Non-real zeroes of real entire functions and their derivatives

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Abstract

A real entire function belongs to the Laguerre-Pólya class LP if it is the limit of a sequence of real polynomials with real zeroes. By building upon results that resolved a long-standing conjecture of Wiman, a number of conditions are established under which a real entire function f must belong to the class LP, or to one of the related classes U^*_{2p}. These conditions typically involve the non-real zeroes of f and its derivatives, or those of the differential polynomial $ff'' - a(f')^2$.

1. *Introduction: the Pólya–Wiman conjectures*

This paper is motivated by a conjecture attributed to Wiman about the zeroes of real entire functions and their derivatives. Here an entire function is said to be real if it takes real values on the real axis. An entire function f belongs to the Laguerre-Pólya class LP if there exists a sequence of real polynomials with only real zeroes that converges locally uniformly to f. Such functions are necessarily real and have only real zeroes unless $f \equiv 0$. It is not difficult to show that LP is closed under differentiation; hence, all derivatives of a transcendental function in LP have only real zeroes. Pólya [**32**] asked whether the converse is true: Must a real entire function belong to LP if $f^{(k)}$ has only real zeroes, for every $k \geq 0$?

The following stronger conjecture due to Wiman [**1**, **2**] dates back to around 1911 and was eventually confirmed in 2002 [**7**]: If f is a real entire function such that ff'' has only real zeroes, then $f \in LP$. Wiman's conjecture therefore implies the striking result that if the zeroes of a transcendental real entire function and its second derivative are real, then the zeroes of all its derivatives are confined to the real axis.

See [**7**] for a history of the proof of Wiman's conjecture. We mention only that important steps were taken by Levin and Ostrovskii [**27**], Sheil-Small [**33**] and Bergweiler, Eremenko and Langley [**7**]. Pólya's conjecture was settled by Hellerstein and Williamson [**15**, **16**].

There are now many theorems related to the Pólya-Wiman conjectures, and the new results presented here are best viewed in this context. Before proceeding, we define a family of classes of real entire functions [**9**, **15**, **16**].

For each integer $p \geq 0$, the class V_{2p} consists of all functions

$$g(z) \exp\left(-az^{2p+2}\right),$$

where $a \geq 0$ and g is a real entire function with real zeroes and genus at most $2p+1$ [**13**, p.29]. The classes U_{2p} are then defined by $U_0 = V_0$ and $U_{2p} = V_{2p} \setminus V_{2p-2}$ for $p \geq 1$. The connection with the Pólya-Wiman conjectures is made clear by the Laguerre-Pólya Theorem that $U_0 = LP$ [**20**, **31**]. We denote by U^*_{2p} the class of real entire functions $f = Pf_0$, where $f_0 \in U_{2p}$ and P is a real polynomial. It follows that every real entire function of finite order with finitely many non-real zeroes belongs to exactly one of the classes U^*_{2p}.

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Theorem 1.1 below was first proved for $f \in U_{2p}$ and $k = 2$ in [**33**]. This result follows a convention that we shall adopt throughout this paper: all counts of zeroes are made with regard to multiplicity unless explicitly stated otherwise. Theorem 1.2 is the corresponding infinite order result, which was proved for $k = 2$ in [**7**], and for $k \geq 3$ in [**22**]. An immediate corollary of these results is that if f is a real entire function and $ff^{(k)}$ has only real zeroes, for some $k \geq 2$, then $f \in LP$. This represents one natural generalisation of Wiman's conjecture.

Theorem 1.1 ([**9**, Corollary 5.2]). *Let f be a real entire function. If $f \in U^*_{2p}$, then $f^{(k)}$ has at least $2p$ non-real zeroes for all $k \geq 2$.*

Theorem 1.2 ([**7**, **22**]). *Let f be a real entire function of infinite order. Then $ff^{(k)}$ has infinitely many non-real zeroes for all $k \geq 2$.*

2. *Statement of results*

In the spirit of the Pólya-Wiman conjectures, the aim of this paper is to seek out conditions under which a real entire function must belong to the class LP or to one of the more general classes U^*_{2p}. These conditions will typically involve the non-real zeroes of the function and its derivatives. The first result below implies that a real entire function f belongs to LP if it has only real zeroes and all the non-real zeroes of f'' are critical points of f.

Theorem 2.1. *Let f be a real entire function with finitely many non-real zeroes. If $f \in U^*_{2p}$, then f'' has at least $2p$ non-real zeroes that are not critical points of f. If instead f is of infinite order, then f'' has infinitely many such zeroes.*

Theorem 2.1 is a mild strengthening of the $k = 2$ cases of Theorems 1.1 and 1.2. Our next result extends these cases in a different direction. It turns out that statements regarding the zeroes of ff'' can sometimes be generalised to ones considering the zeroes of $ff'' - a(f')^2$ for certain values of a. The zeroes of the differential polynomial $ff'' - a(f')^2$ for a general meromorphic f have also been studied in [**4**, **21**, **25**, **26**]. With all this in mind, we remark that if f is entire, then a zero of $ff''/(f')^2 - a$ of multiplicity m is a zero of $ff'' - a(f')^2$ of multiplicity at least m.

Theorem 2.2. *Let $a < 1$ and let f be a real entire function with finitely many non-real zeroes. If $f \in U^*_{2p}$, then $ff''/(f')^2 - a$ has at least $2p$ non-real zeroes. If instead f is of infinite order, then $ff''/(f')^2 - a$ has infinitely many non-real zeroes.*

To see that we cannot take $a \geq 1$ in the above, let $f(z) = \exp(z^{2p})$ for $p \in \mathbb{N}$. Then $f \in U_{2p}$ but $ff''/(f')^2 - a$ has no zeroes if $a = 1$, and only $2p - 2$ non-real zeroes if $a > 1$. However, it follows from [**25**, **26**] that any counterexample to Theorem 2.2 with $a > 1$ must have the form $f = Pe^Q$, where P and Q are polynomials.

The following corollary to Theorem 2.2 is proved in Section 5 by using an argument based on iteration. Note that this time there are no assumptions about the zeroes of the function.

Corollary 2.3. *Let $a \leq \frac{1}{2}$ and let f be a real entire function such that f'/f is of finite lower order. If $ff''/(f')^2 - a$ has only finitely many non-real zeroes, then $f \in U^*_{2p}$ for some p. Moreover, if $ff''/(f')^2 \neq a$ on $\mathbb{C} \setminus \mathbb{R}$, then $f \in LP$.*

Corollary 2.3 is new even for $a = 0$, in which case it shows that a real entire function f must belong to the class LP if f'/f has finite lower order and each non-real zero of ff'' is a critical point of f. It seems reasonable to conjecture that the conclusions of Corollary 2.3 may remain valid even without the assumption that f'/f has finite lower order.

Our next result considers zeroes of higher derivatives.

THEOREM 2.4. *Let $k \geq 2$ and let f be a real entire function such that $f^{(k-1)}/f^{(k-2)}$ is of finite lower order. Suppose that all (respectively, all but finitely many) of the non-real zeroes of $ff^{(k)}$ are also zeroes of $f^{(k-2)}$ and $f^{(k-1)}$. Then $f \in LP$ (respectively, $f \in U_{2p}^*$ for some p).*

The hypothesis that $f^{(k-1)}/f^{(k-2)}$ is of finite lower order is certainly satisfied if either f or f'/f is of finite order. See Lemma 6.4 for a proof of the latter fact.

The results stated above all require that the function under consideration either has only finitely many non-real zeroes or satisfies an order condition. We now seek results that are free of these particular restrictions. Instead, we take integers $M \geq k \geq 2$ and define the following hypotheses for an analytic function f:

(I) all the non-real zeroes of $ff^{(k)}$ are zeroes of f with multiplicity at least k but at most M;

(I′) all but finitely many of the non-real zeroes of $ff^{(k)}$ are zeroes of f with multiplicity at least k but at most M;

(II) $ff'' - a(f')^2$ has no non-real zeroes, for some $a \in \mathbb{C} \setminus \{\frac{1}{2}, 1\}$;

(II′) $ff'' - a(f')^2$ has finitely many non-real zeroes, for some $a \in \mathbb{C} \setminus \{\frac{1}{2}, 1\}$.

Under these hypotheses, the next theorem provides a bound on the Tsuji characteristic for functions defined on a half-plane as developed in [**27**, **35**]. We write $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ for the (open) upper half-plane. We shall say that a function is meromorphic on the closed upper half-plane $\overline{H} \subseteq \mathbb{C}$ to mean that it is meromorphic on some domain containing $\overline{H}$. Let g be meromorphic on $\overline{H}$ and, for $r \geq 1$, let $\mathfrak{n}(r, g)$ denote the number of poles of g, counted with multiplicity, that lie in $\{z : |z - ir/2| \leq r/2,\ |z| \geq 1\}$. The half-plane versions of the integrated counting function and the proximity function are defined as

$$\mathfrak{N}(r, g) = \int_1^r \frac{\mathfrak{n}(t, g)}{t^2}\, dt, \qquad \mathfrak{m}(r, f) = \frac{1}{2\pi} \int_{\sin^{-1}(1/r)}^{\pi - \sin^{-1}(1/r)} \frac{\log^+ |g(r \sin\theta e^{i\theta})|}{r \sin^2\theta}\, d\theta. \tag{2.1}$$

The Tsuji characteristic of g is then the sum $\mathfrak{T}(r, g) = \mathfrak{m}(r, g) + \mathfrak{N}(r, g)$. A useful summary of properties of the Tsuji characteristic may be found in [**7**, p.980], see also [**11**] for more details.

THEOREM 2.5. *If f is analytic on $\overline{H}$ and satisfies either* (I′) *or* (II′) *then, for all $j \geq 0$,*

$$\mathfrak{N}(r, 1/f) = O(\log r) \quad \text{and} \quad \mathfrak{T}\left(r, \frac{f^{(j+1)}}{f^{(j)}}\right) = O(\log r) \quad \text{as} \quad r \to \infty. \tag{2.2}$$

We will apply Theorem 2.5 to obtain the following three results.

THEOREM 2.6. *Let f be a real entire function and take real $a < \frac{1}{2}$ and $M \geq k \geq 2$. Suppose that either*

(i) *all (respectively, all but finitely many) of the non-real zeroes of $ff^{(k-1)}f^{(k)}$ are zeroes of f with multiplicity at least k but at most M; or*

(ii) *$ff'' - a(f')^2$ has no (respectively, finitely many) non-real zeroes and f' has finitely many non-real zeroes.*

Then $f \in LP$ (respectively, $f \in U_{2p}^*$ for some p).

THEOREM 2.7. *Let f be a real entire function.*

(i) *If* (I′) *holds and the zeroes of $f^{(j)}$ have finite exponent of convergence for some $0 \leq j \leq k-1$, then $f \in U_{2p}^*$ for some p. If in addition* (I) *holds, then $f \in LP$.*

(ii) *If* (II′) *holds and the zeroes of f or f' have finite exponent of convergence, then f'/f has finite order. Moreover, if $a < \frac{1}{2}$ then we have $f \in U_{2p}^*$ for some p, and in fact $f \in LP$ if* (II) *also holds.*

The final two results only place a 'finite exponent of convergence' condition on certain *non-real* zeroes. There is no restriction on the frequency of the real zeroes.

