Value Distribution of Meromorphic Functions and their Derivatives

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Abstract

The content of this thesis can be divided into two broad topics. The first half investigates the deficient values and deficient functions of certain classes of meromorphic functions. Here a value is called deficient if a function takes that value less often than it takes most other values. It is shown that the derivative of a periodic meromorphic function has no finite non-zero deficient values, provided that the function satisfies a necessary growth condition.

The classes B and S consist of those meromorphic functions for which the finite critical and asymptotic values form a bounded or finite set. A number of results are obtained about the conditions under which members of the classes B and S and their derivatives may admit rational, or slowly-growing transcendental, deficient functions.

The second major topic is a study of real functions — those functions which are real on the real axis. Some generalisations are given of a theorem due to Hinkkanen and Rossi that characterizes a class of real meromorphic functions having only real zeroes, poles and critical points. In particular, the assumption that the zeroes are real is discarded, although this condition reappears as a conclusion in one result.

Real entire functions are the subject of the final chapter, which builds upon the recent resolution of a long-standing conjecture attributed to Wiman. In this direction, several conditions are established under which a real entire function must belong to the classical Laguerre-Pólya class LP. These conditions typically involve the non-real zeroes of the function and its derivatives.

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Contents

Chapter 1

Introduction

The aim of this chapter is to describe selected parts of the classical theory that underlies the work presented in subsequent chapters. In addition to the the necessary definitions, a number of useful and well-established results are stated, and these may be used later without explicit reference. Proofs will not be reproduced here, rather we shall indicate where they may be found in the literature. Many more background results and concepts will be introduced at appropriate points in the development of this thesis.

1.1 Nevanlinna theory

The value distribution theory of meromorphic functions was greatly developed by Rolf Nevanlinna during the 1920s. In both its scope and its power his approach greatly surpasses previous results, and in his honour the field is now also known as Nevanlinna theory. A pivotal role is played by the Nevanlinna characteristic of a meromorphic function, which conveys information about the function's rate of growth and also gives an indication of the frequency with which different values are taken. The definitive reference for this section is Hayman's monograph [20].

Let f be a meromorphic function, where here and henceforth meromorphic should be taken to mean meromorphic on the whole complex plane, unless explicitly stated otherwise. Before defining the Nevanlinna characteristic (or simply the *characteristic*) of f we introduce some important functionals. Firstly, the *proximity function* is given by

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f(re^{i\theta})\right| d\theta, \quad r > 0,$$

where $\log^+ x = \max\{\log x, 0\}$. This can be thought of as a measure of the extent to which $f(z)$ is large on the circle $|z| = r$. The two *counting functions* count the poles of f: the first, $n(r,f)$, is defined to be the number of poles of f in $\{z : |z| \leq r\}$, where each pole is counted according to its multiplicity. The *integrated counting function* is then defined to be

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t}\,dt + n(0,f)\log r, \quad r > 0.$$

The characteristic of f is now given by the sum

$$T(r,f) = m(r,f) + N(r,f),$$

and is an increasing function of r. The following estimates for the characteristic of the sum or product of two functions are easily obtained by summing similar inequalities involving the proximity and counting functions:

$$T(r,fg) \leq T(r,f) + T(r,g), \qquad T(r,f+g) \leq T(r,f) + T(r,g) + \log 2.$$

The power of this approach to meromorphic function theory is illustrated by the following theorem due to Nevanlinna.

Theorem 1.1 (First Fundamental Theorem)**.** *Let f be a non-constant meromorphic function and let $a \in \mathbb{C}$. Then*

$$T(r,f) = m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) + O(1) = T\left(r, \frac{1}{f-a}\right) + O(1)$$

as $r \to \infty$.

For non-constant f the characteristic $T(r,f)$ tends to infinity with r, and hence for any $a \in \mathbb{C}$ either $m(r, \frac{1}{f-a})$ or $N(r, \frac{1}{f-a})$ must get large. In the latter case we have that f takes the value a often, while the former case corresponds to f being close to a on some part of the circle $|z| = r$. Another viewpoint is to say that the First Fundamental Theorem shows how the characteristic provides an upper bound for the frequency with which f takes any given value. In fact, a value is said to be *deficient* if it is not taken as frequently as is permitted by Theorem 1.1. More precisely, the *deficiency* of a value $a \in \mathbb{C} \cup \{\infty\}$ is defined to be

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,a)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)},$$

where we write $N(r,a)$ for $N(r, \frac{1}{f-a})$ if $a \in \mathbb{C}$, and $N(r,\infty) = N(r,f)$, so that $N(r,a)$ counts the a-points of f. We define $m(r,a)$ similarly. The value a is called *deficient* if $\delta(a,f) > 0$. To quote Hayman [20], "we may regard $\delta(a,f)$ loosely as the proportion by which the number of roots of the equation $f(z) = a$ is less than the maximum permitted number."

In fact deficient values are unusual, as for most values a the counting term $N(r,a)$ will dominate the proximity function $m(r,a)$ in the statement of the First Fundamental

Theorem. This is expressed by the *defect relation*, that for a non-constant function f we have

$$\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta(a,f) \leq 2.$$

In particular, the set of deficient values is countable. Furthermore, as an omitted value always has a deficiency equal to 1, the defect relation is a significant generalisation of Picard's famous theorem that a non-constant meromorphic function can omit at most two values.

The defect relation is a consequence of Nevanlinna's Second Fundamental Theorem, a key ingredient of which is the following lemma. Known as the lemma of the logarithmic derivative, this is an important and very useful result in its own right. It provides an upper bound on the average size of the logarithmic derivative f'/f in terms of the characteristic $T(r,f)$.

Lemma 1.2 (Lemma of the logarithmic derivative)**.** *Let f be meromorphic and non-constant. Then*

$$m\left(r,\frac{f'}{f}\right) = O(\log T(r,f) + \log r),$$

as r tends to infinity outside a set of finite measure.

So far we have been analysing how frequently a function f 'hits' a fixed value, but we shall also be interested in how often our function coincides with a slowly-varying 'moving target'. In this context, a second meromorphic function h satisfying $T(r,h) = o(T(r,f))$ as $r \to \infty$ is said to be a *deficient function* of f if $\delta(0, f-h) > 0$. This means that points where the two functions agree occur at a rate less than the maximum allowed by the First Fundamental Theorem.

The next two results describe certain properties of the behaviour of the Nevanlinna characteristic. The first of these demonstrates that, for an entire function, $T(r,f)$ is comparable to the logarithm of the maximum modulus

$$M(r,f) = \max\{|f(z)| : |z| \leq r\}.$$

Indeed, the maximum modulus has always been a useful tool for studying entire functions, and in some ways the Nevanlinna characteristic represents a powerful evolution of the maximum modulus to the meromorphic setting.

Lemma 1.3. *If f is an entire function then, for $0 < r < R$,*

$$T(r,f) \leq \log^+ M(r,f) \leq \left(\frac{R+r}{R-r}\right) T(R,f).$$

Rational maps are clearly distinguished from transcendental meromorphic functions by the behaviour of their respective characteristic functions.

Lemma 1.4. *Let g be a rational function and let f be a transcendental meromorphic function. Then $T(r,g) = O(\log r)$ while*

$$\frac{T(r,f)}{\log r} \to \infty, \quad \text{as } r \to \infty.$$

Before moving on we introduce two more functionals: the *order* $\rho(f)$ and *lower order* $\lambda(f)$ describe the asymptotic rate of growth of a meromorphic function f. They are defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \qquad \lambda(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

and satisfy $0 \le \lambda(f) \le \rho(f) \le \infty$. For example, the function $\exp(z^n)$ has order n, while rational functions have zero order by Lemma 1.4, and the lower order of $\exp(e^z)$ is infinite.

1.2 Subharmonic functions

Here we describe a class of functions that frequently occur in complex function theory.

Definition. A function $u : D \to [-\infty, \infty)$ on a domain $D \subseteq \mathbb{C}$ is *subharmonic* if it is upper semi-continuous and satisfies the sub-mean-value property; that is, for each $z \in D$ there exists $r_1 > 0$ such that

$$u(z) \le \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, d\theta, \qquad 0 < r \le r_1.$$

We remark that this definition is local in the sense that a function is subharmonic on D if and only if it is subharmonic on some neighbourhood of each point in D. Harmonic functions are always subharmonic and if f is analytic on a domain D then the functions $|f|$ and $\log|f|$ are both subharmonic on D. Furthermore, if u and v are subharmonic then so are $u+v$ and $\max\{u,v\}$. For details see [53, p.28].

1.3 Densities of sets

The *upper linear density* and *upper logarithmic density* of a measurable set $E \subseteq [0,\infty)$ are respectively defined to be

$$\overline{\text{dens}}\, E = \limsup_{r\to\infty} \frac{1}{r} \int_{[1,r]\cap E} dt, \qquad \overline{\text{logdens}}\, E = \limsup_{r\to\infty} \frac{1}{\log r} \int_{[1,r]\cap E} \frac{dt}{t}.$$

The *lower linear density* and *lower logarithmic density* of E, denoted respectively by $\underline{\text{dens}}\, E$ and $\underline{\text{logdens}}\, E$, are obtained by taking the lim inf in place of the lim sup in the above. These densities are related by the following elementary lemma.

Lemma 1.5 ([3]). *The densities of measurable sets $E, F \subseteq [0,\infty)$ satisfy the following:*

(i) $0 \leq \underline{\operatorname{dens}}\, E \leq \underline{\operatorname{logdens}}\, E \leq \overline{\operatorname{logdens}}\, E \leq \overline{\operatorname{dens}}\, E \leq 1,$

(ii) $\underline{\operatorname{logdens}}(E \cup F) \leq \underline{\operatorname{logdens}}\, E + \overline{\operatorname{logdens}}\, F.$

1.4 Singularities of the inverse function

A meromorphic function f has a *critical point* at z if $f'(z) = 0$ or if z is a multiple pole of f. The value taken by f at a critical point is called a *critical value.* A well-known consequence of Rouché's Theorem is that if $w \in \mathbb{C}$ is not a critical value of f, and if $f(z) = w$ for some z, then f is injective on some neighbourhood of z. This means that it is possible to define a branch ϕ of the inverse function f^{-1} on a neighbourhood of w such that $\phi(w) = z$ and $f \circ \phi$ is the identity map near w. This inverse function ϕ turns out to be analytic.

The question now arises of how far ϕ may be analytically continued. From the fact that f fails to be injective near a critical point, it is clear that we cannot necessarily define a continuation of ϕ to a neighbourhood of any critical value of f. Hence the critical values of f are called the *algebraic singularities of* f^{-1}.

However, the critical values of f are not the only barrier to the analytic continuation of the inverse function. Suppose that we wish to analytically continue ϕ along a path $\Gamma(t)$. It may happen that as we approach a point $\alpha = \Gamma(t_0)$, we find that $\phi(\Gamma(t)) \to \infty$ as $t \to t_0$. Assuming that f is transcendental, this occurs if and only if α is an *asymptotic value* of f; that is, there exists a path γ tending to infinity on which $f \to \alpha$. These asymptotic values are the *transcendental singularities of* f^{-1} and will be discussed in much greater detail in Section 6.3.1.

The asymptotic and critical values of f together constitute the *singular values of the inverse function* f^{-1}. These singular values play a significant role in complex dynamics. We denote by B the class of all transcendental meromorphic functions for which the inverse has a bounded set of finite singular values. The subclass S consists of those functions possessing a finite set of singular values of the inverse function.

The exponential function e^z is a member of S because it has no critical points and 0 and ∞ are its only asymptotic values. As another example consider the function $f(z) = e^z + 1/z$. All critical points ζ of this function must satisfy $e^\zeta - 1/\zeta^2 = 0$, so that the critical values are given by $1/\zeta^2 + 1/\zeta$. Since only finitely many of the critical points can lie in $|\zeta| \leq 1$, we see that the set of critical values of f is bounded. Furthermore, the only asymptotic values of f are 0 and ∞, and hence f belongs to the class B.

A series of results on deficient functions of members of the classes B and S and their derivatives will be obtained by exploiting the following lemma, first proved for entire functions by Eremenko and Lyubich.

Lemma 1.6 ([14, 54])**.** *Let f belong to the class B. Then there exist $L > 0$ and $M > 0$ such that, if $|z| > L$ and $|f(z)| > M$, then*

$$\left|\frac{zf'(z)}{f(z)}\right| \geq \frac{\log|f(z)/M|}{C}, \tag{1.4.1}$$

where C is a positive absolute constant.

Chapter 2

Rational deficient functions of certain derivatives

As described in the introductory chapter, the class B consists of those transcendental meromorphic functions that have a bounded set of finite critical and asymptotic values. This chapter considers derivatives of functions in the class B, and demonstrates that under a variety of different conditions these derivatives cannot admit certain rational deficient functions. The deficient values of these derivatives were studied by Langley in [37]. The proofs of all the results of this chapter appeared in [49].

The main result of Section 2.1 will show that if $f \in B$ has finite lower order then any rational deficient function of any derivative of f must vanish at infinity. The results of Section 2.2 restrict to functions in the class S, the subclass of B whose members have a finite number of singularities of the inverse. It is then shown that any rational deficient function of the first derivative f' must have a multiple zero at infinity and, further, must be identically zero if f has finite lower order.

Lemma 1.6 underlies the results of this chapter and the next, while much of the work done in this chapter is focussed on carefully controlling the analytic continuation of a branch of the inverse function.

2.1 Rational deficient functions of derivatives of $f \in B$

The following was proved for h a non-zero constant in [37] and the proof given here is closely based on that paper.

Theorem 2.1. *Let f be a member of the class B of finite lower order, and let n be a positive integer. Let h be a rational function, not zero at infinity. Then $\delta(0, f^{(n)} - h) = 0$.*

The function $e^z + 1/z$ in the class B shows that the hypothesis on h cannot be omitted (see, however, Theorems 2.4 and 2.6 below).

2.1.1 Proof of Theorem 2.1

Several of the results presented in this chapter will rely on the following lemma.

Lemma 2.2 ([37]). *Let g be transcendental and meromorphic in the plane, such that $\delta(0,g) > 2\delta > 0$. Then there exist a sequence $r_k \to \infty$, and for each k an arc Ω_k of the circle $S(0,r_k)$ with centre 0 and radius r_k, such that*

$$\log|g(z)| < -\delta T(r_k, g), \qquad z \in \Omega_k,$$

and such that the angular measure m_k of Ω_k satisfies

$$m_k(\log T(r_k,g))^5 \to \infty.$$

If, in addition, g has finite lower order λ then $m_k \geq m$, in which m is a positive constant depending only on δ and λ.

Suppose that f, h and n are as in the hypothesis of Theorem 2.1, but that the deficiency $\delta(0, f^{(n)} - h) > 2\delta > 0$. For some $N \geq 0$ we may write, without loss of generality,

$$h(z) = \sum_{j=0}^{N} a_j z^j + O(|z|^{-1}), \qquad a_N = \frac{(N+n)!}{N!}.$$

Lemma 2.2 gives a positive constant m, a sequence $r_k \to \infty$, and for each k an arc Ω_k of $S(0,r_k)$ of angular measure at least m such that

$$\left|f^{(n)}(z) - h(z)\right| < \exp\left(-\delta T\left(r_k, f^{(n)} - h\right)\right), \qquad z \in \Omega_k.$$

Therefore,

$$\left|f^{(n)}(z) - \sum_{j=0}^{N} a_j z^j\right| < \frac{A_0}{r_k}, \qquad z \in \Omega_k,$$

where $A_0, A_1, \ldots$ denote positive constants independent of r_k. Integration now gives a monic polynomial $P_k(z) = z^{N+n} + \ldots = \prod_{j=1}^{N+n}(z - d_j)$ such that

$$\left|f^{(q)}(z) - P_k^{(q)}(z)\right| < A_{n-q} r_k^{n-q-1}, \qquad q = 0, \ldots, n, \quad z \in \Omega_k. \tag{2.1.1}$$

Note that this polynomial may depend on r_k and that the monicity follows from our choice of a_N. For sufficiently large r_k, and a small positive constant c independent of r_k, choose

$$z_k \in \Omega_k \setminus \bigcup_{j=1}^{N+n} B(d_j, cr_k),$$

where we write $B(a,r)$ for the open disc centred at a with radius r. Then we have $|P_k(z_k)| \geq (cr_k)^{N+n}$, so that using (2.1.1) gives

$$|f(z_k)| > (cr_k)^{N+n} - A_n r_k^{n-1},$$

and hence $|f(z_k)| \to \infty$. From (2.1.1) we also have that

$$\begin{aligned} \left|\frac{z_k f'(z_k)}{f(z_k)}\right| &< \frac{r_k\left(|P_k'(z_k)| + A_{n-1}r_k^{n-2}\right)}{\left||P_k(z_k)| - A_n r_k^{n-1}\right|} \\ &< \frac{r_k\left(|P_k'(z_k)| + A_{n-1}r_k^{n-2}\right)}{\frac{1}{2}|P_k(z_k)|} \\ &= \frac{2r_k|P_k'(z_k)|}{|P_k(z_k)|} + o(1) = O(1), \qquad \text{as } r_k \to \infty. \end{aligned}$$

This gives a contradiction with Lemma 1.6 and proves the theorem.

2.2 Rational deficient functions of the derivative of $f \in S$

The next result is a partial extension of Theorem 2.1 to functions of arbitrary order.

Theorem 2.3. *Let f be a member of the class S and let h be a rational function, not zero at infinity. Then $\delta(0, f' - h) = 0$.*

This result was proved in [37] for h a non-zero constant. We adapt the proof given there to prove both Theorem 2.3 and also the following theorem.

Theorem 2.4. *Let f belong to the class S and let h be a rational function with a simple zero at infinity; that is, $zh(z)$ tends to a finite non-zero limit as $z \to \infty$. Then $\delta(0, f' - h) = 0$.*

Theorems 2.3 and 2.4 have an immediate consequence.

Corollary 2.5. *If f is a member of the class S then any deficient rational function of f' has a multiple zero at infinity.*

Finally, by imposing a constraint on the order of f, we may rule out altogether the existence of rational deficient functions of the derivative.

Theorem 2.6. *Let f be a member of the class S of finite lower order. Then f' admits no rational deficient functions, except possibly the zero function.*

2.2.1 Some results needed for Theorems 2.3 and 2.4

As in [37], we require a version of the Koebe Distortion Theorem.

Lemma 2.7 ([24, 37])**.** *Let $0 < r < R < \infty$ and let f be analytic and univalent in the disc $B(a, R)$. Then*

$$\max\{|f'(z)| : |z-a| \le r\} \le \frac{2R^3}{(R-r)^3}|f'(a)| \le \frac{16R^4}{(R-r)^4}\min\{|f'(z)| : |z-a| \le r\}.$$

We will write $S(a, r)$ for the circle with centre a and radius r. The next elementary lemma uses Poisson's formula to give a lower bound for the harmonic measure of a small arc of a circle. This will often be used in conjunction with the classical Two Constants Theorem.

Lemma 2.8. *Suppose that $0 < r < R$ and that Σ is an arc of $S(z_0, R)$ of angular measure at least m. Then for $z \in B(z_0, r)$ the harmonic measure of Σ with respect to z and $B(z_0, R)$ satisfies*

$$\omega(z, \Sigma, B(z_0, R)) \geq \frac{m}{2\pi} \cdot \frac{R-r}{R+r}.$$

Proof. Poisson's formula gives that

$$\omega(z, \Sigma, B(z_0, R)) = \frac{1}{2\pi} \int_{\Sigma} \frac{R^2 - |z|^2}{|Re^{it} - z|^2}\, dt \geq \frac{m}{2\pi} \cdot \frac{R^2 - r^2}{(R+r)^2}. \qquad \square$$

Recall the definition of a subharmonic function from Section 1.2. For a proof of the following classical result see, for example, [53, p.101].

Lemma 2.9 (Two Constants Theorem)**.** *Let E be a Borel subset of the boundary of a domain D. Let u be subharmonic on D such that u is bounded above by $M_0 \geq 0$ and*

$$\limsup_{z \to x,\ z \in D} u(z) \leq M_1, \qquad x \in E.$$

Then

$$u(z) \leq M_1 \omega(z, E, D) + M_0, \qquad z \in D.$$

In the next lemma and hereafter, by the *degree* of a rational function g we shall mean $\max\{\deg P, \deg Q\}$ where P, Q are polynomials without common factors and such that $g = P/Q$.

Lemma 2.10. *Let f be a meromorphic function and let g be a rational function, not zero at infinity, and of degree N. Then there exist positive constants κ_1 and κ_2 such that, for small δ,*

$$f(U_\delta) \subseteq V_\delta = \left(\bigcup_{f(z_j) \neq \infty} B\Big(f(z_j), \kappa_1 \delta^{1/N}\Big) \right) \cup \left\{ |w| > \frac{\kappa_2}{\delta^{1/N}} \right\},$$

where

$$U_\delta = \{z \in \mathbb{C} : |g(z)| < \delta\}$$

and the z_j are the zeroes of g.

It shall be useful to note that if ϕ is a branch of f^{-1} and the point w is not in V_δ, then $\phi(w)$ lies outside U_δ and so $|g(\phi(w))| \geq \delta$.

Proof of Lemma 2.10. We can write

$$g(z) = (z - z_j)^{n_j} G_j(z)$$

where $G_j(z_j) \neq 0, \infty$ and $1 \leq n_j \leq N$. Therefore there exists K_j such that, for small δ, the component of U_δ containing z_j lies in a ball of radius $K_j \delta^{1/n_j}$. This holds for each z_j so that

$$U_\delta \subseteq \bigcup B\Big(z_j, K\delta^{1/n_j}\Big) \subseteq \bigcup B\Big(z_j, K\delta^{1/N}\Big)$$

for some K.

If z_j is not a pole of f, then the Taylor expansion gives $K_j' > 0$ such that, for small ρ,

$$|f(z) - f(z_j)| < K_j' \rho \quad \text{when} \quad |z - z_j| < \rho.$$

Take $\kappa_1 = K \max\{K_j'\}$.

If z_j is a pole of f, then there exists $K_j'' > 0$ such that, for small ρ,

$$|f(z)| > K_j''/\rho \quad \text{when} \quad |z - z_j| < \rho.$$

Take $\kappa_2 = \min\{K_j''\}/K$. □

2.2.2 Proof of Theorem 2.3

Let f and h be as in the hypothesis, but assume that $\delta(0, f' - h) > 0$. Then f must have infinite lower order by Theorem 2.1.

