Iterating the minimum modulus

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Joint work with Phil Rippon and Gwyneth Stallard

For any transcendental entire function (tef) $f : \mathbb{C} \to \mathbb{C}$, denote the maximum and minimum modulus by

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)| \quad \text{and} \quad m(r) = m(r, f) = \min_{|z|=r} |f(z)|.$$

- Clearly $m(r) \leq M(r)$ for all $r \geq 0$.
- $M(r)$ strictly increases to ∞ as $r \to \infty$.
- $m(r)$ alternately increases and decreases between values at which $m(r) = 0$.

We denote the iterates of $M(r)$ and $m(r)$ by $M^n(r)$ and $m^n(r)$.
— So, for example, $m^2(r) = m(m(r))$.

The iterated maximum modulus $M^n(r)$ has played a role in complex dynamics for some years. For any tef, if r is large enough then we have

$$M^n(r) \to \infty \text{ as } n \to \infty.$$

This talk surveys the role played by the iterated minimum modulus $m^n(r)$.

After some introductory comments on escaping sets and spiders' webs, the talk has two main parts:

1) Results about entire functions with the property:

$$\text{there exists } r > 0 \text{ such that } m^n(r) \to \infty \text{ as } n \to \infty. \qquad (\star)$$

2) Examples of functions that do, or do not, satisfy this iterated minimum modulus condition $(\star)$.

Fatou, Julia and escaping sets

Let f be a tef and denote its iterates by f^n. The following partition of the complex plane is central to complex dynamics.

Definition

The *Fatou set* of f is

$$F(f) := \{z \in \mathbb{C} : (f^n)_{n\in\mathbb{N}} \text{ is a normal family on some nhd of } z\}.$$

The *Julia set* $J(f) := \mathbb{C} \setminus F(f)$.

In recent decades the escaping set has been studied in detail.

Definition

The *escaping set* $I(f) := \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$.

Eremenko (1989) showed that

- $I(f) \cap J(f) \neq \emptyset$, and
- $J(f) = \partial I(f)$.

Eremenko's (former) conjecture

Eremenko also showed in 1989 that all components of $\overline{I(f)}$ are unbounded.

Eremenko's conjecture

All components of $I(f)$ are unbounded.

Martí-Pete, Rempe and Waterman very recently showed that Eremenko's conjecture does not hold in general — it is possible for $I(f)$ to have a bounded (even singleton) component.

- However, for many families of tefs all components of $I(f)$ are unbounded.
- Moreover, Rippon and Stallard (2005) showed that $I(f)$ always has at least one unbounded component.

Spiders' webs

We will see that for certain families of tefs $I(f)$ has the structure of a "spider's web".

Definition

A set $I \subset \mathbb{C}$ is a *spider's web* if

- I is connected; and
- there exist bounded, simply connected domains G_n such that

$$G_n \subset G_{n+1}, \qquad \partial G_n \subset I, \qquad \text{and} \qquad \bigcup_{n \in \mathbb{N}} G_n = \mathbb{C}.$$

Note: $I(f)$ a spider's web $\Longrightarrow$ $I(f)$ connected
$\Longrightarrow$ Eremenko's conjecture holds for f.

Part 1: Results when $m^n(r) \to \infty$

Our first result concerns tefs for which $m^n(r) \to \infty$ particularly quickly.

Theorem (Rippon, Stallard)

If f is a tef and there exist $r \geq R > 0$ such that

$$m^n(r) \geq M^n(R) \to \infty, \text{ as } n \to \infty,$$

then $I(f)$ is a spider's web (so is connected) and the Fatou set $F(f)$ has no unbounded components.

The hypothesis above is satisfied if any of the following hold:

- f has a multiply-connected Fatou component;
- f grows not too fast and has "regular growth";
- f grows extremely slowly; for example if $\exists k \geq 2$ such that $\log\log M(r) < \frac{\log r}{\log^k r}$ for large r.

A digression on Baker's conjecture

Baker's conjecture (1981)

The Fatou set of a tef f has no unbounded components if the order of f is less than $\frac{1}{2}$, or if f has order $\frac{1}{2}$ minimal type.