THEOREM 2.8. *Let f be an entire function satisfying either* (I′) *or* (II′). *Suppose that the non-real zeroes of $f^{(j)}$ have finite exponent of convergence for some $j \geq 0$. Then $\log\log M(r,f) = O(r \log r)$ as $r \to \infty$.*

The particular estimate for the rate of growth found in Theorem 2.8 has a long history in this area, dating back at least to Levin and Ostrovskii [**27**]. It is through Shen's generalisation [**34**] of one of Levin and Ostrovskii's results that Theorem 2.8 does not require a real function.

Our last theorem extends the theme of Theorem 2.6(i) and Theorem 2.7(i).

THEOREM 2.9. *Let $1 \leq j < k < M < \infty$ and let f be a real entire function such that all (respectively, all but finitely many) of the non-real zeroes of $ff^{(j)}f^{(k)}$ are zeroes of f with multiplicity at least k but at most M. Assume further that these non-real zeroes have finite exponent of convergence. Then $f \in LP$ (respectively, $f \in U_{2p}^*$ for some p).*

Theorems 2.1 and 2.2 are proved in Section 4. A lemma based on iteration theory is introduced in Section 5, and is used to establish Corollary 2.3 and Theorem 2.4. In Section 6, Theorem 2.5 is proved and applied to give Theorems 2.6, 2.7 and 2.8. The proof of Theorem 2.9 is presented in Section 7.

3. *Preliminaries*

We begin with two established lemmas involving the Tsuji characteristic. The first is a version of Hayman's Alternative that goes back essentially to Levin and Ostrovskii [**27**].

LEMMA 3.1. *Let g be meromorphic on $\overline{H}$ and let $c \in \mathbb{C} \setminus \{0\}$. If*

$$\mathfrak{N}(r, 1/g) = O(\log r) \quad \text{and} \quad \mathfrak{N}\left(r, \frac{1}{g'-c}\right) = O(\log r), \qquad r \to \infty,$$

then $\mathfrak{T}(r,g) = O(\log r)$.

The next result will be used to provide a connection between the Nevanlinna and Tsuji proximity functions. We define

$$m_{0\pi}(r,g) = \frac{1}{2\pi}\int_0^{\pi} \log^+ |g(re^{i\theta})|\, d\theta. \tag{3.1}$$

A meromorphic function is said to be real if it maps $\mathbb{R}$ into $\mathbb{R} \cup \{\infty\}$. We note that if g is a real meromorphic function on the plane then $m(r,g) = 2m_{0\pi}(r,g)$.

LEMMA 3.2 ([**27**]). *If g is meromorphic on $\overline{H}$ and $\mathfrak{m}(r,g) = O(\log r)$ as $r \to \infty$, then*

$$\int_R^\infty \frac{m_{0\pi}(r,g)}{r^3}\, dr = O\left(\frac{\log R}{R}\right), \quad R \to \infty.$$

The asymptotic values of a transcendental meromorphic function g are called the *transcendental singularities of* g^{-1} (see [**5**, **30**]). These are further classified as direct or indirect as follows. Suppose that $g(z)$ tends to $\alpha \in \mathbb{C}$ as z goes to infinity along a path γ. For each $\varepsilon > 0$, let $C(\varepsilon)$ denote that component of the set $\{z : |g(z) - \alpha| < \varepsilon\}$ which contains an unbounded subpath of γ. Two different asymptotic paths on which $g \to \alpha$ are considered to determine separate transcendental singularities if and only if the corresponding components $C(\varepsilon)$ are distinct for some $\varepsilon > 0$. The path γ determines an *indirect* transcendental singularity over α if $C(\varepsilon)$ contains infinitely many α-points of g for every $\varepsilon > 0$. Otherwise, the singularity is called *direct* and $C(\varepsilon)$, for all sufficiently small ε, contains no α-points. Transcendental singularities over ∞ are defined and classified by considering $1/g$. A transcendental singularity will be referred to as "lying in a domain D" if $C(\varepsilon) \subseteq D$ for small ε.

In subsequent sections we shall often want to limit the number of singularities of an inverse function found in the upper half-plane. The following three lemmas will be used several times for this purpose.

LEMMA 3.3 (Denjoy-Carleman-Ahlfors Theorem [**30**, §XI.4]). *A meromorphic function of finite lower order has finitely many direct transcendental singularities.*

LEMMA 3.4 ([**24**]). *Let g be a meromorphic function such that $\mathfrak{T}(r,g) = O(\log r)$ as $r \to \infty$. Then there is at most one direct singularity of g^{-1} lying in H.*

LEMMA 3.5 ([**5**, **18**]). *Let g be a meromorphic function of finite lower order. Then any indirect transcendental singularity of g^{-1} must be a limit point of critical values. In particular, if g has finitely many critical values, then g^{-1} has no indirect transcendental singularities.*

Lemma 3.5 is Hinchliffe's extension of the Bergweiler-Eremenko Theorem to include functions of finite lower order.

Nearly half a century after the Pólya-Wiman conjectures were posed, the first significant progress was made by Levin and Ostrovskii [**27**]. They wrote the logarithmic derivative as the product of two functions, one having few poles and one mapping the upper half-plane into itself. Variations of this technique are central to the proofs of Theorems 1.1 and 1.2.

LEMMA 3.6 ([**6**, **7**, **24**]). *Let f be a real entire function with finitely many non-real zeroes. Then the logarithmic derivative has a factorisation*

$$L = \frac{f'}{f} = \phi\psi \tag{3.2}$$

in which ϕ and ψ are real meromorphic functions satisfying the following:

(i) *either $\psi \equiv 1$ or $\psi(H) \subseteq H$;*
(ii) *ψ has a simple pole at each real zero of f, and no other poles;*
(iii) *ϕ has finitely many poles, none of them real;*
(iv) *on each component of $\mathbb{R} \setminus f^{-1}(\{0\})$ the number of zeroes of ϕ is either infinite or even;*

(v) *if* $f \in U_{2p}^*$*, then* ϕ *is a rational function, and if in addition* f *has at least one real zero, then the degree at infinity of* ϕ *is even and satisfies*

$$\deg_\infty(\phi) = \lim_{z\to\infty} \frac{\log|\phi(z)|}{\log|z|} \geq 2p; \tag{3.3}$$

(vi) *if* f *has infinite order, then* ϕ *is transcendental.*

Parts (i)–(v) are proved in [**24**, Lemma 4.2], while part (vi) is [**7**, Lemma 5.1].

The next result is the Carathéodory inequality [**28**, Ch. I.6, Theorem 8′], which is essentially the Schwarz lemma on a half-plane. It shows that away from the real axis ψ is neither too large nor too small, so that in (3.2) the growth of f'/f is dominated by that of ϕ.

Lemma 3.7 ([**28**]). *Let* $\psi : H \to H$ *be analytic. Then*

$$\frac{|\psi(i)|\sin\theta}{5r} < |\psi(re^{i\theta})| < \frac{5r|\psi(i)|}{\sin\theta} \qquad \text{for} \qquad r \geq 1, \quad \theta \in (0,\pi).$$

Lemma 3.8 ([**36**]). *Let* u *be a non-constant continuous subharmonic function on the plane. For* $r > 0$*, let* $\theta^*(r)$ *be the angular measure of that subset of* $S(0,r)$ *on which* $u(z) > 0$*, except that* $\theta^*(r) = \infty$ *if* $u(z) > 0$ *on the whole circle* $S(0,r)$*. Then, for* $r > 0$*,*

$$B(r,u) = \max\{u(z) : |z| = r\} \leq \frac{3}{2\pi}\int_0^{2\pi} \max\{u(2re^{it}), 0\}\, dt$$

and, if $r \leq R/4$ *and* r *is sufficiently large,*

$$B(r,u) \leq 9\sqrt{2}B(R,u)\exp\left(-\pi\int_{2r}^{R/2} \frac{ds}{s\theta^*(s)}\right).$$

4. *Proof of Theorems 2.1 and 2.2*

Theorems 2.1 and 2.2 are proved by making a number of small alterations to the proofs of the $k = 2$ cases of Theorems 1.1 and 1.2. The main difference is that we shall consider a 'relaxed' Newton function $z - hf/f'$, where the constant h is no longer always taken to be 1.

We shall first prove both theorems in the infinite order case, as these results can quickly be deduced from a theorem of Bergweiler, Eremenko and Langley [**7**]. We then tackle the remaining finite order case, where we base our arguments on existing proofs, but cannot so easily quote suitable results from the literature. This section is closely based on a set of notes by Jim Langley that give a unified presentation of the proof of Wiman's conjecture (see also [**24**, §13]).

4.1. *Infinite order case*

The following is the theorem of Bergweiler, Eremenko and Langley mentioned above.

Lemma 4.1 ([**7**]). *Let* $\tilde{L}$ *be a real meromorphic function on the plane such that all but finitely many poles of* $\tilde{L}$ *are real and simple and have positive residues. Suppose that* $\tilde{L} = \tilde{\phi}\psi$*, where* $\tilde{\phi}$ *and* ψ *are real meromorphic functions such that: either* $\psi \equiv 1$ *or* $\psi(H) \subseteq H$*; every pole of* ψ *is real and simple and is a simple pole of* $\tilde{L}$*; and* $\tilde{\phi}$ *is transcendental with finitely many poles. Then* $\tilde{L} + \tilde{L}'/\tilde{L}$ *has infinitely many non-real zeroes.*

Let f be a real entire function of infinite order with only finitely many non-real zeroes. By Lemma 3.6, we have the Levin-Ostrovskii factorisation $L = f'/f = \phi\psi$. For $a < 1$, let

$$\tilde{\phi} = (1-a)\phi \quad \text{and} \quad \tilde{L} = (1-a)L = \tilde{\phi}\psi.$$

Then $\tilde{L}$, $\tilde{\phi}$ and ψ satisfy the hypothesis of Lemma 4.1 by Lemma 3.6(i)–(iii) and (vi). Therefore, Lemma 4.1 gives that

$$\tilde{M} = \tilde{L} + \frac{\tilde{L}'}{\tilde{L}} = (1-a)\frac{f'}{f} + \frac{f}{f'}\left(\frac{ff'' - (f')^2}{f^2}\right) = \frac{f'}{f}\left(\frac{ff''}{(f')^2} - a\right)$$

has infinitely many non-real zeroes. Since $\tilde{M}$ does not vanish at a zero of L, this establishes the infinite order case of Theorem 2.2. Setting $a = 0$ gives the infinite order case of Theorem 2.1.

4.2. *Finite order case*

We assume that $f \in U^*_{2p}$ for some $p \geq 1$, and that $ff''/(f')^2 - a$ has finitely many non-real zeroes, where $a < 1$. To prove Theorem 2.1, make these assumptions with $a = 0$. Let $L = f'/f$.

LEMMA 4.2. *The Tsuji characteristic of the logarithmic derivative L satisfies*

$$\mathfrak{T}(r, L) = O(\log r), \qquad r \to \infty. \tag{4.1}$$

Proof. Write $g = 1/L$; then g has finitely many zeroes in H, and $g' = 1 - ff''/(f')^2$ takes the value $1 - a$ only finitely often in H. The result is now obtained using Lemma 3.1. □

We make the following definitions.

$$h = \frac{1}{1-a} > 0,$$

$$G(z) = z - h\frac{f(z)}{f'(z)} = z - \frac{h}{L(z)}, \qquad G' = h\left(\frac{ff''}{(f')^2} - a\right), \tag{4.2}$$

$$W = \{z \in H : G(z) \in H\}, \qquad Y = \{z \in H : L(z) \in H\}. \tag{4.3}$$

Observe that $Y \subseteq W$, since h is positive. For $h = 1$, this key observation is due to Sheil-Small [**33**], who was the first to consider these sets. He confirmed Wiman's conjecture for functions of finite order by adopting a geometric approach, and studying how the logarithmic derivative f'/f and the Newton function $z - f/f'$ behave as mappings of the upper half-plane.

