Problem class 2

1. a) Let $y = 2\,x - 7$ be an affine line over $k = \mathbb{Q}$. Write out an equation for the projective line L.

   b) Let $L' : 3\,X + 4\,Y - 16\,Z = 0$. Find the point of intersection $L \cap L'$.

   c) Draw an affine picture.

Solution:

a) One equation would be $2\,X - Y - 7\,Z = 0$, but one can multiply this with any scalar $\lambda \in \mathbb{Q}^\times$.

b) We are looking for a non-zero vector (X, Y, Z) solving the equation $M\vec{v} = \vec{0}$ with

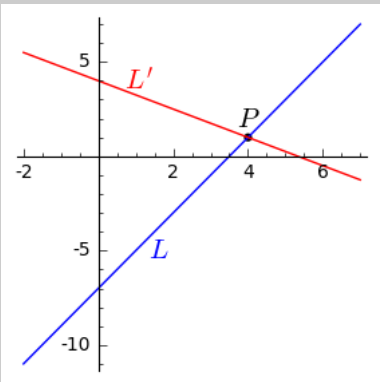
$$M = \begin{pmatrix} 3 & 4 & -16 \\ 2 & -1 & -7 \end{pmatrix}.$$

We can use the usual Gaussian elimination. Here I add the second line four times to the first to get

$$M = \begin{pmatrix} 11 & 0 & -44 \\ 2 & -1 & -7 \end{pmatrix}.$$

The top line reads $11X - 44Z = 0$ which simplifies to $X = 4Z$. Introducing this into the equation of L, one finds $8Z - Y - 7Z = 0$ and hence $Y = Z$. Therefore all solutions are scalar multiples of $(4, 1, 1)$. In other words $(4 : 1 : 1)$ is the intersection point.

c) Here is a picture:



2. Let $P = (X_0 : Y_0 : Z_0)$, $Q = (X_1 : Y_1 : Z_1)$ and $R = (X_2 : Y_2 : Z_2)$ be three distinct points in $\mathbb{P}^2$. Show the following statement: These three points lie on one projective

line if and only if the determinant

$$\det \begin{pmatrix} X_0 & Y_0 & Z_0 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{pmatrix} = 0.$$

Analogously, three distinct lines $a_0X + b_0Y + c_0Z = 0$, $a_1X + b_1Y + c_1Z = 0$ and $a_2X + b_2Y + c_2Z = 0$ meet in one point if and only if the determinant

$$\det \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 0.$$

Solution: To say that the first determinant is zero is equivalent to saying that its three columns $\vec{c}_1$, $\vec{c}_2$, $\vec{c}_3$ are linearly dependent. This is the same as the existence of three a, b, $c \in k$ with at least one of the non-zero such that $a\vec{c}_1 + b\vec{c}_2 + c\vec{c}_3 = \vec{0}$. This is equivalent to $a\,X_i + b\,Y_i + c\,Z_i = 0$ for $i = 0, 1, 2$ and therefore to P, Q and R lying on one line $a\,X + b\,Y + c\,Z = 0$.

The argument is similar for the second determinant. This determinant is zero if and only if its three columns are linearly dependent. This means that there are X_0, Y_0, Z_0 not all zero in k such that $a_i\,X_0 + b_i\,Y_0 + c_i\,Z_0 = 0$ for $i = 0, 1, 2$. That is equivalent to stating that all three lines go through the point $(X_0 : Y_0 : Z_0)$.

Aside we remark that this gives us another way to remember the equation for a line passing through the points P and Q, namely it is

$$\det \begin{pmatrix} X_0 & Y_0 & Z_0 \\ X_1 & Y_1 & Z_1 \\ X & Y & Z \end{pmatrix} = 0.$$

3. For each of the following polynomials in $k[x, y]$ write down their homogenisation and calculate the points at infinity if $k = \mathbb{C}$.

   a) $x^2y - y + 4$
   b) $x^7 + y^2 + xy - y + 1$
   c) $x^n + y^n - 1$ where $n > 1$ is an integer.
   d) $x^4 - 3x^3y + x^2y^2 - 3xy^3$

Solution:

a) The homogenisation is $X^2Y - YZ^3 + 4Z^3 = 0$. The points at infinity are when $Z = 0$. This implies $X^2Y = 0$, so either $X = 0$, which is the point $(0 : 1 : 0)$, or $Y = 0$, which gives $(1 : 0 : 0)$.

b) This times it is $X^7 + Y^2Z^5 + XYZ^5 + YZ^6 + Z^7 = 0$. There is only one point at infinity namely $(0 : 1 : 0)$.

c) On the curve $X^n+Y^n+Z^n=0$ the points at infinity must satisfy $X^n+Y^n=0$ or $X^n=-Y^n$. As $Y\neq 0$, since otherwise X is also zero, we may set $Y=1$. Now we have to solve $X^n=-1=e^{\pi i}$ in $\mathbb{C}$. Hence there are n points, namely $(e^{(1+2k)i\pi/n}:1:0)$ for $k=0,1,\ldots,n-1$.

d) All terms have already degree 4, so the homogenisation is $X^4-3X^3Y-X^2Y^2-3XY^3=0$. This factors as

$$\begin{aligned} X^4-3X^3Y+X^2Y^2-3XY^3 &= X\,(X-3\,Y)(X^2+Y^2)\\ &= X\,(X-3Y)(X-iY)(X+iY) \end{aligned}$$

this curve is in fact the union of four lines all meeting at the origin. The points are infinity are $(0:1:0)$, $(3:1:0)$, $(i:1:0)$ and $(-i:1:0)$.