Recall that the *order* of f is $\rho(f) := \limsup_{r\to\infty} \dfrac{\log\log M(r)}{\log r}$ and that f is said to have *order* $\frac{1}{2}$ *minimal type* if

$$\rho(f) = \frac{1}{2} \quad \text{and} \quad \lim_{r\to\infty} \frac{\log M(r)}{r^{1/2}} = 0.$$

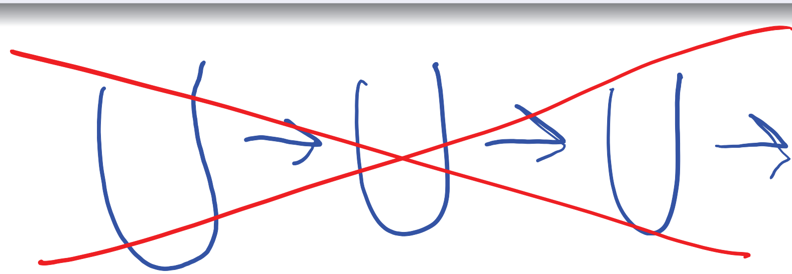
- Baker's conjecture holds for the functions on the previous slide, i.e. satisfying the condition that $\exists r \geq R$ with $m^n(r) \geq M^n(R) \to \infty$.
- However, not all functions of order $< \frac{1}{2}$ satisfy this condition. Not even all functions of order zero!

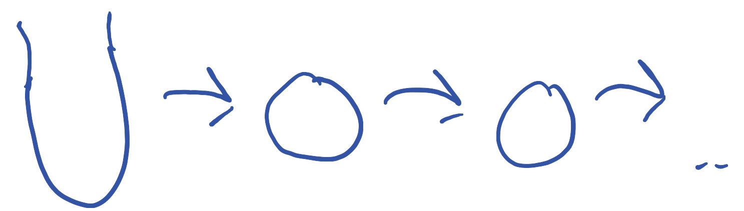
J.-H. Zheng (2000) proved that for functions of order $\leq \frac{1}{2}$ min type, all (pre)periodic components of the Fatou set are bounded. So the remaining case for Baker's conjecture is to rule out unbounded wandering Fatou components.

We have a partial result for *real* entire functions.
Here 'real' means that $f(x) \in \mathbb{R}$ when $x \in \mathbb{R}$, or equivalently $f(\bar{z}) = \overline{f(z)}$.

Theorem (N., Rippon, Stallard)

Let f be a real tef of order less than 1 with only real zeroes.
Then f has no orbits of unbounded wandering Fatou components.





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Then f has no orbits of unbounded wandering Fatou components.

Using Wiman's result that the minimum modulus $m(r)$ is unbounded for functions of order $\leq \frac{1}{2}$ min type, we get:

Corollary

Baker's conjecture holds for real tefs with only real zeroes.

Next we move on from the strong condition $m^n(r) \geq M^n(R)$ to the much weaker condition that

$$\text{there exists } r > 0 \text{ such that } m^n(r) \to \infty \text{ as } n \to \infty. \qquad (\star)$$

Theorem (Osborne, Rippon, Stallard)

Let f be a tef. If $(\star)$ holds, then the set of points with unbounded orbit

$$\{z \in \mathbb{C} : (f^n(z))_{n \in \mathbb{N}} \text{ is unbounded}\}$$

is connected.

Theorem (N., Rippon, Stallard)

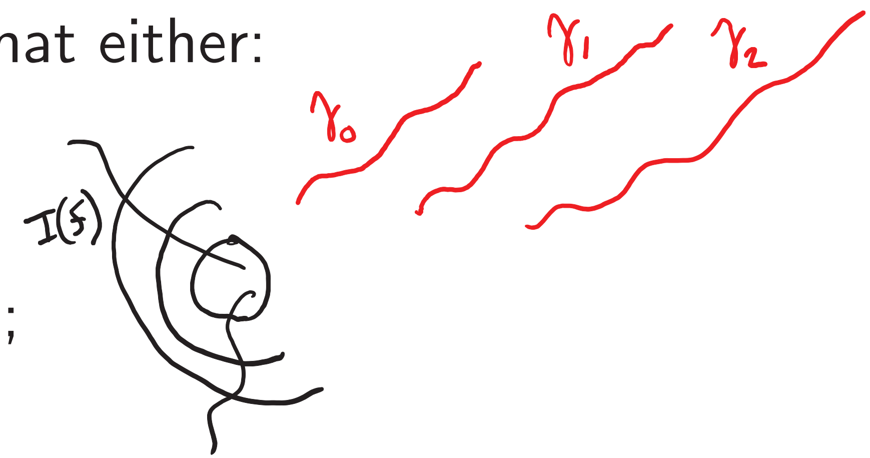
Let f be a real tef of finite order with only real zeros. If $(\star)$ holds, then the escaping set $I(f)$ is a spider's web (so $I(f)$ is connected).