We shall modify Sheil-Small's approach by using a relaxed Newton function $G(z)$ as defined in (4.2). These functions also arise when using the (relaxed) Newton method [**3**, §6] to find the zeroes of f by iterating G (in this context, usually $|h-1| < 1$). Theorems 2.1 and 2.2 will be proved through a detailed study of how G maps components of W into H.

LEMMA 4.3. *The closure of Y contains no real zeroes of f.*

Proof. This is from [**33**, p.181]. If x is a real zero of f, then it is a simple pole of L with positive residue. Then since L is univalent near x and real on the real axis, we see that $\operatorname{Im} L(z) < 0$ for points in H near x. Thus x does not lie in the closure of Y. □

LEMMA 4.4 (cf [**23**, Lemma 4], [**24**, §5]). *The function G has no asymptotic values in $\mathbb{C} \setminus \mathbb{R}$, while the function L has only finitely many.*

Proof. Suppose that $\alpha \in \mathbb{C} \setminus \mathbb{R}$ is an asymptotic value of G. Since G has finitely many non-real critical values by (4.2), the Bergweiler-Eremenko Theorem (Lemma 3.5) shows that α must be a direct transcendental singularity of G^{-1}. Therefore, there exist $\varepsilon \in (0, 1)$ and a component D of the set $\{z \in \mathbb{C} : |G(z) - \alpha| < \varepsilon\}$ such that $G(z) \neq \alpha$ on D. Since G is real meromorphic, we may assume that $D \subseteq H$. We define a continuous subharmonic function on the plane by

$$v(z) = \begin{cases} \log \dfrac{\varepsilon}{|G(z) - \alpha|}, & z \in D \\ 0, & z \in \mathbb{C} \setminus D. \end{cases}$$

Lemma 3.8 and (3.1) give that

$$B(r/2, v) \leq \frac{3}{2\pi} \int_0^{\pi} \log^+ \frac{\varepsilon}{|G(re^{it}) - \alpha|} \, dt \leq 3m_{0\pi}\left(r, \frac{1}{G - \alpha}\right).$$

By (4.1) and (4.2), we have $\mathfrak{T}(r, 1/(G - \alpha)) = O(\log r)$ as $r \to \infty$. Using this and Lemma 3.2, together with the above, now leads to

$$\int_1^{\infty} \frac{B(r/2, v)}{r^3} \, dr \leq \int_1^{\infty} \frac{3m_{0\pi}(r, 1/(G - \alpha))}{r^3} \, dr < \infty. \tag{4.4}$$

We now let δ be small and positive, and claim that

$$G(z) \to \infty \quad \text{as } z \to \infty, \quad \delta < \arg z < \pi - \delta. \tag{4.5}$$

Let $L = f'/f = \phi\psi$ be the Levin-Ostrovskii factorisation described in Lemma 3.6. If f has at least one real zero, then $\deg_{\infty}(\phi) \geq 2$ by (3.3), and then (4.5) follows from Lemma 3.7 and (4.2). Otherwise, f has no real zeroes, and so there exist real polynomials P and Q such that

$$f = Pe^Q, \quad L = P'/P + Q', \quad \deg Q \geq 2p, \tag{4.6}$$

using the fact that $f \in U_{2p}^*$. Hence L is a rational function with a pole at infinity, and again (4.5) follows from (4.2).

By (4.5), the angular measure of $D \cap S(0, r)$ is at most 2δ for all large r. Thus a standard application of Lemma 3.8 gives that $B(R, v) \geq cR^{\pi/2\delta}$, for some positive constant c and all sufficiently large R. As δ is arbitrarily small, this contradicts (4.4), showing that G cannot have an asymptotic value in $\mathbb{C} \setminus \mathbb{R}$.

Suppose now that L has infinitely many non-real asymptotic values. Since L has finitely many non-real poles, it follows from the Phragmén-Lindelöf principle [**36**, p.308] that L^{-1} must have at least two direct transcendental singularities over ∞ lying in H. By (4.1), this stands in contradiction to Lemma 3.4. □

Therefore, G has no asymptotic values in H by the previous lemma, and finitely many critical values in H by (4.2). We use these facts to obtain the next result.

Lemma 4.5. *For each component A of W there is a positive integer k_A such that G maps A onto H with valency k_A; that is, each value $w \in H$ is taken k_A times in A. Furthermore, G' has at least $k_A - 1$ zeroes in A.*

Lemma 4.5 is proved by the following standard argument (see [**7**, p.987–988] or [**22**, §11]). Let $\gamma \subseteq \overline{H}$ be a bounded simple curve such that $H^* = H \setminus \gamma$ is simply-connected and contains no singular values of G^{-1}. Then each component of $G^{-1}(H^*)$ is mapped univalently onto H^* by G, and G maps every component of W onto H with finite valency. The final assertion is proved by an application of the Riemann-Hurwitz formula.

We introduce some more notation before stating our next lemma. Denote by $2q$ the number of distinct non-real zeroes of f and define

$$D(\lambda) = \{z \in H : |z| < \lambda\}, \qquad E(\Lambda) = \{z \in H : |z| > \Lambda\}.$$

LEMMA 4.6. *For sufficiently small positive λ, and sufficiently large positive Λ, there are at least $p+q$ pairs of bounded components $K_j \subseteq V_j \subseteq H$ such that the following conditions are satisfied:*
(i) *K_j is a component of the set $L^{-1}(D(\lambda))$, mapped univalently onto $D(\lambda)$ by L;*
(ii) *V_j is a component of the set $G^{-1}(E(\Lambda))$, mapped univalently onto $E(\Lambda)$ by G;*
(iii) *the V_j are pairwise disjoint;*
(iv) *$\partial K_j \cap \partial V_j$ contains one zero of L.*

Proof. Let Z be a finite set of zeroes of L and let λ and $1/\Lambda$ be small. The arguments of [**22**, Lemma 14.1] and [**24**, Lemma 8.1] contain an elementary analysis of the behaviour of L near its zeroes. This shows that each $\zeta \in Z$ gives rise to pairs $\{K_j, V_j\}$ as in the statement of the lemma, with $\zeta \in \partial K_j \cap \partial V_j$, as follows:

1. If $\zeta \in Z \cap H$ is a zero of L of multiplicity m, then there exist m such pairs $\{K_j, V_j\}$.
2. If $\zeta \in Z \cap \mathbb{R}$ is a zero of even multiplicity m, then there exist $m/2$ such pairs. In this case, the sign of $L(x)$ does not change as real x passes through ζ from left to right.
3. Now suppose that $\zeta \in Z \cap \mathbb{R}$ is a zero of L of odd multiplicity m. If $L^{(m)}(\zeta) > 0$, then there exist $(m+1)/2$ pairs $\{K_j, V_j\}$ and $L(x)$ has a positive sign change at ζ; that is, $L(x)$ changes from negative to positive as x passes through ζ from left to right. If instead $L^{(m)}(\zeta) < 0$, then ζ gives rise to $(m-1)/2$ pairs $\{K_j, V_j\}$ and $L(x)$ has a negative sign change at ζ.

Provided that λ and $1/\Lambda$ are chosen small enough, the components arising from distinct zeroes are disjoint. It remains to show that L has sufficiently many zeroes that we can find at least $p+q$ components K_j. The required argument is exactly the final two paragraphs of the proof of [**24**, Lemma 8.1], using again the factorisation $L = \phi\psi$ from Lemma 3.6. □

We are now ready to complete the proof as in [**24**, p.134-135]. Choose $\theta \in (\pi/4, 3\pi/4)$ such that the ray $\gamma(s) = se^{i\theta}$, $s \in (0, \infty)$, contains no singular values of L^{-1}. This is possible because L has countably many critical values and, by Lemma 4.4, finitely many asymptotic values in H. For each K_j, choose $z_j \in K_j$ with $L(z_j) \in \gamma$, and continue L^{-1} along γ in the direction of infinity. Let Γ_j be the image of this continuation starting at z_j. Then Γ_j is a path in Y on which $L(z) \to \infty$, where Y is defined by (4.3). Hence, Γ_j tends either to infinity or to a pole of L, which must be a zero of f in H by Lemma 4.3. Since $K_j \subseteq Y \subseteq W$, each K_j lies in some component A of W. A component A_ν of W will be called type (α) if there exists $K_j \subseteq A_\nu$ such that Γ_j tends to infinity, and type (β) otherwise.

LEMMA 4.7. *Let n_ν denote the number of K_j contained in a component A_ν of W.*
If A_ν is type (α), then n_ν is at most the number of zeroes of G' in A_ν.
If A_ν is type (β), then n_ν is at most the number of distinct zeroes of f in A_ν.

Proof. First suppose that A_ν is type (α). By Lemma 4.5, it will suffice to show that $n_\nu \leq k_{A_\nu} - 1$. But the fact that the valency k_{A_ν} of G on A_ν exceeds the number n_ν is made clear by the following observation: each of the n_ν sets $K_j \subseteq A_\nu$ corresponds to a bounded component $V_j \subseteq A_\nu$ which is mapped onto $E(\Lambda)$ by G, while we also have a path tending to infinity in A_ν on which $L(z) \to \infty$ and consequently $G(z) \to \infty$, by (4.2).

Now suppose instead that A_ν is type (β). For each K_j contained in A_ν, the path Γ_j must tend to a zero w_j of f in H. Since L has a simple pole at w_j, the w_j corresponding to different K_j must be distinct. Moreover, using (4.2) gives that $G(w_j) = w_j \in H$, so that $w_j \in A_\nu$. □

This completes the proof of Theorem 2.2, since Lemma 4.6 gives $p+q$ components K_j, but by (4.2) and Lemma 4.7, the number of K_j does not exceed q plus the number of zeroes of $ff''/(f')^2 - a$ in H.

To prove Theorem 2.1, we put $a = 0$ and note that G' does not vanish at any zero of f' by (4.2). Hence, using (4.2) again, the number of zeroes of G' in A_ν is at most the number of distinct zeroes of f in A_ν plus the number of zeroes of f'' in A_ν that are not zeroes of f'. Lemma 4.6 still provides $p+q$ components K_j, but now Lemma 4.7 shows that this cannot exceed q plus the number of zeroes of f'' in H that are not critical points of f.

5. *An iteration argument*

A well-known fact from complex dynamics is that, as a meromorphic function is iterated, each attracting fixed point draws in a singularity of the inverse function. We shall use this idea to establish a useful lemma. See [**19**] for an earlier instance of the application of this sort of iteration theory argument to the present context. For a rational function F, we denote by $\mathrm{sing}(F^{-1})$ the set of critical values of F, including ∞ if F has any multiple poles. For a transcendental function, $\mathrm{sing}(F^{-1})$ consists of these critical values together with any finite asymptotic values of F. We now define the sets

$$\mathcal{A}(F) = \{z \in \mathbb{C} \setminus \mathbb{R} : F(z) = z \text{ and either } 0 < |F'(z)| < 1 \text{ or } F'(z) = -1\} \tag{5.1}$$

and

$$\mathcal{C}(F) = \{z \in \mathbb{C} \setminus \mathbb{R} : z \in \mathrm{sing}(F^{-1}),\ |F(z) - z| + |F'(z)| > 0\}, \tag{5.2}$$

so that $\mathcal{C}(F)$ contains the non-real singularities of the inverse function F^{-1} that are not superattracting fixed points of F.

