Problem class 3

1. Let L be the line $Y = 0$ and P the point $(0 : 0 : 1)$ on it. Compute the intersection multiplicity $I_P(C, L)$ for the following curves C.

   a) $C : YZ - X^2 = 0$.

   b) $C : Y^2Z - X^3 - 7XZ^2 = 0$.

   c) $C : XY(Y^2 - X^2)Z^2 + X^8 + Y^8 = 0$.

   d) $C : X^2 + Y^2 + Z^2 = 0$.

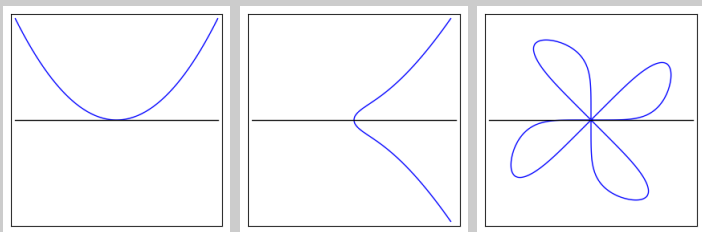
Solution: We best take a parametrisation of L. The simplest is $\ell(s : t) = (t : 0 : s)$ corresponding to the choices P and $(1 : 0 : 0)$ at infinity.

a) We get $G(s : t) = 0 \cdot s - t^2 = -t^2$. So $I_P(C, L) = 2$.

b) Here it is $G(s : t) = -t^3 - 7s^2t$ and so $I_P(C, L) = 1$.

c) This time $G(s : t) = t^8$ and so $I_P(C, L) = 8$.

d) Oh, but $(0 : 0 : 1)$ is not on C, therefore $I_P(C, L) = 0$.



The first three intersections drawn

2. Let C be the curve $Y^2Z - X^3 = 0$. Show that $P = (0 : 0 : 1)$ is a singular point on C by computing $I_P(C, L)$ for all lines L through P.

Solution: Let L be a line though $(0 : 0 : 1)$. It can be given by an equation $aX - bY = 0$ and then by a parametrisation $\ell(s : t) = s(0 : 0 : 1) + t(b : a : 0) = (bt : at : s)$. We get $G(s : t) = (at)^2s - (bt)^3 = a^2st^2 - b^3t^3$. So if $a \neq 0$, then $I_P(C, L) = 2$ and if $a = 0$ then $I_P(C, L) = 3$. In any case it is strictly larger than 1 therefore P is a singular point on C.

3. Let $C : F = 0$ be a projective curve of degree d passing through $P = (0 : 0 : 1)$. Write

$$F(X, Y, Z) = F_1(X, Y) \cdot Z^{d-1} + F_2(X, Y) \cdot Z^{d-2} + \cdots + F_d(X, Y)$$

where $F_i(X,Y) \in k[X,Y]$ is a homogeneous polynomial of degree i in two variables. Prove that P is a singular point if and only if $F_1(X,Y) = 0$.

Assume that P is not singular. Then $F_1(X,Y) = 0$ is the unique tangent at P.

Solution: Let $\ell(s:t) = (bt : at : s)$ be the parametrisation of an arbitrary line through $P = (0:0:1)$. Then

$$G(s:t) = F(\ell(s:t)) = F_1(bt, at)\, s^{d-1} + F_2(bt, at)\, s^{d-2} + \cdots$$

Suppose $F_1 = uX + vY$, then the smallest degree term in t will be $F_1(bt, at)\, s^{d-1} = (bu + av)\, ts^{d-1}$. If $F_1 \neq 0$, then there is a choice of a and b such that $bu + av \neq 0$. Hence $I_P(C, L) = 1$ for that particular L. This shows P is a non-singular point. If $F_1 = 0$, then $I_P(C, L) > 1$ for all L and hence P is singular.

Finally $F_1 \neq 0$, then $I_P(C, L) > 0$ precisely for the line $uX + vY = 0$.

Remark: In general the smallest m such that $F_m(X,Y) \neq 0$ is called the multiplicity of the singularity. Over an algebraically closed field like $\mathbb{C}$, one can factor $F_m(X,Y)$ into m linear factors; these are the tangents of C at P.

4. Let C be a conic defined over $\mathbb{C}$. Show that $C(\mathbb{C})$ cannot be empty.

Solution: Take the general equation of a conic:

$$C: \qquad a_0\, X^2 + a_1\, XY + a_2\, XZ + a_3\, Y^2 + a_4\, YZ + a_5\, Z^2 = 0 \tag{1}$$

We know that at least one of the coefficients a_i is non-zero. Pick one of the variables in the corresponding monomial. Allowing to rename them, we may assume that it is X. If $a_0 \neq 0$, we can pick any non-zero values for Y and Z and solve the degree 2 polynomial in X to get a point on C. Otherwise, we may choose non-zero Y and Z such that $a_1\, X + a_2\, Y \neq 0$ because at least one of these two coefficients is non-zero. In that case we can solve the linear polynomial in X to get a point on $\mathbb{C}$.

(Obviously this fails over fields like $\mathbb{R}$ and $\mathbb{Q}$ because some quadratic polynomials do not have solutions in these fields.)

5. Pick two distinct points P and Q in $\mathbb{Q}^2$. (Yes, I really mean that you pick your favourite rational numbers.) If P, Q and the origin lie on one line then make another choice. Find all conics that pass through P, Q, $(0:0:1)$, $(0:1:0)$ and $(1:0:0)$.

Solution: We start with the general equation of a conic:

$$C: \qquad a_0\, X^2 + a_1\, XY + a_2\, XZ + a_3\, Y^2 + a_4\, YZ + a_5\, Z^2 = 0 \tag{2}$$

To say that $(0:0:1)$ lies on C imposes that $a_5 = 0$. Similarly $(0:1:0) \in C$ and $(1:0:0) \in C$ impose that $a_0 = a_2 = 0$.

Say the two points are $P = (x_P : y_P : 1)$ and $Q = (x_Q : y_Q : 1)$. we are looking for a_1, a_2 and a_4 such that the two equations

$$\begin{aligned} x_P y_P\, a_1 + x_P\, a_2 + y_P\, a_4 &= 0 \\ x_Q y_Q\, a_1 + x_Q\, a_2 + y_Q\, a_4 &= 0 \end{aligned}$$

hold. Because P and Q and the origin do not lie on one line the equations are linearly independent and we find, up to scaling, exactly one choice of a_1, a_2, a_4. Therefore there is a unique conic through these five points. Explicitly,

$$(x_Q y_P - x_P y_Q)\, XY + (x_P - x_Q) y_P y_Q\, XZ + x_P x_Q (y_P - y_Q)\, YZ = 0.$$

For instance for $P = (1, 2)$ and $Q = (3, 4)$, we find the system

$$\begin{aligned} 2\, a_1 + a_2 + 2\, a_4 &= 0 \\ 12\, a_1 + 3\, a_2 + 4\, a_4 &= 0 \end{aligned}$$

Solving this system yields the conic $XY - 8XZ + 3YZ = 0$.

In general there exists precisely one conic going through five points as long as no four of them lie on one line.