Sketch of proof

Let f be real tef, $\rho(f) < \infty$, with only real zeroes. Assume $m^n(r) \to \infty$ for some r. We can show that $\rho(f) \leq 2$ (more on this later).

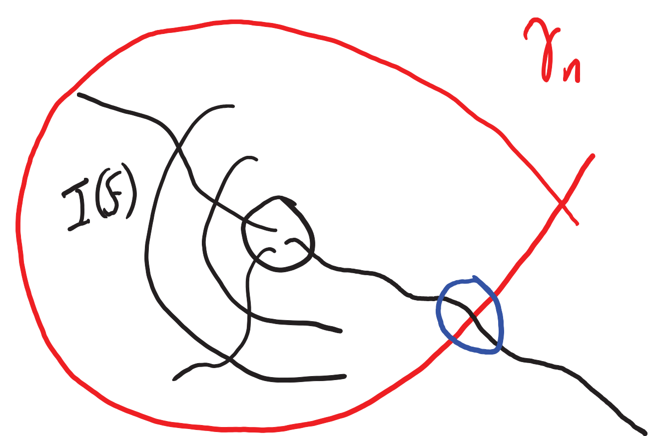
Suppose $I(f)$ is not a spider's web.

- Find a long curve γ_0 that is disjoint from $I(f)$. [Actually some subset]
- Find sequence $\gamma_{n+1} \subset f(\gamma_n)$ such that either:



(I) the γ_n experience repeated radial stretching, escaping to ∞ (so γ_0 meets $I(f)$ — contradiction);

OR

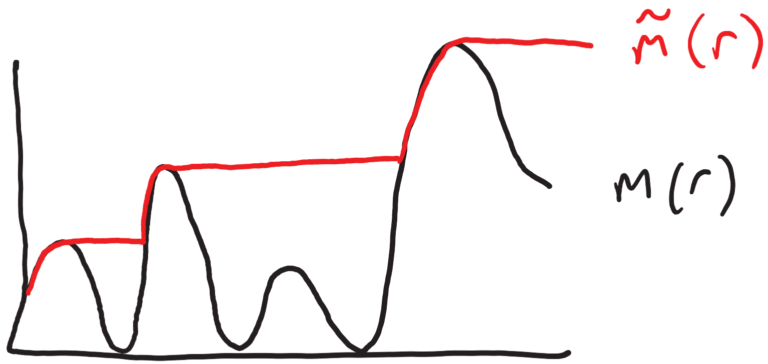


(II) eventually some γ_n winds round 0. But then γ_n meets an unbounded component of $I(f)$, again a contradiction. $\square$

Part 2: For which functions is there r with $m^n(r) \to \infty$?

It is often useful to consider the increasing quantity

$$\tilde{m}(r) := \max_{0 \leq s \leq r} m(s).$$



Part 2: For which functions is there r with $m^n(r) \to \infty$?

It is often useful to consider the increasing quantity

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This leads to equivalent ways to state the $m^n(r) \to \infty$ condition:

Lemma (Osborne, Rippon, Stallard)

Let f be a tef. The following are equivalent:

- There exists $r > 0$ such that $m^n(r) \to \infty$ as $n \to \infty$. $(\star)$
- There exists $R > 0$ such that $\tilde{m}(r) > r$ for all $r \ge R$.
- There exists $r_n \to \infty$ such that $m(r_n) \ge r_{n+1}$.

This lemma often allows one to show that $(\star)$ holds (or does not hold) from function theoretic considerations. For example ...

Theorem (Osborne, Rippon, Stallard)

Let f be a tef. There exists $r > 0$ such that $m^n(r) \to \infty$ if any of the following hold:

(a) The order $\rho(f) < \frac{1}{2}$.

(b) f has a multiply-connected Fatou component.

(c) f has "Hayman gaps" or f has finite order and "Fabry gaps".

Here *Fabry gaps* means that $f(z) = \sum a_k z^{n_k}$ with $n_k/k \to \infty$; while $n_k > k^{1+\varepsilon}$ implies Hayman gaps.

