Why Iwasawa theorists need p-adic L-functions

christian wuthrich

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- $\mathrm{ord}_{(1-qT)} f_E(T) \geqslant \mathrm{rk}\, E(K)$
- If $\text{Ш}(E/K)(p)$ is finite for one prime p, then BSD holds.

Translation to geometry

$$K \quad \leftrightsquigarrow \quad C/k\text{, a smooth projective curve}$$

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E/K	$\leftrightsquigarrow$	$\mathcal{E} \to C$, an elliptic surface
$E(K)$	$\leftrightsquigarrow$	$\mathrm{NS}(\mathcal{E})$
$\mathrm{Sel}_p(E/K)$	$\leftrightsquigarrow$	$\mathrm{H}^2_{\text{ét}}\big(\mathcal{E}, \mathbb{Z}_p(1)\big)$

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$L(E/K, s) = f_E(T)$	$\leftrightsquigarrow$	$\det(1 - \mathrm{Frob}\ T)$ on $\mathrm{H}^2_{\text{ét}}(\overline{\mathcal{E}}, \mathbb{Z}_p(1))$

Here $\mathrm{Frob} \in G = \mathrm{Gal}(\overline{k}/k)$ and $\overline{\mathcal{E}} = \mathcal{E} \times \overline{k}$

$$
\begin{array}{ccccc}
\mathrm{H}^2_{\text{ét}}\big(\overline{\mathcal{E}}, \mathbb{Z}_p(1)\big)^G & \xrightarrow{\ \alpha\ } & \mathrm{H}^2_{\text{ét}}\big(\overline{\mathcal{E}}, \mathbb{Z}_p(1)\big)_G & \xrightarrow{\ \cong\ } & \left(\mathrm{H}^2_{\text{ét}}\big(\overline{\mathcal{E}}, \mu_{p^\infty}\big)^G\right)^\vee \\
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E(K) \otimes \mathbb{Z}_p & & \xrightarrow{\text{Néron-Tate height}} & & \left(E(K) \otimes {}^{\mathbb{Q}_p}/_{\mathbb{Z}_p}\right)^\vee
\end{array}
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Lemma

Let be M a $\mathbb{Z}_p$-module of finite type with a G-action and $\alpha\colon M^G \to M \to M_G$.

- Then $\ker(\alpha)$ and $\operatorname{coker}(\alpha)$ are finite if and only if

$$\operatorname{ord}_T \det(T - 1 + \operatorname{Frob} |M) = \operatorname{rk} M^G.$$

- If so, then the leading term is

$$\frac{\# \operatorname{coker}(\alpha)}{\# \ker(\alpha)},$$

  up to a unit in $\mathbb{Z}_p$.

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- $\mathrm{H}^i_\Sigma(K, \cdot) = \mathrm{H}^i(G_\Sigma(K), \cdot)$

- The fine Selmer group

$$Y_0 \;=\; \ker\Big(\mathrm{H}^2_{\Sigma}(K\;\;, T_pE) \to \bigoplus_{\upsilon\in\Sigma} \mathrm{H}^2(K_\upsilon\;\;, T_pE)\Big)$$

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$$Y_0 = \ker\Big(\mathrm{H}^2_\Sigma(K\ ,T_pE) \to \bigoplus_{\upsilon\in\Sigma} \mathrm{H}^2(K_\upsilon\ ,T_pE)\Big)$$

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- $(Y_0)^\vee$ contains the fine Mordell-Weil group

$$\mathcal{M} = \ker\Big(E(K) \otimes {}^{\mathbb{Q}_p}\!/\!_{\mathbb{Z}_p} \to \bigoplus_{v|p} E(K_v) \otimes {}^{\mathbb{Q}_p}\!/\!_{\mathbb{Z}_p}\Big)$$

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  (probably with finite index)
- $Y = \varprojlim Y_n$

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\begin{array}{ccccc}
\left(\mathrm{H}^1_\Sigma(K_\infty, E_{p^\infty})_\Gamma\right)^\vee & \xrightarrow{\text{duality}} & Y^\Gamma & \xrightarrow{\alpha} & Y_\Gamma \\
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Weak Leopoldt conjecture

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Let be Y a torsion Λ-module and $\alpha\colon Y^\Gamma \to Y_\Gamma$.

- Then $\ker(\alpha)$ and $\operatorname{coker}(\alpha)$ are finite if and only if

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up to a unit in $\mathbb{Z}_p$.

