(E)/2 \to \mathrm{K}_*^M(E(Q))/2) = 0$. Hence the second kernel is zero as well, and so, $\alpha_R|_{k(Q)(A_R)} = 0 \in \mathrm{K}_*^M(E)/2$. By Statement 2, R has a rational point over $k(Q)$. Similarly, Q has a rational point over $k(R)$.

$\square$

Remark If $M(Q)$ or $M(R)$ is undecomposable, then in (2) it is enough to consider only $i = 0$ (see [8], Corollary 3.15).

2 Pure part of the kernel

In this section we will show that for arbitrary quadric Q/k the set of *pure* symbols in $\mathrm{Ker}(\mathrm{K}_*^M(k)/2 \to \mathrm{K}_*^M(k(Q))/2)$ is generated (under the action of the semigroup of *pure* symbols from $\mathrm{K}_*^M(k)/2$) by the *specializations* of one element $\alpha_Q \in \mathrm{K}_{n+1}^M(k(A_Q))$, where α_Q can be taken as α_F for some flag F (and $k(A_Q)$ as $k(A_F)$).

Let me remind (see [7]), that the *cyclic module* over k is a collection of $\mathbb{Z}$-graded abelian groups $M(E)$, where E runs through all finitely generated field extensions E/k, which are connected by the following operations:

(1) For any $\varphi : \mathrm{Spec}(E) \to \mathrm{Spec}(F)$ over k, the homomorphism $\varphi^* : M(F) \to M(E)$ of degree 0 ("pull-back").
(2) For any finite morphism $\varphi : \mathrm{Spec}(E) \to \mathrm{Spec}(F)$ over k, the homomorphism $\varphi_* : M(E) \to M(F)$ of degree 0 ("push-forward").
(3) For any F/k, $M(F)$ has a structure of $\mathrm{K}_*^M(F)$-module, which respects grading on both objects.
(4) For a discrete valuation $\mathfrak{v}$ on F/k, there is the homomorphism $\partial_{\mathfrak{v}} : M(F) \to M(k(\mathfrak{v}))$ of degree -1 ("derivation").

All this data should satisfy natural compatibility rules (see [7], notice, that our notations are a bit different). Typical example is $M(E) = \mathrm{K}_*^M(E)$.

It is not difficult to see, that $M(E) = \mathrm{Ker}_{Q|_E}$ will also be one (really, any submodule of *cyclic module*, closed under operations is a cyclic module; and for a cyclic module M, $M' := M/r$, where r is some fixed integer, is also a cyclic module).

Together with a derivation we can consider "specialization" map. Namely, if π is uniformizing parameter for $\mathfrak{v}$, then we have: $S_{\mathfrak{v}}^{\pi} : M(F) \to M(k(\mathfrak{v}))$ of degree 0, where $S_{\mathfrak{v}}^{\pi}(x) := \partial_{\mathfrak{v}}(\{-\pi\} \cdot x)$. If $\partial_{\mathfrak{v}}(x) = 0$, then $S_{\mathfrak{v}}^{\pi}(x)$ does not depend on the choice of π. If X is a smooth variety, and $\mathrm{Spec}(E)$ is a point on it, given by some $\mathbb{Z}^r$ - valuation, then the composition $S_{\mathfrak{v}_r}^{\pi_r} \circ \cdots \circ S_{\mathfrak{v}_1}^{\pi_1}$ defines a specialization map from $M_{0,E}(k(X))$ to $M(E)$, where $M_{0,E}(k(X)) := \mathrm{Ker}(M(k(X)) \stackrel{\partial_D}{\to} \oplus_{\mathrm{Spec}(E) \in D-div.of X} M(k(D)))$. This map does not depend on the choice of $\mathbb{Z}^r$ - valuation, or on the uniformizers (see [7]). Notice, that the specialization map is a composition of maps of type (3) and (4).

