Problem class 9

1. Consider the curve $y^2 = (x+4)(x+1)(x-5)$ defined over $\mathbb{Q}$. We find a point $P = (-3, 4)$ of infinite order in $E(\mathbb{Q})$. Show that $E(\mathbb{Q})$ has rank 1.

Solution: We take $e_1 = -4$, $e_2 = -1$ and $e_3 = 5$. The only primes dividing the discriminant are 2 and 3. We know that the image of $E(\mathbb{Q})/2E(\mathbb{Q})$ is contained in the group $(a, b, ab) \in H \times H \times H$ where H is generated by -1, 2 and 3. That is $H = \{1, -1, 2, -2, 3, -3, 6, -6\}$.

We can also determine the image of the points we already know.

$$\kappa(O) = (1,1),\ \kappa(T_1) = (3,-3),\ \kappa(T_2) = (3,-2),\ \kappa(T_3) = (1,6), \kappa(P) = (1,-2)$$

Further we get $\kappa(P+T_1) = \kappa(P)\cdot\kappa(T_1) = (3,6)$, $\kappa(P+T_2) = (3,1)$ and $\kappa(P+T_3) = (1,-3)$

To determine which pairs (a,b) are in the image of κ, we have to solve the torsor equations

$$a\,u^2 - b\,v^2 = 3 \qquad \text{and} \qquad a\,u^2 - ab\,w^2 = 9$$

in $u, v, w \in \mathbb{Q}$. Writing $u = U/T$, $v = V/T$ and $w = W/T$ we can solve the equations

$$a\,U^2 - b\,V^2 = 3\,T^2 \qquad \text{and} \qquad a\,U^2 - ab\,W^2 = 9T^2$$

in integers U, V, W and T such that there is no common factor to all four.

Looking at the signs, we conclude that pairs (a,b) with a negative cannot have a solution (either to the first equation when $b > 0$ or to the second equation when $b < 0$).

Here is how the table looks by now

$a\backslash b$	1	-1	2	-2	3	-3	6	-6
1	O			P		$P+T_3$	T_3	
2								
3	$P+T_2$			T_2		T_1	$P+T_1$	
6								

From looking at divisibility by 2, we will show that (a,b) with even a cannot have a solution: If a is even, then the second equation implies that T is even. Looking at the first equation, we now have that $b\,V^2$ is even.

Let us first treat the case when b is odd. Then V is even, which implies, still from the first equation, that aU^2 is divisible by 4. As a is square-free, this means that U is even. In the second equation, we have that abW is divisible by 4 now, but that implies that W is even. This rules out (a,b) with a even and b odd.

Now suppose both a and b are odd. We have already shown that T is even. The second equation implies that $aU^2 = 9T^2 + abW^2$ is divisible by 4. Therefore U is even. From the first equation we see that bV^2 is now divisible by 4 which implies that V is even. Since $aU2$ and bV^2 are now both multiples of 8 we must have that T is divisible by 4. The second equation shows that now that abW^2 is divisible by 8, from which we deduce that W is even, too.

This delete the two rows with $a = 2$ and $a = 6$ in the above table. We are now left with all sixteen pairs (a, b) with $a \in \{1, 3\}$ and $b \in H$. Since the image is a subgroup its number of element is a power of 2. Hence if we can exclude a single further equation, we know that the above is the complete image of κ.

We consider $(1, -1)$. The first equation becomes $U^2+V^2 = 3T^2$. Modulo 3, knowing that only 0 and 1 are squares, we conclude that U and V are both divisible by 3. Then the left hand side is divisible by 9, which implies that $3T^2$ is divisible by 9, too. This shows that T is a multiple of 3.

Now the second equation is $U^2 + W^2 = 9T^2$. Since U is already shown to be a multiple of 3, we see that W is also divisible by 3. Therefore $(1, -1)$ is not in the image of κ.

We conclude that the image of κ is of size 2^3 showing that the rank is 1. (Using heights one could show that $E(\mathbb{Q}) = {}^{\mathbb{Z}}/_{2\mathbb{Z}}(-4, 0) \times {}^{\mathbb{Z}}/_{2\mathbb{Z}}(-1, 0) \times \mathbb{Z}(-3, 4)$.)

2. Let $E : y^2 = (x - e_1)(x - e_2)(x - e_3)$ with $e_1 = 24$, $e_2 = 6$ and $e_3 = -30$. Find a point $P \in E(\mathbb{Q})$ such that $\kappa(P) = (-1, -2, 2)$ and conclude that the rank of $E(\mathbb{Q})$ is at least 1.

Solution: If x_P is the x-coordinate of P then, there are rational numbers u, v, w such that

$$\begin{aligned} x_P - 24 &= (-1)u^2 \\ x_P - 6 &= (-2)v^2 \\ x_P + 30 &= 2w^2 \end{aligned}$$

As usual, write $u = U/T$, $v = V/T$ and $w = W/T$ with integers U, V, W, T without a common divisor. The torsor equations become

$$\begin{aligned} -U^2 + 2V^2 &= -18T^2 \\ -U^2 - 2W^2 &= -54T^2 \end{aligned}$$

From either equation we deduce that U is even. Say $U = 2U'$. The equations then simplify to

$$\begin{aligned} -2U'^2 + V^2 &= -9T^2 \\ -2U'^2 - W^2 &= -27T^2 \end{aligned}$$

If either U' or V is coprime to 3 then both are. But if so $-2U'^2+V^2 \equiv -2\cdot 1+1 \equiv -1 \pmod 3$ is not possible. Hence $U' = 3U''$ and $V = 3V'$. It also follows that W is

divisible by 3, say $W = 3W'$. Now the equations are

$$-2U''^2 + V'^2 = -T^2$$
$$-2U''^2 - W'^2 = -3T^2$$

For these equations is it now easy to find a solution, namely $U'' = V' = W' = T = 1$. This gives $U = 6$ and $u = 6$. Hence $x_P = -u^2 + 24 = -12$. Also $y_P^2 = (-1)u^2 \cdot (-2)v^2 \cdot 2w^2 = (-1) \cdot 6^2 \cdot (-2) \cdot 3^2 \cdot 2 \cdot 3^2$ shows that we can take $y = 2 \cdot 6 \cdot 3 \cdot 3 = 108$. So one point P that maps to $(-1, -2, 2)$ is $(-12, 108)$.

(Of course, there are plenty other points like $\left(\frac{852}{289}, \pm\frac{226044}{4913}\right)$ that map to $(-1, -2, 2)$ under κ. Also $\left(-\frac{261206508}{8958049}, \pm\frac{1063063487196}{26811440657}\right)$ does.)

We can compute the image of $E(\mathbb{Q})_{\text{tors}}$ to be

$$\kappa((24, 0)) = (3, 2, 6) \quad \kappa((6, 0)) = (-2, -2, 1) \quad \kappa((-30, 0)) = (-6, -1, 6)$$

As $(1, -2, 2)$ is not among them P must have infinite order. Indeed $x_{2P} = \frac{105}{4}$ is not an integer.