Proof of (a)

If $\rho(f) < \alpha < \frac{1}{2}$, then by the $\cos \pi\rho$ theorem there is $\varepsilon > 0$ such that for all large r there is $s \in (r^\varepsilon, r)$ such that $m(s) > M(s)^{\cos \pi\alpha}$. So

$$\tilde{m}(r) \geq M(s)^{\cos \pi\alpha} \geq M(r^\varepsilon)^{\cos \pi\alpha} > r$$

for all large r (using $\frac{\log M(r)}{\log r} \to \infty$).
Thus, by the previous lemma, there exists r such that $m^n(r) \to \infty$. □

Examples

Osborne, Rippon and Stallard give the following examples of functions which do or do not have the property that

$$\text{there exists } r > 0 \text{ such that } m^n(r) \to \infty \text{ as } n \to \infty. \qquad (\star)$$

- $\cos\sqrt{z}$ has order $\frac{1}{2}$ and does not satisfy $(\star)$ since $m(r) \le 1$.
- $2z\cos\sqrt{z}$ has order $\frac{1}{2}$ and does satisfy $(\star)$.
- Moreover, for $p \in \mathbb{N}$, $\cos z^p$ does not satisfy $(\star)$, but $2z\cos z^p$ does.
- Functions in the Eremenko-Lyubich class $\mathcal{B}$ have $m(r)$ bounded so do not satisfy $(\star)$.
- $2z(1 + e^{-z})$ satisfies $(\star)$.
- $z + b\sin z$ with $b > 2\pi$ satisfies $(\star)$.
- Fatou's function $z + 1 + e^{-z}$ does not satisfy $(\star)$, but $I(f)$ is a spider's web (Evdoridou).

Order $\frac{1}{2}$ minimal type

Recall that:

- Order $< \frac{1}{2}$ implies $\exists r$ such that $m^n(r) \to \infty$. $(\star)$
- Wiman: order $\frac{1}{2}$ minimal type implies $m(r)$ is unbounded.
- Order $\frac{1}{2}$ min type means $\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \frac{1}{2}$ and $\frac{\log M(r)}{r^{1/2}} \to 0$.

So we might ask: is order $\frac{1}{2}$ minimal type sufficient to imply $(\star)$?

Theorem (N., Rippon, Stallard)

Let f be a tef of order at most $\frac{1}{2}$ minimal type. Then $(\star)$ holds if $\exists r_0$ such that, for $r > r_0$

$$\frac{\log M(r)}{r^{1/2}} \leq \frac{1}{4}\frac{\log M(s)}{s^{1/2}},$$

for some $0 < s < r$ which satisfies $M(s) \geq r^2$.

The condition here says roughly that $\frac{\log M(r)}{r^{1/2}} \to 0$ in a regular manner.

Without some extra condition, the answer to the above question is "no"...

Recall $(\star)$: $\exists r > 0$ such that $m^n(r) \to \infty$.

Theorem (N., Rippon, Stallard)

There exist tefs with order $\frac{1}{2}$ minimal type for which $(\star)$ does not hold. These can be chosen to be real functions with only real zeroes.

Construction of examples is via a generalisation (by R. + S.) of a method of Kjellberg. This produces tefs with slow growth and tight control over $m(r)$ by first making a continuous subharmonic function with the required properties.

$\frac{1}{2} \leq$ Order ≤ 2

Recall ($\star$): $\exists r > 0$ such that $m^n(r) \to \infty$.

Theorem (N., Rippon, Stallard)

For any $\frac{1}{2} \leq \rho \leq 2$, there exist examples of real tefs with only real zeroes and order ρ such that ($\star$) does, and does not, hold.

Examples constructed as infinite products:

- Using very evenly distributed zeroes one can make $m(r)$ bounded, so ($\star$) fails. E.g. for $\frac{1}{2} < \rho < 1$

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^{1/\rho}}\right). \qquad \text{(Hardy, 1905)}$$

- Using very unevenly distributed zeroes (big gaps and high multiplicities) can make examples where ($\star$) holds.

Order > 2

Theorem (N., Rippon, Stallard)

Let f be a tef with $2 < \rho(f) < \infty$ and only real zeroes. Then

(a) there exists θ such that $f(re^{i\theta}) \to 0$ as $r \to \infty$; and

(b) 0 is a deficient value of f.

(a) Proof uses an analysis of the Hadamard factorisation of f.

(b) Follows from a result of Edrei, Fuchs and Hellerstein (1961). □

Recall $(\star)$: $\exists r > 0$ such that $m^n(r) \to \infty$.

- Note that either (a) or (b) implies $m(r) \to 0$ as $r \to \infty$, so $(\star)$ does not hold for such f.
- This is used in the proof of the earlier result that for a real tef of finite order with only real zeroes and $(\star)$, $I(f)$ is a spider's web.

Conjecture: $(\star)$ fails for all tef of infinite order with only real zeroes.