Suppose we have a *cyclic module* M, some (finitely generated) field extension E/k, and some element $x \in M(E)$. Then it is natural to say, that *cyclic submodule generated by* x is the minimal *cyclic submodule* M_x of M, containing x. Since we know, that *cyclic submodules* are precisely submodules, closed under operations, we have $M_x(F)$ consists of those $y \in M(F)$, that y can be obtained from x using operations (1), (2), (3), (4). Similarly, we say, that *submodule weakly generated by* x is the minimal submodule (not nesessarily *cyclic*) M_x^w of M, closed under (1), (3) and (4).

Denote by PKer_Q the ideal in $\mathrm{K}_*^M(k)/2$, generated by *pure* symbols from Ker_Q. Also, we will denote the same way the corresponding submodule in $\mathrm{K}_*^M/2$: $\mathrm{PKer}_Q(E) := \mathrm{PKer}_{Q|_E}$.

Proposition 6.

PKer_Q *coinsides with a submodule of* $\mathrm{K}_*^M/2$ *weakly generated by* α_Q.

Proof

As we know from Statement 2, $\alpha_Q \in \mathrm{PKer}_Q(k(A_Q))$. Also, it is clear, that operations (1) and (3) preserve PKer_Q.

Lemma 7.

Operation (4) (acting on K^M_*) maps pure symbols to pure ones.

Proof

Really, for a valuation $\mathfrak{v}$ on a filed F, any *pure* symbol in F can be written as $\beta = \{\pi^{k_1} \cdot b_1, \ldots, \pi^{k_n} \cdot b_n\}$, where π is uniformizing parameter, and b_j's are units for $\mathfrak{v}$.

Applying the fact, that $\{\pi^k \cdot a, \pi^l \cdot b\} = \{\pi^k \cdot a, -\pi^{l-k} \cdot b/a\}$, we get, that $\beta = \{\pi^d \cdot c_1, c_2, \ldots, c_n\}$, where $d = g.c.d.(k_j; j = 1, \ldots, n)$, and c_m's are units.

Then $\partial_{\mathfrak{v}}(\beta) = \{c_2^d, c_3, \ldots, c_n\}$ is *pure*.

□

So, we have, that $(\mathrm{K}^M_* /2)^w_{\alpha_Q} \subset \mathrm{PKer}_Q$.

Before we prove the opposite inclusion, let us generalize a bit the construction of *Universal symbol*.

Suppose we have two quadratic forms $p' \subset p''$, full flag G of forms "between them": $p' = p_0 \subset p_1 \subset \cdots \subset p_s = p''$, where $\dim(p_{t+1}) - \dim(p_t) = 1$, and an inclusion of p' into some Pfister form $\langle\langle\gamma\rangle\rangle$. Let $p_j = p_0 \perp \langle -c_1, \ldots, -c_j\rangle$. For any $0 \le j \le t \le s$, we will denote as $G^{[j,t]}$ the subflag $p_j \subset \cdots \subset p_t$.

Then we can construct some purely transcendental field extension $k(A_{G,\gamma})$, and some pure symbol $\alpha_{G,\gamma}$ in the following way.

Let $\langle\langle\gamma\rangle\rangle = p_0 \perp q_{G^{[0,0]},\gamma}$, then take: $k(A_{G^{[0,1]},\gamma}) := k(V_{q_{G^{[0,0]},\gamma} \perp \langle c_1\rangle})$, and take $\alpha_{G^{[0,1]},\gamma} := \gamma \cdot \{q_{G^{[0,0]},\gamma} \perp \langle c_1\rangle(\mathfrak{v})\}$, where $q_{G^{[0,0]},\gamma} \perp \langle c_1\rangle(\mathfrak{v})$ is the *generic value* of $q_{G^{[0,0]},\gamma} \perp \langle c_1\rangle$. We have: $p_1|_{k(A_{G^{[0,1]},\gamma})} \subset \langle\langle\alpha_{G^{[0,1]},\gamma}\rangle\rangle$.