Without loss of generality we may write

$$h(z) = z^n g(z), \qquad g(\infty) = 1, \tag{2.2.1}$$

where $n \geq 0$ and g is a rational function. Let N be the degree of g and let $z_1, \ldots, z_N$ be the zeroes of g, possibly with repetition. Denote by a_j the finite elements of the finite set

$$\big\{\text{singular values of } f^{-1}\big\} \cup \{f(z_1), \ldots, f(z_N)\}.$$

By applying Lemma 2.2 to $f' - h$, we obtain a sequence $r_k \to \infty$ and, for each k, an arc Ω_k' of $S(0, r_k)$ of arc length

$$32\varepsilon_k = \frac{32}{(\log T(r_k, f' - h))^5}, \tag{2.2.2}$$

such that

$$\left|\frac{f'(z)}{z^n} - g(z)\right| \leq |f'(z) - h(z)| < \exp(-c_1 T(r_k, f' - h)), \qquad z \in \Omega_k'. \tag{2.2.3}$$

Here and throughout this proof c_j, C_j, d_j denote positive constants independent of r_k. Using (2.2.1) and (2.2.3) shows that

$$\left|\frac{f'(z)}{z^n} - 1\right| = o(1), \qquad z \in \Omega'_k. \tag{2.2.4}$$

We recall that Ω'_k has arc length $32\varepsilon_k$, and hence we may denote its endpoints by α'_k and $\beta'_k = \alpha'_k e^{32i\varepsilon_k/r_k}$. Then since ε_k/r_k is small,

$$\begin{aligned} \left|(\alpha'_k)^{n+1} - (\beta'_k)^{n+1}\right| &= |\alpha'_k - \beta'_k| \left|(\alpha'_k)^n \left(1 + e^{32i\varepsilon_k/r_k} + \ldots + e^{32in\varepsilon_k/r_k}\right)\right| \\ &\geq |\alpha'_k - \beta'_k| r_k^n. \end{aligned}$$

We have from (2.2.4) that $f'(z) = z^n(1 + o(1))$ for $z \in \Omega'_k$. Integrating this now gives that

$$f(\alpha'_k) - f(\beta'_k) = \frac{1}{n+1}\left((\alpha'_k)^{n+1} - (\beta'_k)^{n+1}\right)(1 + o(1)).$$

Hence, by the above we can pick $\alpha_k, \beta_k \in \Omega'_k$ such that

$$|f(\alpha_k) - f(\beta_k)| = 16\varepsilon_k. \tag{2.2.5}$$

Let Ω_k be that subarc of Ω'_k joining α_k to β_k. Since $\varepsilon_k \to 0$ and there are only finitely many a_j, there is no loss of generality in assuming that

$$|f(\alpha_k) - a_j| \geq 8\varepsilon_k \qquad \text{for all } j. \tag{2.2.6}$$

The aim of this proof is to analytically continue a branch of the inverse function f^{-1} satisfying an asymptotic differential equation. By extending sufficiently far, we shall uncover a contradiction with the Eremenko-Lyubich Lemma of page 6. We begin with:

Lemma 2.11. *Let ϕ be that branch of f^{-1} mapping $f(\alpha_k)$ to α_k. Then ϕ extends analytically and univalently to $B(f(\alpha_k), 2\varepsilon_k)$ and satisfies there*

$$\left|\phi^n(w)\phi'(w) - \frac{1}{g(\phi(w))}\right| < \exp(-c_2 T(r_k, f' - h)). \tag{2.2.7}$$

Proof. By (2.2.6) the function ϕ extends to be analytic and univalent on $B(f(\alpha_k), 8\varepsilon_k)$.

Using (2.2.4), and always assuming that r_k is sufficiently large,

$$|\phi'(f(\alpha_k))| = \left|\frac{\alpha_k^n}{f'(\alpha_k)}\right| \frac{1}{|\alpha_k^n|} < \frac{2}{r_k^n}, \tag{2.2.8}$$

so that the Distortion Theorem (Lemma 2.7) gives

$$|\phi'(w)| \leq \frac{2(8\varepsilon_k)^3}{(4\varepsilon_k)^3}|\phi'(f(\alpha_k))| < \frac{32}{r_k^n}, \qquad w \in B(f(\alpha_k), 4\varepsilon_k). \tag{2.2.9}$$

Integrating this leads to

$$|\phi(w) - \alpha_k| < \frac{128\varepsilon_k}{r_k^n} < 1, \qquad w \in B(f(\alpha_k), 4\varepsilon_k),$$

so that $|\phi(w)|$ is large on $B(f(\alpha_k), 4\varepsilon_k)$, implying that $|g(\phi(w))| > \frac{1}{2}$ there, by (2.2.1). Therefore,

$$\left|\phi^n(w)\phi'(w) - \frac{1}{g(\phi(w))}\right| < (r_k+1)^n \frac{32}{r_k^n} + 2 = O(1), \quad w \in B(f(\alpha_k), 4\varepsilon_k). \tag{2.2.10}$$

Furthermore, (2.2.5) shows that there exists a simple subarc L_k of $f(\Omega_k)$ joining $f(\alpha_k)$ to $S(f(\alpha_k), 4\varepsilon_k)$. For $w \in L_k$, by writing $z = \phi(w)$ the estimates (2.2.3) and (2.2.4) give that

$$\left|\phi^n(w)\phi'(w) - \frac{1}{g(\phi(w))}\right| = \frac{\left|\frac{f'(z)}{z^n} - g(z)\right|}{\left|\frac{f'(z)}{z^n}\right| |g(z)|} < 4\exp(-c_1 T(r_k, f'-h)). \tag{2.2.11}$$

Using (2.2.10), (2.2.11) and the standard harmonic measure estimate

$$\omega(w, L_k, B(f(\alpha_k), 4\varepsilon_k) \setminus L_k) \geq C_1, \qquad w \in B(f(\alpha_k), 2\varepsilon_k) \setminus L_k,$$

an application of the Two Constants Theorem now establishes (2.2.7). To see this, apply Lemma 2.9 to the subharmonic function

$$u(w) = \log\left|\phi^n(w)\phi'(w) - \frac{1}{g(\phi(w))}\right|$$

on the domain $B(f(\alpha_k), 4\varepsilon_k) \setminus L_k$. This yields

$$u(w) \leq -C_1 c_1 T(r_k, f'-h) + O(1) < -c_2 T(r_k, f'-h)$$

for $w \in B(f(\alpha_k), 2\varepsilon_k) \setminus L_k$ and a suitable choice of c_2. □

We now assert that

$$|f(\alpha_k)| < r_k^{n+1} \tag{2.2.12}$$

with at most finitely many exceptions which we discard. Otherwise, $|f(\alpha_k)| \geq r_k^{n+1}$ infinitely often and (1.4.1), (2.2.4) give a contradiction, since

$$\frac{1}{C}\log\frac{r_k^{n+1}}{M} \leq \left|\frac{\alpha_k f'(\alpha_k)}{f(\alpha_k)}\right| \leq \frac{r_k(2r_k^n)}{r_k^{n+1}} = 2,$$

where C, M are as in Lemma 1.6.

Define

$$\eta = \min\{|a_j - a_{j'}| : a_j \neq a_{j'}\}$$

and let σ be positive but small compared to $\min\{1, \eta\}$. Following [37], it is now claimed that for all sufficiently large k there exists ζ_k with

$$|\zeta_k - f(\alpha_k)| = \sigma \tag{2.2.13}$$

such that ϕ may be analytically continued to $B(\zeta_k, \sigma + 2\varepsilon_k)$. To show this, first let a_ν be the nearest a_j to $f(\alpha_k)$. Choose ζ_k satisfying (2.2.13) so that a_ν, $f(\alpha_k)$, ζ_k are collinear, with $f(\alpha_k)$ separating a_ν from ζ_k. If there then exists $a_\mu \in B(\zeta_k, \sigma + 2\varepsilon_k)$, it must satisfy $|f(\alpha_k) - a_\mu| < 3\sigma$ and so $a_\mu = a_\nu$ since σ is small compared to η. This contradicts the fact that $|a_\nu - \zeta_k| \geq \sigma + 8\varepsilon_k$ by (2.2.6) and the choice of ζ_k. Hence no a_j lies in $B(\zeta_k, \sigma + 2\varepsilon_k)$, so ϕ may be extended analytically and univalently into this disc.

Lemma 2.12. *For $w \in B(\zeta_k, \sigma)$,*

$$\left|\phi^n(w)\phi'(w) - \frac{1}{g(\phi(w))}\right| < \exp\left(\frac{-c_3 T(r_k, f' - h)}{(\log T(r_k, f' - h))^{10}}\right) \tag{2.2.14}$$

and

$$|\phi(w) - \alpha_k| < 1. \tag{2.2.15}$$

Proof. Note that (2.2.12), (2.2.13) imply that

$$B(\zeta_k, \sigma + \varepsilon_k) \subseteq B(0, |f(\alpha_k)| + 2\sigma + \varepsilon_k) \subseteq B(0, 2r_k^{n+1}). \tag{2.2.16}$$

As discussed above, no a_j lies in $B(\zeta_k, \sigma + 2\varepsilon_k)$ so

$$\mathrm{dist}(B(\zeta_k, \sigma + \varepsilon_k), f(z_j)) \geq \varepsilon_k \qquad \text{for all } f(z_j) \neq \infty.$$

Therefore, taking

$$\delta = \delta_k = \min\left\{\left(\frac{\varepsilon_k}{\kappa_1}\right)^N, \left(\frac{\kappa_2}{2r_k^{n+1}}\right)^N\right\}$$

we have

$$V_{\delta_k} \subseteq \left(\bigcup_{f(z_j)\neq\infty} B(f(z_j), \varepsilon_k)\right) \cup \{|w| > 2r_k^{n+1}\},$$

so that

$$V_{\delta_k} \cap B(\zeta_k, \sigma_k + \varepsilon_k) = \emptyset$$

where κ_1, κ_2 and V_δ are as in Lemma 2.10. The remark following Lemma 2.10 then gives

$$\frac{1}{|g(\phi(w))|} \leq \frac{1}{\delta_k} < \left(\frac{\kappa_1}{\varepsilon_k}\right)^N + \left(\frac{2r_k^{n+1}}{\kappa_2}\right)^N, \qquad w \in B(\zeta_k, \sigma + \varepsilon_k). \tag{2.2.17}$$

Using (2.2.8), (2.2.13) and the Distortion Theorem (Lemma 2.7) yields

$$|\phi'(w)| < d_1 \varepsilon_k^{-4}, \qquad w \in B(\zeta_k, \sigma + \varepsilon_k).$$

Integrating this,

$$|\phi(w) - \alpha_k| < d_2 \varepsilon_k^{-4}, \qquad |\phi(w)| < d_2 \varepsilon_k^{-4} + r_k$$

for $w \in B(\zeta_k, \sigma + \varepsilon_k)$. Together with (2.2.17) this gives that

$$\left| \phi^n(w)\phi'(w) - \frac{1}{g(\phi(w))} \right| < C_2 \left(\frac{r_k}{\varepsilon_k} \right)^{N_0}, \qquad w \in B(\zeta_k, \sigma + \varepsilon_k), \tag{2.2.18}$$

for some integer N_0 depending only on n and N.

By (2.2.13) the disc $B(f(\alpha_k), 2\varepsilon_k)$ meets the circle $S(\zeta_k, \sigma + \varepsilon_k)$ on an arc Σ_k of angular measure at least $d_3\varepsilon_k$. Furthermore, (2.2.7) holds on Σ_k and Lemma 2.8 shows that

$$\omega(w, \Sigma_k, B(\zeta_k, \sigma + \varepsilon_k)) \geq d_4\varepsilon_k^2, \qquad w \in B(\zeta_k, \sigma).$$

Using this, (2.2.2), (2.2.7), (2.2.18) and applying the Two Constants Theorem now gives

$$\log \left| \phi^n(w)\phi'(w) - \frac{1}{g(\phi(w))} \right| \leq O(\log r_k) + O(\log\log T(r_k, f' - h)) - \frac{c_2 d_4 T(r_k, f' - h)}{(\log T(r_k, f' - h))^{10}}$$

for $w \in B(\zeta_k, \sigma)$. Recalling that f has infinite lower order, the estimate (2.2.14) follows.

Suppose now that (2.2.15) fails, so that by (2.2.13) we can pick $w_0 \in B(\zeta_k, \sigma)$ such that

$$|\phi(w_0) - \alpha_k| = 1$$

but $|\phi(w) - \alpha_k| < 1$ for all w on the linear path γ joining $f(\alpha_k)$ to w_0. Then $\phi(w)$ is large on γ so that $|\phi^n(w)\phi'(w)| \leq 2$ there by (2.2.1) and (2.2.14). Hence

$$1 \leq \frac{1}{n+1} |\phi^{n+1}(w_0) - \alpha_k^{n+1}| = \left| \int_\gamma \phi^n(w)\phi'(w)\, dw \right| \leq 2|w_0 - f(\alpha_k)| \leq 4\sigma,$$

the first inequality being shown when $n \geq 1$ by writing $v = \phi(w_0) - \alpha_k$ and observing that $|(\alpha_k + v)^{n+1} - \alpha_k^{n+1}| \geq r_k^n \geq n + 1$ for r_k large enough. Since σ is small this contradiction establishes (2.2.15). □

We continue to follow [37]: Let τ be positive, but small compared to σ/q, where q is the number of a_j. Choose

$$y_k \in \left[\mathrm{Im}(\zeta_k) - \frac{\sigma}{4}, \mathrm{Im}(\zeta_k) + \frac{\sigma}{4} \right]$$

such that the strip $\{w \in \mathbb{C} : |\,\mathrm{Im}(w) - y_k| < 4\tau\}$ contains none of the a_j. Then ϕ extends analytically and univalently to this strip, starting from the point

$$W_k = \mathrm{Re}(\zeta_k) + iy_k \in B(\zeta_k, \sigma).$$

Choose K large so that $|a_j| < K$ for all j and define the rectangular domains

$$\begin{aligned} D_k &= \{w \in \mathbb{C} : |\,\mathrm{Re}(w) - \mathrm{Re}(\zeta_k)| < 4K,\ |\,\mathrm{Im}(w) - y_k| < \tau\}, \\ D'_k &= \{w \in \mathbb{C} : |\,\mathrm{Re}(w) - \mathrm{Re}(\zeta_k)| < 8K,\ |\,\mathrm{Im}(w) - y_k| < 2\tau\}. \end{aligned}$$

Lemma 2.13. *For large r_k and $w \in D_k$,*

$$|\phi(w) - \alpha_k| < C_3, \tag{2.2.19}$$

and

$$\left|\phi^n(w)\phi'(w) - \frac{1}{g(\phi(w))}\right| < \exp\left(\frac{-c_4 T(r_k, f' - h)}{(\log T(r_k, f' - h))^{10}}\right). \tag{2.2.20}$$

Proof. From (2.2.15) we know that $|\phi(W_k)| > r_k - 1$, so that for r_k large enough (2.2.1) and (2.2.14) imply that $|\phi'(W_k)| \leq 2$. Hence, repeated use of the Distortion Theorem yields

$$|\phi'(w)| \leq C_4, \qquad w \in D_k'. \tag{2.2.21}$$

Using (2.2.15) and integrating (2.2.21) establishes (2.2.19) for $w \in D_k'$:

$$|\phi(w) - \alpha_k| < |\phi(w) - \phi(W_k)| + 1 < C_3.$$

Therefore, $|\phi(w)|$ is large on D_k' and so $|g(\phi(w))| > \frac{1}{2}$ there. Hence

$$\left|\phi^n(w)\phi'(w) - \frac{1}{g(\phi(w))}\right| \leq |\phi(w)|^n|\phi'(w)| + \frac{1}{|g(\phi(w))|} = O(r_k^n), \qquad w \in D_k',$$

and since (2.2.14) holds on the line $w = \mathrm{Re}(\zeta_k) + iy, |y - y_k| \leq 2\tau$, the Two Constants Theorem gives (2.2.20). □

Let

$$A_k = W_k - 3K - \frac{r_k^{n+1}}{8(n+1)}, \qquad \Delta_k = B\left(A_k, \frac{r_k^{n+1}}{8(n+1)}\right),$$

$$\tilde{A}_k = W_k + 3K + \frac{r_k^{n+1}}{8(n+1)}, \qquad \tilde{\Delta}_k = B\left(\tilde{A}_k, \frac{r_k^{n+1}}{8(n+1)}\right).$$

Then $\mathrm{dist}(\Delta_k, \tilde{\Delta}_k) = 6K$, and so one of the discs Δ_k and $\tilde{\Delta}_k$ must lie in the region $\{w \in \mathbb{C} : |w| > 3K\}$. The argument is the same in either case, so we shall assume that this holds for Δ_k.

Let

$$\Delta_k' = B\left(A_k, \frac{r_k^{n+1}}{8(n+1)} + K\right), \qquad \Delta_k'' = B\left(A_k, \frac{r_k^{n+1}}{8(n+1)} + 2K\right)$$

and observe that since none of the a_j lie in the disc Δ_k'', we may extend ϕ analytically and univalently to Δ_k'' starting from $W_k - 3K \in D_k$. See Figure 2.1.

Lemma 2.14. *For $w \in \Delta_k$,*

$$|\phi^{n+1}(w) - \alpha_k^{n+1}| < \frac{r_k^{n+1}}{2}, \tag{2.2.22}$$

and

$$\phi^n(w)\phi'(w) = 1 + o(1). \tag{2.2.23}$$

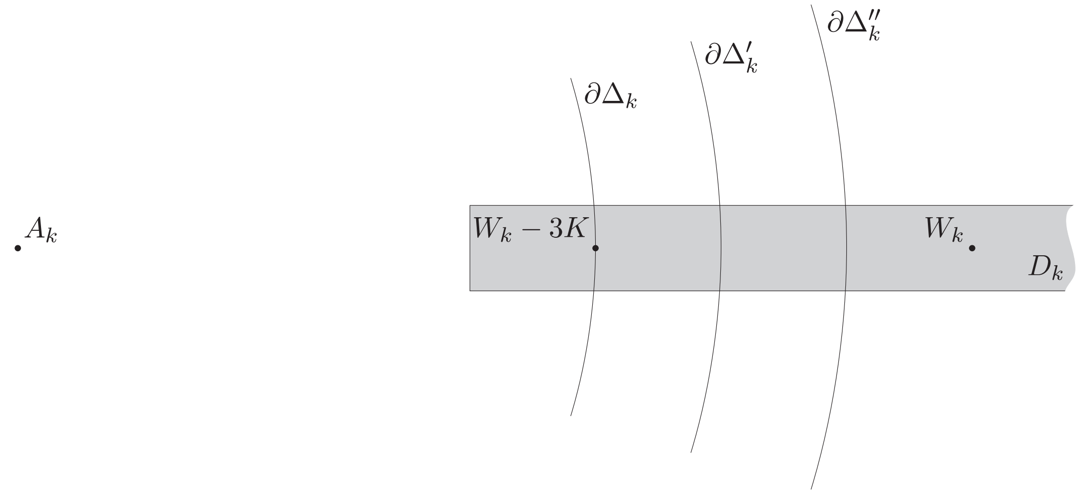

Figure 2.1: Arrangement of domains.

Proof. We will again use Lemma 2.10 to bound $g(\phi(w))$.

Recalling that $W_k \in B(\zeta_k, \sigma)$, we have from (2.2.16) that $|W_k| < 2r_k^{n+1}$ and so

$$\Delta_k' \subseteq B(0, 3r_k^{n+1}).$$

Furthermore, since no a_j lie in Δ_k'',

$$\mathrm{dist}(\Delta_k', f(z_j)) \geq K \quad \text{for all } f(z_j) \neq \infty.$$

Using the above and taking

$$\delta = \delta_k = \left(\frac{\kappa_2}{3r_k^{n+1}}\right)^N,$$

gives V_{δ_k} disjoint from Δ_k', and an application of Lemma 2.10 and the subsequent remark gives that

$$\frac{1}{|g(\phi(w))|} \leq \left(\frac{3r_k^{n+1}}{\kappa_2}\right)^N, \qquad w \in \Delta_k'. \tag{2.2.24}$$

From (2.2.1), (2.2.19) and (2.2.20) we have that $|\phi'(W_k - 3K)| \leq 2$, so that the Distortion Theorem yields

$$|\phi'(w)| \leq C_5 r_k^{4(n+1)}, \qquad w \in \Delta_k'. \tag{2.2.25}$$

Using (2.2.19) and integrating the above gives, for $w \in \Delta_k'$,

$$\begin{aligned} |\phi(w)| &< |\phi(w) - \phi(W_k - 3K)| + r_k + C_3 \\ &< \left(C_5 r_k^{4(n+1)}\right)\left(\frac{r_k^{n+1}}{4(n+1)} + 2K\right) + r_k + C_3 < C_6 r_k^{5(n+1)}, \end{aligned}$$

which combines with (2.2.24) and (2.2.25) to give

$$\left|\phi^n(w)\phi'(w) - \frac{1}{g(\phi(w))}\right| < C_7 r_k^{N_1}, \qquad w \in \Delta_k', \tag{2.2.26}$$

where the integer N_1 depends only on n and N.

Since D_k intersects $\partial\Delta'_k$ in an arc Γ_k of angular measure at least d_5/r_k^{n+1}, Lemma 2.8 implies that

$$\omega(w, \Gamma_k, \Delta'_k) \geq \frac{d_6}{r_k^{2n+2}}, \qquad w \in \Delta_k. \tag{2.2.27}$$

As (2.2.20) holds on Γ_k, the Two Constants Theorem, (2.2.26) and (2.2.27) give that

$$\log\left|\phi^n(w)\phi'(w) - \frac{1}{g(\phi(w))}\right| < O(\log r_k) - \frac{c_4 d_6 T(r_k, f'-h)}{r_k^{2n+2}(\log T(r_k, f'-h))^{10}}, \qquad w \in \Delta_k.$$

Note that the right hand side tends to $-\infty$ because $f' - h$ has infinite lower order. Hence

$$\left|\phi^n(w)\phi'(w) - \frac{1}{g(\phi(w))}\right| = o(1), \qquad w \in \Delta_k. \tag{2.2.28}$$

Suppose now that (2.2.22) fails, so that we can pick $w_0 \in \Delta_k$ such that

$$|\phi^{n+1}(w_0) - \alpha_k^{n+1}| = \frac{r_k^{n+1}}{2} \tag{2.2.29}$$

but $|\phi^{n+1}(w) - \alpha_k^{n+1}| < r_k^{n+1}/2$ for all w on the linear path γ joining $W_k - 3K$ to w_0 (this is possible by (2.2.19) for r_k sufficiently large). Observe that, for w on γ,

$$\begin{aligned} |\phi^{n+1}(w) - \alpha_k^{n+1}| < \frac{r_k^{n+1}}{2} \quad &\Rightarrow \quad |\phi(w)| \geq \frac{r_k}{2} \\ &\Rightarrow \quad |\phi^n(w)\phi'(w)| < \frac{3}{2}, \end{aligned}$$

the second implication assuming r_k is large and using (2.2.1) and (2.2.28). Hence

$$\begin{aligned} |\phi^{n+1}(W_k - 3K) - \phi^{n+1}(w_0)| &= \left|\int_\gamma (n+1)\phi^n(w)\phi'(w)\, dw\right| \\ &\leq \frac{3(n+1)}{2}|w_0 - (W_k - 3K)| \\ &\leq \frac{3r_k^{n+1}}{8}, \end{aligned} \tag{2.2.30}$$

but also,

$$\begin{aligned} |\phi^{n+1}(W_k - 3K) - \phi^{n+1}(w_0)| &\geq |\phi^{n+1}(w_0) - \alpha_k^{n+1}| - |\phi^{n+1}(W_k - 3K) - \alpha_k^{n+1}| \\ &\geq \frac{r_k^{n+1}}{2} - O(r_k^n) \end{aligned}$$

using (2.2.29) and (2.2.19). This contradicts (2.2.30) if r_k is sufficiently large. Therefore (2.2.22) holds and together with (2.2.1) and (2.2.28) gives (2.2.23). □

We complete the proof of the theorem by picking $\hat{w} \in \Delta_k$ with

$$|\hat{w}| \geq \frac{r_k^{n+1}}{16(n+1)}$$

and setting $\hat{z} = \phi(\hat{w})$. By Lemma 2.14,

$$\frac{r_k^{n+1}}{2} \leq |\hat{z}|^{n+1} \leq \frac{3r_k^{n+1}}{2} \qquad \text{and} \qquad |\phi^n(\hat{w})\phi'(\hat{w})| \geq \frac{1}{2}.$$

Then using Lemma 1.6,

$$\begin{aligned} \frac{1}{48(n+1)} = \frac{r_k^{n+1}/16(n+1)}{2\left(\frac{3}{2}r_k^{n+1}\right)} &\leq \frac{|\hat{w}|}{2|\phi(\hat{w})|^{n+1}} \\ &\leq \left|\hat{w}\frac{\phi'(\hat{w})}{\phi(\hat{w})}\right| = \left|\frac{f(\hat{z})}{\hat{z}f'(\hat{z})}\right| \leq \frac{C}{\log\left|\frac{\hat{w}}{M}\right|} \leq \frac{C}{\log\left(\frac{r_k^{n+1}}{16(n+1)M}\right)}. \end{aligned}$$

This contradiction proves Theorem 2.3.

