Problem class 4

1. Let $C : X^3 + Y^3 + Z^3 = 0$ defined over $\mathbb{Q}$. Show that C is smooth.

   Also find the tangent to C at the point $(1 : -1 : 0)$.

2. Let k be a field, $a \in k$ and $g \in k[x]$ of degree $d > 0$. Let C be the projective closure of the affine curve $y^2 = (x-a)^2 \cdot g(x)$. Show that $(a : 0 : 1)$ is a singular point on C.

3. Let k be a field, $n > 1$ an integer and $g \in k[x]$ of degree $d > n + 1$. Let C be the projective closure of the affine curve $y^n = g(x)$. Show that $(0 : 1 : 0)$ is a singular point on C.

4. Let k be a field whose characteristic is not 2. Let $e_1, e_2, \ldots, e_d$ be distinct elements in k. Let C be the projective closure of the affine curve $y^2 = (x-e_1)(x-e_2)\cdots(x-e_d)$. Show that there are no singular point with $Z \neq 0$ on C.

5. Let k be a field of characteristic different from 2. Show that the projective closure C of the curve $y^2 = (x-e_1)(x-e_2)(x-e_3)$ defined over k is smooth if and only the e_i are distinct.