Lemma 5.1. *Let F be a real meromorphic function on the plane. If $\mathcal{C}(F)$ is finite, then so is $\mathcal{A}(F)$ and $|\mathcal{A}(F)| \le |\mathcal{C}(F)|$.*

Proof. Let $z_j \in \mathcal{A}(F)$. We suppose first that $|F'(z_j)| < 1$. It then follows that z_j lies in a component C_j of the Fatou set of F (see [**3**, **29**]), and that

$$F^n(z) \to z_j \quad \text{as} \quad n \to \infty, \qquad z \in C_j, \tag{5.3}$$

where F^n denotes the nth iterate of F. If we suppose instead that $F'(z_j) = -1$, then there must exist at least two components of the Fatou set on which $F^n(z) \to z_j$ and which include z_j in their boundary [**29**, §10]. These components are Leau domains, and in this case we let C_j be the union of all these Leau domains, so that we again have (5.3).

It follows from (5.3) that distinct points $z_j \in \mathcal{A}(F)$ give rise to disjoint subsets C_j of the Fatou set. Since F is real, we see also that no C_j can meet the real axis, and that $\infty \notin C_j$.

It is well known [**3**, §4.3] that each set C_j must contain a point of $\mathrm{sing}(F^{-1})$, say w_j. By the previous paragraph, $w_j \in \mathbb{C} \setminus \mathbb{R}$. If w_j is a fixed point of F, then $w_j = z_j$ by (5.3), in which case $|F'(w_j)| = |F'(z_j)| > 0$ since $z_j \in \mathcal{A}(F)$. Hence, $w_j \in \mathcal{C}(F)$ and the result follows. □

5.1. *Proof of Corollary 2.3*

As in the statement of the corollary, we take $a \leq \frac{1}{2}$ and let f be a real entire function such that f'/f is of finite lower order and $ff''/(f')^2 - a$ has only finitely many non-real zeroes.

We aim to show that f has only finitely many non-real zeroes. Let G be defined by (4.2), where $h = (1-a)^{-1}$ and so $0 < h \leq 2$. By (4.2) and our hypotheses on f, we see that G has finite lower order and G' has finitely many non-real zeroes. Lemmas 3.3 and 3.5 now show that G^{-1} has finitely many direct, and no indirect, transcendental singularities over $\mathbb{C} \setminus \mathbb{R}$. Thus the set $\mathcal{C}(G)$ is finite, and Lemma 5.1 implies that $\mathcal{A}(G)$ is also finite.

If $\zeta \in \mathbb{C} \setminus \mathbb{R}$ is a zero of f of multiplicity m, then $G(\zeta) = \zeta$ and

$$G'(\zeta) = 1 - h\left(\frac{f}{f'}\right)'(\zeta) = 1 - \frac{h}{m} \in [-1, 1).$$

Hence, assuming that ζ is not one of the finitely many non-real zeroes of G', we have that $\zeta \in \mathcal{A}(G)$. We therefore deduce that f has a finite number of non-real zeroes. Theorem 2.2 now gives that $f \in U_{2p}^*$ for some p.

Next suppose that $ff''/(f')^2 \neq a$ on H. Then by (4.2), the zeroes of G' are all real. Since $f \in U_{2p}^*$, Lemma 4.4 applies and shows that G has no asymptotic values in $\mathbb{C} \setminus \mathbb{R}$. Thus $\mathcal{C}(G)$ is empty. Therefore, Lemma 5.1 shows that $\mathcal{A}(G)$ is also empty, and so f cannot have any zeroes $\zeta \in \mathbb{C} \setminus \mathbb{R}$. Hence, f must belong to the class LP by Theorem 2.2 (recall that $U_0 = LP$).

5.2. *Proof of Theorem 2.4*

Let $k \geq 2$ and let f be a real entire function such that $f^{(k-1)}/f^{(k-2)}$ is of finite lower order (we exclude the case where $f^{(k-2)} \equiv 0$). Assume that all but finitely many of the non-real zeroes of $ff^{(k)}$ are also zeroes of $f^{(k-2)}$ and $f^{(k-1)}$. Write

$$F_k(z) = z - \frac{f^{(k-2)}(z)}{f^{(k-1)}(z)}, \qquad F_k' = -\frac{f^{(k-2)}f^{(k)}}{(f^{(k-1)})^2}. \tag{5.4}$$

Then F_k is the Newton function of $f^{(k-2)}$ and is, in particular, a real meromorphic function of finite lower order.

Lemma 5.2. *If ζ is a non-real zero of $f^{(k-2)}$ of multiplicity $m \geq 2$, then $\zeta \in \mathcal{A}(F_k)$, where $\mathcal{A}(F_k)$ is defined by* (5.1) *and* (5.4).

Proof. Observe that $F_k(\zeta) = \zeta$, and calculate that $F_k'(\zeta) = 1 - 1/m$. Hence, $\frac{1}{2} \leq F_k'(\zeta) < 1$ and thus $\zeta \in \mathcal{A}(F_k)$. □

By Lemma 5.2, all but finitely many of the non-real zeroes of $ff^{(k)}$ are members of $\mathcal{A}(F_k)$, because of our assumption about these zeroes.

Our next task is to prove that $\mathcal{C}(F_k)$ is finite. The main observation here is that by (5.4), all but finitely many of the non-real critical points of F_k are also fixed points of F_k. Using (5.2), this immediately implies that $\mathcal{C}(F_k)$ contains only a finite number of critical values of F_k. A second consequence of our observation is that the set of critical values of F_k can have no limit points in $\mathbb{C} \setminus \mathbb{R}$. Therefore, by Lemma 3.5 there are no indirect transcendental singularities of F_k^{-1} lying in $\mathbb{C} \setminus \mathbb{R}$. Hence, using Lemma 3.3 we see that F_k has only finitely many non-real asymptotic values, and so $\mathcal{C}(F_k)$ is indeed finite.

An application of Lemma 5.1 now shows that $ff^{(k)}$ has finitely many non-real zeroes. Theorem 1.2 then implies that f has finite order, and hence $f \in U_{2p}^*$ for some p.

We prove next that $f \in LP$ if all the non-real zeroes of $ff^{(k)}$ are zeroes of $f^{(k-2)}$ and $f^{(k-1)}$. For such a function f, Lemma 5.2 shows that all the non-real zeroes of $ff^{(k)}$ lie in

$\mathcal{A}(F_k)$. Therefore, by Theorem 1.1 and Lemma 5.1, it will suffice to show that $\mathcal{C}(F_k) = \emptyset$ in this case. By (5.4), a non-real zero of F_k' is now necessarily a zero of $f^{(k-2)}$, and so a fixed point of F_k. Using (5.2), it follows that no critical values of F_k belong to $\mathcal{C}(F_k)$, and it just remains to show that F_k has no non-real asymptotic values. Since the class U_{2p}^* is closed under differentiation [**9**, Corollary 2.12], we have that $f^{(k-2)} \in U_{2p}^*$. Therefore all the statements made when proving Theorems 2.1 and 2.2 in Section 4.2 remain valid with $f^{(k-2)}$ in place of f and $a = 0$. In particular, if we replace f with $f^{(k-2)}$ and set $a = 0$, then the function G defined in (4.2) becomes F_k. The result we require is then provided by Lemma 4.4.

6. *Theorem 2.5 and applications*

In this section, we will establish Theorem 2.5 and apply it to prove Theorems 2.6, 2.7 and 2.8.

6.1. *Proof of Theorem 2.5*

We shall first obtain a normal families result for functions satisfying (I′) or (II′). This leads to a lower bound for the distance between the distinct zeroes of such functions, from which a careful counting argument gives the first estimate of (2.2). The half-plane versions of some standard value distribution results then complete the proof of Theorem 2.5.

LEMMA 6.1. *Let $k \geq 2$, let $a \in \mathbb{C} \setminus \{\frac{1}{2}, 1\}$ and let $\mathcal{G}$ be a family of functions analytic on a domain Ω. Suppose that for each $g \in \mathcal{G}$, either*
(i) *every zero of $gg^{(k)}$ in Ω is a zero of g with multiplicity at least k; or*
(ii) *$gg'' - a(g')^2$ has no zeroes in Ω.*
Then $\mathcal{F} = \{g/g' : g \in \mathcal{G}\}$ is a normal family on Ω.

REMARK. In fact, Lemma 6.1 holds for a family of meromorphic functions provided that every member satisfies condition (ii) and $\frac{1}{a-1} \notin \mathbb{N}$ as well as $a \neq \frac{1}{2}, 1$. The proof needs only minor modification, but we will not need this result.

Proof of Lemma 6.1. This is an application of a theorem of Bergweiler and Langley from [**8**], where the differential operators Ψ_k are defined by

$$\Psi_1(y) = y, \qquad \Psi_{k+1}(y) = y\Psi_k(y) + \frac{d}{dz}(\Psi_k(y)).$$

We repeat the observation made in [**8**] that an easy proof by induction yields $\Psi_k(g'/g) = g^{(k)}/g$.

We may assume that either every $g \in \mathcal{G}$ satisfies (i) or that every $g \in \mathcal{G}$ satisfies (ii). Suppose first that each $g \in \mathcal{G}$ satisfies condition (i) and let $G = g'/g$. Then we see that $\Psi_k(G) = g^{(k)}/g$ does not vanish in Ω. Moreover, the poles of G are simple and have integer residues not less than k. Therefore, the family $\mathcal{F}_0 = \{g'/g : g \in \mathcal{G}\}$ satisfies the hypotheses of [**8**, Theorem 1.3], and hence both $\mathcal{F}_0$ and $\mathcal{F}$ are normal on Ω.

Next, suppose instead that each $g \in \mathcal{G}$ satisfies condition (ii). We may assume that $a \neq 0$, otherwise every $g \in \mathcal{G}$ satisfies (i) with $k = 2$. This time we set $G = (1-a)g'/g$. We see that

$$\Psi_2(G) = G^2 + G' = (1-a)\left((1-a)\left(\frac{g'}{g}\right)^2 + \frac{gg'' - (g')^2}{g^2}\right) = \frac{1-a}{g^2}\,(gg'' - a(g')^2),$$

and so $\Psi_2(G)$ has no zeroes in Ω. Condition (ii) implies that g has only simple zeroes, so that G has only simple poles, each with residue $1 - a$. Since $1 - a$ is neither zero nor one, and $2(1 - a) \neq 1$, we find that the family $\mathcal{F}_1 = \{(1-a)g'/g : g \in \mathcal{G}\}$ satisfies the hypotheses of [**8**, Theorem 1.3] with $k = 2$. Therefore, $\mathcal{F}_1$ is normal on Ω and the result follows. □

The next lemma is essentially contained in [**10**, Lemma 2.1]. We use the notation

$$D(w,r)=\{z\in\mathbb{C}:|z-w|<r\},\qquad E(R)=\{z\in H:|z|>R\}. \tag{6.1}$$

Lemma 6.2. *Let $R\geq 0$, $d>0$ and $0<c<1$. Suppose that u is meromorphic on $\overline{H}$ such that the family*

$$\left\{\frac{u(z_0+(c\,\mathrm{Im}\,z_0)z)}{c\,\mathrm{Im}\,z_0}:z_0\in E(R)\right\}$$

is normal on the unit disc, and $|u'(\zeta)|\geq d$ whenever $u(\zeta)=0$ with $\zeta\in E(R)$.