Now, by definition, $k(A_{G^{[0,i+1]},\gamma}) := k(A_{G^{[0,i]},\gamma})(A_{G^{[i,i+1]},\alpha_{G^{[0,i]},\gamma}})$, and $\alpha_{G^{[0,i+1]},\gamma} := \alpha_{G^{[i,i+1]},\alpha_{G^{[0,i]},\gamma}}$. Also, by definition: $\langle\langle\alpha_{G^{[0,l]},\gamma}\rangle\rangle = p_l|_{k(A_{G^{[0,l]},\gamma})} \perp q_{G^{[0,l]},\gamma}$.

Let us return now to the proof of Proposition 6.

Suppose we have some *pure* symbol β in $\mathrm{Ker}(\mathrm{K}^M_*(k)/2 \to \mathrm{K}^M_*(k(Q))/2)$.

Let F be full flag of quadrics: $Q_0 \subset Q_1 \subset \cdots \subset Q_n = Q$, $q_i = \langle 1, -a_0, \ldots, -a_i\rangle$, which we used in the construction of $\alpha_Q = \alpha_F$. Notice, that $q_0 = \langle\langle a_0\rangle\rangle$. Then it is easy to see, that $k(A_F) = k(A_{F,\{a_0\}})$, and $\alpha_F = \alpha_{F,\{a_0\}}$.

We have $q_n = q \subset \langle\langle\beta\rangle\rangle$ for some $\beta \in \mathrm{K}^M_*(k)/2$ (since q represents 1). Take $\beta_{Q_0} := \{a_0\}$. We have: $q_0 \subset \langle\langle\beta_{Q_0}\rangle\rangle \subset \langle\langle\beta\rangle\rangle$.

Suppose $q_i \subset \langle\langle\beta_{Q_i}\rangle\rangle \subset \langle\langle\beta\rangle\rangle$, $deg(\beta_{Q_i}) = j$, then $deg(\alpha_{F^{[i,n]},\beta_{Q_i}}) = j+n-i$.

We have: $\alpha_{F^{[i,n]},\beta_{Q_i}} = \beta_{Q_i}\cdot\{y_{i+1},\dots,y_n\}$, where y_l is a pull-back of $(q_{F^{[i,l-1]},\beta_{Q_i}} \perp \langle a_l\rangle)(\mathfrak{v}_l)$ to $k(A_{F^{[i,n]},\beta_{Q_i}})$.

Let $q_i \subset \langle\langle\beta_{Q_i}\rangle\rangle$, and $\langle\langle\beta_{Q_i}\rangle\rangle = q_i \perp q_{F^{[i,i]},\beta_{Q_i}}$. We have two possibilities:
1)$q_{F^{[i,i]},\beta_{Q_i}} \perp \langle a_{i+1}\rangle$ is isotropic, i.e. there are t_0, z_0, that $q_{F^{[i,i]},\beta_{Q_i}}(t_0) + a_{i+1}z_0^2 = 0$.
2)$q_{F^{[i,i]},\beta_{Q_i}} \perp \langle a_{i+1}\rangle$ is anisotropic.

1) In this case we have: divisor $y_{i+1} = 0$ on $A_{F^{[i,i+1]},\beta_{Q_i}} = V_{q_{F^{[i,i]},\beta_{Q_i}}\perp\langle a_{i+1}\rangle}$ has rational point (t_0, z_0), and $q_{i+1} \subset \langle\langle\beta_{Q_i}\rangle\rangle$. So, we can construct $k(A_{F^{[i+1,n]},\beta_{Q_i}})$, and $\alpha_{F^{[i+1,n]},\beta_{Q_i}}$. Let $Y_{i+1} = 0$ be the preimage of the divisor $y_{i+1} = 0$ under the natural projection $A_{F^{[i,n]},\beta_{Q_i}} \to V_{q_{F^{[i,i]},\beta_{Q_i}}\perp\langle a_{i+1}\rangle}$.

Let p'_l be the *specialization* of $q_{F^{[i,l]},\beta_{Q_i}}$ over $k(A_{F^{[i,l]},\beta_{Q_i}})$ at the generic point E_l of the fiber of $A_{F^{[i,l]},\beta_{Q_i}}$ over the rational point (t_0, z_0) of $V_{q_{F^{[i,i]},\beta_{Q_i}}\perp\langle a_{i+1}\rangle}$.