2.2.3 Proof of Theorem 2.4

Let f and h be as in the hypothesis, but assume that $\delta(0, f' - h) > 0$. This proof is again closely based on [37] and will be similar to that of Theorem 2.3.

Let $g(z) = zh(z)$; then g is rational and without loss of generality

$$g(z) \to 1 \quad \text{as} \quad z \to \infty. \tag{2.2.31}$$

Let N be the degree of g and let $z_1, \ldots, z_N$ be the zeroes of g, possibly with repetition. Denote by a_j the finite elements of the finite set

$$\left\{\text{singular values of } f^{-1}\right\} \cup \{f(0), f(z_1), \ldots, f(z_N)\}.$$

Since the function f is single-valued, $w = f(0)$ is the only point at which any branch of the inverse function f^{-1} can take the value zero. Hence, any branch of f^{-1} is non-zero on any domain containing none of the a_j.

By applying Lemma 2.2 to $f' - h$, we obtain a sequence $r_k \to \infty$ and, for each k, an arc Ω_k of $S(0, r_k)$ joining α_k to β_k, of angular measure

$$32\varepsilon_k = \frac{32}{(\log T(r_k, f' - h))^5}, \tag{2.2.32}$$

such that

$$|f'(z) - h(z)| < \exp(-c_1 T(r_k, f' - h)), \qquad z \in \Omega_k. \tag{2.2.33}$$

Here and throughout this proof c_j, C_j denote positive constants independent of r_k. From (2.2.31) and (2.2.33) we have that

$$|zf'(z) - g(z)| < \exp(-c_2 T(r_k, f' - h)), \qquad z \in \Omega_k \tag{2.2.34}$$

and

$$\left|f'(z) - \frac{1}{z}\right| = o\left(\frac{1}{|z|}\right), \qquad z \in \Omega_k.$$

Integrating this last expression,

$$\left|f(\alpha_k) - f(\beta_k) - \log\frac{\alpha_k}{\beta_k}\right| = |f(\alpha_k) - f(\beta_k) \pm 32\varepsilon_k i| = o(\varepsilon_k),$$

where the choice of sign depends on the choice of labelling of the endpoints of Ω_k as α_k and β_k. For either choice,

$$16\varepsilon_k \leq |f(\alpha_k) - f(\beta_k)| \leq 64\varepsilon_k, \tag{2.2.35}$$

and so since $\varepsilon_k \to 0$ there is no loss of generality in assuming that

$$|f(\alpha_k) - a_j| \geq 8\varepsilon_k \qquad \text{for all } j. \tag{2.2.36}$$

Let the constants M, C and L be as in Lemma 1.6 and choose A large enough that

$$A\log\frac{A}{M} > 2C. \tag{2.2.37}$$

It is now asserted that

$$|f(\alpha_k)| < A \tag{2.2.38}$$

for all but finitely many α_k, which we discard. If not, then $|f(\alpha_k)| \geq A$ infinitely often and we can find $\hat{z} = \alpha_k$ such that

$$|\hat{z}| = r_k \geq L, \qquad |f(\hat{z})| \geq A \geq M \qquad \text{and} \qquad \left|\alpha_k f'(\alpha_k)\right| \leq 2,$$

using (2.2.31) and (2.2.34). But then Lemma 1.6 gives

$$\frac{A}{2} \leq \left|\frac{f(\hat{z})}{\hat{z}f'(\hat{z})}\right| \leq \frac{C}{\log|f(\hat{z})/M|} \leq \frac{C}{\log(A/M)}, \tag{2.2.39}$$

contradicting (2.2.37) and so proving the assertion.

Lemma 2.15. *Let ϕ be that branch of the inverse function f^{-1} mapping $f(\alpha_k)$ to α_k. Then ϕ extends to be analytic and univalent on $B(f(\alpha_k), 2\varepsilon_k)$ and satisfies there*

$$|\phi(w)| > C_1 r_k \tag{2.2.40}$$

and

$$\left|\frac{\phi'(w)}{\phi(w)} - \frac{1}{g(\phi(w))}\right| < \exp(-c_3 T(r_k, f' - h)). \tag{2.2.41}$$

Proof. By (2.2.36) and the discussion following the definition of the a_j, the function ϕ extends to be analytic, univalent and non-zero on $B(f(\alpha_k), 8\varepsilon_k)$. This implies that $\log\phi$ is also analytic and univalent there.

Using (2.2.31) and (2.2.34) gives that

$$\left|\frac{\phi'(f(\alpha_k))}{\phi(f(\alpha_k))}\right| = \frac{1}{|\alpha_k f'(\alpha_k)|} \leq 2, \tag{2.2.42}$$

assuming as always that r_k is sufficiently large. Applying the Distortion Theorem to $\log \phi$ now shows that

$$\left| \frac{\phi'(w)}{\phi(w)} \right| \leq C_2, \qquad w \in B(f(\alpha_k), 4\varepsilon_k). \tag{2.2.43}$$

Integrating this for $w \in B(f(\alpha_k), 4\varepsilon_k)$ leads to

$$C_2 \geq \left| \int_w^{f(\alpha_k)} \frac{\phi'(t)}{\phi(t)} \, dt \right| = |\log \phi(f(\alpha_k)) - \log \phi(w)| = \left| \log \frac{\alpha_k}{\phi(w)} \right| \geq \log \frac{r_k}{|\phi(w)|}$$

which establishes (2.2.40). This means that $|\phi(w)|$ is large for $w \in B(f(\alpha_k), 4\varepsilon_k)$, implying that $|g(\phi(w))| > \frac{1}{2}$ by (2.2.31). Therefore,

$$\left| \frac{\phi'(w)}{\phi(w)} - \frac{1}{g(\phi(w))} \right| \leq C_2 + 2, \qquad w \in B(f(\alpha_k), 4\varepsilon_k).$$

Furthermore, (2.2.35) shows that there exists a simple subarc L_k of $f(\Omega_k)$ joining $f(\alpha_k)$ to $S(f(\alpha_k), 4\varepsilon_k)$ and, for $w \in L_k$, (2.2.34) and (2.2.43) give

$$\begin{aligned} \left| \frac{\phi'(w)}{\phi(w)} - \frac{1}{g(\phi(w))} \right| &= \left| \frac{\phi'(w)}{\phi(w)} \right| \frac{1}{|g(\phi(w))|} \left| \frac{\phi(w)}{\phi'(w)} - g(\phi(w)) \right| \\ &\leq 2C_2 \exp(-c_2 T(r_k, f' - h)). \end{aligned}$$

The last two statements, together with the standard harmonic measure estimate

$$\omega(w, L_k, B(f(\alpha_k), 4\varepsilon_k) \setminus L_k) \geq C_3, \qquad w \in B(f(\alpha_k), 2\varepsilon_k) \setminus L_k,$$

are now sufficient to give (2.2.41) by applying the Two Constants Theorem to the subharmonic function $\log \left| \frac{\phi'(w)}{\phi(w)} - \frac{1}{g(\phi(w))} \right|$. □

Lemma 2.16. *There exist a small positive constant σ and a sequence ζ_k satisfying*

$$|\zeta_k - f(\alpha_k)| = \sigma \tag{2.2.44}$$

such that ϕ extends to be analytic, univalent and non-zero on $B(\zeta_k, \sigma)$ and satisfies there

$$\left| \frac{\phi'(w)}{\phi(w)} - \frac{1}{g(\phi(w))} \right| = o(1). \tag{2.2.45}$$

Proof. By the argument preceding Lemma 2.12, we can choose σ and ζ_k satisfying (2.2.44) such that $B(\zeta_k, \sigma + 2\varepsilon_k)$ contains none of the a_j (we use the fact that (2.2.36) is the same as (2.2.6), while (2.2.44) is (2.2.13)). Hence we can extend ϕ analytically and univalently to the disc $B(\zeta_k, \sigma + 2\varepsilon_k)$. Furthermore, ϕ is non-zero there and so $\log \phi$ is also analytic and univalent. This means that the Distortion Theorem may be applied to $\log \phi$, using (2.2.42) and (2.2.44) to give

$$\left| \frac{\phi'(w)}{\phi(w)} \right| \leq \frac{C_4}{\varepsilon_k^4}, \qquad w \in B(\zeta_k, \sigma + \varepsilon_k). \tag{2.2.46}$$

By our choice of a_j, σ and ζ_k we have

$$\operatorname{dist}(B(\zeta_k, \sigma + \varepsilon_k), f(z_j)) \geq \varepsilon_k \qquad \text{for all } f(z_j) \neq \infty, \tag{2.2.47}$$

and (2.2.38), (2.2.44) give that

$$B(\zeta_k, \sigma + \varepsilon_k) \subseteq B(0, A + 1). \tag{2.2.48}$$

Let κ_1, κ_2 and V_δ be as in Lemma 2.10 and take

$$\delta = \delta_k = \left(\frac{\varepsilon_k}{\kappa_1}\right)^N.$$

Since $\varepsilon_k \to 0$, we can assume that $\delta_k < \left(\frac{\kappa_2}{A+1}\right)^N$. Therefore,

$$V_{\delta_k} \subseteq \left(\bigcup_{f(z_j) \neq \infty} B(f(z_j), \varepsilon_k)\right) \cup \{|w| > A + 1\}$$

and so using (2.2.47) and (2.2.48),

$$V_{\delta_k} \cap B(\zeta_k, \sigma + \varepsilon_k) = \emptyset.$$

Applying Lemma 2.10 and the subsequent remark, we obtain

$$\frac{1}{|g(\phi(w))|} \leq \frac{1}{\delta_k} = \frac{\kappa_1^N}{\varepsilon_k^N}, \qquad w \in B(\zeta_k, \sigma + \varepsilon_k).$$

Combined with (2.2.46) this yields

$$\left|\frac{\phi'(w)}{\phi(w)} - \frac{1}{g(\phi(w))}\right| \leq \frac{C_5}{\varepsilon_k^{N_0}}, \qquad w \in B(\zeta_k, \sigma + \varepsilon_k), \tag{2.2.49}$$

where $N_0 = \max\{N, 4\}$.

By (2.2.44), the disc $B(f(\alpha_k), 2\varepsilon_k)$ intersects the circle $S(\zeta_k, \sigma + \varepsilon_k)$ in an arc Σ_k of angular measure at least $c_4 \varepsilon_k$, and hence Lemma 2.8 gives

$$\omega(w, \Sigma_k, B(\zeta_k, \sigma + \varepsilon_k)) \geq c_5 \varepsilon_k^2, \qquad w \in B(\zeta_k, \sigma).$$

Since (2.2.41) holds on Σ_k, we apply the Two Constants Theorem, using (2.2.32) and (2.2.49), to obtain

$$\begin{aligned} \log\left|\frac{\phi'(w)}{\phi(w)} - \frac{1}{g(\phi(w))}\right| &\leq \log\left(\frac{C_5}{\varepsilon_k^{N_0}}\right) - c_3 c_5 \varepsilon_k^2 T(r_k, f' - h) \\ &\leq O(\log\log T(r_k, f' - h)) - \frac{c_3 c_5 T(r_k, f' - h)}{(\log T(r_k, f' - h))^{10}}. \end{aligned}$$

Noting that this last expression tends to $-\infty$ establishes (2.2.45). □

Following [37], let τ be positive, but small compared to σ/q, where q is the number of a_j. Choose

$$y_k \in \left[\mathrm{Im}(\zeta_k) - \frac{\sigma}{4}, \mathrm{Im}(\zeta_k) + \frac{\sigma}{4}\right]$$

such that the strip $\{w \in \mathbb{C} : |\,\mathrm{Im}(w) - y_k| < 4\tau\}$ contains none of the a_j. Then starting from the point

$$W_k = \mathrm{Re}(\zeta_k) + iy_k \in B(\zeta_k, \sigma),$$

we may analytically continue ϕ to give a non-zero, univalent function on this strip. Define the rectangular domains

$$\begin{aligned} D_k &= \{w \in \mathbb{C} : |\,\mathrm{Re}(w) - \mathrm{Re}(\zeta_k)| < 2A,\ |\,\mathrm{Im}(w) - y_k| < \tau\}, \\ D'_k &= \{w \in \mathbb{C} : |\,\mathrm{Re}(w) - \mathrm{Re}(\zeta_k)| < 4A,\ |\,\mathrm{Im}(w) - y_k| < 2\tau\}. \end{aligned}$$

Lemma 2.17. *For $w \in D_k$,*

$$\left|\frac{\phi'(w)}{\phi(w)} - \frac{1}{g(\phi(w))}\right| = o(1). \tag{2.2.50}$$

Proof. By the choice of y_k,

$$\mathrm{dist}(D'_k, f(z_j)) \geq 2\tau \quad \text{for all } f(z_j) \neq \infty,$$

and by (2.2.38) and (2.2.44),

$$D'_k \subseteq B(0, C_6).$$

It then follows from Lemma 2.10 that

$$|g(\phi(w))| > c_6, \qquad w \in D'_k, \tag{2.2.51}$$

because D'_k does not meet V_δ for small δ. Considering this and (2.2.45) shows that

$$\left|\frac{\phi'(W_k)}{\phi(W_k)}\right| \leq \frac{2}{c_6},$$

so that repeated use of the Distortion Theorem applied to $\log\phi$ gives

$$\left|\frac{\phi'(w)}{\phi(w)}\right| \leq C_7, \qquad w \in D'_k.$$

Hence, using (2.2.51) again,

$$\left|\frac{\phi'(w)}{\phi(w)} - \frac{1}{g(\phi(w))}\right| < C_7 + \frac{1}{c_6}, \qquad w \in D'_k.$$

Noting that (2.2.45) holds on the line $w = \mathrm{Re}(\zeta_k) + iy$, $|y - y_k| \leq 2\tau$, we can now obtain (2.2.50) by once more applying the Two Constants Theorem to $\log\left|\frac{\phi'(w)}{\phi(w)} - \frac{1}{g(\phi(w))}\right|$. □

Choose $R \geq L$ so large that, by (2.2.31),

$$|g(z) - 1| < \frac{1}{4} \qquad \text{for } |z| > R. \tag{2.2.52}$$

Lemma 2.18. *For r_k sufficiently large,*

$$|\phi(w)| > R, \qquad w \in Y_k = D_k \cup B(\zeta_k, \sigma). \tag{2.2.53}$$

Proof. Using (2.2.44) we may pick $w_k \in B(\zeta_k, \sigma) \cap B(f(\alpha_k), \varepsilon_k)$, then by (2.2.40),

$$|\phi(w_k)| > C_1 r_k > R. \tag{2.2.54}$$

Let X be the component of the open set $\{w \in Y_k : |\phi(w)| > R\}$ that contains w_k. Note that for $w \in X$ we have, using (2.2.52), that

$$\left| \frac{1}{g(\phi(w))} \right| < \frac{3}{2},$$

and so by (2.2.45) and (2.2.50),

$$\left| \frac{\phi'(w)}{\phi(w)} \right| < 2, \qquad w \in X. \tag{2.2.55}$$

Suppose now that (2.2.53) fails to hold. Then $\partial X \cap Y_k \neq \emptyset$. We consider two cases as shown in Figure 2.2.

Case 1: $\partial X \cap B(\zeta_k, \sigma) \neq \emptyset$

Pick $v \in \partial X \cap B(\zeta_k, \sigma)$ such that the straight line segment joining w_k to v lies in X. Let γ be this line segment.

Case 2: $\partial X \cap B(\zeta_k, \sigma) = \emptyset$

Pick $v \in \partial X \cap Y_k$ such that the line segment joining W_k to v lies in X. Let γ be the line from w_k to W_k followed by the line from W_k to v.

In either case $|\phi(v)| = R$ and the path γ from w_k to v satisfies

$$\gamma \subseteq X \quad \text{and} \quad \text{length}(\gamma) \leq C_8. \tag{2.2.56}$$

Now using (2.2.54), (2.2.55) and (2.2.56) gives

$$\begin{aligned} \log \frac{C_1 r_k}{R} &\leq \left| \log \left| \frac{\phi(w_k)}{\phi(v)} \right| \right| \\ &\leq |\log \phi(w_k) - \log \phi(v)| = \left| \int_\gamma \frac{\phi'(w)}{\phi(w)} \, dw \right| \\ &\leq C_8 \sup_{w \in \gamma} \left| \frac{\phi'(w)}{\phi(w)} \right| \leq 2C_8. \end{aligned}$$

This is clearly a contradiction for r_k large enough. □

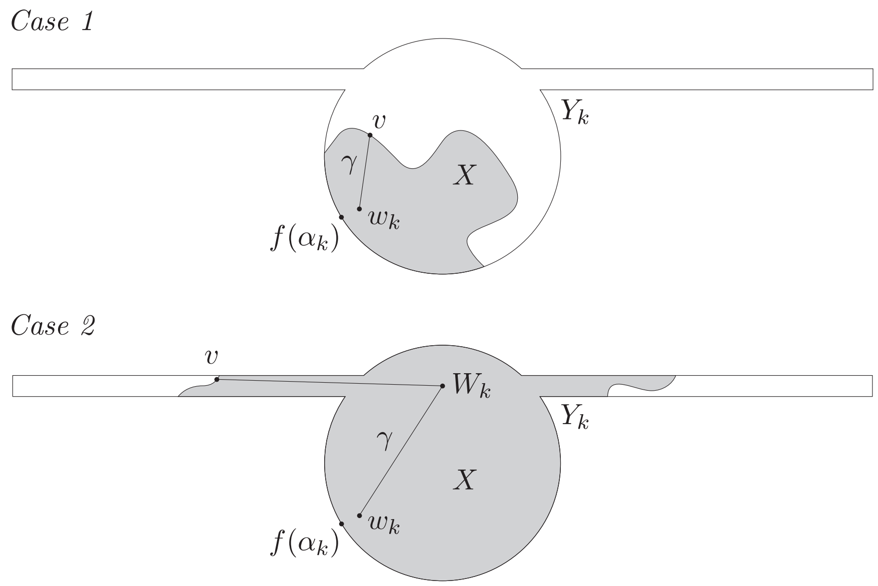

Figure 2.2: Choosing v and γ.

To complete the proof of Theorem 2.4 we observe that, for r_k sufficiently large, we can find $\hat{w} \in D_k$ and $\hat{z} = \phi(\hat{w})$ such that

$$|\hat{w}| \geq A > M$$

and by (2.2.53),

$$|\hat{z}| = |\phi(\hat{w})| > R \geq L.$$

Then using (2.2.50) and (2.2.52),

$$\frac{1}{|\hat{z}f'(\hat{z})|} = \left|\frac{\phi'(\hat{w})}{\phi(\hat{w})}\right| \geq \frac{1}{2}.$$

Hence, we can again use Lemma 1.6 to obtain (2.2.39) in contradiction to (2.2.37).

2.2.4 Proof of Theorem 2.6

Suppose that $f \in S$ is of finite lower order, but that $h \not\equiv 0$ is a rational deficient function of f'. Then by Theorems 2.3 and 2.4 we may take $n \geq 2$ such that, without loss of generality,

$$h(z) = z^{-n}(1 + o(1)), \quad \text{as } |z| \to \infty. \tag{2.2.57}$$

Lemma 2.2 gives a positive constant m, a sequence $r_k \to \infty$ and, for each k, an arc Ω_k of $S(0, r_k)$ of angular measure m on which

$$|f' - h| < \exp(-\delta T(r_k, f' - h)) = o(1/r_k^n), \tag{2.2.58}$$

where δ is a positive constant. The second estimate of (2.2.58) makes use of the fact that $T(r, f' - h)/\log r \to \infty$ since $f' - h$ is transcendental. Combining (2.2.57) and

(2.2.58) shows that, for $z \in \Omega_k$,

$$f'(z) = \frac{1+o(1)}{z^n}, \quad \text{and so} \quad f(z) = c_k - \frac{1+o(1)}{(n-1)z^{n-1}} \quad \text{as } r_k \to \infty, \tag{2.2.59}$$

for some sequence c_k. The c_k cannot tend to infinity, as this would lead to a contradiction with Lemma 1.6 because zf'/f would tend to zero on the arcs Ω_k. Hence we may assume that $c_k \to c$. Applying Lemma 1.6 to $1/(f-c) \in S$ now gives that, for $z \in \Omega_k$,

$$\frac{zf'}{f-c} \to \infty, \quad \text{as } r_k \to \infty. \tag{2.2.60}$$

By (2.2.59),

$$\frac{zf'}{f-c} = \frac{z^n f'}{z^{n-1}(f-c)} = \frac{1+o(1)}{(c_k-c)z^{n-1} - \frac{1+o(1)}{n-1}}, \quad z \in \Omega_k,$$

as $r_k \to \infty$, and so (2.2.60) implies that

$$(c_k - c)z^{n-1} = \frac{1}{n-1} + o(1), \quad z \in \Omega_k.$$

Therefore, $c_k \neq c$ for large r_k. However, the argument of the left-hand side of this last expression varies by $(n-1)m$ over Ω_k, while that of the right-hand side varies only by $o(1)$. This contradiction completes the proof of Theorem 2.6.

2.2.5 Remark on multiple zeroes at infinity

It is worth mentioning that the method of Theorems 2.3 and 2.4 does not seem to extend to prohibit the derivative of $f \in S$ from having a rational deficient function with a double (or higher order) zero at infinity. In the proofs given above, we obtain the asymptotic differential equations $\phi'\phi^n = 1 + o(1)$ and $\phi'/\phi \approx 1$. By analytic continuation, we find points where both $|w|$ and $|\phi(w)|$ are large and so the Eremenko-Lyubich Lemma (Lemma 1.6) applies. Considering a rational function with a double zero at infinity leads to the equation $\phi'/\phi^2 \approx 1$. Comparison with the solution $\phi(w) = \frac{1}{w_0 - w}$ to the exact equation $\phi'/\phi^2 = 1$ suggests that it is not possible to find points where both $|w|$ and $|\phi(w)|$ are large in this case.

Another viewpoint on this is to note that both proofs consider regions where $|f' - h|$ is small. In Theorem 2.3 we have $h(z) \sim z^n$ for non-negative n, and so f asymptotically behaves like z^{n+1}. Similarly, in Theorem 2.4 we have that $h(z) \sim 1/z$ and f behaves like $\log z$. In both cases, $|f(z)|$ is large when $|z|$ is large and it is possible to find points satisfying the hypothesis of the Eremenko-Lyubich Lemma. If, however, h has a multiple zero at infinity, then $h(z) \sim z^{-m}$ for $m \geq 2$, and this leads to f behaving like $a + z^{1-m}$. In particular, $|f(z)|$ may remain bounded when $|z|$ is large and the Eremenko-Lyubich Lemma on which the method relies does not apply.

Whether the derivative of an infinite order function in the class S can have a deficient rational function with a multiple zero at infinity remains an open question.

Chapter 3

Slowly growing deficient functions of members of the class B

The classes B and S are of interest in iteration theory because of the significant role played by the singular values of the inverse function [4, 14, 46, 54]. Motivated by a desire to investigate the frequency of fixed points of mappings in the class B, Langley and Zheng [43] studied their small deficient functions. In this chapter, we extend a result of [43] by giving a number of conditions on transcendental deficient functions of members of the classes B and S. The proofs presented here have been published in [48].

Theorem 3.1. *Let f be a member of the class B of finite lower order, and let h be a zero order transcendental meromorphic function with deficient poles; that is, $\delta(\infty, h) > 0$. Then $\delta(0, f-h) = 0$.*

We shall obtain the following related result for deficient functions of positive order.