Then there exists $b>0$ with the following property: any pair $z_1,z_2\in H$ of distinct zeroes of u satisfies $|z_1-z_2|\geq b\,\mathrm{Im}\,z_1$.

Proof. Let $z_1\in H$ be a zero of u. Since u has only a finite number of zeroes lying in $\{z\in H:|z|\leq R\}$, there is no loss of generality in assuming that $z_1\in E(R)$. By equicontinuity, there exists a positive constant δ, independent of the choice of z_1, such that

$$\left|\frac{u(z_1+(c\,\mathrm{Im}\,z_1)z)}{c\,\mathrm{Im}\,z_1}\right|\leq 1\quad\text{for}\quad z\in D(0,2\delta);$$

equivalently, $|u(z)|\leq c\,\mathrm{Im}\,z_1$ for $z\in D(z_1,2\delta c\,\mathrm{Im}\,z_1)$. Now assume that z_2 is a zero of u with $0<|z_1-z_2|\leq\delta c\,\mathrm{Im}\,z_1$. The function

$$h(z)=\frac{u(z)}{(z-z_1)(z-z_2)}$$

is analytic on $D(z_1,2\delta c\,\mathrm{Im}\,z_1)$, and satisfies

$$|h(z)|\leq\frac{c\,\mathrm{Im}\,z_1}{(2\delta c\,\mathrm{Im}\,z_1)(\delta c\,\mathrm{Im}\,z_1)}$$

on the boundary of $D(z_1,2\delta c\,\mathrm{Im}\,z_1)$, and so on the whole disc by the maximum principle. Thus,

$$d\leq|u'(z_1)|=|(z_1-z_2)h(z_1)|\leq\frac{|z_1-z_2|}{2\delta^2c\,\mathrm{Im}\,z_1},$$

which gives the required lower bound for $|z_1-z_2|$. □

Lemma 6.3. *Let $b>0$ and suppose that u is meromorphic on $\overline{H}$ such that any pair $z_1,z_2\in H$ of distinct zeroes of u satisfies $|z_1-z_2|\geq b\,\mathrm{Im}\,z_1$. If the zeroes of u have bounded multiplicities, then $\mathfrak{N}(r,1/u)=O(\log r)$ as $r\to\infty$.*

Proof. We begin by claiming that, for $r>1$,

$$\left\{z:|z|\geq 1,\ \left|z-\frac{ir}{2}\right|\leq\frac{r}{2}\right\}\subseteq D_r=\left\{x+iy:\frac{1}{r}\leq y\leq r,\ |x|\leq\sqrt{ry}\right\}. \tag{6.2}$$

To prove this claim, suppose that $x+iy$ lies in the set on the left-hand side of (6.2). By calculating that $S(0,1)$ intersects $S(ir/2,r/2)$ at points with imaginary part $1/r$, we get that $1/r\leq y\leq r$. We also have

$$x^2+\left(y-\frac{r}{2}\right)^2\leq\frac{r^2}{4},\quad\text{which implies that}\quad |x|\leq\left|ry-y^2\right|^{1/2}\leq\sqrt{ry}.$$

Cover the upper half-plane with squares

$$A_{p,q}=\left\{z:2^{p-1}\leq\mathrm{Im}\,z\leq 2^p,\ |\mathrm{Re}\,z-2^{p-1}q|\leq 2^{p-2}\right\},\quad p,q\in\mathbb{Z}.$$

Observe that each square $A_{p,q}$ has side length 2^{p-1} and contains at most N zeroes of u, where N is independent of p and q. This is because the distinct zeroes in $A_{p,q}$ are separated by a distance of at least $2^{p-1}b$ and have bounded multiplicities. It now follows from (6.2) that $\mathfrak{n}(r, 1/u)$ is at most N times the number of squares $A_{p,q}$ that meet D_r. To count these squares, first note that row p meets D_r if and only if $2^p \geq 1/r$ and $2^{p-1} \leq r$; or equivalently, $-L \leq p \leq L+1$, where L is the greatest integer not exceeding $\log_2 r$. When row p meets D_r, the square $A_{p,q}$ does so if and only if

$$2^{p-1}\left(|q| - \frac{1}{2}\right) \leq \sqrt{r2^p},$$

and there can be at most $4\left(2^{-p/2}\sqrt{r}\right) + 2$ such integers q. Hence, the number of squares $A_{p,q}$ that intersect D_r does not exceed

$$\sum_{p=-L}^{L+1}\left(4\left(2^{-p/2}\sqrt{r}\right)+2\right) \leq 4\sqrt{r}\frac{2^{L/2}}{1-2^{-1/2}} + 4L + 4 \leq \frac{4r}{1-2^{-1/2}} + 4\log_2 r + 4.$$

Therefore, $\mathfrak{n}(r, 1/u) = O(r)$ as $r \to \infty$, and recalling definition (2.1) completes the proof. □

We are now able to apply the preceding sequence of lemmas to establish Theorem 2.5. To this end, let f be analytic on $\overline{H}$ and satisfy either (I′) or (II′). Fix $c \in (0,1)$. Then for a sufficiently large choice of R, the family

$$\mathcal{G} = \{f(z_0 + (c\,\mathrm{Im}\,z_0)z) : z_0 \in E(R)\} \tag{6.3}$$

of analytic functions on the unit disc satisfies the hypothesis of Lemma 6.1. Hence,

$$\mathcal{F} = \left\{\frac{f(z_0 + (c\,\mathrm{Im}\,z_0)z)}{(c\,\mathrm{Im}\,z_0)f'(z_0 + (c\,\mathrm{Im}\,z_0)z)} : z_0 \in E(R)\right\} \tag{6.4}$$

is a normal family on the unit disc by Lemma 6.1.

We note that the multiplicities of the non-real zeroes of f are bounded above by some constant M_0. In case (II′) this follows from the fact that f has only finitely many non-real multiple zeroes. We now write $u = f/f'$. If ζ is a non-real zero of u, then ζ must also be a zero of f, say of multiplicity m, and so $u'(\zeta) = 1/m \geq 1/M_0$. Therefore Lemma 6.2 applies to u with $d = 1/M_0$, since we have shown that (6.4) is normal on the unit disc. Upon combining the conclusion of Lemma 6.2 with the observation that u has only simple zeroes, we obtain from Lemma 6.3 that

$$\mathfrak{N}(r, 1/f) \leq M_0\mathfrak{N}(r, 1/u) = O(\log r), \quad r \to \infty. \tag{6.5}$$

This establishes the first estimate of (2.2).

We now assert that

$$\mathfrak{T}(r, f'/f) = O(\log r), \quad r \to \infty. \tag{6.6}$$

In the case that f satisfies (II′), we can use Hayman's Alternative (Lemma 3.1) to deduce (6.6) because $\mathfrak{N}(r, 1/u) = O(\log r)$ by (6.5), while the function $u' - (1-a)$ has finitely many non-real zeroes by (II′).

Now suppose instead that f satisfies (I′). Observe that it will suffice to show that (6.6) holds as $r \to \infty$ outside a set of finite measure, because the Tsuji characteristic differs from a non-decreasing continuous function by a bounded additive term [**11**, p.27]. Hence, if (6.6) fails to hold, then there must exist a set J of infinite measure such that $\log r = o(\mathfrak{T}(r, f'/f))$ as $r \to \infty$ through values in J. Since f is analytic on $\overline{H}$ and satisfies (I′), we get from (6.5) that

$$\mathfrak{N}(r, 1/f) + \mathfrak{N}(r, 1/f^{(k)}) + \mathfrak{N}(r, f) = O(\log r) = o(\mathfrak{T}(r, f'/f)) \quad \text{as } r \to \infty \text{ on } J.$$

Since the lemma of the logarithmic derivative holds for the Tsuji characteristic (see [**27**, p.332] or [**11**, p.108]), we can now apply the standard Tumura-Clunie argument [**13**, Thm 3.10, p.74] on J to obtain a contradiction. Here we use the fact that all the exceptional sets arising in the proof have finite measure, and that the exceptional cases encountered all imply (6.6) anyway. See also Lemma 1 of [**17**] and the remark of [**17**, p.476].

Write $L_j = f^{(j+1)}/f^{(j)}$, so that $\mathfrak{T}(r, L_0) = O(\log r)$ as $r \to \infty$, by (6.6). Using the relation

$$L_{j+1} = L_j + \frac{L_j'}{L_j}, \tag{6.7}$$

together with the lemma of the logarithmic derivative on a half-plane [**11**, p.108], a proof by induction now shows that $\mathfrak{T}(r, L_j) = O(\log r)$ as $r \to \infty$, which is the second estimate of (2.2).

Remark. Theorem 2.5 states that f has very few zeroes from the viewpoint of the Tsuji characteristic. However, f could have many non-real zeroes in the Nevanlinna sense; in fact, these zeroes could have infinite exponent of convergence. This difference can arise when the zeroes are concentrated near the real axis. We remark, however, that the condition on the separation of the zeroes in Lemma 6.3 is not strong enough to conclude that the zeroes form an A-set as studied, for example, in [**17**] and [**34**].

6.2. *Proof of Theorem 2.6*

The proofs of parts (i) and (ii) are similar but for clarity they are presented separately.

Part (i). Suppose that all but finitely many of the non-real zeroes of $ff^{(k-1)}f^{(k)}$ are zeroes of f with multiplicity at least k but at most M. To show that $f \in U_{2p}^*$ for some p, it will suffice by Theorem 1.2 to show that $ff^{(k)}$ has only finitely many non-real zeroes. Define F_k by (5.4), and note that all but finitely many of the non-real zeroes of $ff^{(k)}$ belong to $\mathcal{A}(F_k)$ by Lemma 5.2. Hence, by Lemma 5.1 it will suffice to prove that $\mathcal{C}(F_k)$ is finite. As in Section 5.2, the hypothesis on f and (5.4) imply that all but a finite number of the non-real critical points of F_k are fixed points of F_k, so that $\mathcal{C}(F_k)$ contains only finitely many critical values of F_k by (5.2). It remains to show that F_k does not have infinitely many non-real asymptotic values.

The function f satisfies condition (I$'$), so Theorem 2.5 and (5.4) give that

$$\mathfrak{T}(r, F_k) = \mathfrak{T}(r, f^{(k-1)}/f^{(k-2)}) + O(1) = O(\log r) \quad \text{as } r \to \infty.$$

Lemma 3.4 now shows that there is at most one direct transcendental singularity of F_k^{-1} lying in H. Observe that our hypothesis on f implies that F_k has a finite number of poles in H. Hence, using the Phragmén-Lindelöf principle [**36**, p.308], it follows that between any pair of paths on which F_k tends to distinct finite values there must lie a direct transcendental singularity over infinity. Therefore, F_k has at most four non-real finite asymptotic values. This completes the proof that $f \in U_{2p}^*$.

