Problem class 4

1. Let $C : X^3 + Y^3 + Z^3 = 0$ defined over $\mathbb{Q}$. Show that C is smooth.

   Also find the tangent to C at the point $(1 : -1 : 0)$.

Solution: Suppose $(X : Y : Z)$ is a singular point in $C(K)$ for some field K containing $\mathbb{Q}$. Then

$$\frac{\partial F}{\partial X} = 3\,X^2 = 0$$
$$\frac{\partial F}{\partial Y} = 3\,Y^2 = 0$$
$$\frac{\partial F}{\partial Z} = 3\,Z^2 = 0$$

We deduce $X = Y = Z = 0$ which is not a projective point. Therefore there are no singular points in $C(K)$ and hence C is smooth.

The tangent at $(1 : -1 : 0)$ is $3X + 3Y + 0\,Z = 0$. So it is $X + Y = 0$.

By the way, over a field of characteristic 3 instead, every point, like $(1 : 1 : 1)$, is singular. Well, $X^3 + Y^3 + Z^3 = (X + Y + Z)^3$ in characteristic 3; so things went pear-shaped anyway.

2. Let k be a field, $a \in k$ and $g \in k[x]$ of degree $d > 0$. Let C be the projective closure of the affine curve $y^2 = (x - a)^2 \cdot g(x)$. Show that $(a : 0 : 1)$ is a singular point on C.

Solution: The projective closure is the curve

$$C : F = -Y^2\,Z^d + (X - aZ)^2 \cdot G(X, Z) = 0$$

where G is the homogenisation of g. First of all $(a : 0 : 1)$ is indeed a point in $C(k)$. The partial derivatives are

$$\frac{\partial F}{\partial X} = 2(X - aZ) \cdot G(X, Z) + (X - aZ)^2 \cdot \frac{\partial G}{\partial X}(X, Z)$$
$$\frac{\partial F}{\partial Y} = -2Y\,Z^d$$
$$\frac{\partial F}{\partial X} = -dY^2 Z^{d-1} + 2(-a)(X - aZ) \cdot G(X, Z) + (X - aZ)^2 \cdot \frac{\partial G}{\partial Z}(X, Z)$$

At the point $(a : 0 : 1)$ we have $Y = 0$ and $X - aZ = 0$. So all three above vanish there; therefore it is a singular point.

3. Let k be a field, $n > 1$ an integer and $g \in k[x]$ of degree $d > n+1$. Let C be the projective closure of the affine curve $y^n = g(x)$. Show that $(0:1:0)$ is a singular point on C.

Solution: The projective closure is the curve

$$C : F = -Y^n Z^{d-n} + G(X,Z) = 0$$

where G is the homogenisation of g. First of all $(0:1:0)$ is indeed a point in $C(k)$ as $G(0,0) = 0$ and $d > n$. The partial derivatives are

$$\begin{aligned}
\frac{\partial F}{\partial X} &= \frac{\partial G}{\partial X}(X,Z) \\
\frac{\partial F}{\partial Y} &= -nY^{n-1}Z^{d-n} \\
\frac{\partial F}{\partial X} &= -(d-n)\,Y^n\,Z^{d-n-1} + \frac{\partial G}{\partial Z}(X,Z)
\end{aligned}$$

At the point $(0:1:0)$ we have $X = Z = 0$. Therefore the three above vanish if $d-n > 0$ and $d-n-1 > 0$. Hence it is a singular point by the assumption on d.

4. Let k be a field whose characteristic is not 2. Let $e_1, e_2, \ldots, e_d$ be distinct elements in k. Let C be the projective closure of the affine curve $y^2 = (x-e_1)(x-e_2)\cdots(x-e_d)$. Show that there are no singular point with $Z \neq 0$ on C.

Solution: The projective closure is the curve

$$C : F = -Y^2 Z^{d-2} + (X - e_1 Z)(X - e_2 Z)\cdots(X - e_d Z) = 0.$$

Suppose $(X:Y:Z)$ with $Z \neq 0$ is a singular point. Then

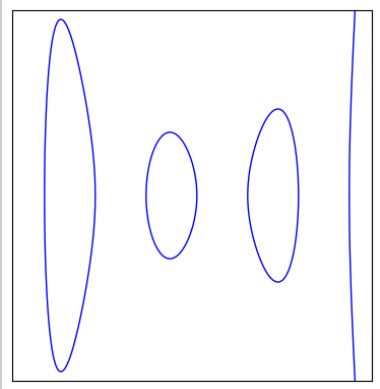
$$\begin{aligned}
0 = \frac{\partial F}{\partial X} &= (X - e_2 Z)(X - e_3 Z)\cdots(X - e_d Z) + \\
&\quad + (X - e_1 Z)(X - e_3 Z)\cdots(X - e_d Z) + \cdots \\
&\quad + (X - e_1 Z)(X - e_2 Z)\cdots(X - e_{d-1} Z) \\
0 = \frac{\partial F}{\partial Y} &= -2YZ^{d-2}
\end{aligned}$$

Since the characteristic is not 2 and $Z \neq 0$, we deduce that $Y = 0$. Then from $F = 0$ we deduce that one of the factors $X - e_i Z$ must vanish. So we are at the point $(e_i : 0 : 1)$ for some i. Looking at the derivative with respect to X, we see that all terms except the one where the term $X - e_i Z$ was dropped will vanish. We find

$$\frac{\partial F}{\partial X}(e_i : 0 : 1) = (e_i - e_1)(e_i - e_2)\cdots(e_i - e_{i-1})(e_i - e_{i+1})\cdots(e_i - e_d)$$

and this is non zero because the e_i are distinct.

Note that the previous exercise shows that this would be wrong if $d > 3$ or if the e_i were not distinct.



Example plot for $y^2 = x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$

5. Let k be a field of characteristic different from 2. Show that the projective closure C of the curve $y^2 = (x-e_1)(x-e_2)(x-e_3)$ defined over k is smooth if and only the e_i are distinct.

Solution: Assume that the e_i are distinct. Then by the previous exercise, there are no singular points with $Z \neq 0$. We only need to check if there are singular points at infinity. The points at infinity satisfy $Z = 0$ and hence $X = 0$. Therefore there is a single point $(0:1:0)$ at infinity. The value of

$$\frac{\partial F}{\partial Z} = -Y^2 - e_1(X-e_2Z)(X-e_3Z) - e_2(X-e_1Z)(X-e_3Z) - e_3(X-e_1Z)(X-e_2Z)$$

at $(0:1:0)$ is -1 hence the point is also non-singular. Hence C is smooth.

Conversely assume that C is smooth. By the second problem, we cannot have repeated linear factors as otherwise $(a:0:1)$ is singular. Hence all the e_i are distinct.