4. Fix a field k and three elements α, β, γ, all of them different from 1. Consider the following six points

$$P=(1:1:1),\ Q=(\alpha:1:1),\ R=(1:0:0)$$
$$P'=(\alpha\beta:\beta:1),\ Q'=(1:1:\gamma),\ R'=(0:1:0)$$

(a) Show that the point P, Q and R lie on one line.

Solution: The line is $Y-Z=0$ which is an equation that is obviously satisfied by all three points. Alternatively one computes the determinant

$$\det\begin{pmatrix}1&1&1\\ \alpha&1&1\\ 1&0&0\end{pmatrix}=0$$

for instance by noticing that the last two columns are equal.

(b) Show that it is equivalent that P', Q' and R' lie on one line as to say that $\alpha\beta\gamma=1$.

Solution: The three points lie on one line if and only if the determinant

$$\det\begin{pmatrix}\alpha\beta&\beta&1\\ 1&1&\gamma\\ 0&1&0\end{pmatrix}$$

is zero. That is, if and only if $1-\alpha\beta\gamma=0$.

(c) Assume now $\alpha\beta\gamma=1$. Let P'' be the intersection of the line through Q and R' with the line through Q' and R. Let Q'' be the intersection of the line through P and R' with the line through P' and R. Let R'' be the intersection of the line through P and Q' with the line through P' and Q. Compute P'', Q'' and R'' and show that they lie on one line.

(See at the back of this page for a picture.)

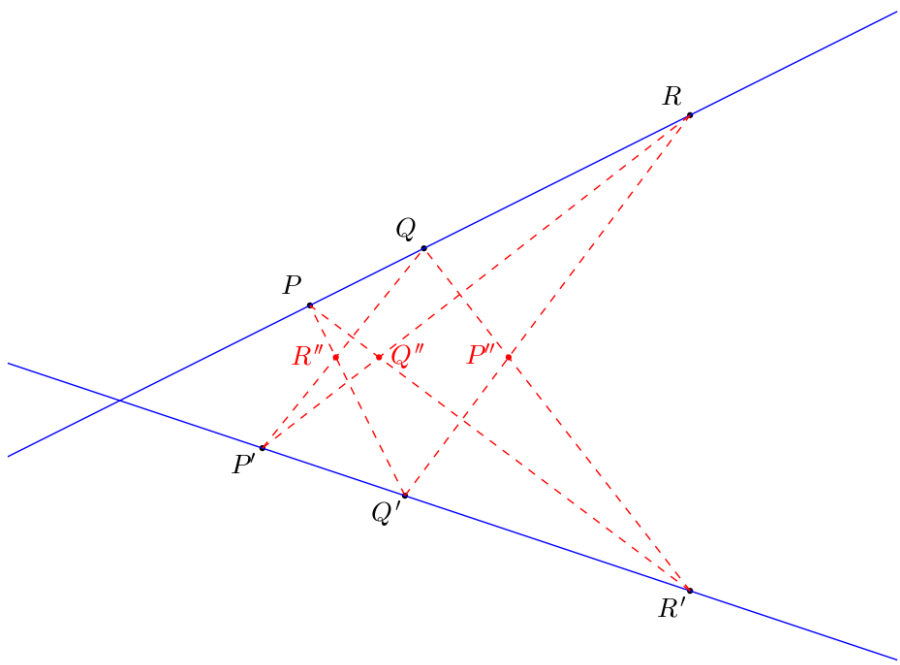
Solution: The line through Q and R' is $X - \alpha Z = 0$ and the line through Q' and R is $\gamma Y - Z$. Hence P'' is the point $(\alpha\gamma : 1 : \gamma)$.

The line through P and R' is given by $X - Z = 0$ and the line through P' and R by $Y - \beta Z = 0$. Therefore $Q'' = (1 : \beta : 1)$.

Finally the line through P and Q' is $X - Y = 0$, while the line through P' and Q is $X - \alpha Y = 0$. Thus $R'' = (0 : 0 : 1)$.

We are left to check that P'', Q'' and R'' lie on one line. Since

$$\det \begin{pmatrix} \alpha\gamma & 1 & \gamma \\ 1 & \beta & 1 \\ 0 & 0 & 1 \end{pmatrix} = \alpha\beta\gamma - 1 = 0$$

this is indeed the case.

(We have in fact checked a special case of a theorem by Pappus. Even better, one can use this argument to prove the theorem as one has the freedom to pick coordinates in the projective plane in such a way that this computation falls out.)