Now assume that all of the non-real zeroes of $ff^{(k-1)}f^{(k)}$ are zeroes of f with multiplicity at least k but at most M. We have already shown that $f \in U_{2p}^*$, so f has finite order and, in particular, $f^{(k-1)}/f^{(k-2)}$ must have finite lower order. We conclude that $f \in LP$ by Theorem 2.4.

Part (ii). Suppose that f' and $ff'' - a(f')^2$ both have only finitely many non-real zeroes. We aim to show that the zeroes of f are real with finitely many exceptions, so that $f \in U_{2p}^*$ for some p by Theorem 2.2. We define G by (4.2) with $h = (1-a)^{-1}$, so that $h \in (0, 2)$. Then G has finitely many non-real critical points by (4.2) and our assumptions on f. Note that if $\zeta \in \mathbb{C} \setminus \mathbb{R}$ is a zero of f, but is not one of the finitely many non-real zeroes of G' or f', then by (4.2),

$$G(\zeta) = \zeta \quad \text{and} \quad G'(\zeta) = 1 - h\left(\frac{f}{f'}\right)'(\zeta) = 1 - h \in (-1, 1),$$

and so $\zeta \in \mathcal{A}(G)$. Therefore, to show that f has finitely many non-real zeroes, it will again suffice by Lemma 5.1 to show that $\mathcal{C}(G)$ is finite. As G has a finite number of non-real critical values, we only need to bound the number of non-real asymptotic values.
Using the fact that f satisfies condition (II′), we deduce from Theorem 2.5 that $\mathfrak{T}(r, G) = O(\log r)$ as $r \to \infty$. The proof that G has at most four non-real asymptotic values is now exactly as in part (i), using the fact that non-real poles of G can only occur at the finitely many non-real zeroes of f'. This completes the proof that $f \in U^*_{2p}$, and we note that this certainly implies that f'/f is of finite lower order. Under the stronger assumption that $ff'' - a(f')^2$ has no non-real zeroes, Corollary 2.3 immediately gives that $f \in LP$.

6.3. *Proof of Theorem 2.7*

We will use the following simple lemma.

Lemma 6.4. *Let g be a meromorphic function on the plane and let $L_j = g^{(j+1)}/g^{(j)}$. Then the order of L_{j+1} does not exceed the order of L_j.*

Proof. Assume that L_j has finite order, otherwise there is nothing to prove. Equation (6.7) holds for the L_j, and so $T(r, L_{j+1}) \leq 4T(r, L_j) + O(\log r)$ as $r \to \infty$, using the lemma of the logarithmic derivative. □

To prove Theorem 2.7, suppose that f is a real entire function such that either

(i) condition (I′) holds and the zeroes of $f^{(j)}$ have finite exponent of convergence for some $0 \leq j \leq k-1$; or
(ii) condition (II′) holds and the zeroes of $f^{(j)}$ have finite exponent of convergence for $j = 0$ or $j = 1$.

In either case, let $L^* = f'/f$ if $j = 0$, and let $L^* = f^{(j-1)}/f^{(j)}$ if $j > 0$. Then the poles of L^* have finite exponent of convergence, and so there exists $K \geq 3$ such that

$$I_1 = \int_1^\infty \frac{N(t, L^*)}{t^K}\, dt < \infty.$$

Theorem 2.5 gives that $\mathfrak{T}(r, L^*) = O(\log r)$ as $r \to \infty$. Hence, by Lemma 3.2 and the sentence preceding it, we have

$$I_2 = \int_1^\infty \frac{m(t, L^*)}{t^3}\, dt < \infty.$$

Since $T(r, L^*)$ is an increasing function of r, we see that for $r \geq 1$,

$$\frac{T(r, L^*)}{(2r)^K}\, r \leq \int_r^{2r} \frac{T(t, L^*)}{t^K}\, dt \leq I_1 + I_2,$$

from which we deduce that L^* has finite order.

In case (ii), the function L^* is either f'/f or f/f', and so f'/f must have finite order. The proof is then completed by applying Corollary 2.3.

To conclude the proof in case (i), we first appeal to Lemma 6.4 to establish that the order of $f^{(k-1)}/f^{(k-2)}$ does not exceed that of L^*. Then $f^{(k-1)}/f^{(k-2)}$ certainly has finite lower order, and the required results follow from Theorem 2.4.

6.4. *Proof of Theorem 2.8*

As in the statement of the theorem, suppose that f is an entire function satisfying either (I′) or (II′), and assume that the non-real zeroes of $f^{(j)}$ have finite exponent of convergence for some $j \geq 0$.

There exists an entire function Π whose zeroes are precisely the non-real zeroes of $f^{(j)}$, and whose order is equal to the exponent of convergence of these zeroes and so is finite. (Here Π may be formed as a Weierstrass product, see [**13**, p.24–30].) Pick three distinct values $a_1, a_2, a_3 \in \mathbb{C}$. Checking a straightforward set inclusion shows that $\mathfrak{n}(r, 1/(\Pi - a_\nu)) \leq n(r, 1/(\Pi - a_\nu))$, and since Π has finite order, it follows that there exists $K > 0$ such that

$$\mathfrak{N}(r, 1/(\Pi - a_\nu)) \leq N(r, 1/(\Pi - a_\nu)) = O(r^K).$$

The second fundamental theorem for the Tsuji characteristic now gives

$$\mathfrak{T}(r, \Pi) \leq \sum_{\nu=1}^{3} \mathfrak{N}(r, 1/(\Pi - a_\nu)) + O(\log r + \log \mathfrak{T}(r, \Pi)) = O(r^K)$$

as $r \to \infty$ outside a set of finite measure. It follows that in fact $\mathfrak{T}(r, \Pi) = O(r^K)$ as $r \to \infty$ without an exceptional set, because $\mathfrak{T}(r, \Pi)$ differs from a non-decreasing continuous function by a bounded additive term [**11**, p.27]. Using this, the lemma of the logarithmic derivative [**11**, p.108] gives that $\mathfrak{m}(r, \Pi'/\Pi) = O(\log r)$ as $r \to \infty$.

Define the entire function g by $f^{(j)} = \Pi g$, so that g has only real zeroes. Then

$$\mathfrak{m}\left(r, \frac{g'}{g}\right) \leq \mathfrak{m}\left(r, \frac{f^{(j+1)}}{f^{(j)}}\right) + \mathfrak{m}\left(r, \frac{\Pi'}{\Pi}\right) + O(1) = O(\log r), \tag{6.8}$$

as $r \to \infty$, by using the above and Theorem 2.5. Since g satisfies (6.8) and has only real zeroes, Theorem 1A of [**34**] states that $\log\log M(r, g) = O(r \log r)$ as $r \to \infty$. As Π has finite order, it follows that

$$\log\log M(r, f^{(j)}) = O(r \log r), \quad r \to \infty.$$

After integrating a total of j times, this leads to the required estimate for $M(r, f)$.

7. *Proof of Theorem 2.9*

For functions of finite order, Theorem 2.9 follows immediately from Theorem 2.4. Therefore to prove Theorem 2.9 in full, it will suffice to show that any function satisfying the more general hypotheses has finite order. Note further that the $j = k - 1$ case of Theorem 2.9 is contained in Theorem 2.6(i).

Henceforth, we shall assume that f is an infinite order function that satisfies the more general hypotheses of Theorem 2.9 with $j \leq k - 2$. We aim to demonstrate a contradiction with Theorem 1.2 by showing that $ff^{(k)}$ has only finitely many non-real zeroes. The proof will then be complete.

We will again study a suitable Newton function. Let

$$F(z) = z - \frac{f^{(k-2)}(z)}{f^{(k-1)}(z)}, \qquad F' = \frac{f^{(k)} f^{(k-2)}}{(f^{(k-1)})^2}. \tag{7.1}$$

The next result is absolutely central to Theorem 2.9, but we postpone its proof to Section 7.2. Instead, we first describe how we may obtain the desired contradiction from it by applying the ideas of Section 5.

Proposition 7.1. *F^{-1} has no indirect transcendental singularities over $\mathbb{C} \setminus \mathbb{R}$.*

In fact, once Proposition 7.1 is established, it is easy to show that F has only a finite number of non-real asymptotic values. To do this, observe that f satisfies condition (I′), so that $\mathfrak{T}(r, F) = O(\log r)$ by Theorem 2.5. Then Lemma 3.4 gives that F^{-1} has at most two direct transcendental singularities over $\mathbb{C} \setminus \mathbb{R}$.

Using (7.1) and the hypotheses on f, we see that all but finitely many of the non-real critical points of F are also fixed points. Hence, the set $\mathcal{C}(F)$ defined by (5.2) is finite. Lemma 5.1 now gives that the set $\mathcal{A}(F)$ of (5.1) must also be finite. Since Lemma 5.2 applies to zeroes of f with multiplicity at least k, we find that the non-real zeroes of $ff^{(k)}$ belong to $\mathcal{A}(F)$ with only finitely many exceptions. This leads us to deduce that $ff^{(k)}$ has only finitely many non-real zeroes. As indicated earlier, this contradiction with Theorem 1.2 is enough to complete the proof of Theorem 2.9.

7.1. *An estimate required for Proposition 7.1*

Write, for $m = 0, 1, \ldots, k$,

$$L_m = \frac{f^{(m+1)}}{f^{(m)}}.$$

Then because f satisfies condition (I′), we get from Theorem 2.5 that

$$\mathfrak{T}(r, L_m) = O(\log r), \quad \text{as } r \to \infty. \tag{7.2}$$

This section is devoted to proving the following result, which will later be used in the proof of Proposition 7.1. Both these proofs will use many ideas from [**22**].

PROPOSITION 7.2. *Let $\delta > 0$ and $P > 0$. Then, for $0 \leq m \leq k$, we have*

$$|L_m(z)| = \left| \frac{f^{(m+1)}(z)}{f^{(m)}(z)} \right| > |z|^P, \qquad |z| = r,\ \delta \leq \arg z \leq \pi - \delta, \tag{7.3}$$

on a set of values of r with logarithmic density 1.

In fact, Proposition 7.2 holds for any real entire function f of infinite order that satisfies (I′) and has non-real zeroes with finite exponent of convergence.

Following [**22**, §4], we use a Levin-Ostrovskii factorisation

$$L_m = \phi_m \psi_m \tag{7.4}$$

similar to that described for L_0 in Lemma 3.6, but where ϕ_m may have infinitely many poles.

LEMMA 7.3. *For each $0 \leq m \leq k$, there exist real meromorphic functions ϕ_m and ψ_m satisfying* (7.4) *such that*

(i) *either $\psi_m \equiv 1$ or $\psi_m(H) \subseteq H$;*
(ii) *ϕ_m has only simple poles, all of which are zeroes of $f^{(m)}$ and only finitely many of which are real; and*
(iii) *ϕ_m has finite order.*

Proof. The real meromorphic function ψ_m is formed as a product using the real zeroes of $f^{(m)}$ as in [**22**, §4]. The function ϕ_m is then defined by (7.4), and properties (i) and (ii) follow from this construction.