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Selmer groups

**p-adic
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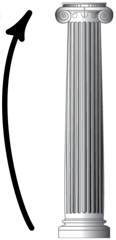
p-adic L-function
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p-adic L-function

There exists a p-adic L-function $\mathcal{L}_\alpha \in \Lambda \otimes \mathbb{Q}_p$ such that

$$\mathcal{L}_\alpha(\chi) = \chi(\mathcal{L}_\alpha) = \frac{p^n}{\alpha^n} \cdot \frac{L\big(E, \chi^{-1}, 1\big)}{\tau(\chi^{-1}) \cdot \Omega}$$

for all characters $\mathbb{1} \neq \chi$ on Γ of conductor p^n.

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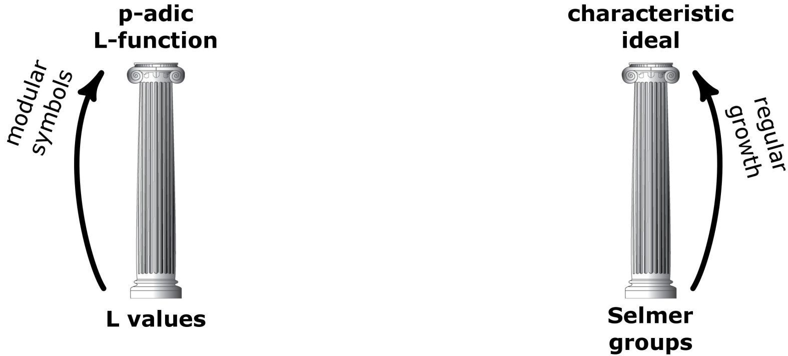
- Written in the basis of eigenvectors e_α, e_β such that $e_\alpha + e_\beta = \omega$ on $D_p(E)$, we get $\mathcal{L}_p = \mathcal{L}_\alpha e_\alpha + \mathcal{L}_\beta e_\beta$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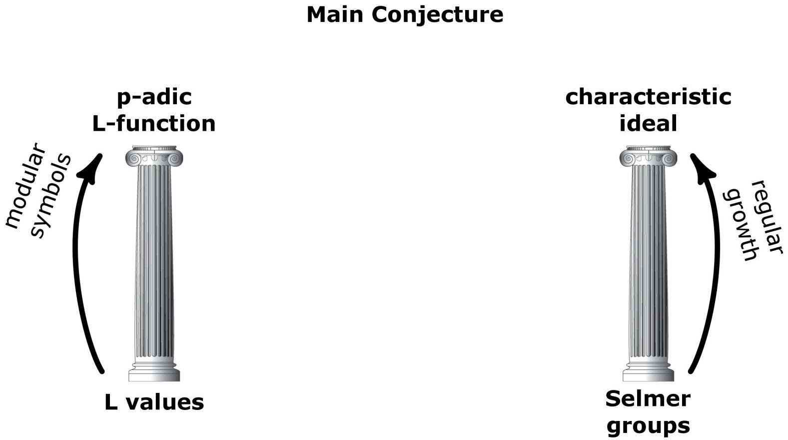
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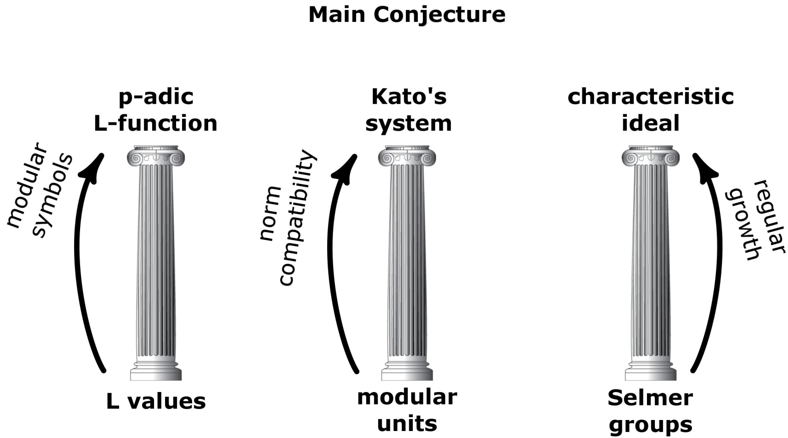
Main conjecture

