Problem class 3

1. Let L be the line $Y = 0$ and P the point $(0 : 0 : 1)$ on it. Compute the intersection multiplicity $I_P(C, L)$ for the following curves C.

   a) $C : YZ - X^2 = 0$.

   b) $C : Y^2Z - X^3 - 7XZ^2 = 0$.

   c) $C : XY(Y^2 - X^2)Z^2 + X^8 + Y^8 = 0$.

   d) $C : X^2 + Y^2 + Z^2 = 0$.

2. Let C be the curve $Y^2Z - X^3 = 0$. Show that $P = (0 : 0 : 1)$ is a singular point on C by computing $I_P(C, L)$ for all lines L through P.

3. Let $C : F = 0$ be a projective curve of degree d passing through $P = (0 : 0 : 1)$. Write

$$F(X, Y, Z) = F_1(X, Y) \cdot Z^{d-1} + F_2(X, Y) \cdot Z^{d-2} + \cdots + F_d(X, Y)$$

   where $F_i(X, Y) \in k[X, Y]$ is a homogeneous polynomial of degree i in two variables. Prove that P is a singular point if and only if $F_1(X, Y) = 0$.

   Assume that P is not singular. Then $F_1(X, Y) = 0$ is the unique tangent at P.

4. Let C be a conic defined over $\mathbb{C}$. Show that $C(\mathbb{C})$ cannot be empty.

5. Pick two distinct points P and Q in $\mathbb{Q}^2$. (Yes, I really mean that you pick your favourite rational numbers.) If P, Q and the origin lie on one line then make another choice. Find all conics that pass through P, Q, $(0 : 0 : 1)$, $(0 : 1 : 0)$ and $(1 : 0 : 0)$.