From part (i) and Lemma 3.7, we get that $m_{0\pi}(r, 1/\psi_m) = O(\log r)$ as $r \to \infty$, where $m_{0\pi}(r, 1/\psi_m)$ is defined by (3.1). Using this, (7.2) and (7.4), and applying Lemma 3.2, gives that

$$\int_1^\infty \frac{m_{0\pi}(r,\phi_m)}{r^3}\,dr \le \int_1^\infty \frac{m_{0\pi}(r,L_m)+m_{0\pi}(r,1/\psi_m)}{r^3}\,dr < \infty. \tag{7.5}$$

Following [**22**, Lemma 4.1], we now claim that there exists $q \ge 1$ such that, for $0 \le m \le k$,

$$n(r,\phi_m) \le \sum_{0\le\mu<m} n(r,1/\phi_\mu) + O(r^q) \quad \text{as } r\to\infty. \tag{7.6}$$

To prove this we need only consider the non-real poles of ϕ_m, since ϕ_m has only finitely many real poles by part (ii). When $m=0$, the estimate (7.6) follows from noting that the (simple) non-real poles of ϕ_0 are non-real zeroes of f, and so have finite exponent of convergence. Now suppose that $m\ge 1$ and z_0 is a non-real pole of ϕ_m. Then z_0 is a simple pole of ϕ_m and a zero of $f^{(m)}$. Let $0\le p\le m$ be the least integer such that $f^{(p)}(z_0)=0$. Then either $p\ge 1$ and $\phi_{p-1}(z_0)=0$; or else z_0 is a non-real zero of f, and these have finite exponent of convergence. This completes the proof of (7.6), as claimed.

We now prove part (iii) of the lemma by induction on m. Suppose that ϕ_ν has finite order for $\nu=0,1,\dots,m-1$ (we assume nothing when $m=0$). Then from (7.6) we have that, for some $q_m\ge 1$,

$$N(r,\phi_m) = O(r^{q_m}) \quad \text{as } r\to\infty.$$

Hence, using (7.5) and the fact that ϕ_m is a real function,

$$\int_1^\infty \frac{T(r,\phi_m)}{r^{q_m+2}}\,dr \le \int_1^\infty \frac{2m_{0\pi}(r,\phi_m)}{r^3}\,dr + \int_1^\infty \frac{N(r,\phi_m)}{r^{q_m+2}}\,dr < \infty.$$

Since $T(r,\phi_m)$ is increasing, it follows that ϕ_m has finite order. □

As each of the functions ϕ_m has finite order, we can apply [**12**, Corollary 2] to show that, for $0\le m\le k$,

$$\log^+\left|\frac{\phi_m'(z)}{\phi_m(z)}\right| = O(\log r) \tag{7.7}$$

as $|z|=r\to\infty$ outside a set of finite logarithmic measure.

Lemma 7.3(i) states that if $\psi_m\not\equiv 1$, then $\psi_m(H)\subseteq H$. In this case, an analytic branch of $\log\psi_m$ may be defined on H. By Bloch's Theorem, the image of $D\big(z,\frac{\operatorname{Im} z}{2}\big)$ under $\log\psi_m$ must contain a disc of radius at least $C_B|(\log\psi_m)'(z)|\frac{\operatorname{Im} z}{2}$, where C_B is Bloch's Constant. As this image is contained in $\log H$, the radius of such a disc cannot exceed $\pi/2$ and therefore

$$\left|\frac{\psi_m'(z)}{\psi_m(z)}\right| \le \frac{\pi}{C_B \operatorname{Im} z}. \tag{7.8}$$

Using (7.4) and the definition of the L_m, we obtain the identity

$$L_m = L_{m-1} + \frac{L_{m-1}'}{L_{m-1}} = L_{m-1} + \frac{\phi_{m-1}'}{\phi_{m-1}} + \frac{\psi_{m-1}'}{\psi_{m-1}},$$

which immediately leads to

$$\log^+|L_m(z)| \ge \log^+|L_{m-1}(z)| - \log^+\left|\frac{\phi_{m-1}'(z)}{\phi_{m-1}(z)}\right| - \log^+\left|\frac{\psi_{m-1}'(z)}{\psi_{m-1}(z)}\right| - \log 3. \tag{7.9}$$

If we now take z with $|z|=r$ and $\delta\le\arg z\le\pi-\delta$, and repeatedly use (7.9) together with (7.7) and (7.8), then we conclude that

$$\log^+|L_m(z)| \ge \log^+|L_0(z)| - O(\log r) \tag{7.10}$$

as $r\to\infty$ outside a set of finite logarithmic measure. As a result of (7.10), we see that it will suffice to prove Proposition 7.2 with $m=0$. We shall now concentrate on this particular case.

Let Π be a real entire function of finite order whose zeroes are precisely the non-real zeroes of f. For example, Π may be formed as a Weierstrass product [**13**, p.24–30] because the non-real zeroes of f are assumed to have finite exponent of convergence. Define g by

$$f = \Pi g;$$

then g is real entire and has only real zeroes. We take the Levin-Ostrovskii factorisation $g'/g = \phi\psi$ as described in Lemma 3.6. The function ϕ is entire by Lemma 3.6(i) and (iii), as g has no non-real zeroes. Moreover, ϕ is transcendental by Lemma 3.6(vi) because f, and hence also g, are of infinite order. Observe that

$$L_0 = \frac{f'}{f} = \frac{\Pi'}{\Pi} + \frac{g'}{g} = \frac{\Pi'}{\Pi} + \phi\psi. \tag{7.11}$$

We show next that the order of ϕ does not exceed 1. The characteristic $T(r,\phi)$ of the real entire function ϕ is equal to $2m_{0\pi}(r,\phi)$, and because this is increasing we obtain the inequality

$$\frac{T(R,\phi)}{(2R)^3}\, R \le \int_R^{2R} \frac{2m_{0\pi}(r,\phi)}{r^3}\, dr. \tag{7.12}$$

Since Π has finite order, we can use the Tsuji half-plane versions of the second fundamental theorem and the lemma of the logarithmic derivative to show that $\mathfrak{m}(r,\Pi'/\Pi) = O(\log r)$, as in Section 6.4. Together with (7.2) and (7.11), this gives that $\mathfrak{m}(r,\phi\psi) = O(\log r)$. We see from Lemma 3.6(i) and Lemma 3.7 that $m_{0\pi}(r,1/\psi) = O(\log r)$. Using these estimates and applying Lemma 3.2 to $\phi\psi$, we deduce that

$$\int_R^\infty \frac{m_{0\pi}(r,\phi)}{r^3}\, dr \le \int_R^\infty \frac{m_{0\pi}(r,\phi\psi) + m_{0\pi}(r,1/\psi)}{r^3}\, dr = O\left(\frac{\log R}{R}\right)$$

as $R \to \infty$. Comparing this estimate with (7.12) reveals that $T(R,\phi) = O(R\log R)$, so that the order of ϕ is indeed no greater than 1.

By combining the next lemma with the fact that ϕ is transcendental, we are able to find points of large modulus that satisfy the inequality in Proposition 7.2 when $m = 0$.

LEMMA 7.4. *Given $\varepsilon > 0$ and $\delta > 0$, we can find $\sigma \in (0,\delta]$ and a set $E_1 \subseteq [1,\infty)$ of upper logarithmic density at most ε with the following property. For each $r \notin E_1$, there exists $\theta = \theta(r) \in (\sigma, \pi - \sigma)$ such that*

$$\log|L_0(re^{i\theta})| > \frac{T(r,\phi)}{2} - O(\log r) \quad \text{as } r \to \infty.$$

Proof. We begin by calling upon two standard growth estimates that both hold outside small exceptional sets. As the function Π has finite order, [**12**, Corollary 2] tells us that

$$\log^+\left|\frac{\Pi'(z)}{\Pi(z)}\right| = O(\log r) \tag{7.13}$$

as $|z| = r \to \infty$ outside a set of finite logarithmic measure. Meanwhile, the order of the entire function ϕ does not exceed 1, and so a standard result of [**14**] gives that, provided $C > 1$,

$$\log M(r,\phi) \le 3T(2r,\phi) \le 3CT(r,\phi) \tag{7.14}$$

outside a set of upper logarithmic density at most $\log 2/\log C$. We set $C = 2^{1/\varepsilon}$; then (7.13) and (7.14) both hold outside a set E_1 with upper logarithmic density at most ε. We now take $\sigma = \min\left\{\frac{\pi}{24C}, \delta\right\}$ and claim that, for each $r \notin E_1$, we can pick $\theta \in (\sigma, \pi - \sigma)$ such that

$$\log|\phi(re^{i\theta})| > \frac{T(r,\phi)}{2}. \tag{7.15}$$

Otherwise, if no such θ exists, then we could obtain a contradiction as follows, by using (7.14) and the fact that ϕ is a real function:

$$T(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\phi\left(re^{it}\right)\right| dt \le \frac{4\sigma}{2\pi}\, 3CT(r,\phi) + \frac{T(r,\phi)}{2} \le \frac{3T(r,\phi)}{4}.$$

We can now complete the proof of the lemma by using (7.11), (7.13), (7.15) and Lemma 3.7,

$$\begin{aligned}\log|L_0(re^{i\theta})| &\ge \log|\phi(re^{i\theta})| + \log|\psi(re^{i\theta})| - \log^+\left|\frac{\Pi'(re^{i\theta})}{\Pi(re^{i\theta})}\right| - \log 2 \\ &> \frac{T(r,\phi)}{2} - O(\log r)\end{aligned}$$

as $r \to \infty$ outside E_1. □

LEMMA 7.5. *Given $\varepsilon > 0$ and $\sigma > 0$, we can find $\lambda > 1$ such that $ff^{(k)}$ has no zeroes in*

$$A(r) = \{z : r/\lambda < |z| < \lambda r,\ \sigma/2 < \arg z < \pi - \sigma/2\}$$

for all r outside a set E_2 of upper logarithmic density at most ε.

Proof. Fix $c \in (0,1)$ and let $\mathcal{G}$ and $\mathcal{F}$ be the families of functions on the unit disc given by (6.3) and (6.4) respectively, where $E(R)$ is as in (6.1). As f satisfies condition (I′), a sufficiently large choice of R ensures that each member of $\mathcal{G}$ satisfies hypothesis (i) of Lemma 6.1, and so we deduce that $\mathcal{F}$ is normal on the unit disc. We now write $u = f/f'$. The argument following (6.4) shows that u satisfies the hypothesis of Lemma 6.2.

Denote by $z_1, z_2, \ldots$ those distinct zeroes of $ff^{(k)}$ that lie in $\{z : \sigma/2 < \arg z < \pi - \sigma/2\}$. Applying Lemma 6.2 to u gives $b > 0$ such that, if z_p, z_q are distinct zeroes of f, then

$$|z_p - z_q| \ge b\,\mathrm{Im}\, z_p \ge b \sin(\sigma/2)|z_p|.$$

Since all but finitely many of the z_n are zeroes of f, we may assume that the above inequality holds for all distinct pairs z_p, z_q by reducing b if necessary. It follows that the number of the z_n that lie in any annulus $\{z : r < |z| < 2r\}$ has an upper bound independent of r. Therefore, we can find a constant B such that

$$\#\{z_n : |z_n| < r\} \le B \log r, \qquad r \ge 2.$$

We now take $\lambda = \exp(\varepsilon/2B)$ and

$$E_2 = \bigcup_{n=1}^{\infty} \left[\frac{|z_n|}{\lambda},\ \lambda|z_n|\right].$$

Denoting the upper logarithmic density of E_2 by l, we have

$$l \le \limsup_{r\to\infty} \frac{1}{\log r} \sum_{|z_n|<\lambda r} \int_{|z_n|/\lambda}^{\lambda|z_n|} \frac{dt}{t} \le \limsup_{r\to\infty} \frac{B\log\lambda r}{\log r}\, 2\log\lambda = \varepsilon.$$

It just remains to note that if $w \in A(r)$ and $ff^{(k)}(w) = 0$, then $w = z_n$ for some n. In this case, $r/\lambda < |z_n| < \lambda r$ and hence $r \in E_2$. □

LEMMA 7.6 ([**22**, Lemma 2.4]). *Let $s > 0$ and let h be analytic on $D(0, 2s)$ with $h(z)h^{(k)}(z) \ne 0$ there. Then $G = h'/h$ satisfies*

$$\log M(s, G) \le c_0(1 + \log^+|G(0)|),$$

in which $c_0 > 0$ depends only on s.