Theorem 3.2. *Let $0 < \delta, \nu < 1$ and let f be a member of the class B of finite lower order λ. Then there exists $\rho > 0$ with the following property. For all transcendental meromorphic functions h of order less than ρ, and satisfying $\delta(\infty, h) > 26\rho(h)^{1-\nu}$, we have*

$$\delta(0, f-h) < \delta.$$

Moreover, for $\varepsilon > 0$ we may take $\rho = \delta^{(1+\varepsilon)/\nu}$ provided that $\delta \leq \delta_0(\varepsilon, \lambda)$ where δ_0 is positive and depends only on ε and λ.

The next result partially extends Theorem 3.1 to functions $f \in B$ of arbitrary order.

Theorem 3.3. *Let f belong to the class B and let h be transcendental and meromorphic with deficient poles, and such that*

$$T(r, h) = O(\log r)^P \quad \text{as } r \to \infty$$

for some P. Then $\delta(0, f-h) = 0$.

Theorems 3.1, 3.2 and 3.3 together substantially improve a result from [43], in which it was shown that if f is the class B and h is transcendental meromorphic with finitely many poles, and such that $T(r,h) = o(\log r)^2$ as $r \to \infty$, then $\delta(0, f-h) = 0$.

For a function h to be called a deficient function of f, it is normally required that $T(r,h) = o(T(r,f))$ as $r \to \infty$, but this is not necessary for Theorems 3.1, 3.2 or 3.3. Thus in each case we are also considering whether $f \in B$ can be a deficient function of h. Note, however, that $f - h$ is non-constant, as we shall see that the deficiency of the poles of h ensures that $h \notin B$.

We can modify the hypotheses of the above three results by using the following observation on deficient functions, the proof of which is given later.

Lemma 3.4. *If f and h are meromorphic functions such that either*

$$T(r,h) = o(T(r,f)) \quad \text{or} \quad T(r,f) = o(T(r,h)) \quad \text{as } r \to \infty \tag{3.0.1}$$

then

$$\delta\left(0, \frac{1}{f-a} - \frac{1}{h-a}\right) = \delta(0, f-h) \qquad \text{for all } a \in \mathbb{C}.$$

By applying this, Theorems 3.1, 3.2 and 3.3 immediately give the following corollary.

Corollary 3.5. *Let $a \in \mathbb{C}$ and let f be a transcendental meromorphic function such that the set of singular values of the inverse function f^{-1} does not accumulate at a.*

(i) If f has finite lower order and h is a zero order transcendental meromorphic function satisfying (3.0.1) and with deficient value a, then $\delta(0, f-h) = 0$.

(ii) Suppose that $0 < \delta, \nu < 1$ and that f has finite lower order. Then there exists $\rho > 0$ such that, for all transcendental meromorphic functions h satisfying (3.0.1) with order less than ρ and $\delta(a,h) > 26\rho(h)^{1-\nu}$, we have $\delta(0, f-h) < \delta$.

(iii) If h is a transcendental meromorphic function satisfying (3.0.1), with deficient value a, and such that $T(r,h) = O(\log r)^P$ as $r \to \infty$ for some P, then we have $\delta(0, f-h) = 0$.

If f is in the class S then it satisfies the condition in the above corollary for any value of a, because a finite set of singular values has no accumulation points. Note also that the condition is equivalent to the function $1/(f-a)$ belonging to the class B.

We mention that it was shown in [43] that a non-constant rational function cannot be a deficient function of a member of the class S. Further, it has been pointed out to the author by Alexandre Eremenko that, by combining a recent deep result of Yamanoi with a result about the class S due to Teichmüller, it could be shown that functions in the class S never admit non-constant small deficient functions.

3.1 Preliminaries

The following is included for completeness.

Proof of Lemma 3.4. We may assume that $T(r,h) = o(T(r,f))$ as $r \to \infty$, and because $(f-a)-(h-a) = f-h$ we may also assume that $a = 0$. We need two simple facts; the first of these is the straightforward estimate

$$\begin{aligned} T(r, 1/f - 1/h) &\geq T(r,1/f) - T(r,1/h) - \log 2 \\ &\geq T(r,f)(1+o(1)) \geq T(r,f-h)(1+o(1)). \end{aligned}$$

Secondly, since the function $hf/(f-h)$ has poles only where f and h both have poles or where $f-h=0$, we have

$$n\left(r, \frac{hf}{f-h}\right) \leq n(r, 1/(f-h)) + 2n(r,h).$$

Hence,

$$\begin{aligned} \delta\left(0, \frac{1}{f} - \frac{1}{h}\right) &= 1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{hf}{f-h}\right)}{T(r, 1/f - 1/h)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{N(r,1/(f-h)) + 2N(r,h)}{T(r,f-h)(1+o(1))} \\ &= 1 - \limsup_{r\to\infty} \frac{N(r,1/(f-h))}{T(r,f-h)} = \delta(0, f-h), \end{aligned}$$

and since $1/(1/f) = f$ we get equality. □

Recall from Section 1.3 the definition of the logarithmic density of a measurable set.

Lemma 3.6 ([21]). *Let $S(r)$ be an unbounded positive non-decreasing function on $[r_0, \infty)$, continuous from the right, of order ρ and lower order λ. Let $A > 1$ and $B > 1$. Then*

$$S(Ar) < BS(r)$$

outside an exceptional set G satisfying

$$\overline{\operatorname{logdens}}\, G \leq \rho\left(\frac{\log A}{\log B}\right), \qquad \underline{\operatorname{logdens}}\, G \leq \lambda\left(\frac{\log A}{\log B}\right).$$

The next result provides a lower bound on the minimum modulus, which is defined by

$$L(r,h) = \min\{|h(z)| : |z| = r\}.$$

Lemma 3.7 ([18]). *Let h be a meromorphic function of order ρ. If $\rho < \sigma < 1/2$ then*

$$\log L(r,h) > \cos(\pi\sigma) m(r,h) - \pi\sigma \sin(\pi\sigma) T(r,h), \qquad r \in E,$$

where the set E has lower logarithmic density at least $1 - \rho/\sigma$.

In particular, it follows from Lemma 3.7 that if h has deficient poles and order zero, then there exists a positive constant d such that

$$\log L(r, h) > dT(r, h) \tag{3.1.1}$$

on a set of logarithmic density 1.

The following standard argument shows that a function h which satisfies (3.1.1) for arbitrarily large r cannot belong to the class B. Nevanlinna [47, p.287] proved that if $h \in B$ then, for sufficiently large R, all the components of

$$W = \{z \in \mathbb{C} : |h(z)| > R\}$$

are simply-connected. We assume that $R > |h(0)|$ and choose r_1 large enough that $L(r_1, h) > R$ by (3.1.1). Then $S(0, r_1)$ lies in a simply-connected component of W and so $0 \in W$, which is a contradiction.

Lemma 3.8 (Cartan's Lemma, [23, p.366])**.** *Let $x_1, \ldots, x_M$ be real numbers, not necessarily distinct, and define $\mu(r, t) = \#\{m : |x_m - r| < t\}$. Then for $A > 2e$ and $h > 0$ we have that*

$$\mu(r, t) < \frac{Mt}{eh}, \qquad 0 < t < \infty,$$

for r outside an exceptional set of linear measure less than $2Ah$.

Cartan's Lemma is used in the proof of Fuchs' small arcs lemma [16], of which the next result is a routine consequence. The version stated here is derived from [23, p.721] and is stated explicitly in [39].

Lemma 3.9 ([39])**.** *Let g be a non-constant meromorphic function and let $0 < \eta < 1$.*

(i) There exist a constant $K(\eta) \geq 1$ depending only on η, and a subset $I_\eta \subseteq [0, \infty)$ of lower logarithmic density at least $1 - \eta$, such that if $r \in I_\eta$ is large and F_r is a subinterval of $[0, 2\pi]$ of length m, then

$$\int_{F_r} \left| \frac{rg'(re^{i\theta})}{g(re^{i\theta})} \right| d\theta \leq K(\eta) T(er, g) m \log\left(\frac{2\pi e}{m}\right).$$

(ii) Suppose that the function g has finite lower order (respectively finite order). Then there exist a positive constant L, and a subset $J_\eta \subseteq [0, \infty)$ of upper (respectively lower) logarithmic density at least $1 - \eta$, such that if $r \in J_\eta$ is large and F_r is a subinterval of $[0, 2\pi]$ of length m, then

$$\int_{F_r} \left| \frac{rg'(re^{i\theta})}{g(re^{i\theta})} \right| d\theta \leq LT(r, g) m \log\left(\frac{2\pi e}{m}\right).$$

In fact, the second part of Lemma 3.9 follows from the first part and Lemma 3.6.

The following Fuchs type result is key to the proof of Theorems 3.1 and 3.2.

Lemma 3.10. *Let h be a meromorphic function.*

(i) Suppose that h has order zero (respectively lower order zero) and let $\delta_1, \delta_2 \in (0,1)$. Then

$$\int_0^{2\pi} \left| \frac{rh'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta < \delta_1 T(r,h)$$

for all r outside an exceptional set E of upper (respectively lower) logarithmic density at most δ_2.

(ii) There exists a positive absolute constant K_0 such that if the order of h satisfies $0 < \rho(h) < \frac{1}{32}$, then

$$\int_0^{2\pi} \left| \frac{rh'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta < K_0 \rho(h) T(r,h)$$

for all r outside an exceptional set of upper logarithmic density at most $\frac{1}{4}$.

Remark. It is straightforward to show (using for example [39, Lemma 6]) that part (i) of Lemma 3.10 actually implies that

$$\int_0^{2\pi} \left| \frac{rh'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta = o(T(r,h))$$

as $r \to \infty$ outside a set of zero logarithmic density (respectively zero lower logarithmic density).

Proof of Lemma 3.10. For $0 < |z| = r < R$, the differentiated Poisson-Jensen formula [32, p.65] gives

$$\left| \frac{h'(z)}{h(z)} \right| \le \frac{4R}{(R-r)^2} \left(T(R,h) + O(1)\right) + \sum_{|c_k|<R} \frac{2}{|z-c_k|},$$

where the c_k are the zeroes and poles of h repeated according to multiplicity. Integrating this leads to

$$\int_0^{2\pi} \left| \frac{rh'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta \le \frac{8\pi Rr}{(R-r)^2}\left(T(R,h) + O(1)\right) + 2 \sum_{|c_k|<R} H_k, \tag{3.1.2}$$

where

$$H_k = r \int_0^{2\pi} \frac{d\theta}{|re^{i\theta} - c_k|} = 2r \int_0^{\pi} \frac{d\theta}{|re^{i\theta} - |c_k||}. \tag{3.1.3}$$

We proceed to estimate the H_k. Defining $\gamma_k = |r - |c_k||/r$, for a given r, and following Fuchs [16], we divide the c_k into two classes:

(I) those c_k for which $\gamma_k < \pi/2$, i.e. $|r - |c_k|| < \pi r/2$,

(II) those c_k for which $\gamma_k \ge \pi/2$, i.e. $|r - |c_k|| \ge \pi r/2$.

For $c_k \in$ (II), we have the following straightforward estimates:

$$H_k \le 2r \int_0^{\pi} \frac{d\theta}{|r - |c_k||} = \frac{2\pi r}{|r - |c_k||} \le 4,$$

$$\sum_{\substack{|c_k|<R \\ c_k \in (\mathrm{II})}} H_k \le 4n(R), \tag{3.1.4}$$

where $n(R) = n(R, h) + n(R, 1/h)$ is the number of c_k lying in $|z| \le R$.

Now consider $c_k \in$ (I). Using (3.1.3),

$$\begin{aligned} H_k &\le 2r \int_0^{\gamma_k} \frac{d\theta}{|r - |c_k||} + 2r \int_{\gamma_k}^{\pi/2} \frac{d\theta}{|\operatorname{Im}(re^{i\theta} - |c_k|)|} + 2r \int_{\pi/2}^{\pi} \frac{d\theta}{r} \\ &= \frac{2r\gamma_k}{|r - |c_k||} + 2r \int_{\gamma_k}^{\pi/2} \frac{d\theta}{r \sin\theta} + \pi \\ &\le 2 + \pi + \pi \int_{\gamma_k}^{\pi/2} \frac{d\theta}{\theta} \\ &= 2 + \pi + \pi \log \frac{\pi r}{2|r - |c_k||}, \qquad \text{for } c_k \in (\mathrm{I}). \end{aligned} \tag{3.1.5}$$

To count the number of $|c_k|$ near r, we define

$$\mu(r, t) = \#\{|c_k| < R : |r - |c_k|| < t\},$$

counting with multiplicities. Set $R = \alpha 2^n$ for $\alpha > 2$. An application of Cartan's Lemma (Lemma 3.8) with $A = 6$ and $h_n = 2^{n-3}\delta_2/3$ gives that

$$\mu(r, t) < \frac{n(R)t}{eh_n} = \frac{48n(R)t}{2^{n+1}e\delta_2}, \qquad 0 < t < \infty, \tag{3.1.6}$$

for $r \in [2^n, 2^{n+1}]$ outside an exceptional set E_n of linear measure at most $12h_n = 2^{n-1}\delta_2$. Combining (3.1.5) and (3.1.6) yields

$$\begin{aligned} \sum_{\substack{|c_k|<R \\ c_k \in (\mathrm{I})}} H_k &\le \int_{t=0}^{\pi r/2} \Big(2 + \pi + \pi \log \frac{\pi r}{2t}\Big)\, d\mu(r, t) \\ &\le (2 + \pi)\mu\Big(r, \frac{\pi r}{2}\Big) + \pi \int_0^{\pi r/2} \frac{\mu(r, t)}{t}\, dt \\ &\le \frac{24(2 + \pi)\pi r n(R)}{2^{n+1}e\delta_2} + \pi \int_0^{\pi r/2} \frac{48n(R)}{2^{n+1}e\delta_2}\, dt \\ &\le \frac{48\pi(1 + \pi)}{e\delta_2} n(R), \end{aligned} \tag{3.1.7}$$

for $r \in [2^n, 2^{n+1}] \setminus E_n$. Observe that

$$n(R) \le n(\alpha r) = n(\alpha r, h) + n(\alpha r, 1/h) \le \frac{2}{\log \alpha}(T(\alpha^2 r, h) + O(1)).$$

Using this, (3.1.4) and (3.1.7), the estimate (3.1.2) becomes

$$\begin{aligned} I &= \int_0^{2\pi} \left| \frac{rh'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta \\ &\leq \frac{8\pi Rr}{(R-r)^2}(T(R,h)+O(1)) + \left(8 + \frac{96\pi(1+\pi)}{e\delta_2}\right) n(R) \\ &\leq \left(\frac{8\pi(\alpha 2^n)2^{n+1}}{(2^n(\alpha-2))^2} + \left(8+\frac{96\pi(1+\pi)}{e\delta_2}\right)\frac{2}{\log\alpha}\right)(T(\alpha^2 r,h)+A) \\ &= 16\left(\frac{\pi\alpha}{(\alpha-2)^2} + \frac{1}{\log\alpha}\left(1+\frac{12\pi(1+\pi)}{e\delta_2}\right)\right)(T(\alpha^2 r,h)+A), \qquad (3.1.8) \end{aligned}$$

for $r \in [2^n, 2^{n+1}] \setminus E_n$ and some constant A. Hence, for $2^m < s \leq 2^{m+1}$, inequality (3.1.8) holds for all $r \in [1, s]$ outside $\bigcup_{n=0}^m E_n$, which has linear measure at most

$$\delta_2(2^{-1} + 1 + 2 + \ldots + 2^{m-1}) < \delta_2 s.$$

Therefore, (3.1.8) holds for all $r > 0$ outside an exceptional set E' with upper linear density at most δ_2. By Lemma 1.5(i), the upper logarithmic density of E' is also at most δ_2. We now prove the two parts of the lemma separately.

(i) Assume that h has order zero (respectively lower order zero). Then Lemma 3.6 gives that

$$T(\alpha^2 r, h) + A \leq 2T(r,h)$$

outside a set E'' of upper (respectively lower) logarithmic density zero. Now let $E = E' \cup E''$. By Lemma 1.5(ii), the upper (respectively lower) logarithmic density of E is at most δ_2. By the above, for $r \notin E$,

$$\int_0^{2\pi} \left| \frac{rh'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta \leq 32\left(\frac{\pi\alpha}{(\alpha-2)^2} + \frac{1}{\log\alpha}\left(1+\frac{12\pi(1+\pi)}{e\delta_2}\right)\right) T(r,h).$$

The proof of part (i) is thus completed by choosing α sufficiently large.

(ii) Assume now that the order of h satisfies $0 < \rho(h) < \frac{1}{32}$. Applying Lemma 3.6 gives that

$$T(\alpha^2 r, h) + A \leq eT(r,h) + A \leq 3T(r,h)$$

outside a set E'' of upper logarithmic density at most $2\rho(h)\log\alpha$. Thus taking $\delta_2 = \frac{1}{8}$ and $\log\alpha = \frac{1}{16\rho(h)} > 2$, we have from (3.1.8) that

$$\int_0^{2\pi} \left| \frac{rh'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta \leq \frac{48}{\log\alpha}\left(\frac{\pi\alpha\log\alpha}{(\alpha-2)^2} + 1 + \frac{96\pi(1+\pi)}{e}\right) T(r,h)$$

for $r \notin E' \cup E''$. The upper logarithmic density of this exceptional set does not exceed

$$\delta_2 + 2\rho(h)\log\alpha = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

Since the term

$$\frac{\pi\alpha\log\alpha}{(\alpha-2)^2}$$

is bounded for $\alpha > e^2$, we can find an absolute constant K_0 such that

$$48(16\rho(h))\left(\frac{\pi\alpha\log\alpha}{(\alpha-2)^2}+1+\frac{96\pi(1+\pi)}{e}\right)T(r,h)\leq K_0\rho(h)T(r,h). \qquad \square$$

3.2 Proof of Theorem 3.1

As in the hypothesis, let $f \in B$ be of finite lower order and let h be a transcendental meromorphic function of zero order with deficient poles, but suppose that $\delta(0, f-h) > 0$.

Lemma 3.11. *There exist positive constants m and c, and a set J of positive upper logarithmic density such that, for $r \in J$,*

$$\log|f(z)-h(z)| < -cT(r,f-h) \tag{3.2.1}$$

on a subset Σ_r of $S(0,r)$ of angular measure at least m. Furthermore, for $z \in \Sigma_r$,

$$zf'(z) = zh'(z) + o(1) \quad \text{as } r\to\infty \text{ in } J. \tag{3.2.2}$$

Proof. Since $\delta(0, f-h) > 0$, we can pick z_0 with $|z_0| = r$, for all sufficiently large r, such that

$$\log|f(z_0)-h(z_0)| < -\frac{1}{2}\delta(0,f-h)T(r,f-h).$$

Applying Lemma 3.9(ii) to $f-h$ gives a constant $L > 0$ and a set J of positive upper logarithmic density such that, for $r \in J$ and F_r any subinterval of $[0, 2\pi]$ of length m,

$$\int_{F_r}\left|\frac{r(f'(re^{i\theta})-h'(re^{i\theta}))}{f(re^{i\theta})-h(re^{i\theta})}\right|d\theta \leq LT(r,f-h)m\log\left(\frac{2\pi e}{m}\right).$$

Choose m so small that

$$Lm\log\left(\frac{2\pi e}{m}\right)\leq\frac{1}{4}\delta(0,f-h).$$

Let Ω_r be that arc of $S(0,r)$ with midpoint z_0 and angular measure $2m$. Then for $r \in J$ the estimate (3.2.1) holds on Ω_r with $c = \delta(0, f-h)/4$. Furthermore, by considering $F_r = \{\arg z : z \in \Omega_r\}$ in the above, we see that

$$\int_{\Omega_r}\left|\frac{f'(z)-h'(z)}{f(z)-h(z)}\right||dz|\leq 2LT(r,f-h)m\log\left(\frac{\pi e}{m}\right),$$

so that the subset of Ω_r on which

$$\left|\frac{f'(z)-h'(z)}{f(z)-h(z)}\right|\leq\frac{2LT(r,f-h)}{r}\log\left(\frac{\pi e}{m}\right)$$

must have angular measure at least m. Let Σ_r be this subset. Using (3.2.1) now yields, for $z \in \Sigma_r$,

$$|zf'(z) - zh'(z)| \leq 2LT(r, f-h)\log(\pi e/m)\exp(-cT(r, f-h)) = o(1)$$

as $r \to \infty$ in J. □

The remark following Lemma 3.7 shows that we can find a positive constant d such that

$$\log L(r,h) > dT(r,h) \tag{3.2.3}$$

on a set of logarithmic density 1. Let J' be that subset of J on which (3.2.3) holds and note that J' has positive upper logarithmic density. In particular,

$$h(z) \to \infty \quad \text{as} \quad |z| = r \to \infty \text{ in } J'.$$

For $z \in \Sigma_r$, Lemma 3.11 gives that

$$f(z) = h(z) + o(1), \quad \text{as } r \to \infty \text{ in } J', \tag{3.2.4}$$

implying that

$$f(z) \to \infty \quad \text{and} \quad \frac{1}{f(z)} = \frac{1+o(1)}{h(z)} \quad \text{as } r \to \infty \text{ in } J'. \tag{3.2.5}$$

Together with (3.2.2) this gives, for $z \in \Sigma_r$,

$$\frac{zf'(z)}{f(z)} = \frac{zh'(z) + o(1)}{h(z)}(1+o(1)), \quad \text{as } r \to \infty \text{ in } J'. \tag{3.2.6}$$

Let M and C be as in Lemma 1.6, the hypothesis of which is satisfied by f and $z \in \Sigma_r$ for all sufficiently large $r \in J'$ by (3.2.5). Therefore, using Lemma 1.6, (3.2.3), (3.2.4) and (3.2.6) yields

$$\begin{aligned} dT(r,h) < \log|h(z)| &= (1+o(1))\log|f(z)| \\ &\leq (1+o(1))\left(C\left|\frac{zf'(z)}{f(z)}\right| + \log M\right) \\ &= (1+o(1))\left(C\left|\frac{zh'(z)+o(1)}{h(z)}\right| + \log M\right) \end{aligned}$$

for $z \in \Sigma_r$ as $r \to \infty$ in J'. Hence there exists $K > 0$ such that, for all large $r \in J'$ and $z \in \Sigma_r$,

$$\left|\frac{zh'(z)}{h(z)}\right| > KT(r,h).$$

Since the angular measure of Σ_r is at least m, this leads to

$$\int_0^{2\pi} \left|\frac{rh'(re^{i\theta})}{h(re^{i\theta})}\right| d\theta \geq mKT(r,h)$$

for large $r \in J'$. This contradicts Lemma 3.10(i), thus proving the theorem.

3.3 Proof of Theorem 3.2

Let f, δ, ν and λ be as in the hypothesis. Assume that the transcendental meromorphic function h satisfies $\delta(\infty, h) > 26\rho(h)^{1-\nu}$, but that

$$\delta(0, f-h) \geq \delta.$$

Let K_0 be as in Lemma 3.10(ii) and let K_1 be the constant $K(\frac{1}{8})$ of Lemma 3.9(i); then $K_1 \geq 1$. Define the constants

$$C_1 = 16CK_0K_1 \quad \text{and} \quad C_2 = \pi/2CK_0,$$

where C is as in Lemma 1.6. We may assume that $C_2 < \frac{1}{16}$, since Lemma 3.10(ii) continues to hold if we demand that $K_0 > 8\pi/C$. The function

$$\phi(x) = C_1 e^{4\lambda+1} x \log\left(\frac{C_2 e}{x}\right)$$

is strictly increasing for $0 < x < C_2$, and $\phi(C_2/2) \geq 4\pi K_1 > \delta$ so that we may define $\rho^\nu < C_2/2$ by $\phi(\rho^\nu) = \delta$.