$$\mathcal{L}_p\,\Lambda = I$$

Main Conjecture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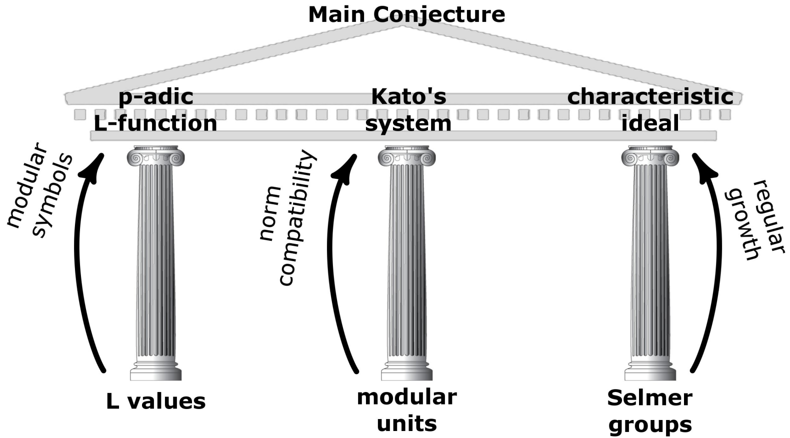
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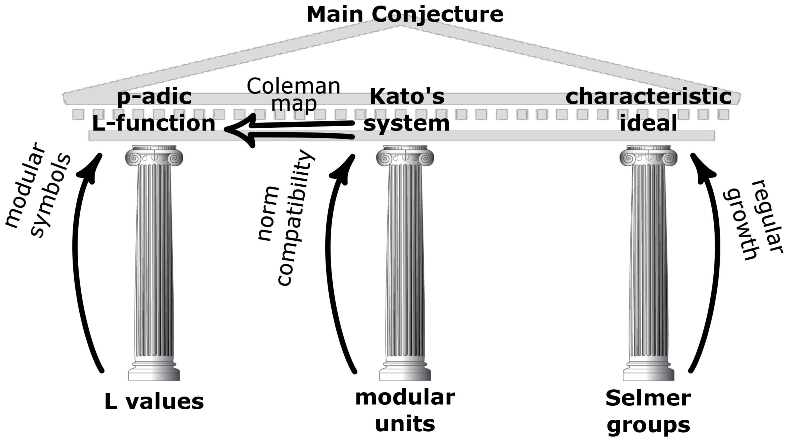
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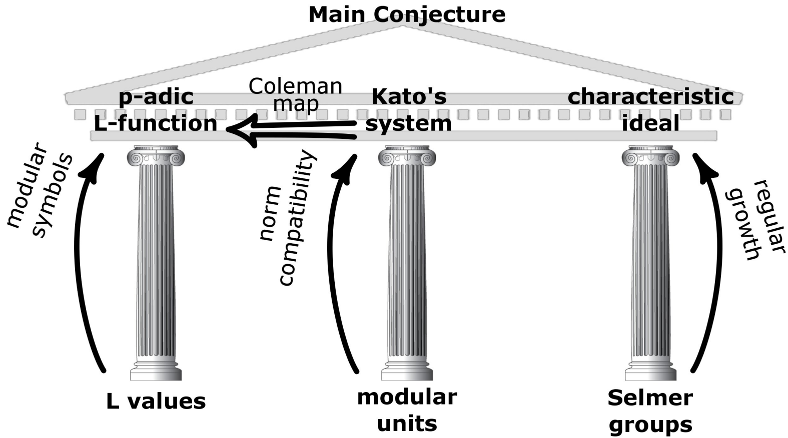
Kato

$$\mathrm{Col}(\mathbf{c}) = \mathcal{L}_p$$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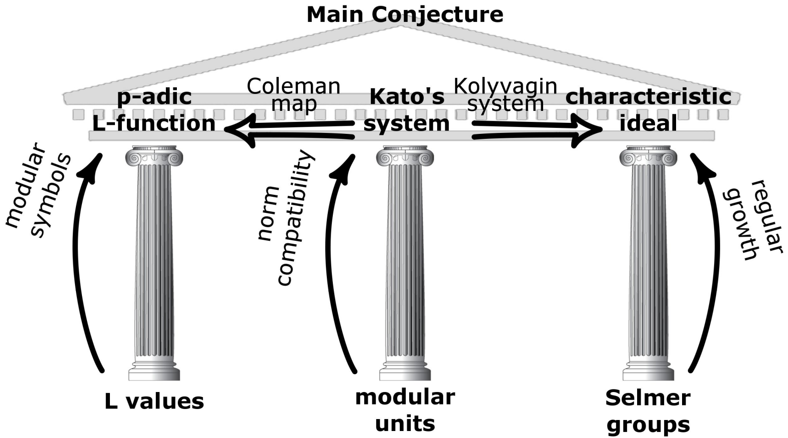
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Kato's Theorem

