Problem class 1

1. A triple of natural number (X, Y, Z) is called a pythagorean triple if $X^2+Y^2 = Z^2$ as they are the sides of a triangle with a right angle. Consider $x = X/Z$ and $y = Y/Z$. Now we are looking for positive rational numbers x, y satisfying $x^2+y^2 = 1$. Use the ideas in Chapter 1 to find plenty of pythagorean triples.

2. Draw a picture of the solution set $C(k)$ in k^2 for the equation

$$2\,y + 3\,x + 5 = 0$$

and the fields $k = \mathbb{R}$, $k = \mathbb{F}_7$, $k = \mathbb{F}_5$, $k = \mathbb{F}_3$, $k = \mathbb{F}_2$ and $k = \mathbb{Q}$.

3. Let $(A, +)$ and $(B, \#)$ be abelian groups and let $f : A \to B$ be a map such that $f(-a) = -f(a)$ for all $a \in A$ and

$$a + a' + a'' = 0 \qquad \Rightarrow \qquad f(a)\#f(a')\#f(a'') = 0.$$

Prove that f is a group homomorphism.

4. Let k be a field and $n > 0$. On the set $X = k^{n+1} \setminus \big\{(0, 0, \ldots, 0)\big\}$ define the following relation:

$$(x_0, x_1, \ldots, x_n) \sim (y_0, y_1, \ldots, y_n)$$
$$\text{if and only if there exists } \lambda \in k^\times \text{ such that}$$
$$x_0 = \lambda \cdot y_0,\ x_1 = \lambda \cdot y_1,\ \ldots,\ x_n = \lambda \cdot y_n$$

Prove that this is an equivalence relation. Recall, this means that we need to check

Reflexive: $a \sim a$ for all $a \in X$.

Symmetric: If $a \sim b$ for some $a, b \in X$ then $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$ for some $a, b, c \in X$, then $a \sim c$.