We aim to show that $\rho(h) \geq \rho$. We will then be done, because for $\varepsilon > 0$ and δ less than some positive $\delta_0(\varepsilon, \lambda)$ we see that $\phi(\delta^{1+\varepsilon}) \leq \delta$, and this implies that $\delta^{(1+\varepsilon)/\nu} \leq \rho$.

By Theorem 3.1 we have that $\rho(h) > 0$, and since $\rho < C_2/2$ we may assume that $\rho(h) < \frac{1}{32}$. It follows that the lower order $\lambda(f-h)$ is less than $\lambda + \frac{1}{32}$. Applying Lemma 3.7 to h, and taking $\sigma = 8\rho(h)$ in the notation there, now leads to

$$\underline{\text{logdens}}\left\{r > 0 : \log L(r,h) > \frac{\sqrt{2}}{2}\left(\frac{m(r,h)}{T(r,h)} - 8\pi\rho(h)\right)T(r,h)\right\} \geq \frac{7}{8}.$$

Therefore, recalling that $\delta(\infty, h) > 26\rho(h)^{1-\nu}$ and calculating $\sqrt{2}(26 - 8\pi) \approx 1.2$, we get that

$$\log L(r,h) > \frac{\sqrt{2}}{2}(26 - 8\pi\rho(h)^\nu)\rho(h)^{1-\nu}T(r,h) > \frac{\rho(h)^{1-\nu}T(r,h)}{2} \tag{3.3.1}$$

on a set of lower logarithmic density at least $\frac{7}{8}$. Hence, $h \notin B$ by the argument given after (3.1.1), and so $f - h$ is non-constant.

Applying Lemma 3.9(i) to $f - h$ with $\eta = \frac{1}{8}$ gives a set $I_{1/8}$ of lower logarithmic density at least $\frac{7}{8}$ such that, for $r \in I_{1/8}$,

$$\int_{F_r}\left|\frac{r(f'(re^{i\theta}) - h'(re^{i\theta}))}{f(re^{i\theta}) - h(re^{i\theta})}\right| d\theta \leq K_1 T(er, f-h) m \log\left(\frac{2\pi e}{m}\right),$$

where F_r is any interval of length m. An application of Lemma 3.6 now yields

$$\int_{F_r}\left|\frac{r(f'(re^{i\theta}) - h'(re^{i\theta}))}{f(re^{i\theta}) - h(re^{i\theta})}\right| d\theta \leq K_1 e^{4\lambda+1} T(r, f-h) m \log\left(\frac{2\pi e}{m}\right) \tag{3.3.2}$$

for $r \in H \subseteq I_{1/8}$, where the upper logarithmic density of H is at least $\frac{5}{8}$. To see this, take $S(r) = T(r, f-h)$, $A = e$ and $B = e^{4\lambda(f-h)}$ in the notation of Lemma 3.6 and make use of Lemma 1.5(ii). Let H' be that subset of H on which (3.3.1) holds; then H' has upper logarithmic density at least $\frac{1}{2}$.

Choose $m = C_1\rho^\nu/4K_1 = 2\pi\rho^\nu/C_2 < \pi$. Then

$$K_1 e^{4\lambda+1} m \log\left(\frac{2\pi e}{m}\right) = \frac{\phi(\rho^\nu)}{4} = \frac{\delta}{4}. \tag{3.3.3}$$

The next lemma and its proof are very similar to Lemma 3.11.

Lemma 3.12. *There exist $c > 0$ and, for each $r \in H$, a subset Σ_r of $S(0,r)$ of angular measure at least m on which*

$$\log|f(z) - h(z)| < -cT(r, f-h) \tag{3.3.4}$$

and

$$zf'(z) = zh'(z) + o(1) \quad \textit{as } r \to \infty \textit{ in } H. \tag{3.3.5}$$

Proof. Since $\delta(0, f-h) \geq \delta$, we can pick z_0 with $|z_0| = r$, for all large r, such that

$$\log|f(z_0) - h(z_0)| < -\frac{1}{2}\delta T(r, f-h). \tag{3.3.6}$$

Let Ω_r be that arc of $S(0,r)$ with midpoint z_0 and angular measure $2m$. Using (3.3.2), (3.3.3) and (3.3.6), we see that for $r \in H$, the estimate (3.3.4) holds on Ω_r with $c = \delta/4$. By considering $F_r = \{\arg z : z \in \Omega_r\}$ in (3.3.2), with m replaced by $2m$, we see that

$$\int_{\Omega_r} \left|\frac{f'(z) - h'(z)}{f(z) - h(z)}\right| |dz| \leq 2K_1 e^{4\lambda+1} T(r, f-h) m \log\left(\frac{\pi e}{m}\right),$$

so that the subset of Ω_r on which

$$\left|\frac{f'(z) - h'(z)}{f(z) - h(z)}\right| \leq \frac{2K_1 e^{4\lambda+1} T(r, f-h)}{r} \log\left(\frac{\pi e}{m}\right)$$

must have angular measure at least m. Let Σ_r be this subset. For $z \in \Sigma_r$, using (3.3.4) now yields

$$|zf'(z) - zh'(z)| \leq 2K_1 e^{4\lambda+1} T(r, f-h) \log(\pi e/m) \exp(-cT(r, f-h)) = o(1)$$

as $r \to \infty$ in H. □

It follows from (3.3.1) and (3.3.4) that, for $z \in \Sigma_r$,

$$f(z) \to \infty \quad \text{and} \quad \frac{1}{f(z)} = \frac{1 + o(1)}{h(z)}, \quad \text{as } r \to \infty \text{ in } H'. \tag{3.3.7}$$

Together with (3.3.5) this gives, for $z \in \Sigma_r$,

$$\frac{zf'(z)}{f(z)} = \frac{zh'(z) + o(1)}{h(z)}(1 + o(1)), \quad \text{as } r \to \infty \text{ in } H'. \tag{3.3.8}$$

The hypothesis of Lemma 1.6 is now satisfied by f and $z \in \Sigma_r$ for all sufficiently large $r \in H'$ by (3.3.7). Therefore Lemma 1.6, (3.3.1), (3.3.4) and (3.3.8) now yield

$$\begin{aligned}\frac{\rho(h)^{1-\nu}T(r,h)}{2} < \log|h(z)| &= (1+o(1))\log|f(z)| \\ &\leq (1+o(1))\left(C\left|\frac{zf'(z)}{f(z)}\right| + \log M\right) \\ &\leq 2C\left|\frac{zh'(z)}{h(z)}\right|\end{aligned}$$

for $z \in \Sigma_r$ as $r \to \infty$ in H'. Since Σ_r has angular measure at least m, integrating the above leads to

$$\int_0^{2\pi}\left|\frac{rh'(re^{i\theta})}{h(re^{i\theta})}\right| d\theta \geq \frac{m\rho(h)^{1-\nu}}{4C}T(r,h) \quad \text{for all large } r \in H'.$$

But, by Lemma 3.10(ii), there exist large $r \in H'$ for which

$$\int_0^{2\pi}\left|\frac{rh'(re^{i\theta})}{h(re^{i\theta})}\right| d\theta < K_0\rho(h)T(r,h).$$

Comparing these last two inequalities, we must have that

$$\rho(h)^{\nu} > \frac{m}{4CK_0} = \rho^{\nu}$$

by the choice of m and C_1.

3.4 Proof of Theorem 3.3

Most of the proof of Theorem 3.3 will be contained in the next three lemmas, the first of which builds upon the result of Hayman stated as Lemma 3.6.

Lemma 3.13. *Suppose that for $j = 1, \ldots, N$ the functions $\psi_j(r)$ are positive and non-decreasing on $[e, \infty)$, continuous from the right, and such that $\psi_j(r) = O(\log r)^P$ as $r \to \infty$, for some P. Let $\alpha > 1$ and $\delta > 0$. Then there exist a constant B and a set E of lower logarithmic density at most δ such that, for $r \notin E$,*

$$\psi_j(r^{\alpha}) \leq B\psi_j(r)$$

for each $j = 1, \ldots, N$.

Proof. For $s \geq 1$, define

$$\phi_j(s) = \psi_j(e^s) = O(s^P).$$

Then Lemma 3.6 applies to ϕ_j (we may assume that ψ_j is unbounded) to give

$$\phi_j(\alpha s) < B\phi_j(s)$$

for s outside an exceptional set G_j. The constant B is chosen so large that

$$\overline{\operatorname{logdens}}\, G_j \leq P\left(\frac{\log \alpha}{\log B}\right) \leq \frac{\delta}{N}.$$

Now let $G = \bigcup G_j$ and note that by Lemma 1.5,

$$\underline{\operatorname{logdens}}\, G \leq \delta. \tag{3.4.1}$$

Taking $E = \{r \geq e : \log r \in G\}$ and $r = e^s \notin E$, we now have that, for each j,

$$\psi_j(r^\alpha) = \phi_j(\alpha s) < B\phi_j(s) = B\psi_j(r).$$

Suppose now that $\underline{\operatorname{logdens}}\, E > l > \delta$. Let χ_E be the characteristic function of E. Then

$$L(r) = \int_e^r \chi_E(t)\frac{dt}{t} > l \log r - c$$

for some constant c and all $r \geq e$. We now calculate

$$\begin{aligned}
\int_{[1,s]\cap G} \frac{d\tau}{\tau} &= \int_e^r \chi_E(t)\frac{dt}{t\log t} = \int_e^r \frac{dL(t)}{\log t} \\
&= \frac{L(r)}{\log r} + \int_e^r \frac{L(t)}{t(\log t)^2}\, dt \\
&> \frac{L(r)}{\log r} + \int_e^r \left(\frac{l}{t\log t} - \frac{c}{t(\log t)^2}\right) dt \\
&> l - \frac{c}{\log r} + l\log\log r + \frac{c}{\log r} - c = l\log s + l - c,
\end{aligned}$$

so that $\underline{\operatorname{logdens}}\, G \geq l > \delta$ contradicting (3.4.1). Hence, the lower logarithmic density of E does not exceed δ and the lemma is proved. □

We apply the previous lemma to obtain the following pointwise estimate for the logarithmic derivative of a slowly-growing meromorphic function.

Lemma 3.14. *Let h be meromorphic such that $T(r,h) = O(\log r)^P$ for some P, and let $0 < \delta \leq 1$. Then*

$$M\left(r, \frac{zh'}{h}\right) = o(T(r,h))$$

as $r \to \infty$ outside a set of lower logarithmic density δ.

We remark that we can in fact take $\delta = 0$ in the above statement, by applying, for example, [39, Lemma 6].

Proof of Lemma 3.14. We may assume that the function h is transcendental. Define $n(r) = n(r,h) + n(r,1/h)$. Then

$$n(r) \leq \frac{2T(r^2,h)}{\log r} + o(1) = O(\log r)^{P-1}. \tag{3.4.2}$$

Using (3.4.2) and applying Lemma 3.13 to $n(r)$ and $T(r,h)$, we obtain a constant B and a set E of lower logarithmic density at most $\delta/2$ such that, for $r \notin E$,

$$T(r^2,h) \le BT(r,h), \qquad n(r^2) \le Bn(r).$$

In particular, by using (3.4.2) again,

$$n(r^2) = O\left(\frac{T(r,h)}{\log r}\right) \quad \text{for } r \notin E. \tag{3.4.3}$$

Since h has order zero, we see from the standard product representation for meromorphic functions of order less than 1 [20, p.21] that

$$\left|\frac{h'(z)}{h(z)}\right| \le \sum \frac{1}{|z-a_k|} \le \sum \frac{1}{|r-|a_k||}, \tag{3.4.4}$$

where $r = |z|$, and the a_k are the zeroes and poles of h repeated according to multiplicity. Suppose that $r \in [2^{n-1}, 2^n)$ and let $s = 2^n$ and

$$\mu(r,t) = \#\{|a_k| < s(\log s)^P : |r-|a_k|| < t\}.$$

Cartan's Lemma (Lemma 3.8) gives, with $A = 6$ and $h_n = \delta s/96$,

$$\frac{\mu(r,t)}{t} < \frac{96n(s(\log s)^P)}{e\delta s}$$

for $0 < t < \infty$ and $r \in [2^{n-1}, 2^n) \setminus F_n$, where the exceptional set F_n has measure at most $\delta s/8$. Since μ is integer-valued, we have

$$\mu(r,t) = 0 \quad \text{for} \quad t \le t_0 = \frac{e\delta s}{96n(s(\log s)^P)}.$$

Therefore, for $r \in [2^{n-1}, 2^n) \setminus F_n$,

$$\begin{aligned}
\sum_{|a_k|\le r(\log r)^P} |r-|a_k||^{-1} &\le \sum_{|a_k|<s(\log s)^P} |r-|a_k||^{-1} = \int_{t_0}^{s(\log s)^P} \frac{d\mu(r,t)}{t} \\
&= \frac{\mu(r,s(\log s)^P)}{s(\log s)^P} + \int_{t_0}^{s(\log s)^P} \frac{\mu(r,t)}{t^2}\,dt \\
&\le \frac{96n(s(\log s)^P)}{e\delta s}\left(1 + \int_{t_0}^{s(\log s)^P} \frac{dt}{t}\right) \\
&= \frac{96n(s(\log s)^P)}{e\delta s}\left(1 + \log\frac{96n(s(\log s)^P)(\log s)^P}{e\delta}\right).
\end{aligned}$$

Noting that $s(\log s)^P \le 2r(\log 2r)^P \le r^2$, for r at least some large R_0, now gives

$$\sum_{|a_k|\le r(\log r)^P} |r-|a_k||^{-1} \le \frac{96n(r^2)}{e\delta r}\left(1 + \log\frac{96n(r^2)(\log 2r)^P}{e\delta}\right) \tag{3.4.5}$$

for $r \in [2^{n-1}, 2^n) \setminus F_n$. Hence if $2^{m-1} < R < 2^m$, then (3.4.5) holds for $r \in [R_0, R]$ outside the set $\bigcup_{n=1}^m F_n$, which has measure at most $\delta(\frac{1}{4} + \frac{1}{2} + \ldots + 2^{m-3}) < \delta R/2$. Therefore, (3.4.5) holds for $r \notin F$, where $\overline{\text{logdens}}\, F \leq \overline{\text{dens}}\, F \leq \delta/2$. Using (3.4.2) and (3.4.3), this gives

$$\sum_{|a_k| \leq r(\log r)^P} |r - |a_k||^{-1} = O\left(\frac{T(r,h)\log\log r}{r\log r}\right) = o\left(\frac{T(r,h)}{r}\right) \tag{3.4.6}$$

as $r \to \infty$ outside $E \cup F$. Furthermore, $\underline{\text{logdens}}(E \cup F) \leq \delta$ by Lemma 1.5.

We now consider those a_k for which $|a_k| > r(\log r)^P$. For such a_k, we have

$$|r - |a_k|| > \frac{|a_k|}{2}$$

provided r is large. Using this,

$$\begin{aligned} \sum_{|a_k| > r(\log r)^P} |r - |a_k||^{-1} &\leq \int_{r(\log r)^P}^{\infty} \frac{2}{t}\, dn(t) \\ &\leq 2\int_{r(\log r)^P}^{\infty} \frac{n(t)}{t^2}\, dt \leq C\int_{r(\log r)^P}^{\infty} \frac{(\log t)^{P-1}}{t^2}\, dt \end{aligned} \tag{3.4.7}$$

for some constant C by (3.4.2).

It is now claimed that, for $q \in \mathbb{R}$,

$$I_q = \int_R^{\infty} \frac{(\log t)^q}{t^2}\, dt = O\left(\frac{(\log R)^q}{R}\right), \quad \text{as } R \to \infty.$$

For $q \leq 0$ this is trivial, and

$$I_q = \int_R^{\infty} \left[\frac{q(\log t)^{q-1}}{t^2} - \frac{d}{dt}\left(\frac{(\log t)^q}{t}\right)\right] dt = qI_{q-1} + \frac{(\log R)^q}{R},$$

so that the claim holds for all q by induction. Using this and (3.4.7) now gives that

$$\sum_{|a_k| > r(\log r)^P} |r - |a_k||^{-1} = O\left(\frac{(\log(r(\log r)^P))^{P-1}}{r(\log r)^P}\right) = o\left(\frac{1}{r}\right). \tag{3.4.8}$$

Putting together (3.4.4), (3.4.6) and (3.4.8) now yields

$$\left|\frac{zh'(z)}{h(z)}\right| = o(T(r,h))$$

as $|z| = r \to \infty$ outside a set of lower logarithmic density not exceeding δ. □

The proof of the next lemma is due to James Langley.

Lemma 3.15. *Let G be a transcendental meromorphic function of positive lower order and suppose that 0 is a deficient value of G. Then, for all r outside a set of finite logarithmic measure, there exists some z with $|z| = r$ such that*

$$|G(z)| = o(1) \quad \textit{and} \quad |zG'(z)| = o(1)$$

as $r \to \infty$.

Proof. Write $T(r) = T(r, G)$ and let $p(s) = T(e^s)^{\frac{1}{2}}$. Applying Borel's Lemma [20, Lemma 2.4] to $p(s)$ gives

$$T\Big(\exp\Big(s + T(e^s)^{-\frac{1}{2}}\Big)\Big)^{\frac{1}{2}} = p\bigg(s + \frac{1}{p(s)}\bigg) < 2p(s) = 2T(e^s)^{\frac{1}{2}}$$

outside a set of values of s of finite linear measure. Taking $r = e^s$ and $R = r\exp(T(r)^{-\frac{1}{2}})$, this becomes

$$T(R) < 4T(r) \tag{3.4.9}$$

for r outside a set of finite logarithmic measure. Let

$$H_r = \left\{t \in [0, 2\pi] : \log|G(re^{it})| < -\frac{1}{2}\delta(0, G)T(r)\right\}.$$

Then

$$\frac{1}{2\pi}\int_{[0,2\pi]\setminus H_r} \log^+\frac{1}{|G(re^{it})|}\,dt \le \frac{1}{2}\delta(0, G)T(r),$$

so that by the definition of deficiency

$$\frac{1}{2\pi}\int_{H_r} \log^+\frac{1}{|G(re^{it})|}\,dt \ge \frac{1}{2}\delta(0, G)T(r)(1 - o(1)). \tag{3.4.10}$$

Let $m(r)$ be the measure of H_r. Lemma III of [11] gives that

$$\frac{1}{2\pi}\int_{H_r} \log^+\frac{1}{|G(re^{it})|}\,dt \le \frac{11R}{R - r}m(r)\left(1 + \log^+\frac{1}{m(r)}\right)T(R, 1/G). \tag{3.4.11}$$

Observe that

$$\frac{R}{R - r} = \frac{\exp(T(r)^{-\frac{1}{2}})}{\exp(T(r)^{-\frac{1}{2}}) - 1} = (1 + o(1))\left(T(r)^{-\frac{1}{2}} + O\big(T(r)^{-1}\big)\right)^{-1} = T(r)^{\frac{1}{2}}(1 + o(1))$$

and that for small $m(r)$,

$$1 + \log^+\frac{1}{m(r)} < \frac{1}{m(r)^{\frac{1}{4}}}.$$

Using (3.4.9), (3.4.10) and the above, the inequality (3.4.11) becomes

$$\frac{1}{2}\delta(0, G)(1 - o(1)) \le 44m(r)^{\frac{3}{4}}T(r)^{\frac{1}{2}}(1 + o(1)),$$

and it follows that $m(r) > T(r)^{-\frac{3}{4}}$ for all r outside a set of finite logarithmic measure. Now consider

$$H'_r = \left\{t \in H_r : \log\left|\frac{G'(re^{it})}{G(re^{it})}\right| > T(r)^{\frac{7}{8}}\right\}.$$

If $H'_r = H_r$, then

$$m(r, G'/G) \ge \frac{1}{2\pi}\int_{H'_r} \log^+\left|\frac{G'(re^{it})}{G(re^{it})}\right|\,dt \ge \frac{m(r)}{2\pi}T(r)^{\frac{7}{8}} > \frac{T(r)^{\frac{1}{8}}}{2\pi},$$

but for r outside a set of finite measure, this contradicts the lemma of the logarithmic derivative (Lemma 1.2) as G has positive lower order. Therefore, we can pick $z = re^{it}$ with $t \in H_r \setminus H_r'$, and this z satisfies

$$\log |G(z)| < -\frac{1}{2}\delta(0, G)T(r),$$

$$\log |zG'(z)| < \log r + T(r)^{\frac{7}{8}} - \frac{1}{2}\delta(0, G)T(r). \qquad \square$$

We now proceed to prove Theorem 3.3. Let f and h be as in the hypothesis, but assume that $\delta(0, f-h) > 0$. By Theorem 3.1, we may assume that f has infinite lower order. By the remark following Lemma 3.7, there exists a positive constant d such that

$$\log L(r, h) > dT(r, h)$$

on a set of logarithmic density 1. Applying Lemma 3.15 to $f-h$ gives, for each r outside a set of finite logarithmic measure, a point $z = z_r$ with $|z| = r$, such that

$$f(z) = h(z) + o(1) \quad \text{and} \quad zf'(z) = zh'(z) + o(1)$$

as $r \to \infty$. Arguing as in the proofs of Theorems 3.1 and 3.2, this leads to

$$\frac{zf'(z)}{f(z)} = \frac{zh'(z) + o(1)}{h(z)}(1 + o(1)), \quad \text{for } z = z_r,$$

as $r \to \infty$ on a set of logarithmic density 1. Combining Lemmas 1.6 and 3.14 with the above now gives, for $z = z_r$,

$$dT(r, h) < \log |h(z)| = \log |f(z) + o(1)| = O\left(\left|\frac{zf'(z)}{f(z)}\right|\right) = O\left(\left|\frac{zh'(z)}{h(z)}\right|\right) = o(T(r, h))$$

as $r \to \infty$ outside a set of small lower logarithmic density. This contradiction completes the proof of the theorem.

Chapter 4

Deficient values of periodic derivatives

Entire periodic functions can have deficient values; for example, the function $e^z + a$ omits the value a. However, the derivative of this example has no non-zero finite deficient values. Theorem 4.1 below shows that this holds in general for any derivative of a periodic meromorphic function of finite lower order. Some counterexamples of infinite lower order are constructed in Section 4.1. The results of this chapter have previously been published in [48].

Theorem 4.1. *Let f be a periodic meromorphic function of finite lower order. Then f' has no non-zero finite deficient values.*

The proof of Theorem 4.1 will use the following elementary lemma.

Lemma 4.2. *For $r > 0$ and small positive m, let $L(\phi)$ be the length of the interval*

$$\left\{\mathrm{Re}\left(re^{i\theta}\right) : \theta \in [\phi, \phi + m]\right\}.$$

Then $L(\phi) \geq r\left(1 - \cos\frac{m}{2}\right)$.

Proof.

$$L(\phi) = \begin{cases} r(1 - \cos(\phi + m)), & \phi \in \left[-\frac{m}{2}, 0\right] \\ r(\cos\phi - \cos(\phi + m)), & \phi \in \left[0, \frac{\pi}{2} - \frac{m}{2}\right]. \end{cases}$$

L is clearly increasing over $\left[-\frac{m}{2}, 0\right]$. For $\phi \in \left(0, \frac{\pi}{2} - \frac{m}{2}\right)$,

$$L'(\phi) = r(\sin(\phi + m) - \sin\phi) \geq 0$$

and so L is in fact increasing on $\left[-\frac{m}{2}, \frac{\pi}{2} - \frac{m}{2}\right]$. By symmetry considerations, we see that this implies that $L(\phi) \geq L\left(-\frac{m}{2}\right) = r\left(1 - \cos\frac{m}{2}\right)$ for all ϕ. □

We now establish the main result.