The estimate for L_0 provided by Lemma 7.4 is valid at only one point for each value of the modulus r. We now aim to use Lemmas 7.5 and 7.6 to extend this estimate to a large arc of the circle $|z| = r$.

Choose $\varepsilon > 0$ small, let σ and E_1 be as in Lemma 7.4, and let λ and E_2 be as in Lemma 7.5. Let $r \geq 1$ with $r \notin E_1 \cup E_2$, and take $\theta = \theta(r)$ as given by Lemma 7.4. Define the scaled functions

$$f_r(z) = f(rz), \qquad G_r(z) = \frac{f_r'(z)}{f_r(z)} = rL_0(rz). \tag{7.16}$$

Lemma 7.5 gives that $ff^{(k)}$ has no zeroes in $A(r)$, and so $f_r f_r^{(k)}$ is non-zero on $A(1)$. Therefore, repeated application of Lemma 7.6 gives a constant c_1, depending only on λ and σ, such that

$$\log^+|G_r(e^{i\theta})| \leq c_1(1 + \log^+|G_r(e^{it})|)$$

for all $t \in [\delta, \pi - \delta]$. It is clear from (7.16) that $|L_0(rz)| \leq |G_r(z)| \leq r|L_0(rz)|$, and so we can re-write the above as

$$\log^+|L_0(re^{i\theta})| \leq c_1(1 + \log r + \log^+|L_0(re^{it})|), \qquad t \in [\delta, \pi - \delta].$$

Combining this with the result of Lemma 7.4 gives that

$$\log^+|L_0(re^{it})| \geq c_2 T(r, \phi) - O(\log r), \qquad t \in [\delta, \pi - \delta], \tag{7.17}$$

as $r \to \infty$ outside $E_1 \cup E_2$, and where the constant c_2 is independent of r and t.

By recalling (7.10) and the fact that ϕ is transcendental, the estimate (7.17) shows that (7.3) holds for r outside an exceptional set with upper logarithmic density at most 2ε. Since ε may be chosen arbitrarily small, this completes the proof of Proposition 7.2.

7.2. *Proof of Proposition 7.1*

Assume that F^{-1} has an indirect transcendental singularity over some $\alpha \in H$. Our strategy for demonstrating a contradiction is based upon [**22**, §10] and will be as follows. First, we find a sequence of asymptotic values β_n such that $F(z) \to \beta_n$ as z tends to infinity on a path Γ_n. From (7.1), we have that

$$L_{k-2}(z) = \frac{f^{(k-1)}(z)}{f^{(k-2)}(z)} = \frac{1}{z - F(z)}. \tag{7.18}$$

Hence, Proposition 7.2 shows that $F(z) \approx z$ in most of the plane. It follows that the region where F is near β_n must be narrow, and this fact can be used to deduce that $F \to \beta_n$ quickly on Γ_n. Via (7.18), this leads to a good description of how L_{k-2} behaves like $(z - \beta_n)^{-1}$ on Γ_n. By integrating this, we discover the asymptotics of $f^{(k-2)}$ on Γ_n, and then also of $f^{(j)}$ and $f^{(j-1)}$ by further integration. The hypothesis on the zeroes of $f^{(j)}$ implies that $1/L_{j-1} = f^{(j-1)}/f^{(j)}$ has only finitely many non-real poles. This lack of poles, together with our asymptotic knowledge of this function, allows us to show that $1/L_{j-1}$ grows rapidly in the upper half-plane. The contradiction between this fast rate of growth and the estimate of (7.2) will ultimately establish Proposition 7.1.

Following the above outline, the details of the proof will now be presented under the assumption that F^{-1} has an indirect transcendental singularity over $\alpha \in H$. We are guided by [**22**, §10] throughout.

Recall that the non-real critical values of F form a discrete set because, by (7.1), all but finitely many of the non-real critical points are fixed points. The proof of [**22**, Lemma 10.3] uses this fact to show that, for $n = 0, 1, 2, \ldots$, there exist pairwise distinct $\beta_n \in H$ and pairwise disjoint simple paths to infinity $\Gamma_n \subseteq H$ such that

$$F(z) \to \beta_n \quad \text{as} \quad z \to \infty \quad \text{on} \quad \Gamma_n.$$

We now appeal to the argument of Lemmas 10.4, 10.5 and 10.6 of [**22**] — these rely on [**22**, Lemma 9.2], the conclusion of which is provided in our case by Proposition 7.2 and (7.18). By doing so, we are able to find constants $A_n \in \mathbb{C} \setminus \{0\}$ and error functions τ_n such that

$$f^{(k-2)}(z) = A_n(z - \beta_n) + \tau_n(z), \qquad \tau_n(z) = O(|z|^{-1}), \tag{7.19}$$

as $z \to \infty$ on Γ_n (this is Lemma 10.4 and (42) of [**22**]). Furthermore, for any $K \in \mathbb{N}$,

$$\int_{\Gamma_n} |u^K \tau_n(u)||du| < \infty. \tag{7.20}$$

This assertion is part of [**22**, Lemma 10.6] and means that the error term τ_n decays quickly on Γ_n. The next lemma is essentially Lemma 10.7 of [**22**].

LEMMA 7.7. *Let $0 \le m \le k-2$. Then, as $z \to \infty$ on Γ_n,*

$$f^{(m)}(z) = \frac{A_n(z-\beta_n)^{k-m-1}}{(k-m-1)!} + O(|z|^{k-m-3}).$$

Proof. If $m = k-2$, then the result is an immediate consequence of (7.19). Now assume that $m \le k-3$. Fix $z_0 \in \Gamma_n$ and write

$$h(z) = f^{(m)}(z) - \frac{A_n(z-\beta_n)^{k-m-1}}{(k-m-1)!}.$$

Then (7.19) gives that $h^{(k-m-2)}(z) = \tau_n(z)$. Taylor's formula with the integral form of the remainder gives a polynomial Q of degree at most $k-m-3$ such that

$$h(z) = Q(z) + \int_{z_0}^{z} \frac{(z-u)^{k-m-3}}{(k-m-3)!}\, \tau_n(u)\, du.$$

Using (7.20) now shows that $h(z) = O(|z|^{k-m-3})$ as $z \to \infty$ on Γ_n, as required. □

Recalling our assumption that $1 \le j \le k-2$, we apply Lemma 7.7 with $m = j-1$ and $m = j$ to show that, as $z \to \infty$ on Γ_n,

$$\frac{f^{(j-1)}(z)}{f^{(j)}(z)} = \frac{(z-\beta_n)^{k-j} + O(|z|^{k-j-2})}{(k-j)(z-\beta_n)^{k-j-1} + O(|z|^{k-j-3})} = \frac{z-\beta_n}{k-j} + O(|z|^{-1}). \tag{7.21}$$

By the hypothesis on the non-real zeroes of $f^{(j)}$, there exists a large r_1 such that $E(r_1) = \{z \in H : |z| > r_1\}$ contains no poles of $f^{(j-1)}/f^{(j)}$. We can now choose simple paths Γ_n^* in $E(r_1)$, each tending to infinity and pairwise disjoint apart from a common starting point, such that (7.21) holds as $z \to \infty$ on Γ_n^*. Relabelling if necessary, we obtain pairwise disjoint simply-connected subdomains $D_1, D_2, \dots$ of $E(r_1)$, with D_n bounded by Γ_{n-1}^* and Γ_n^*. Set

$$H_n(z) = \frac{f^{(j-1)}(z)}{f^{(j)}(z)} - \frac{z-\beta_n}{k-j}. \tag{7.22}$$

The construction of the D_n shows that H_n is analytic on the closure $\overline{D_n}$. Furthermore, by considering (7.21), we see that H_n tends to zero as $z \to \infty$ on Γ_n^*, while H_n tends to the non-zero value $\frac{\beta_n - \beta_{n-1}}{k-j}$ as $z \to \infty$ on Γ_{n-1}^*. Therefore, H_n must be unbounded on D_n by the Phragmén-Lindelöf principle [**36**, p.308].

Let N be a large integer. Take $c^* > 0$ large, and for $n = 1, \dots, N$ define

$$u_n(z) = \begin{cases} \log^+ \left| \dfrac{H_n(z)}{c^*} \right|, & z \in D_n \\ 0, & z \in \mathbb{C} \setminus D_n. \end{cases}$$

Then each u_n is a continuous subharmonic function on the plane that is both non-negative and non-constant. Let $\theta_n(s)$ be the angular measure of the intersection of D_n with the circle $S(0,s)$. Applying Lemma 3.8 to u_n, with r_2 large and $1 \leq n \leq N$, gives

$$\begin{aligned}\int_{r_2}^{r} \frac{\pi\, ds}{s\theta_n(s)} \ &\leq \ \log B(2r, u_n) + O(1) \leq \log\left(\frac{1}{2\pi}\int_0^{\pi} u_n(4re^{it})\, dt\right) + O(1)\\ &\leq \ \log\left(m_{0\pi}(4r, H_n)\right) + O(1)\\ &\leq \log^+\left(m_{0\pi}\left(4r, \frac{f^{(j-1)}}{f^{(j)}}\right)\right) + o(\log r)\end{aligned}$$

as $r \to \infty$, using (7.22). Summing this over n, and combining with the the Cauchy-Schwarz inequality

$$N^2 \leq \sum_{n=1}^{N} \theta_n(s) \sum_{n=1}^{N} \frac{1}{\theta_n(s)} \leq \sum_{n=1}^{N} \frac{\pi}{\theta_n(s)},$$

yields

$$N^2 \log r \leq N \log^+\left(m_{0\pi}\left(4r, \frac{f^{(j-1)}}{f^{(j)}}\right)\right) + o(\log r), \qquad r \to \infty.$$

Since $f^{(j)}/f^{(j-1)} = L_{j-1}$, this implies that

$$(N - o(1)) \log r \leq \log^+(m_{0\pi}(4r, 1/L_{j-1})), \qquad r \to \infty,$$

and so, for all large r,

$$m_{0\pi}(r, 1/L_{j-1}) \geq r^{N-1}. \tag{7.23}$$

However, (7.2) gives that $\mathfrak{T}(r, 1/L_{j-1}) = O(\log r)$ as $r \to \infty$. Therefore, by Lemma 3.2 the integral

$$\int_1^{\infty} \frac{m_{0\pi}(r, 1/L_{j-1})}{r^3}\, dr$$

converges. As N is large, this clear contradiction with (7.23) is enough to complete the proof of Proposition 7.1.

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