The iterated minimum modulus and Eremenko's conjecture

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We study the iteration of a transcendental entire function (tef) $f\colon \mathbb{C} \to \mathbb{C}$

Definition

The *escaping set* $I(f) := \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$.

Eremenko (1989) showed that

- $I(f) \cap J(f) \neq \emptyset$, where $J(f)$ is the Julia set;
- $J(f) = \partial I(f)$;
- all components of $\overline{I(f)}$ are unbounded.

Eremenko's conjecture

All components of $I(f)$ are unbounded.

It is now known that $I(f)$ always has at least one unbounded component.

Eremenko's conjecture is open in general, but known for a wide range of examples:

- for many tefs, including the exponential family, $I(f)$ is a "Cantor bouquet" of uncountably many unbounded curves;
- for many other families $I(f)$ has the structure of a "spider's web".

Definition

A set $I \subset \mathbb{C}$ is a *spider's web* if

- I is connected; and
- there exist bounded, simply connected domains G_n such that

$$G_n \subset G_{n+1}, \qquad \partial G_n \subset I, \qquad \text{and} \qquad \bigcup_{n\in\mathbb{N}} = \mathbb{C}.$$

Note: $I(f)$ a spider's web $\Longrightarrow$ $I(f)$ connected
$\Longrightarrow$ Eremenko's conjecture holds for f.

For example, $I(f)$ is a spider's web if any of the following hold:

- f has a multiply-connected Fatou component;
- f grows not too fast and has "regular growth";
- f grows extremely slowly; namely $\exists k \geq 2$ such that $\log\log M(r) < \frac{\log r}{\log^k r}$ for large r.

We denote the *maximum modulus* and *minimum modulus* of f by

$$M(r) = \max_{|z|=r} |f(z)| \quad \text{and} \quad m(r) = \min_{|z|=r} |f(z)|.$$

Much of the above relies on finding $r > R$ such that $m^n(r) > M^n(R) \to \infty$, which implies that $I(f)$ is a spider's web.

...but there exist functions of order 0 for which there are no such r, R.

Using a new approach, we show $I(f)$ is a spider's web under a condition based on $m^n(r)$ only, without any regularity assumptions.

We focus on the class of real tefs of finite order with only real zeroes.

- f is called *real* if $f(x) \in \mathbb{R}$ when $x \in \mathbb{R}$ (equivalently $f(\overline{z}) = \overline{f(z)}$),
- the *order* of f is $\rho(f) := \limsup_{r\to\infty} \dfrac{\log\log M(r)}{\log r}$.

Theorem 1 (N., Rippon, Stallard)

Let f be a real tef of finite order with only real zeros. If

$$\exists r > 0 \text{ such that } m^n(r) \to \infty \text{ as } n \to \infty, \qquad (\star)$$

then $I(f)$ is a spider's web (so $I(f)$ is connected).

- All tef with order $< \frac{1}{2}$ satisfy $(\star)$.
- $\cos\sqrt{z}$ has order $= \frac{1}{2}$, real zeroes, and does not satisfy $(\star)$.
- $2z\cos\sqrt{z}$ has order $= \frac{1}{2}$, real zeroes, and does satisfy $(\star)$.
- When order > 2 we prove Theorem 1 by showing that $(\star)$ is never satisfied...

Theorem 2 (N., Rippon, Stallard)

Let f be a tef with $2 < \rho(f) < \infty$ and only real zeroes. Then

(a) there exists θ such that $f(re^{i\theta}) \to 0$ as $r \to \infty$; and

(b) 0 is a deficient value of f.

Note that both (a) and (b) imply $m(r) \to 0$ as $r \to \infty$, so $(\star)$ does not hold for such f.

Proof.

(a) Uses an analysis of the Hadamard factorisation of f.

(b) Follows from a result of Edrei, Fuchs and Hellerstein (1961).

□

Conjecture: $(\star)$ fails for all tef of infinite order with only real zeroes.

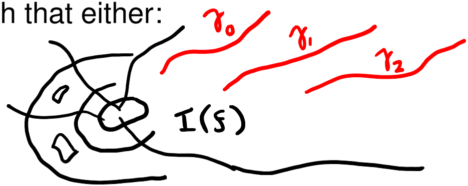
Sketch of proof of Theorem 1

Let f be real tef, $\rho(f) < \infty$, with only real zeroes. Assume $m^n(r) \to \infty$ for some r. Note $\rho(f) \leq 2$ by Theorem 2.

Suppose $I(f)$ is not a spider's web.

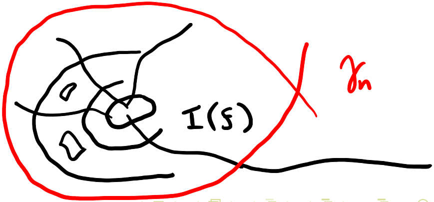
- $\mathbb{C} \setminus I(f)$ has an unbounded component, so take a long curve γ_0 with $\gamma_0 \cap I(f) = \emptyset$.
- Find sequence $\gamma_{n+1} \subset f(\gamma_n)$ such that either:

(I) the γ_n experience repeated radial stretching, escaping to ∞ (so γ_0 meets $I(f)$ — contradiction);



OR

(II) eventually some γ_n winds round 0. But then γ_n meets an unbounded component of $I(f)$, again a contradiction. □



Recall $(\star)$: $\exists r > 0$ such that $m^n(r) \to \infty$.

We've seen that if f is a real tef of finite order with only real zeroes, then $(\star)$ always holds if $\rho(f) < \frac{1}{2}$ and never holds if $\rho(f) > 2$.

Theorem 3 (N., Rippon, Stallard)

For any $\frac{1}{2} \leq \rho \leq 2$, there exist examples of real tefs with only real zeroes and order ρ such that $(\star)$ does, and does not, hold.

Examples constructed as infinite products:

- Using very evenly distributed zeroes one can make $m(r)$ bounded, so $(\star)$ fails. E.g. for $\frac{1}{2} < \rho < 1$

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^{1/\rho}}\right). \qquad \text{(Hardy, 1905)}$$

- Using very unevenly distributed zeroes (big gaps and high multiplicities) can make examples where $(\star)$ holds.