Proof of Theorem 4.1. Let f be a periodic meromorphic function of finite lower order and suppose that f' has a non-zero finite deficient value. Without loss of generality we may take both the period and the deficient value to be 1. Let δ be such that $\delta(1, f') > 3\delta > 0$.

Using Fuchs' small arcs lemma (Lemma 3.9), we find a small positive m and a set $J \subseteq [0, \infty)$ of upper logarithmic density at least $\frac{1}{2}$ such that, if $r \in J$ is large and F_r is a subinterval of $[0, 2\pi]$ of length m, then

$$\int_{F_r} \left| \frac{rf''(re^{i\theta})}{f'(re^{i\theta}) - 1} \right| d\theta < \delta T(r, f'). \tag{4.0.1}$$

Fix $r \in J$ large such that

$$r\left(1 - \cos\frac{m}{2}\right) > 2,$$

$$m\left(r, \frac{1}{f' - 1}\right) > 3\delta T(r, f'),$$

and

$$2mr \exp(-\delta T(r, f')) < 1.$$

Here we can satisfy the second inequality by the definition of deficiency, and the third by using Lemma 1.4 and the fact that f' must be transcendental. Choose z_0 satisfying $|z_0| = r$ and $\log |f'(z_0) - 1| \leq -3\delta T(r, f')$.

Let Ω be an arc of $S(0, r)$ with endpoint z_0 and angular measure m. Then using (4.0.1) we see that

$$\log |f'(z) - 1| < -2\delta T(r, f'), \qquad z \in \Omega. \tag{4.0.2}$$

For $n \in A = \mathbb{Z} \cap [-2r, 2r] \backslash \{0\}$, the circle $S(0, r)$ intersects $S(n, r)$ at one or two points with real part $\frac{n}{2}$. By Lemma 4.2 and the choice of r, the interval $\{\operatorname{Re} z : z \in \Omega\}$ has length at least $r\big(1 - \cos\frac{m}{2}\big) > 2$, and so it must contain $\frac{N-1}{2}, \frac{N}{2}$ for some $N - 1, N \in A$. Hence Ω meets $S(N - 1, r)$ and $S(N, r)$. We pick points of intersection α and β respectively, as shown in Figure 4.1. Note that $\alpha + 1 \in S(N, r)$, and that reflection of Ω in the line $\operatorname{Re} z = \frac{N}{2}$ gives an arc Ω' of $S(N, r)$ that contains $\alpha + 1$ and β. Using (4.0.1) and the periodicity of f' and f'' we have, for some θ_0,

$$\int_{\Omega'} \left| \frac{f''(z)}{f'(z) - 1} \right| |dz| = \int_{\theta_0}^{\theta_0 + m} \left| \frac{f''(N + re^{i\theta})}{f'(N + re^{i\theta}) - 1} \right| r d\theta < \delta T(r, f').$$

Since $\beta \in \Omega \cap \Omega'$, the above and (4.0.2) yield

$$\log |f'(z) - 1| < -\delta T(r, f'), \qquad z \in \Omega \cup \Omega'.$$

Let γ be the path joining α to β along Ω followed by the path from β to $\alpha + 1$ along Ω'.

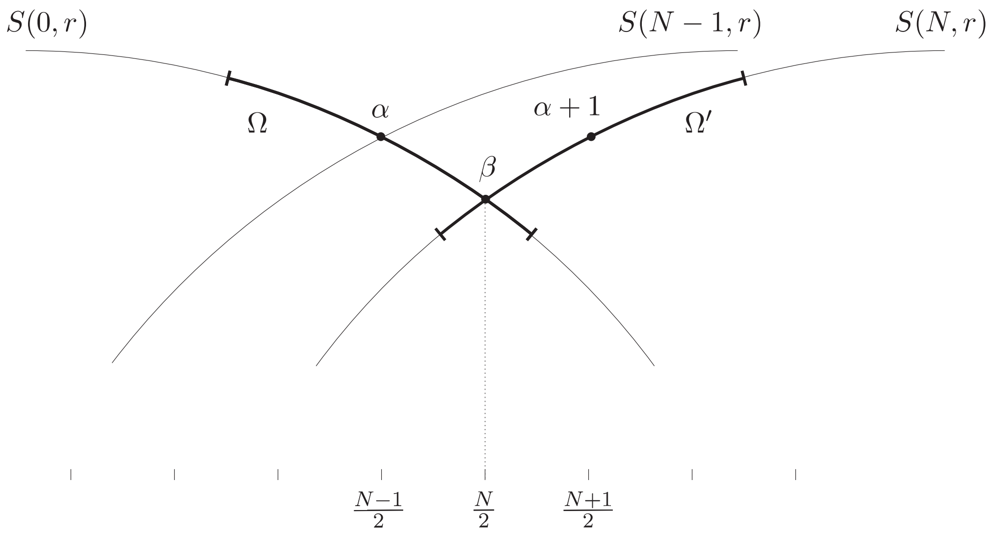

Figure 4.1: Arrangement of circles, arcs and points.

Then the length of γ is at most $2mr$ and so

$$\left|\int_\gamma (f'(z)-1)\,dz\right| \le \text{length}(\gamma)\max_{z\in\gamma}\{|f'(z)-1|\} < 2mr\exp(-\delta T(r,f')) < 1,$$

by recalling our choice of r. However, this is a contradiction since

$$\int_\gamma (f'(z)-1)\,dz = f(\alpha+1)-(\alpha+1)-(f(\alpha)-\alpha) = -1. \qquad \square$$

4.1 Infinite order counterexamples

The periodic entire function

$$\int_0^{e^z} \frac{1-e^t}{t}\,dt$$

has derivative $1-e^{e^z}$, which omits the value 1. In fact, there exist derivatives of periodic entire functions having arbitrarily many deficient values. The rest of this section is devoted to constructing such an example.

For an integer $q \ge 2$, define

$$F(z) = \int_0^{e^z} \frac{1}{w}\left(\int_0^w e^{-t^q}dt\right)dw.$$

Then F is entire, periodic and has derivative

$$F'(z) = \int_0^{e^z} e^{-t^q}dt. \tag{4.1.1}$$

It shall be useful to define the function $G(z) = e^{-e^{qz}}$ and the set S as the union of the sectors

$$S_k = \left\{z : \left|\arg z - \frac{2\pi k}{q}\right| \le \frac{\pi}{2q}\right\}.$$

Lemma 4.3. *Taking $G(z)$ and S_k as above, the contribution to $m(r,1/G)$ of the set where $e^z \in S_k$ is*

$$J_k = \frac{T(r,G)}{q}(1+o(1)), \quad \text{as } r \to \infty.$$

We delay the proof of Lemma 4.3. To exhibit the deficiencies of F', let $\omega = e^{2\pi i/q}$ and, for integer k, let

$$I_k = \int_0^{\omega^k \infty} e^{-t^q}\,dt = \omega^k I_0, \tag{4.1.2}$$

where the path of integration is given by $t = \omega^k s$ for $s \in [0,\infty)$. Note that $I_k \neq 0, \infty$ and $I_j \neq I_k$ for $0 \le j < k < q$. By Cauchy's Theorem,

$$F'(z) = I_k - \int_{\gamma_k} e^{-t^q}\,dt,$$

where γ_k follows the circular arc from e^z to $\omega^k|e^z|$ and then the ray $\omega^k s$ for $s \in [|e^z|, \infty)$.

Suppose now that $e^z \in S_k$. For t lying on γ_k, we have that

$$\left|e^{-t^q}\right| = e^{-\operatorname{Re}(t^q)} \le e^{-\operatorname{Re}(e^{qz})} = |G(z)|$$

(since $\gamma_k \subseteq S_k$ and $t \mapsto t^q$ maps S_k to the right half-plane). Writing

$$e^{-t^q} = \frac{qt^{q-1}e^{-t^q}}{qt^{q-1}}$$

and integrating by parts yields

$$\int_{\gamma_k} e^{-t^q}\,dt = \frac{e^{-e^{qz}}}{qe^{(q-1)z}} - \frac{q-1}{q}\int_{\gamma_k} \frac{e^{-t^q}}{t^q}\,dt.$$

Hence, when $e^z \in S_k$,

$$|F'(z) - I_k| \le |G(z)| \left(\frac{e^{|(q-1)z|}}{q} + \frac{q-1}{q}\int_{\gamma_k} \frac{|dt|}{|t|^q} \right) = O(e^{qr}|G(z)|)$$

as $|z| = r \to \infty$. Using this together with Lemma 4.3 now leads to

$$\frac{T(r,G)}{q}(1+o(1)) \le m\left(r, \frac{1}{F' - I_k}\right) + O(r), \quad \text{as } r \to \infty. \tag{4.1.3}$$

If $e^z \in S$ and t lies on the straight line joining the origin to e^z, then $|e^{-t^q}| \le 1$ so that $|F'(z)| \le |e^z|$ by (4.1.1). If instead $e^z \notin S$ and t lies on the straight line joining the origin to e^z, we see that $|e^{-t^q}| \le |G(z)|$ so that by (4.1.1) we have $|F'(z)| \le |e^z G(z)|$. Therefore,

$$T(r,G) \ge T(r,F') - r. \tag{4.1.4}$$

The function $G(z) = e^{-e^{qz}}$ has infinite lower order (see (4.1.5) below), hence the $O(r)$ terms in inequalities (4.1.3) and (4.1.4) are certainly $o(T(r,G))$. Comparing (4.1.3) with (4.1.4) now reveals that $\delta(I_k, F') \ge 1/q$ for $k = 0, \dots, q-1$. Since F' is entire, the sum of the deficiencies over finite values cannot exceed 1, and so we must have equality here.

Proof of Lemma 4.3. We first observe that if $e^z \notin S$, then $\mathrm{Re}(e^{qz}) \le 0$ and so $|G(z)| \ge 1$. Hence, these points contribute nothing to $m(r, 1/G)$, and so

$$T(r, G) = J_0 + \ldots + J_{q-1} + O(1).$$

Thus it will suffice to prove that $J_k = J_l + o(T(r, G))$.

We remark that

$$e^z \in S_k \quad \Leftrightarrow \quad \left|\mathrm{Im}(z) - 2\pi\left(n + \frac{k}{q}\right)\right| \le \frac{\pi}{2q} \;\text{ for some integer } n.$$

From [20, p.7] we have that

$$T(r, G) \sim \frac{e^{qr}}{\sqrt{2\pi^3 qr}}. \tag{4.1.5}$$

Calculate, for $z = re^{i\theta} \in S_k$,

$$\log^+ \frac{1}{|G(re^{i\theta})|} = \log^+ \left|e^{e^{qz}}\right| = \mathrm{Re}(e^{qz}) = e^{qr\cos\theta}\cos(qr\sin\theta) \tag{4.1.6}$$

and fix a small angle $\alpha > 0$. Then for $\theta \in [\alpha, 2\pi - \alpha]$,

$$\log^+ \frac{1}{|G(re^{i\theta})|} = O(e^{qr\cos\alpha}) = o(T(r, G))$$

by (4.1.5). Note also that the angular measure of $\{z : |\operatorname{Im} z| \le 4\pi\}$ with respect to $S(0, r)$ is $O(1/r)$, so that the contribution to J_k from this region is $O(e^{qr}/r) = o(T(r, G))$.

Let J_k^+ and J_k^- denote the contributions to J_k from the upper and lower half-planes respectively. It now follows from all of the above that, for $k = 0, \ldots, q$,

$$J_k^+ = \sum_{n=1}^{N} H_{k,n} + o(T(r, G)),$$

where $H_{k,n}$ is the contribution to J_k from

$$E_{k,n} = S(0, r) \cap \left\{z : \mathrm{Re}(z) > 0, \left|\mathrm{Im}(z) - 2\pi\left(n + \frac{k}{q}\right)\right| \le \frac{\pi}{2q}\right\}$$

and N is the least integer exceeding $1 + (r/2\pi)\sin\alpha$. In particular, $2\pi N \approx r\sin\alpha$ and N is independent of k. See Figure 4.2.

Using (4.1.6) and changing from the angular variable θ to the scaled imaginary part $t = qr\sin\theta$ shows that

$$\begin{aligned} H_{k,n} &= \int_{\{\theta : re^{i\theta} \in E_{k,n}\}} e^{qr\cos\theta}\cos(qr\sin\theta)\, d\theta \\ &= \int_{2\pi(nq+k)-\pi/2}^{2\pi(nq+k)+\pi/2} \cos t \frac{e^{\sqrt{q^2r^2 - t^2}}}{\sqrt{q^2r^2 - t^2}}\, dt. \end{aligned} \tag{4.1.7}$$

For $0 < \theta < 2\alpha$, the variable t is positive but small compared to qr, and therefore

$$1 < \sqrt{q^2r^2 - (t + 2\pi)^2} < \sqrt{q^2r^2 - t^2}.$$

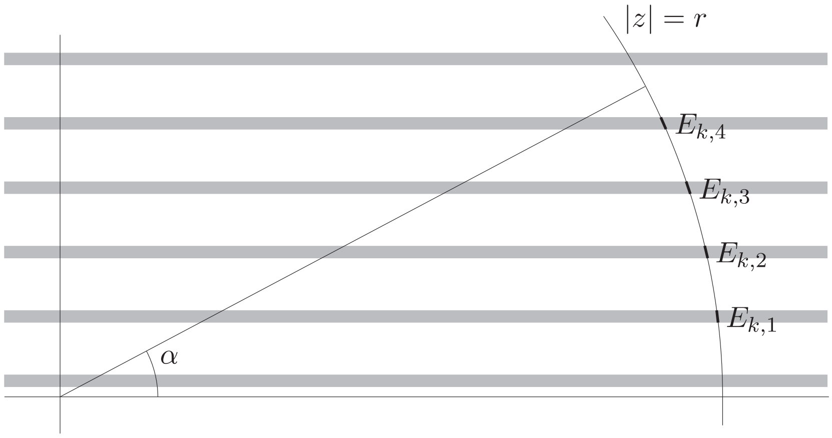

Figure 4.2: The sets $E_{k,n}$ shown for $N = 4$. The shaded set is $\{z : e^z \in S_k\}$.

Since e^x/x is increasing for $x > 1$, this implies that

$$\frac{e^{\sqrt{q^2r^2-(t+2\pi)^2}}}{\sqrt{q^2r^2-(t+2\pi)^2}} < \frac{e^{\sqrt{q^2r^2-t^2}}}{\sqrt{q^2r^2-t^2}}. \tag{4.1.8}$$

Hence $H_{k+1,n} \leq H_{k,n}$ by (4.1.7), and therefore

$$J^+_{k+1} \leq J^+_k + o(T(r,G)) \quad \text{for} \quad k = 0, \ldots, q-1.$$

However, $J^+_0 = J^+_q$ because $S_0 = S_q$, and so we must have that $J^+_k = J^+_l + o(T(r,G))$ for all k, l.

This argument can be repeated to show that $J^-_{k+1} \geq J^-_k + o(T(r,G))$, and hence we have equality (in this case t is negative so inequality (4.1.8) is reversed). □

Chapter 5

Real meromorphic functions

A meromorphic function is said to be real if $f(z)$ is real or infinite whenever z is real. Many functions from real analysis extend to real meromorphic functions on the complex plane; for example, $\sin z$, e^z and rational functions with real coefficients. It is easily seen that any real meromorphic function f satisfies the reflection property $f(\overline{z}) = \overline{f(z)}$.

The study of real entire functions has a long history and will be the subject of Chapter 6. The starting point for this chapter is the following theorem of Hinkkanen and Rossi [31].

Theorem 5.1 ([31])**.** *Suppose that f is a non-entire real transcendental meromorphic function with only real poles, and that the zeroes of f and f' are real. If f' omits a non-zero value α, then the omitted value is real and*

$$f(z) = \alpha z - \lambda \tan(cz + d) + A, \tag{5.0.1}$$

where λ, c, d and A are real and $\lambda, c \neq 0$. Furthermore, the zeroes of f'' are real.

This result arose from an endeavour to determine all meromorphic functions f with only real poles for which f, f' and f'' each have only real zeroes. Hellerstein, Shen and Williamson [25, 26, 27] settled this question for all entire functions and for those meromorphic functions that are not a constant multiple of a real function. The problem remains open for real meromorphic functions, although there are some other partial results similar to Theorem 5.1. The real entire case is discussed in more detail in Section 6.1.1.

We aim to generalise Theorem 5.1 by adopting weaker hypotheses: the functions studied in the sequel are permitted arbitrary zeroes and finitely many non-real poles and critical points. In addition, the derivative must either take some non-zero value only finitely often (Theorems 5.2 and 5.3), or at least have a non-zero deficient value (Corollary 5.5). The results and proofs of this chapter appeared in [50].

5.1 Two characterization theorems

The following theorem characterizes all functions that fail to satisfy Hinkkanen and Rossi's hypothesis at only finitely many points. In this case, the restriction on the zeroes of f is shown to be a consequence rather than a prerequisite.

Theorem 5.2. *Suppose that f is a real transcendental meromorphic function such that all but finitely many of the zeroes and poles of f' are real, and $f'(z) = \alpha$ only finitely often for some finite non-zero α. Then f can be written in the form*

$$f(z) = \alpha z + i\lambda \frac{P(z)e^{icz} - \overline{P(\overline{z})}e^{-icz}}{P(z)e^{icz} + \overline{P(\overline{z})}e^{-icz}} + A, \tag{5.1.1}$$

where α, λ and A are real constants, $\alpha\lambda \neq 0$, $c > 0$ and P is a polynomial with zeroes $a_1, \ldots, a_N$ (repeated to multiplicity) such that $a_j \neq \overline{a_k}$.

In the converse direction, if f is given by (5.1.1) then all but finitely many of the zeroes and poles of f and f'' are real, and the equation $f'(z) = \alpha$ has at most $2N$ solutions, counting with multiplicities. Moreover, all but finitely many of the zeroes of f' are real if and only if either $0 < \lambda c/\alpha < 1$ or

$$\lambda c = \alpha \quad \text{and} \quad \sum_{j=1}^{N} \frac{\operatorname{Im} a_j}{|x - a_j|^2} < 0 \quad \text{as real } x \to \pm\infty. \tag{5.1.2}$$

Lemma 5.10 below shows that if $\lambda c = \alpha$ then the condition (5.1.2) is satisfied if $\sum \operatorname{Im} a_j < 0$, and is not satisfied if $\sum \operatorname{Im} a_j > 0$.

Before proceeding we briefly consider some examples. If we take $P(z) \equiv e^{id}$, then we see that (5.1.1) simply reduces to (5.0.1). Choosing instead $P(z) = z + i$ and $c = 1$ gives

$$f(z) = \alpha z + \lambda \frac{z \sin z + \cos z}{\sin z - z \cos z} + A, \qquad f'(z) = \alpha - \lambda \frac{z^2}{(\sin z - z \cos z)^2}.$$

In this case the derivative omits α, showing that the relevant part of Theorem 5.2 cannot be changed to "$f'(z) = \alpha$ has $2N$ solutions".

Kohs and Williamson proved in [33] that Hinkkanen and Rossi's Theorem 5.1 essentially continues to hold without the demand that f is real and transcendental. By an extension of the method of Kohs and Williamson, we show that in the statement of Theorem 5.2 we may replace the assumption that the function is real by the condition that it has infinitely many poles.

Theorem 5.3. *Let g be a transcendental meromorphic function such that all but finitely many of the zeroes and poles of g' are real, and $g'(z) = \beta$ only finitely often for some finite non-zero β. Then all but finitely many of the zeroes of g'' are real, and either* *(i) we have $g = \beta f + d$, where d is a constant and f is a real function satisfying the hypothesis of Theorem 5.2 with $\alpha = 1$; or*

(ii) we have $g(z) = R(z)e^{icz} + \beta z + d$, where R is a rational function, c and d are constants and c is real.

The following example demonstrates that case (ii) can occur, and hence also that Theorem 5.2 may fail for strictly non-real functions with finitely many poles. Let α be non-zero and take

$$f(z) = \alpha z + \frac{3 - iz}{z - i}\,\alpha e^{iz}.$$

Then f has only one pole and clearly cannot be written in the form (5.1.1). However, the derivative

$$f'(z) = \alpha + \left(\frac{z+i}{z-i}\right)^2 \alpha e^{iz}$$

only takes the value α at one point and has finitely many non-real zeroes. To establish this last claim, write

$$\frac{(z-i)^2 f'(z)}{\alpha e^{iz/2}} = (z-i)^2 e^{-iz/2} + (z+i)^2 e^{iz/2}.$$

It will be shown in Lemma 5.16 that functions of this form have only finitely many non-real zeroes.

5.2 An asymptotic result

We now weaken the hypotheses of Theorems 5.1 and 5.2 by allowing $f'(z) = \alpha$ infinitely often, and just requiring α to be a deficient value of f'. Under these conditions, f has the same asymptotic behaviour away from the real axis as was found in the two earlier theorems. We shall prove this as a corollary to the following result.

Theorem 5.4. *Let g be a real transcendental meromorphic function of positive lower order. Assume that g has a non-zero finite deficient value α, and that all but finitely many of the zeroes, poles and α-points of g are real.*

(i) If α is real, then for $\varepsilon > 0$,

$$g(z) \sim \alpha \quad \textit{as } z \to \infty \quad \textit{with} \ \ \varepsilon < |\arg z| < \pi - \varepsilon.$$

(ii) If α is non-real, then g takes the values α and $\overline{\alpha}$ only finitely often and

$$g(z) = \mathrm{Re}(\alpha) + i\ \mathrm{Im}(\alpha)\frac{P(z)e^{icz} - \overline{P(\overline{z})}e^{-icz}}{P(z)e^{icz} + \overline{P(\overline{z})}e^{-icz}},$$

where c is real and P is a polynomial.

Remark.

1. In case (ii) of the above, the function g is asymptotic to α in one component of $\varepsilon < |\arg z| < \pi - \varepsilon$, and is asymptotic to $\overline{\alpha}$ in the other component.

2. If g has zero lower order and a deficient value α, then by a result of [18] similar to Lemma 3.7, there exist a positive constant d, and a set of radii r with upper logarithmic density one, such that $\log |g(re^{i\theta}) - \alpha| < -dT(r, g)$. That is, $g(z) \sim \alpha$ on whole circles of suitable radius. It follows that g has no other deficient values, and that if g is a real function then α must be real.

Using the fact that f and f' have equal lower order [21], we establish a corollary to Theorem 5.4. As a transcendental derivative cannot take two finite values only finitely often (see Lemma 5.9 below), applying Theorem 5.4 to f' and then integrating yields the following result.

Corollary 5.5. *Let f be a real transcendental meromorphic function of positive lower order. Assume that f' has a non-zero finite deficient value α, and that all but finitely many of the poles of f, and the zeroes and α-points of f', are real. Then α is real and, for $\varepsilon > 0$,*

$$f(z) \sim \alpha z \quad \text{as } z \to \infty \quad \text{with } \varepsilon < |\arg z| < \pi - \varepsilon.$$

We present an example of a function that satisfies the hypothesis of Corollary 5.5 but not that of Theorem 5.2. Let the real transcendental function h be given by

$$h(z) = \frac{1}{3}\tan^3 z - 3\tan z + 4z.$$

Observe that h has only real poles and that the derivative

$$h'(z) = \tan^2 z \sec^2 z - 3\sec^2 z + 4 = (\tan^2 z - 1)^2$$

has only real zeroes. Recalling that $\tan^2 z$ omits -1, we see that $h'(z) = 4$ if and only if $\tan z = \pm\sqrt{3}$. As all the zeroes of $\tan z \pm \sqrt{3}$ are real and simple, it follows that $h'(z) = 4$ only for real z, and that

$$n\left(r, \frac{1}{h' - 4}\right) = \frac{4r}{\pi} + O(1), \quad r \to \infty.$$

By calculating

$$T(r, h') = 2T(r, \tan^2 z - 1) = 4T(r, \tan z) + O(1) = \frac{8r}{\pi} + O(1), \quad r \to \infty,$$

we find that

$$\delta(4, h') = 1 - \limsup_{r\to\infty} \frac{N(r, 1/(h' - 4))}{T(r, h')} = \frac{1}{2},$$

so that 4 is a deficient value of h'.

