Real Meromorphic Functions

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Real entire functions

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What can be done for real meromorphic functions?

A classification question

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Classify meromorphic f such that f, f' and f'' have only real zeros and poles.

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Theorem (HSW 1977-83)

If f is entire and f, f', f'' have only real zeros, then f is one of

- Ae^{Bz}
- $A(e^{icz} - e^{id})$
- $A\exp\left(e^{i(cz+d)}\right)$
- $A\exp\left\{K\left(i(cz+d) - e^{i(cz+d)}\right)\right\}$
- $Az^m e^{-az^2+bz}\prod\left(1-\frac{z}{z_n}\right)e^{z/z_n}$

$A, B \in \mathbb{C}$
$c, d, K \in \mathbb{R}$
$a \geq 0, \quad b, z_n \in \mathbb{R}$

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Theorem (HSW)

A meromorphic strictly non-real f with real poles (at least one) such that f, f', f'' have real zeros is either

$$\frac{Ae^{-i(cz+d)}}{\sin(cz+d)} \quad \text{or} \quad \frac{A\exp\left\{-2i(cz+d)-2e^{2i(cz+d)}\right\}}{\sin^2(cz+d)}$$

where A is complex, c, d are real and $Ac \neq 0$.

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$$\alpha z + \lambda \tan(cz + d) + A \qquad \text{with } \alpha, \lambda, c, d, A \text{ real.}$$

- Other examples: $(\tan z + 4)\tan z$ and $\tan^3 z - 9\tan z$.

The derivative of the first example never takes the value α:

$$\frac{d}{dz}\Big(\alpha z + \lambda \tan(cz + d) + A\Big) = \alpha + \lambda c \sec^2(cz + d).$$

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Are there other examples that satisfy the extra condition $f' \neq \alpha$?
No other transcendental examples if $\alpha = 0\ldots$

Theorem (Hellerstein, Shen and Williamson)

If f is real transcendental meromorphic with real zeros and poles (at least one of each), $f' \neq 0$ and f'' has real zeros then

$$f(z) = \lambda \tan(cz+d) + A \qquad \lambda, c, d, A \in \mathbb{R}.$$

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Theorem (Hinkkanen and Rossi)

Suppose f is real transcendental meromorphic with real poles (at least one) and f, f' have real zeros. If f' omits some non-zero value α, then α is real and

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- We will extend the above in a different direction.

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Theorem 1 (N. '08)

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$$f(z) = \alpha z + i\lambda \frac{P(z)e^{icz} - \overline{P(\overline{z})}e^{-icz}}{P(z)e^{icz} + \overline{P(\overline{z})}e^{-icz}} + A \qquad (1)$$

where λ, c, A are real and P is a polynomial.

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- $P(z) \equiv e^{id}$ then (1) becomes $\alpha z - \lambda \tan(cz + d) + A$.
- $P(z) = z + i$, $c = 1$ get $f(z) = \alpha z + \lambda \dfrac{z \sin z + \cos z}{\sin z - z \cos z} + A$.

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$$w'' + \frac{1}{2}S(z)w = 0.$$

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- Find solutions to DE on a domain D. Re-arranging gives

$$f(z) = \alpha z + \frac{kPe^{icz} + lQe^{-icz}}{Pe^{icz} + Qe^{-icz}} \qquad \text{on } D, \tag{2}$$

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- Show that P and Q are polynomials, and so (2) holds on $\mathbb{C}$.
- As f is a real function can now show (2) gives required form.
- Finally, find enough real zeros of f, f'' that there cannot be infinitely many other (non-real) zeros.

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Theorem 2 (N. '08)

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The second case can occur. For example,

$$f(z) = \alpha z + \frac{3 - iz}{z - i}\alpha e^{iz}, \qquad f'(z) = \alpha + \left(\frac{z + i}{z - i}\right)^2 \alpha e^{iz}.$$

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Corollary

For f as above, all but finitely many of the zeros of f'' are real.