Coursework 3 on chapter 6 and 7

1. Let E be an elliptic curve in Weierstrass form over a field k. Prove the following: If $O \neq P = (x_P : y_P : 1)$ is a point of order 3, then
$$3\,x_P^4 + 6\,A\,x_P^2 + 12\,B\,x_P - A^2 = 0.$$

2. Let E be the elliptic curve over $\mathbb{Q}$ defined by $y^2 = x^3 + 9\,x - 18$. With the help of the reduction modulo 5, or otherwise, determine $E(\mathbb{Q})_{\text{tors}}$.

3. Let E be the elliptic curve $y^2 = x^3 + 3\,x + 1$ defined over $\mathbb{Q}$. Prove that $E(\mathbb{Q})$ has no non-zero torsion point.

4. Let $p \geqslant 5$ be a prime and consider the elliptic curve $E:\ y^2 = x^3 + p\,x$. Use the theorem of Lutz-Nagell to find all torsion point in $E(\mathbb{Q})$.

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5. Let E be the elliptic curve $y^2 = x^3 - x$ defined over $\mathbb{Q}$.
   (a) Consider the torsor equations associated to E for given $a, b \in \mathbb{Q}^\times$.
$$a\,u^2 - b\,v^2 = 1$$
$$a\,u^2 - ab\,w^2 = 2$$
   Show that there are no rational solutions in (u, v, w) when $a < 0$. Show that there are no rational solutions when $a = 1$ and $b = 2$.
   (b) Show that $E(\mathbb{Q})$ is finite.

6. Determine the group structure of $E(\mathbb{Q})$ for $E : y^2 = x^3 - 25x$.

7. Let E be an elliptic curve given by an equation $y^2 = (x - e_1)(x - e_2)(x - e_3)$ with e_i three distinct elements in $\mathbb{F}_p$ for some odd prime p. Consider the Kummer map $\kappa\colon E(\mathbb{F}_p) \to {}^{\mathbb{F}_p^\times}/_{\square} \times {}^{\mathbb{F}_p^\times}/_{\square} \times {}^{\mathbb{F}_p^\times}/_{\square}$. Determine the image of κ.

8. Let E be the elliptic curve $y^2 = (x + 130)(x - 41)(x - 89)$. Show that the following three points are all of infinite order
$$P_1 = (95, 270) \qquad P_2 = (98, -432) \qquad P_3 = (-\tfrac{4066}{49}, \tfrac{343440}{343})$$
and use them further to prove that the rank of $E(\mathbb{Q})$ is at least 3.

9. Let E be the elliptic curve $y^2 = (x + 12)(x + 10)(x - 22)$ defined over $\mathbb{Q}$. We are given that the rank of $E(\mathbb{Q})$ is zero. Conclude that the system of equations
$$U^2 - 2\,V^2 = 2\,T^2 \qquad \text{and} \qquad U^2 - 2\,W^2 = 34\,T^2$$
has no solution in integers U, V, W, T except $(0, 0, 0, 0)$.

   Show that there are real solutions and solutions modulo $m = 16$ and $m = 17$. (One could in fact check that there are solutions modulo all integers m and hence the corresponding projective curve in $\mathbb{P}^3$ is a counter-example to the Hasse principle.)