5.3 Proof of Theorem 5.2

5.3.1 Preliminaries

The first lemma given here is contained in a more general result due to Edrei [9].

Lemma 5.6 ([9]). *Let f be meromorphic with only finitely many non-real zeroes and poles, and only finitely many non-real roots of $f^{(n)}(z) = \alpha$, for some $\alpha \in \mathbb{C} \setminus \{0\}$ and $n \geq 0$. If*

$$\delta(0, f) + \delta(\infty, f) + \delta(\alpha, f^{(n)}) > 0,$$

then the order of f does not exceed one.

Lemma 5.7 ([19, Corollary 1]). *Let f be meromorphic of finite order ρ, let $\varepsilon > 0$ and let*

$$H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_m, j_m)\}$$

be a finite set of pairs of integers that satisfy $k_q > j_q \geq 0$ for $q = 1, \ldots, m$. Then for all $\psi \in [0, 2\pi)$ outside a set of zero measure, there exists $R(\psi) > 1$ with the following property: for all z satisfying $\arg z = \psi$ *and $|z| \geq R(\psi)$, and for all $(k, j) \in H$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}.$$

We now state a version of the classical Phragmén-Lindelöf principle.

Lemma 5.8 ([57, Theorem 5.61]). *Let $R > 0$ and let $-\pi \leq a < b \leq \pi$. Let f be analytic on a domain containing*

$$S = \{z : |z| \geq R,\ a \leq \arg z \leq b\}.$$

Assume that f is bounded on the boundary of S and that

$$\log |f(z)| < |z|^{\sigma}$$

for all large z in S, where $\sigma < \pi/(b-a)$. Then f is bounded in S.

In connection with Lemma 5.8, it shall be useful to note that if a function f is meromorphic on the plane with finitely many poles and finite order $\rho < \sigma$, then after factoring out the poles, Lemma 1.3 shows that $\log |f(z)| < |z|^{\sigma}$ for all large z.

The next lemma is a well-known consequence of Nevanlinna's Second Fundamental Theorem.

Lemma 5.9 ([20, p.59]). *The derivative of a transcendental meromorphic function takes every finite value infinitely often, with at most one exception.*

Lemma 5.10. *Let $z_1, \ldots, z_n \in \mathbb{C}$ and $y_1, \ldots, y_n \in \mathbb{R}$. Then*

$$\sum_{j=1}^{n} \frac{y_j}{|z - z_j|^2} = |z|^{-2} \sum_{j=1}^{n} y_j + O(|z|^{-3}) \quad \text{as } |z| \to \infty.$$

Proof. This is trivial, we simply write

$$\frac{y_j}{|z - z_j|^2} = \frac{y_j}{|z|^2(1 + O(|z|^{-1}))} = \frac{y_j}{|z|^2} + O(|z|^{-3}). \qquad \square$$

5.3.2 Wiman-Valiron theory

The Wiman-Valiron theory can be used to describe the behaviour of an entire function, and its derivatives, near points where the function attains its maximum modulus. The results stated in this section may all be found in [22], and represent only a small part of this powerful theory.

Given a transcendental entire function F, the Wiman-Valiron technique is based on the function's power series,

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

For $r > 0$, we define the *maximum term*

$$\mu(r, F) = \max\{|a_n| r^n : n = 0, 1, 2, \ldots\}.$$

The *central index* $\nu(r, F)$ is then defined to be the largest n for which $|a_n| r^n = \mu(r, F)$. It can be shown that $\nu(r, F)$ is a non-decreasing function of r, and that $\nu(r, F) \to \infty$ as $r \to \infty$.

There is a connection between the rates of growth of $\nu(r, F)$ and $T(r, f)$. In particular, the order $\rho(F)$ as defined on page 4 satisfies

$$\rho(F) = \limsup_{r \to \infty} \frac{\log \nu(r, F)}{\log r}. \tag{5.3.1}$$

We now state part of the main theorem of Wiman-Valiron theory.

Lemma 5.11 ([22])**.** *Let F be a transcendental entire function and let $k \in \mathbb{N}$. If $|z_0| = r$ and $|F(z_0)| = M(r, F)$, then*

$$\frac{F^{(k)}(z_0)}{F(z_0)} \sim \frac{\nu(r, F)^k}{{z_0}^k}$$

as $r \to \infty$ outside a set of finite logarithmic measure.

5.3.3 Hille's method

The proof of Theorem 5.2 involves studying solutions of differential equations of the form

$$w'' + b(z)w = 0 \tag{5.3.2}$$

where $b(z)$ is a rational function. Hille's method [29, §5.6] can be used to give an asymptotic description of these solutions if $b(z) \sim dz^n$ as $z \to \infty$, where $n \geq -1$. We shall only consider the $n = 0$ case, so that

$$b(z) = d + O(|z|^{-1}), \quad z \to \infty,$$

for some non-zero constant d.

The *critical rays* are defined to be those rays $\arg z = \theta$ for which

$$\theta = -\frac{\arg d}{2} \quad \text{or} \quad \theta = \pi - \frac{\arg d}{2}.$$

Assume that $\arg z = \theta_0$ is a critical ray, let $\delta > 0$ and let R_1 be large and positive. Define the region

$$S_1 = \{z : |z| > R_1, \ |\arg z - \theta_0| < \pi - \delta\}$$

and the transformation

$$Z = \int_{R_1 e^{i\theta_0}}^{z} b(t)^{1/2}\, dt = d^{1/2} z + O(\log|z|), \quad z \in S_1, \ z \to \infty.$$

There then exist principal solutions $u_+(z)$ and $u_-(z)$ of (5.3.2) on S_1 given by

$$u_\pm(z) = b(z)^{-1/4} \exp(\pm iZ + o(1)).$$

These principal solutions are analytic in S_1 and have no zeroes there. However, any linear combination $\mu u_+ + \nu u_-$, where μ and ν are non-zero constants, has infinitely many zeroes near the critical ray $\arg z = \theta_0$. Another significant feature of the critical rays is that the dominant $d^{1/2} z$ term in Z is real on these rays.

5.3.4 Proof of Theorem 5.2 – Part one

Let f be a real transcendental meromorphic function such that all but finitely many of the zeroes and poles of f' are real, and $f'(z) = \alpha$ only finitely often for some finite non-zero α. This section is devoted to proving that f can be written in the form (5.1.1) with α, λ and A real, $\lambda \neq 0$, $c > 0$ and P a polynomial without a pair of complex conjugate roots.

It is immediate that α is real, since otherwise the real transcendental derivative f' only takes the values α and $\overline{\alpha}$ finitely often, contradicting Lemma 5.9. Let

$$H(z) = f(z) - \alpha z \tag{5.3.3}$$

and note that by Lemma 5.6 the order of H satisfies $\rho(H) = \rho(f') \leq 1$.

Our aim is to write f in the form (5.1.1) by expressing H as a quotient of solutions to the differential equation (5.3.2), in which the function $b(z)$ is equal to half the Schwarzian derivative of H.

Lemma 5.12. *The Schwarzian derivative*

$$S(H) = \frac{H'''}{H'} - \frac{3}{2}\left(\frac{H''}{H'}\right)^2$$

is rational.

Proof. Since H has finite order, the lemma of the logarithmic derivative gives that

$$m(r, S(H)) = O(\log r).$$

Recall that the Schwarzian derivative $S(H)$ has poles only at the multiple points of H. Therefore, to show that $S(H)$ is rational, we shall show that H has only finitely many multiple points. As $H' = f' - \alpha$ has finitely many zeroes, our task is reduced to showing that H has only finitely many multiple poles.

Define the real function $g(z)$ by

$$f' = \alpha + 1/g. \tag{5.3.4}$$

Denote by $a_1, \ldots, a_N$ the poles of g, and by $b_1, \ldots, b_M$ and $c_1, c_2, \ldots$ respectively the non-real and real zeroes of $g + 1/\alpha$, all repeated according to multiplicity. The sequence c_n must be infinite because, by Lemma 5.9, the transcendental derivative f' cannot take the values 0 and α both only finitely often. Using Lemma 5.6 gives $\rho(g) = \rho(f') \leq 1$, so that we have the Weierstrass product representation [20, p.21]

$$g(z) + \frac{1}{\alpha} = z^p e^{az+b} \frac{\prod_{n=1}^{M}(z - b_n)}{\prod_{n=1}^{N}(z - a_n)} \prod_{\substack{n=1 \\ c_n \neq 0}}^{\infty} \left(1 - \frac{z}{c_n}\right) e^{z/c_n}$$

for some real constants a and b, and $p = \#\{n : c_n = 0\}$. We calculate

$$\frac{g'}{g + 1/\alpha} = a - \sum_{n=1}^{N} \frac{1}{z - a_n} + \sum_{n=1}^{M} \frac{1}{z - b_n} + \sum_{\substack{n=1 \\ c_n \neq 0}}^{\infty} \left(\frac{1}{z - c_n} + \frac{1}{c_n}\right) + \frac{p}{z},$$

$$\left(\frac{g'}{g + 1/\alpha}\right)' = \sum_{n=1}^{N} \frac{1}{(z - a_n)^2} - \sum_{n=1}^{M} \frac{1}{(z - b_n)^2} - \sum_{n=1}^{\infty} \frac{1}{(z - c_n)^2}. \tag{5.3.5}$$

We now restrict z to real values with $|z|$ large, and see from (5.3.5) that

$$\left(\frac{g'}{g + 1/\alpha}\right)' = \sum_{n=1}^{\infty} \frac{-1}{|z - c_n|^2} + O\left(\frac{1}{|z|^2}\right) < 0, \tag{5.3.6}$$

the final inequality coming from Lemma 5.10 by truncating the sum in (5.3.6) to a large number of terms.

By (5.3.3) and (5.3.4), the multiple poles of H correspond to zeroes of g of order greater than 2. At these zeroes the left-hand side of (5.3.6) vanishes, and hence there can only be finitely many of them on the real axis. Since H has only finitely many non-real poles, this completes the proof. □

Let

$$b(z) = \frac{1}{2} S(H)(z). \tag{5.3.7}$$

Theorem 6.1 of [35] states that if $D \subseteq \mathbb{C}$ is a simply-connected domain on which b is analytic, then (5.3.2) has two linearly independent analytic solutions w_1, w_2 on D such that $H = w_1/w_2$ there. We may assume that these solutions are normalised by $w_1 w_2' - w_1' w_2 = 1$. It follows that, on D,

$$H' = \frac{-1}{{w_2}^2}, \qquad \frac{H'}{H} = \frac{-1}{w_1 w_2}, \qquad \frac{H'}{H^2} = \frac{-1}{{w_1}^2},$$

and therefore ${w_1}^2$, $w_1 w_2$ and ${w_2}^2$ all have meromorphic extensions to the complex plane. Hence, if v is any solution of (5.3.2) on D, then v^2 extends meromorphically to the whole complex plane. Furthermore, this extension has order at most one, and has poles only at the (finitely many) poles of b. The latter claim can be proved by noting that v^2 is a solution of $4b(z)w^2 + 2ww'' - (w')^2 = 0$.

It is through studying equation (5.3.2) and its solutions that we will be able to express $f = H + \alpha z$ in the form (5.1.1).

Lemma 5.13. *The rational function $b(z)$ has a non-zero real value at infinity.*

Proof. That $b(z)$ is both a rational function and a real function follows from Lemma 5.12 and (5.3.7). Moreover, $b(z)$ does not vanish identically because H is not a Möbius map. Hence, we must show that $b(\infty) \neq 0, \infty$. As the order of H does not exceed one, Lemma 5.7 gives a ray on which

$$\left| \frac{H''}{H'} \right| \leq |z|^{\varepsilon}, \qquad \left| \frac{H'''}{H'} \right| \leq |z|^{2\varepsilon}.$$

Therefore, using (5.3.7) again, the rational function $b(z)$ must be finite at infinity.

Suppose now that $b(\infty) = 0$, so that

$$S(H)(z) = 2b(z) = O(|z|^{-1}) \quad \text{as } z \to \infty. \tag{5.3.8}$$

We shall use Wiman-Valiron theory to show that in this case the order of H is at most $\frac{1}{2}$. This leads to a contradiction as follows: By hypothesis, $H' = f' - \alpha$ has finitely many zeroes, however, it was proved in [13] that the derivative of any transcendental function of order less than 1 must have infinitely many zeroes.

We now prove the assertion that (5.3.8) implies that $\rho(H) \leq \frac{1}{2}$. Since H' has finitely many zeroes, we can write

$$g = \frac{1}{H'} = \frac{F}{P}, \tag{5.3.9}$$

where F is a transcendental entire function and P is a polynomial. We calculate

$$\begin{aligned} S(H) &= \frac{1}{2}\left(\frac{g'}{g}\right)^2 - \frac{g''}{g} \\ &= \frac{1}{2}\left(\frac{F'}{F} - \frac{P'}{P}\right)^2 - \left(\frac{F''}{F} - 2\frac{P'}{P}\frac{F'}{F} + 2\left(\frac{P'}{P}\right)^2 - \frac{P''}{P}\right) \\ &= \frac{1}{2}\left(\frac{F'}{F}\right)^2 - \frac{F''}{F} + O\left(\left|\frac{F'}{zF}\right| + \frac{1}{|z|^2}\right) \qquad \text{as } z \to \infty. \end{aligned}$$

For each $r > 0$, choose z_0 such that $|z_0| = r$ and $|F(z_0)| = M(r,F)$. Applying Lemma 5.11 to the above gives that, as $r \to \infty$ outside a set of finite logarithmic measure,

$$\begin{aligned} S(H)(z_0) &= \frac{1}{2}\left(\frac{\nu(r,F)}{z_0}(1+o(1))\right)^2 - \frac{\nu(r,F)^2}{{z_0}^2}(1+o(1)) + O\left(\frac{\nu(r,F)}{r^2} + \frac{1}{r^2}\right) \\ &= -\frac{\nu(r,F)^2}{2{z_0}^2}(1+o(1)). \end{aligned}$$

The last equality here uses the fact that the central index $\nu(r,F)$ tends to infinity with r. It now follows that (5.3.8) implies that

$$\nu(r,F) = O\left(r^{1/2}\right) \tag{5.3.10}$$

as $r \to \infty$ outside a set of finite logarithmic measure. Since $\nu(r,F)$ is a non-decreasing function of r, we deduce that in fact (5.3.10) holds as $r \to \infty$ without an exceptional set. Using (5.3.1), (5.3.9) and (5.3.10) now establishes that $\rho(H) = \rho(F) \le \frac{1}{2}$. ☐

Let C be the non-zero real value taken by b at infinity, and choose c so that $c^2 = C$. We now apply Hille's method as described in Section 5.3.3 to find solutions of (5.3.2). Let $\arg z = \theta_0$ be a critical ray and let

$$S_1 = \{z : |z| > R_1,\ |\arg z - \theta_0| < \pi - \delta\},$$

where R_1 is large and $0 < \delta < \pi/4$. By Hille's method, principal solutions of (5.3.2) on S_1 are given by

$$u_\pm(z) = b(z)^{-1/4}\exp(\pm icz + O(\log|z|)), \quad z \to \infty. \tag{5.3.11}$$

These solutions are analytic and non-zero on S_1.

The next lemma shows that we may take c to be real and positive.

Lemma 5.14. *The value C is positive.*

Proof. Suppose that $C < 0$ and so c is purely imaginary. In this case, the critical ray $\arg z = \theta_0$ lies along the imaginary axis and if μ, ν are non-zero constants, then $\mu u_+ + \nu u_-$ has infinitely many zeroes near this critical ray.

By the discussion of (5.3.7) above, $H = w_1/w_2$ and $H' = -1/{w_2}^2$ on S_1, where w_1 and w_2 are linear combinations of u_+ and u_-. Since H has only finitely many non-real poles, w_2 must be a multiple of a principal solution, $w_2 = \kappa u_\pm$. Then using (5.3.11), we see that $H'(z) = -1/(\kappa u_\pm)^2$ tends to either zero or infinity as $|z| \to \infty$ with z real. Hence, $H'(z) + \alpha = 0$ has only finitely many real roots. On recalling that $f' = H' + \alpha$ has only finitely many non-real zeroes, we uncover a contradiction with Lemma 5.9: the transcendental derivative f' takes both of the values 0 and α only finitely often. □

We now choose $c = \sqrt{C} > 0$.

Lemma 5.15. *We can write*

$$H(z) = \frac{kP(z)e^{icz} + lQ(z)e^{-icz}}{P(z)e^{icz} + Q(z)e^{-icz}} \tag{5.3.12}$$

where $k, l \in \mathbb{C}$ and P and Q are polynomials without common zeroes.

Proof. For $z \in S_1$, let

$$v_\pm(z) = u_\pm(z)e^{\mp icz}.$$

Referring again to the discussion preceding Lemma 5.13, we find that the functions $v_\pm^2 = (u_\pm e^{\mp icz})^2$ extend to be meromorphic on the plane, with finitely many poles and orders not exceeding one. Also, (5.3.11) gives that

$$v_\pm^2(z) = O(|z|^M), \quad z \in S_1,$$

for some M. Applying the Phragmén-Lindelöf principle (see Lemma 5.8 and the following remark) to the functions $v_\pm^2/z^M$ now shows that these functions are bounded near infinity. Hence, the functions $v_\pm^2$ must be rational. Moreover, as $v_\pm$ is analytic on S_1, we can write

$$v_\pm^2 = \frac{r_\pm}{s_\pm} \tag{5.3.13}$$

where $r_\pm$ and $s_\pm$ are polynomials and $s_\pm$ has no zeroes in S_1. In particular, we may define an analytic branch of $(s_+s_-)^{1/2}$ on S_1.

The discussion of (5.3.7) above gives that, on S_1, we can write H as a quotient of solutions of (5.3.2),

$$H = \frac{\mu_1 u_+ + \nu_1 u_-}{\mu_2 u_+ + \nu_2 u_-} = \frac{\mu_1 v_+ e^{icz} + \nu_1 v_- e^{-icz}}{\mu_2 v_+ e^{icz} + \nu_2 v_- e^{-icz}}. \tag{5.3.14}$$

Multiplying through by a factor $(s_+s_-)^{1/2}$, and then taking $P = \mu_2 v_+ (s_+s_-)^{1/2}$ and $Q = \nu_2 v_- (s_+s_-)^{1/2}$, we see that (5.3.14) becomes (5.3.12) on S_1. These functions P and Q are analytic on S_1, and by (5.3.13) both P^2 and Q^2 are polynomial. Neither P nor Q can vanish identically, since if $\mu_2\nu_2 = 0$ then $H'(z) = r(z)e^{\pm 2icz}$ for some rational function $r(z)$, and this contradicts the reality of H.

We may assume that the polynomials P^2 and Q^2 have no common zeroes in the plane. To see this, first suppose that $P^2(z_0) = Q^2(z_0) = 0$. If $z_0 \in S_1$, then P and Q are analytic at z_0 and we may divide both by $(z - z_0)$. Otherwise, $z_0 \notin S_1$ and we may divide both P and Q by a branch of $(z - z_0)^{1/2}$ that is analytic on S_1.

We complete the proof by showing that P and Q are themselves polynomial, so that (5.3.12) must hold on the whole plane by the Identity Theorem. We shall prove that P and Q may be analytically continued along any path, and then the Monodromy Theorem gives that P and Q are analytic, and hence polynomial, on the plane.

Let

$$\gamma : [0, \infty) \to \mathbb{C}, \quad \gamma(0) \in S_1$$

be a path starting in S_1. Suppose that $0 < t_0 < \infty$ is maximal such that both P and Q can be analytically continued along the path $\gamma(t)$ for $0 \leq t < t_0$. As P^2 and Q^2 are polynomial, the point $\gamma(t_0)$ must be a zero of either P^2 or Q^2. Suppose that $P(\gamma(t_0))^2 = 0$ (the proof being identical if instead $Q(\gamma(t_0))^2 = 0$). Then $\gamma(t_0)$ is not a zero of Q^2, and so Q admits analytic continuation along $\gamma(t)$ for $t < t_0 + \varepsilon$. Since H is meromorphic on the plane, (5.3.12) defines a meromorphic continuation of P along $\gamma(t)$ for $t < t_0 + \varepsilon$; namely,

$$P(\gamma(t)) = \frac{l - H(\gamma(t))}{H(\gamma(t)) - k} Q(\gamma(t)) e^{-2ic\gamma(t)}.$$

As P^2 is a polynomial this continuation must be analytic, contradicting the maximality of t_0. □

The function H is real and satisfies (5.3.12), so we must have that

$$\operatorname{Im}\left(k|P(x)|^2 + l|Q(x)|^2 + kP(x)\overline{Q(x)}e^{2icx} + l\overline{P(x)}Q(x)e^{-2icx}\right) = 0, \quad x \in \mathbb{R}. \quad (5.3.15)$$

Write $k = k_r + ik_i$ and $l = l_r + il_i$, where $k_r, k_i, l_r, l_i \in \mathbb{R}$, and let

$$R(x) = \operatorname{Re}\left(P(x)\overline{Q(x)}\right) \quad \text{and} \quad I(x) = \operatorname{Im}\left(P(x)\overline{Q(x)}\right).$$

Observe that R and I are real polynomials, not both vanishing identically. Now (5.3.15) becomes

$$k_i|P(x)|^2 + l_i|Q(x)|^2 + [(k_r - l_r)R(x) - (k_i + l_i)I(x)] \sin 2cx + \\ + [(k_r - l_r)I(x) + (k_i + l_i)R(x)] \cos 2cx = 0,$$

and because P, Q, R and I are polynomials, this leads to

$$k_i|P(x)|^2 + l_i|Q(x)|^2 = 0, \quad (5.3.16)$$

$$(k_r - l_r)R(x) - (k_i + l_i)I(x) = 0, \quad (5.3.17)$$

$$(k_r - l_r)I(x) + (k_i + l_i)R(x) = 0. \quad (5.3.18)$$

Inspection of (5.3.17) and (5.3.18) yields $k_r = l_r$ and $k_i = -l_i$. Hence, $l = \overline{k}$ and k must be non-real, otherwise H would be constant. Now (5.3.16) shows that, for real z,

$$P(z)\overline{P(\overline{z})} = Q(z)\overline{Q(\overline{z})}, \tag{5.3.19}$$

and in fact this holds on the whole plane, as both sides are polynomials in z. Since P and Q have no common zeroes, it follows that z_0 is a zero of P if and only if $\overline{z_0}$ is a zero of Q of equal multiplicity. Therefore,

$$\overline{P(\overline{z})} = \beta Q(z)$$

for some β, and (5.3.19) gives that $|\beta| = 1$. Using the fact that $\overline{\beta^{1/2}} = \beta^{-1/2}$ allows us to assume that $\beta = 1$, by replacing P and Q by $P_1 = \beta^{1/2}P$ and $Q_1 = \beta^{1/2}Q$, and re-labelling.

By writing $k = \bar{l} = A + \lambda i$ and using (5.3.3), equation (5.3.12) now becomes (5.1.1).

5.3.5 Proof of Theorem 5.2 – Part two

In this section, f is assumed to be given by (5.1.1) where α, λ and A are real, $\alpha\lambda \neq 0$, $c > 0$ and P is a polynomial with zeroes $a_1, \ldots, a_N$ (repeated to multiplicity) such that $a_j \neq \overline{a_k}$. We aim to prove that f and f'' have only finitely many non-real zeroes and poles, and that the equation $f'(z) = \alpha$ has at most $2N$ solutions, counting with multiplicities. We show further that all but finitely many of the zeroes of f' are real if and only if either $0 < \lambda c/\alpha < 1$ or condition (5.1.2) is satisfied.

