Problem class 2

1. a) Let $y = 2\,x - 7$ be an affine line over $k = \mathbb{Q}$. Write out an equation for the projective line L.
   b) Let $L' : 3\,X + 4\,Y - 16\,Z = 0$. Find the point of intersection $L \cap L'$.
   c) Draw an affine picture.

2. Let $P = (X_0 : Y_0 : Z_0)$, $Q = (X_1 : Y_1 : Z_1)$ and $R = (X_2 : Y_2 : Z_2)$ be three distinct points in $\mathbb{P}^2$. Show the following statement: These three points lie on one projective line if and only if the determinant

$$\det \begin{pmatrix} X_0 & Y_0 & Z_0 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{pmatrix} = 0.$$

   Analogously, three distinct lines $a_0X + b_0Y + c_0Z = 0$, $a_1X + b_1Y + c_1Z = 0$ and $a_2X + b_2Y + c_2Z = 0$ meet in one point if and only if the determinant

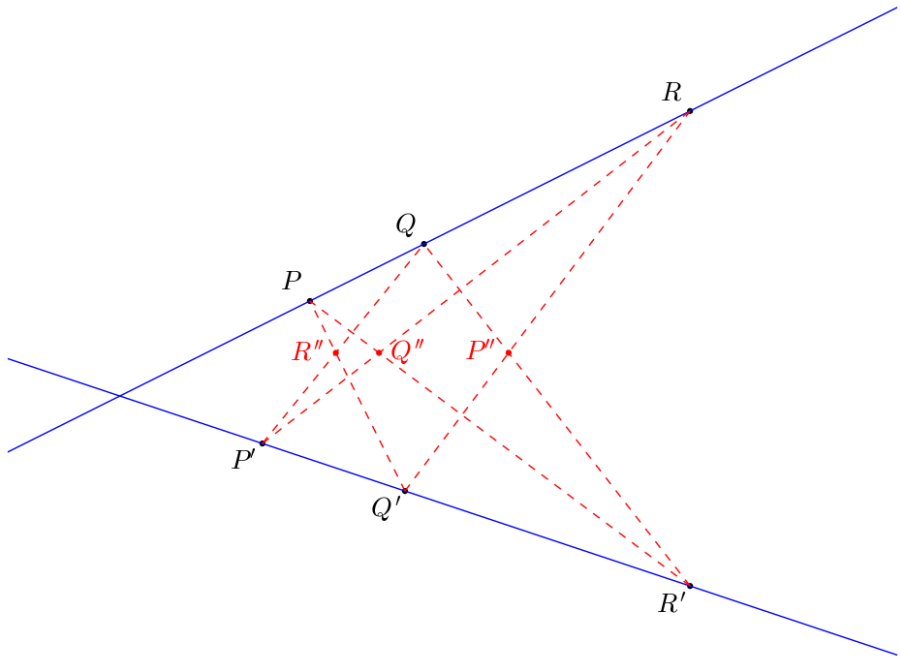
$$\det \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 0.$$

3. For each of the following polynomials in $k[x, y]$ write down their homogenisation and calculate the points at infinity if $k = \mathbb{C}$.
   a) $x^2y - y + 4$
   b) $x^7 + y^2 + xy - y + 1$
   c) $x^n + y^n - 1$ where $n > 1$ is an integer.
   d) $x^4 - 3x^3y + x^2y^2 - 3xy^3$

4. Fix a field k and three elements α, β, γ, all of them different from 1. Consider the following six points

$$P = (1 : 1 : 1),\ Q = (\alpha : 1 : 1),\ R = (1 : 0 : 0)$$
$$P' = (\alpha\beta : \beta : 1),\ Q' = (1 : 1 : \gamma),\ R' = (0 : 1 : 0)$$

   (a) Show that the point P, Q and R lie on one line.
   (b) Show that it is equivalent that P', Q' and R' lie on one line as to say that $\alpha\beta\gamma = 1$.
   (c) Assume now $\alpha\beta\gamma = 1$. Let P'' be the intersection of the line through Q and R' with the line through Q' and R. Let Q'' be the intersection of the line through P and R' with the line through P' and R. Let R'' be the intersection of the line through P and Q' with the line through P' and Q. Compute P'', Q'' and R'' and show that they lie on one line.
   (See at the back of this page for a picture.)


R
Q
P
R''
Q''
P''
P'
Q'
R'