Then E_l is naturally a purely transcendental extension of $k(A_{F^{[i+1,l]},\beta_{Q_i}})$, and if $j : \mathrm{Spec}(E_l) \to \mathrm{Spec}(k(A_{F^{[i+1,l]},\beta_{Q_i}}))$ is corresponding morphism, then we have: $j^*(q_{F^{[i+1,l]},\beta_{Q_i}})$ is a nondegenerate part of p'_l (it can be easily proved by induction on l).

Hence, $\alpha_{F^{[i+1,n]},\beta_{Q_i}} \in \mathrm{K}^M_{j+n-i-1}(k(A_{F^{[i+1,n]},\beta_{Q_i}}))/2$ is a specialization of the derivative $\partial_{Y_{i+1}=0}(\alpha_{F^{[i,n]},\beta_{Q_i}})$.

We can take $\beta_{Q_{i+1}} = \beta_{Q_i}$. Then $q_{i+1} \subset \langle\langle\beta_{Q_{i+1}}\rangle\rangle \subset \langle\langle\beta\rangle\rangle$.

2) We have: $q_i \subset \langle\langle\beta_{Q_i}\rangle\rangle \subset \langle\langle\beta\rangle\rangle$, and $q_i \subset q_{i+1} \subset \langle\langle\beta\rangle\rangle$. Since $q_{F^{[i,i]},\beta_{Q_i}} \perp \langle a_{i+1}\rangle$ is anisotropic, there exist some subform p of $\langle\langle\beta\rangle\rangle$ of dimension $= \dim(\langle\langle\beta_{Q_i}\rangle\rangle) + 1$, which contains both $\langle\langle\beta_{Q_i}\rangle\rangle$ and q_{i+1}. That means, that there exist (t_0, z_0), s.t. $p = \langle\langle\beta_{Q_i}\rangle\rangle \perp \langle q_{F^{[i,i]},\beta_{Q_i}}(t_0) + a_{i+1}z_0^2\rangle$.

Take: $\beta_{Q_{i+1}} := \beta_{Q_i}\cdot\{q_{F^{[i,i]},\beta_{Q_i}}(t_0) + a_{i+1}z_0^2\} \in \mathrm{K}^M_{j+1}(k)/2$. Then $q_{i+1} \subset \langle\langle\beta_{Q_{i+1}}\rangle\rangle \subset \langle\langle\beta\rangle\rangle$.

Evidently, $\beta_{Q_{i+1}}$ is a specialization of $\alpha_{F^{[i,i+1]},\beta_{Q_i}} \in \mathrm{K}^M_{j+1}(k(V_{q_{F^{[i,i]},\beta_{Q_i}}\perp\langle a_{i+1}\rangle}))/2$ at the rational point (t_0, z_0).

Also, we have natural identification: $k(A_{F^{[i+1,l]},\beta_{Q_{i+1}}})$ with E_l, where E_l is defined precisely as in the case (1). Under this identification, $q_{F^{[i+1,l]},\beta_{Q_{i+1}}}$ is isomorphic to the specialization of $q_{F^{[i,l]},\beta_{Q_i}}$ at the generic point (E_l) of the fiber of the natural projection $A_{F^{[i,l]},\beta_{Q_i}} \to V_{q_{F^{[i,i]},\beta_{Q_i}}\perp\langle a_{i+1}\rangle}$ over the point (t_0, z_0).

Hence, $\alpha_{F^{[i+1,n]},\beta_{Q_{i+1}}} \in \mathrm{K}^M_{j+n-i}(k(A_{F^{[i+1,n]},\beta_{Q_{i+1}}}))/2$ is a *specialization* of $\alpha_{F^{[i,n]},\beta_{Q_i}} \in \mathrm{K}^M_{j+n-i}(k(A_{F^{[i,n]},\beta_{Q_i}}))/2$.