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Theorem

- $\operatorname{ord}_T \mathcal{L}_p \geqslant \operatorname{rk} E(\mathbb{Q})$.
- If we have equality and the p-adic height is non-degenerate, then leading term gives an upper bound for $\#Ш(E/\mathbb{Q})(p)$.

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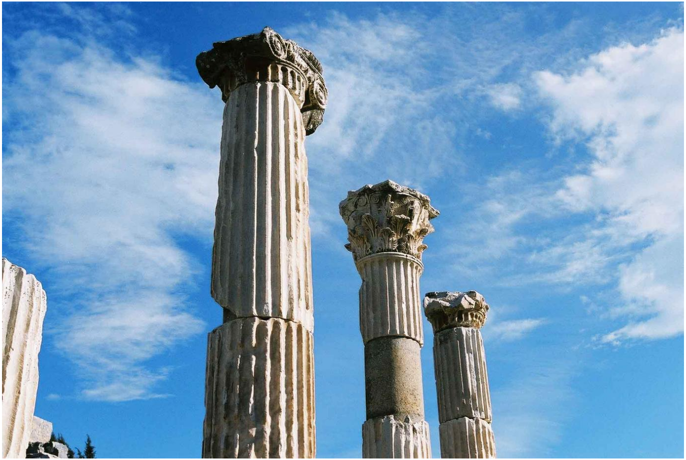
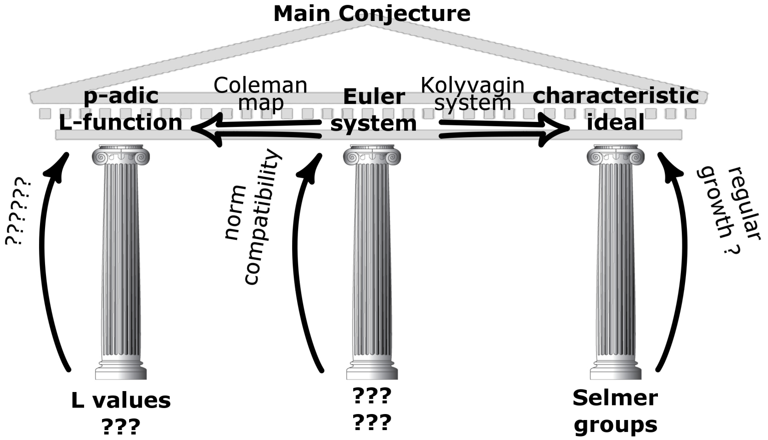
Iwasawa theory for Galois representations

V a finite dimensional vector space with an action by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Suppose V is crystalline.

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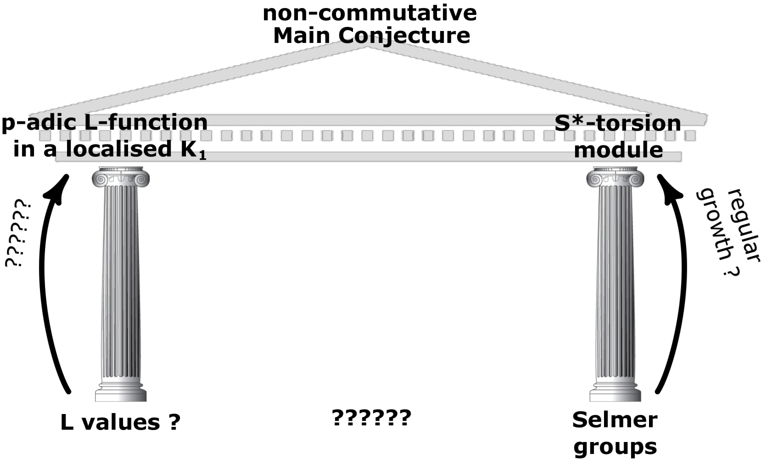
Non-commutative Iwasawa theory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