Problem class 7

1. Let E be the elliptic curve over $\mathbb{Q}$ defined by $y^2 = x^3 - 16\,x + 16$. Use the reductions at $p = 3$ and $p = 5$ to prove that $E(\mathbb{Q})_{\text{tors}}$ is trivial. Then show that $E(\mathbb{Q})$ is infinite.

2. Show that the torsion subgroup of $E(\mathbb{Q})$ for $E : y^2 = x^3 - 11\,x - 5$ is trivial.

3. Determine $E(\mathbb{Q})_{\text{tors}}$ for the curve $E : y^2 = x^3 - 5005$.

4. Let B be a square-free integer with $B \neq \pm 1$. Consider the curve $E : y^2 = x^3 + B$.
   (a) Show that $E(\mathbb{Q})$ does not contain a 2-torsion point.
   (b) Let $P = (x_P, y_P)$ be a torsion point in $E(\mathbb{Q})$. Using Lutz-Nagell show that y_P divides 27.
   (c) $(*)$ Let p be a prime dividing B. Suppose $p \neq 3$. Use the reduction modulo p to show that $pP = O$ for all torsion points $P \in E(\mathbb{Q})$.
   (d) Show that $\tilde{E}(\mathbb{F}_5)$ is cyclic of order 6 if $5 \nmid B$.
   (e) Conclude that $E(\mathbb{Q})$ has no non-zero torsion points.

Here a list of elliptic curves showing that all possible abelian groups in Mazur's theorem can occur as torsion subgroups of $E(\mathbb{Q})$:

Curve A, B	$E(\mathbb{Q})_{\text{tors}}$
-1, 1	O
1, 0	$\mathbb{Z}/2\mathbb{Z}$
0, 4	$\mathbb{Z}/3\mathbb{Z}$
-2, 1	$\mathbb{Z}/4\mathbb{Z}$
-1, 0	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
-432, 8208	$\mathbb{Z}/5\mathbb{Z}$
0, 1	$\mathbb{Z}/6\mathbb{Z}$
-3483, 121014	$\mathbb{Z}/7\mathbb{Z}$
-5211, 319734	$\mathbb{Z}/8\mathbb{Z}$
-5211, -31050	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$
-17739, 1205766	$\mathbb{Z}/9\mathbb{Z}$
-58347, 3954150	$\mathbb{Z}/10\mathbb{Z}$
-157707, 78888006	$\mathbb{Z}/12\mathbb{Z}$
-24003, 1296702	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$
-1386747, 368636886	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$