Continuing this way, we get: $q = q_n \subset \langle\langle \beta_{Q_n} \rangle\rangle \subset \langle\langle \beta \rangle\rangle$, and for each $0 \leq i < n$, $\alpha_{F[i+1,n],\beta_{Q_{i+1}}} \in \mathrm{K}^M_{j+n-i}(k(A_{F[i+1,n],\beta_{Q_{i+1}}}))/2$ is obtained from $\alpha_{F[i,n],\beta_{Q_i}} \in \mathrm{K}^M_{j+n-i}(k(A_{F[i,n],\beta_{Q_i}}))/2$ using derivations and specializations.

But, evidently, $k(A_{F[n,n],\beta_{Q_n}}) = k$, and $\alpha_{F[n,n],\beta_{Q_n}} = \beta_{Q_n}$.

On the other hand, as we saw, $\alpha_{F[0,n],\beta_{Q_0}} = \alpha_F$.

Hence, β is divisible by some pure symbol, obtained from α_F using derivations and specializations.

So, $\mathrm{PKer}_Q \subset (\mathrm{K}^M_* /2)^w_{\alpha_Q}$.

Proposition is proven.

□

Remark 1 In particular we see, that for any *pure symbol* $\beta \in \mathrm{Ker}(\mathrm{K}^M_*(k)/2 \to \mathrm{K}^M_*(k(Q))/2)$, β is divisible by a *pure* symbol from the kernel of degree $\leq \dim(Q) + 1$. This result was known to M.Knebusch, as E.R.Gentile and D.B.Shapiro point out (see [1], Remark after Corollary 8).

Remark 2 Actually, we could make our *universal symbol* α_Q a bit more "universal". Namely, we can consider $F(\mathbb{P}^{n+1})$ - variety of full flags in $\mathbb{P}^{n+1} \supset Q$.

Then taking the generic point $k(F(\mathbb{P}^{n+1}))$ of later variety with generic flag F_{gen}, we would get corresponding *universal symbol* $\alpha_{gen,Q} := \alpha_{F_{gen}} \in \mathrm{K}^M_{n+1}(k(F(\mathbb{P}^{n+1}))(A_{F_{gen}}))/2$.

This way, $\alpha_{gen,Q}$ does not depend on anything but Q itself, and our usual α_{F_0} is just a specialization of $\alpha_{gen,Q}$ at the generic point of the fiber of A_F over k-rational point F_0 of $F(\mathbb{P}^{n+1})$ (notice, that we can consider A_F as a fibration over $F(\mathbb{P}^{n+1})$ with rational fibers).

From Proposition 6 it follows that $(\mathrm{K}^M_* /2)^w_{\alpha_{F_0}} = (\mathrm{K}^M_* /2)^w_{\alpha_{gen,Q}}$. Really, the first module is a submodule of the later (since α_{F_0} is a specialization of $\alpha_{gen,Q}$), and the later module is, evidently, a submodule in PKer_Q.

Since the same is true for any extension E/k, we get: $(\mathrm{K}^M_* /2)_{\alpha_{F_0}} = (\mathrm{K}^M_* /2)_{\alpha_{gen,Q}}$ as well (see [7]).

In the light of Proposition 6, our Question 1 is equivalent to the following:

Question 8.

Is the following true?

a) $\mathrm{Ker}_Q = (\mathrm{K}^M_* /2)_{\alpha_Q}$.

b) PKer_Q *is stable under transfers.*

Notice, that Question 8)b) is trivial for $\dim(Q) \leq 2$; to the contrary - Question 8)a) for $\dim(Q) = 0$ is more or less equivalent to Milnor's conjecture (on e'tale cohomology). Hopefully, due to V.Voevodsky's methods we know that the answer is positive for Q - *Pfister quadric* (see [6]). Natural strategy for Question 8)a) would be to reduce the case of arbitrary quadric Q/k to that of *Pfister* one $Q_{\alpha_Q}/k(A_Q)$, using the universality of the symbol α_Q.

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