Together with the result established in the previous section, this completes the proof of Theorem 5.2.

It will be useful to write $Q(z) = \overline{P(\overline{z})}$ and to differentiate (5.1.1) to obtain

$$f' - \alpha = 2i\lambda \frac{P'Q - PQ' + 2icPQ}{(Pe^{icz} + Qe^{-icz})^2} \tag{5.3.20}$$

and

$$f'' = \frac{p_0(z)e^{icz} + p_1(z)e^{-icz}}{(Pe^{icz} + Qe^{-icz})^3}, \tag{5.3.21}$$

where p_0, p_1 are polynomials, by using the quotient rule $(G/H^2)' = (HG' - 2GH')/H^3$. From the reality of both f'' and the denominator of (5.3.21), we have

$$p_0(z)e^{icz} + p_1(z)e^{-icz} = \overline{p_0(\overline{z})}e^{-icz} + \overline{p_1(\overline{z})}e^{icz},$$

which implies that $p_1(z) = \overline{p_0(\overline{z})}$.

The assertion that the equation $f'(z) = \alpha$ has at most $2N$ solutions is proved simply by observing that the numerator of the right-hand side of (5.3.20) is a polynomial of degree $2N$.

From (5.1.1), we see that if z_0 is a pole of f then z_0 satisfies

$$P(z_0)e^{icz_0} + \overline{P(\overline{z_0})}e^{-icz_0} = 0,$$

and if z_1 is a zero of f then z_1 satisfies

$$(\alpha z_1 + A + i\lambda)P(z_1)e^{icz_1} + (\alpha z_1 + A - i\lambda)\overline{P(\overline{z_1})}e^{-icz_1} = 0.$$

Similarly, from (5.3.21) we see that if z_2 is a zero of f'' then z_2 satisfies

$$p_0(z_2)e^{icz_2} + \overline{p_0(\overline{z_2})}e^{-icz_2} = 0.$$

Therefore, the fact that f and f'' have only finitely many non-real zeroes and poles follows from the next lemma.

Lemma 5.16. *If $p(z) \not\equiv 0$ is a polynomial, then*

$$F(z) = p(z)e^{iz} + \overline{p(\overline{z})}e^{-iz} \tag{5.3.22}$$

has only finitely many non-real zeroes.

Proof. For real x,

$$F(x) = 2\,\mathrm{Re}(p(x))\cos x - 2\,\mathrm{Im}(p(x))\sin x. \tag{5.3.23}$$

Let m be a large positive or negative integer. If $\mathrm{Re}(p(x)) \not\equiv 0$, then (5.3.23) shows that $F(x)$ changes sign over the interval $[m\pi, (m+1)\pi]$. Otherwise, $\mathrm{Im}(p(x)) \not\equiv 0$ and $F(x)$ changes sign over $\left[\left(m - \frac{1}{2}\right)\pi, \left(m + \frac{1}{2}\right)\pi\right]$. In either case, we see that F has at least $2t/\pi - O(1)$ real zeroes in $\{z : |z| \leq t\}$.

We calculate that $T(r, F) = 2r/\pi + O(\log r)$ as $r \to \infty$, using (5.3.22) and the fact that $p(z)e^{iz}$ is large where $\overline{p(\overline{z})}e^{-iz}$ is small, and vice versa. Denoting by $n(t)$ the number of non-real zeroes of F in $\{z : |z| \leq t\}$, we have

$$n(t, 1/F) \geq n(t) + \frac{2t}{\pi} - O(1)$$

and so

$$\begin{aligned}\int_0^r \frac{n(t)}{t}\,dt &\leq N(r, 1/F) - \frac{2r}{\pi} + O(\log r)\\ &\leq T(r, F) - \frac{2r}{\pi} + O(\log r) = O(\log r), \quad r \to \infty.\end{aligned}$$

This implies that $n(t)$ is bounded, and so F has finitely many non-real zeroes. □

Lemma 5.17. *All but finitely many of the zeroes of f' are real if and only if either $0 < \lambda c/\alpha < 1$ or condition* (5.1.2) *holds.*

Proof. Define the real functions

$$g_1 = \frac{P}{Q}\, e^{2icz} + \frac{Q}{P}\, e^{-2icz} \tag{5.3.24}$$

and

$$\begin{aligned} g_2 &= \frac{2\lambda i}{\alpha}\left(\frac{Q'}{Q} - \frac{P'}{P} - 2ic\right) - 2 \\ &= \frac{4\lambda c}{\alpha} - 2 + \frac{4\lambda}{\alpha}\sum_{j=1}^{N} \frac{\operatorname{Im} a_j}{(z-a_j)(z-\overline{a_j})}. \end{aligned} \tag{5.3.25}$$

Then by (5.3.20),

$$f' = \frac{\alpha PQ}{(Pe^{icz} + Qe^{-icz})^2}\,(g_1 - g_2),$$

so that f' and $g_1 - g_2$ have the same zeroes with finitely many exceptions. To see this, note that $g_1(z) = -2$ at a zero of $Pe^{icz} + Qe^{-icz}$, but that $g_2(z) = -2$ only finitely often.

Fix an analytic branch of $\log(P/Q)$ on the simply-connected domain

$$D = \{z : |z| > R,\ \operatorname{Im} z < 1\},$$

where R is large. We can choose a real number ϕ such that

$$\varepsilon(z) = \phi - i\log\left(\frac{P(z)}{Q(z)}\right) = o(1) \quad \text{as } z \to \infty \text{ in } D. \tag{5.3.26}$$

The function $\varepsilon(z)$ is analytic and real, since $|P(x)/Q(x)| = 1$ for real x. For each large positive or negative integer n, we can find a real number x_n such that

$$2cx_n + \varepsilon(x_n) = n\pi + \phi.$$

Using (5.3.24) and (5.3.26), we can write

$$g_1(z) = 2\cos(2cz - \phi + \varepsilon(z)), \quad z \in D. \tag{5.3.27}$$

We now have that

$$g_1(x_n) = 2(-1)^n \quad \text{and} \quad x_n = \frac{n\pi + \phi}{2c} + o(1) \quad \text{as } n \to \pm\infty. \tag{5.3.28}$$

Assume now that either $0 < \lambda c/\alpha < 1$ or condition (5.1.2) holds. Then (5.3.25) gives $|g_2(x)| < 2$ for all large real x, and so (5.3.28) shows that $g_1 - g_2$ changes sign over $[x_n, x_{n+1}]$. Therefore, $g_1 - g_2$ has at least $4ct/\pi - O(1)$ real zeroes in $\{z : |z| \le t\}$, and the same is true of f'. Using (5.3.20), we calculate

$$T(r, f') = 2T(r, Pe^{icz} + Qe^{-icz}) + O(\log r) = \frac{4cr}{\pi} + O(\log r), \quad r \to \infty,$$

using the fact that Pe^{icz} is large where Qe^{-icz} is small, and vice versa. By an argument similar to that used at the end of the proof of Lemma 5.16, this is sufficient to show that all but finitely many of the zeroes of f' are real.

We tackle the proof of the converse in two cases.

(i) Suppose first that either $\lambda c/\alpha < 0$ or $\lambda c/\alpha > 1$. Then, by (5.3.25) and (5.3.27), for real x of large absolute value we have that $|g_1(x)| \leq 2$ and $|g_2(x)| > 2$. Therefore, both $g_1 - g_2$ and f' have only finitely many real zeroes. Hence, f' must have infinitely many non-real zeroes. This is because the derivative f' cannot have only finitely many zeroes and α-points in the plane, by Lemma 5.9.

(ii) Suppose instead that $\lambda c = \alpha$ but that (5.1.2) fails to hold. Then

$$\sum_{j=1}^{N} \frac{\operatorname{Im} a_j}{|x - a_j|^2} > 0 \quad \text{and so} \quad g_2(x) > 2$$

either for all large positive x or for all large negative x. For such x, we have $|g_1(x)| \leq 2$ by (5.3.27). Hence, $g_1 - g_2$ either has only finitely many positive zeroes, or only finitely many negative zeroes.

Using (5.3.25) and (5.3.28) gives that

$$g_1(x_{2n}) - 2 = 0 \quad \text{and} \quad g_2(z) - 2 = o(1), \quad \text{as } z \to \infty,$$

and we see from (5.3.27) that $|g_1 - 2|$ is bounded away from zero on a small circle about x_{2n}. Hence, it follows from Rouché's Theorem that $g_1 - g_2$ has at least one zero near each point x_{2n}, for $|n|$ sufficiently large. Combining this with (5.3.28) and the result of the previous paragraph shows that $g_1 - g_2$ has infinitely many non-real zeroes, and the same is true of f'. □

5.4 Proof of Theorem 5.3

The following lemma is the key to the proof of Theorem 5.3.

Lemma 5.18. *Let F be meromorphic such that all but finitely many of the zeroes and poles of F are real, and $F(z) = 1$ only finitely often. If F has infinitely many multiple poles, then F is real.*

Proof. The order of F does not exceed one by Lemma 5.6. Hence, we can write

$$F(z) = \frac{h(z)P_1(z)e^{Az}}{k(z)P_2(z)},$$

where: h and k are real entire functions of order at most one with only real zeroes and no common zeroes; the polynomials P_1 and P_2 have no real zeroes; and A is a constant. Furthermore, there exists an unbounded real sequence (x_n) of multiple zeroes of k. Since $F - 1$ has only finitely many zeroes, but the same poles as F, we can also write

$$F(z) = 1 + \frac{P_3(z)e^{(A+B)z}}{k(z)P_2(z)}, \tag{5.4.1}$$

where P_3 is a polynomial and B is a constant. Equating these two expressions for $F(z)$ yields

$$h(z)P_1(z) = k(z)P_2(z)e^{-Az} + P_3(z)e^{Bz}. \tag{5.4.2}$$

Evaluating (5.4.2) and its derivative at each of the points x_n gives

$$h(x_n)P_1(x_n) = P_3(x_n)e^{Bx_n} \tag{5.4.3}$$

and

$$h'(x_n)P_1(x_n) + h(x_n)P_1'(x_n) = (P_3'(x_n) + BP_3(x_n))e^{Bx_n},$$

which lead to

$$\frac{h'(x_n)}{h(x_n)} + \frac{P_1'(x_n)}{P_1(x_n)} = \frac{P_3'(x_n)}{P_3(x_n)} + B.$$

Therefore, B must be real because h is a real function and $P_j'(x_n)/P_j(x_n) \to 0$ as $|x_n| \to \infty$. Now (5.4.3) shows that $P_1(x_n)/P_3(x_n)$ is real for every x_n, and therefore P_1/P_3 is a real function (since the rational function $P_1(z)/P_3(z) - \overline{P_1(\overline{z})/P_3(\overline{z})}$ must be identically zero). Dividing equation (5.4.2) by P_3 gives that the function

$$\frac{P_2(z)e^{-Az}}{P_3(z)}$$

is real, and hence (5.4.1) shows that F must also be real. □

Let the function g be as in the hypothesis of Theorem 5.3. Assume first that g has infinitely many poles and apply Lemma 5.18 with $F = g'/\beta$. This gives that on the real axis g/β has constant imaginary part. It then follows immediately that we have case (i) of the theorem.

Now suppose instead that g has only finitely many poles. By Lemma 5.6, the order of g' is at most one and it follows that

$$g'(z) - \beta = R_1(z)e^{icz}$$

for some rational function $R_1 \not\equiv 0$. We show next that c is real. Suppose not, then $g'(x)$ tends to either β or infinity as real $x \to \pm\infty$, and so g' must have finitely many real zeroes. But then g' takes each of the values 0, β and ∞ only finitely often, implying that g' is rational and hence $c = 0$.

Write $R_1(z) = P(z) + \sum_{k=1}^{n} \frac{a_k}{(z-z_k)^{m_k}}$, where P is a polynomial, the m_k are positive and the z_k need not be distinct. Observe that

$$\int P(z)e^{icz}\,dz = \frac{P(z)e^{icz}}{ic} - \int \frac{P'(z)e^{icz}}{ic}\,dz + \text{constant}$$

and, for $m \geq 2$,

$$\int \frac{e^{icz}}{(z-z_k)^m}\,dz = \frac{e^{icz}}{(1-m)(z-z_k)^{m-1}} - \int \frac{ice^{icz}}{(1-m)(z-z_k)^{m-1}}\,dz + \text{constant}.$$

Hence, repeated integration by parts yields

$$g(z) - \beta z = \int R_1(z)e^{icz}\, dz + d' = R(z)e^{icz} + \int \left(\sum_{k=1}^{n} \frac{A_k e^{icz}}{z - z_k} \right) dz + d, \qquad (5.4.4)$$

where $R(z)$ is rational and $d, d', A_1, \ldots, A_n$ are constants. Then the sum

$$\sum_{k=1}^{n} \frac{A_k e^{icz}}{z - z_k} = \frac{d}{dz}\left(g(z) - \beta z - R(z)e^{icz} - d\right)$$

is the derivative of a meromorphic function, and so must be identically zero as it cannot have any simple poles. Now (5.4.4) shows that we have case (ii) of the theorem.

Finally, the assertion about the zeroes of g'' follows from Theorem 5.2 in case (i) and by straightforward differentiation in case (ii).

5.5 Proof of Theorem 5.4

Let g be as in the hypothesis of Theorem 5.4. Then by Lemma 5.6, the order of g does not exceed one.

Suppose initially that α is non-real. Then since g is real it has no real α-points, and so g takes the values α and $\overline{\alpha}$ only finitely often. Hence, we may write

$$\frac{g(z) - \overline{\alpha}}{\alpha - g(z)} = \frac{P(z)}{Q(z)} e^{2icz}, \qquad (5.5.1)$$

where c is a complex constant and P and Q are polynomials with zeroes at the $\overline{\alpha}$-points and α-points of g respectively. Since g is real, it follows that $Q(z)$ and $\overline{P(\overline{z})}$ have the same zeroes according to multiplicity, and so $Q(z) = \beta\overline{P(\overline{z})}$ for some constant β. Furthermore,

$$\frac{P(z)}{\beta\overline{P(\overline{z})}} e^{2icz} = \frac{g(z) - \overline{\alpha}}{\alpha - g(z)} = \overline{\left(\frac{\alpha - g(\overline{z})}{g(\overline{z}) - \overline{\alpha}}\right)} = \frac{\overline{\beta}P(z)}{\overline{P(\overline{z})}} e^{2i\overline{c}z},$$

which implies that $e^{2i(c-\overline{c})z} = |\beta|^2$. Therefore, c is real and $|\beta| = 1$. Using $\overline{\beta^{1/2}} = \beta^{-1/2}$ allows us to assume that $\beta = 1$, by replacing P and Q by $P_1 = \beta^{-1/2}P$ and $Q_1 = \beta^{-1/2}Q$ and re-labelling. Rearranging (5.5.1) now yields

$$g(z)\left[\overline{P(\overline{z})}e^{-icz} + P(z)e^{icz}\right] = \alpha P(z)e^{icz} + \overline{\alpha}\,\overline{P(\overline{z})}e^{-icz},$$

which gives the required form for g.

We now turn our attention to the case where α is real, so that without loss of generality we may henceforth assume $\alpha = 1$. The next lemma provides a simple estimate of the logarithmic derivative without the exceptional set that occurs in Lemma 5.7.

Lemma 5.19. *Let F be a meromorphic function of order at most ρ with all but finitely many of its zeroes and poles real. Let $\delta > 0$ and $\eta > 0$. Then*

$$\left|\frac{F'(z)}{F(z)}\right| = o(|z|^{\rho-1+\eta}) \quad \textit{as } z \to \infty,\ \delta < |\arg z| < \pi - \delta.$$

Proof. First note that

$$m(r,F) + m(r,1/F) + n(r,F) + n(r,1/F) = o(r^{\rho+\eta}), \quad r \to \infty.$$

Let z be such that $|z| = r$ and $\delta < |\arg z| < \pi - \delta$. The differentiated Poisson-Jensen formula [32, p.65] gives

$$\left|\frac{F'(z)}{F(z)}\right| \leq \frac{4}{r}\left(m(2r,F) + m(2r,1/F)\right) + \sum_{|z_j|<2r} \frac{2}{|z - z_j|},$$

where the z_j are the zeroes and poles of F repeated according to multiplicity. For the finitely many non-real z_j, we have $|z - z_j|^{-1} = O(r^{-1})$ as $r \to \infty$, while for the real z_j we have $|z - z_j| \geq |\operatorname{Im} z| \geq r \sin \delta$. Therefore, as $r \to \infty$,

$$\left|\frac{F'(z)}{F(z)}\right| \leq o(r^{\rho-1+\eta}) + \frac{2}{r \sin \delta}\left(n(2r,F) + n(2r,1/F)\right) = o(r^{\rho-1+\eta}). \qquad \square$$

Since the order of g is at most one, taking $0 < \varepsilon_1 < \varepsilon/4$ and $\eta > 0$ both small and applying Lemma 5.19 gives that

$$\left|\frac{g'(z)}{g(z)}\right| = o(|z|^{\eta}) \quad \text{as } z \to \infty, \quad \varepsilon_1 < |\arg z| < \pi - \varepsilon_1. \tag{5.5.2}$$

Define $\sigma \in (1,2)$ by

$$\sigma = 1 + \frac{\lambda \sin(\varepsilon/2)}{8}, \tag{5.5.3}$$

where $\lambda = \lambda(g)$ is the lower order of g. Applying Lemma 3.9(ii) to $g - 1$, we can find a small positive constant m, and a set J of lower logarithmic density greater than $1/\sigma$, such that if $r \in J$ is large and F_r is a subinterval of $[0, 2\pi]$ of length m, then

$$\int_{F_r} \left|\frac{rg'(re^{i\theta})}{g(re^{i\theta}) - 1}\right| d\theta \leq \frac{\delta(1,g)}{4} T(r,g).$$

By the definition of deficiency, for large $r \in J$ there exists z_0 with $|z_0| = r$ and

$$\log|g(z_0) - 1| \leq -\frac{\delta(1,g)}{2} T(r,g).$$

It follows that g is near 1 on any arc of angular measure m with z_0 as one endpoint. In particular, because g is real and ε_1 is small we can find, for large $r \in J$, an arc

$$\Omega(r) \subseteq A(r) = \{z : |z| = r,\ 2\varepsilon_1 < \arg z < \pi - 2\varepsilon_1\}$$

of angular measure $m/2$ on which

$$\log|g(z)-1| < -c_1 T(r,g), \tag{5.5.4}$$

denoting by $c_1, c_2, \ldots$ positive constants not depending on r.

It is now claimed that we can choose by induction a sequence (r_k) in J satisfying $2r_k < r_{k+1} < {r_k}^\sigma$. Otherwise, there exists a large $r_k \in J$ such that $(2r_k, {r_k}^\sigma) \cap J = \emptyset$. Taking l such that $1/\sigma < l < \underline{\mathrm{logdens}}\, J$ then leads to the following contradiction:

$$l \log {r_k}^\sigma < \int_{[1,{r_k}^\sigma]\cap J} \frac{dt}{t} \leq \int_1^{2r_k} \frac{dt}{t} = (1/\sigma)\log {r_k}^\sigma + \log 2.$$

We deduce immediately that

$$\bigcup_{k=1}^{\infty} (r_k, {r_k}^\sigma) \quad \text{contains all large } r. \tag{5.5.5}$$

Define two sequences of arcs by $\Omega_k = \Omega(r_k)$ and $A_k = A(r_k)$. Applying Lemma 5.19 to $g-1$ gives that, on Ω_k,

$$\left|\frac{g'(z)}{g(z)-1}\right| = o({r_k}^\eta) \qquad \text{as } r_k \to \infty,$$

so that on Ω_k using (5.5.4) twice yields

$$\begin{aligned} \log\left|\frac{g'(z)}{g(z)}\right| &\leq \log|g'(z)| + o(1) \\ &\leq \log|g(z)-1| + O(\log r_k) < -c_2 T(r_k, g), \quad r_k \to \infty. \end{aligned}$$

We show next that a similar bound holds on the whole of the arc A_k. To do this, note that by conformal invariance,

$$\omega(z, \Omega_k, D_k \setminus \Omega_k) > c_3, \quad z \in A_k,$$

where $D_k = \{z : r_k/2 < |z| < 2r_k,\ \varepsilon_1 < \arg z < \pi - \varepsilon_1\}$. Using (5.5.2) and the above, the Two Constants Theorem now gives

$$\log\left|\frac{g'(z)}{g(z)}\right| < -c_4 T(r_k, g), \quad z \in A_k. \tag{5.5.6}$$

Let

$$\begin{aligned} S_k &= \{z : r_k < |z| < {r_k}^2,\ 2\varepsilon_1 < \arg z < \pi - 2\varepsilon_1\}, \\ S'_k &= \{z : r_k < |z| < {r_k}^\sigma,\ \varepsilon < \arg z < \pi - \varepsilon\}. \end{aligned}$$

Lemma 5.20. *For large k, the harmonic measure of the arc A_k satisfies*

$$\omega(z, A_k, S_k) \geq \frac{1}{2\pi {r_k}^{4(\sigma-1)/\sin(\varepsilon/2)}} = \frac{1}{2\pi {r_k}^{\lambda/2}}, \qquad z \in S'_k.$$

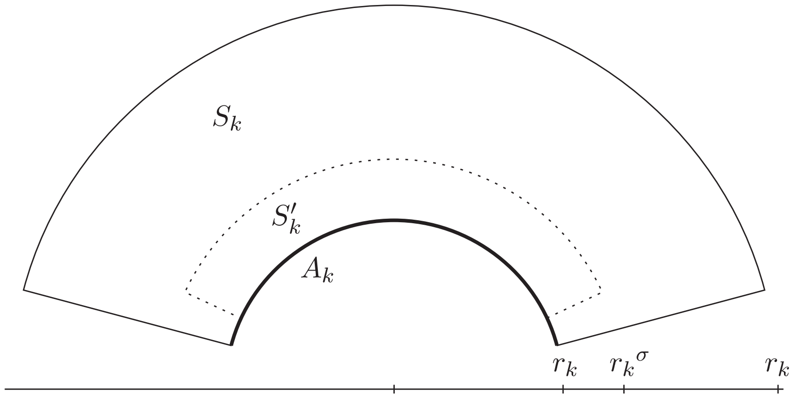

Figure 5.1: The domain S_k with subdomain S'_k and boundary arc A_k.

Remark. In fact, for $z \in S'_k$ it is true that $\omega(z, A_k, S_k) \geq c_5 r_k{}^{-\pi(\sigma-1)/(\pi-4\varepsilon_1)}$, and this can be shown in a number of ways. For example, an explicit series representation for $\omega(z, A_k, S_k)$ can be obtained by conformally mapping S_k onto a rectangle. Another method involves comparing $\omega(z, A_k, S_k)$ with the harmonic measure of the interval $[-r_k, r_k]$ with respect to the upper half-plane, which can itself be estimated via a mapping to the unit circle. However, Lemma 5.20 will suffice for our purpose.

The proof of Lemma 5.20 is simply an application of the following lemma that goes back to Nevanlinna.

Lemma 5.21 ([10, Lemma E])**.** *Let D be a domain bounded by a Jordan curve $\mathcal{C}$ consisting of a Jordan arc $\mathcal{A}$ and its complement $\mathcal{B}$ in $\mathcal{C}$. Let Γ be a rectifiable curve in D joining a point $a \in \mathcal{A}$ to a point in $\mathcal{B}$. Let z be a point on Γ and let $\rho_{\mathcal{B}}(z)$ denote the distance of z from $\mathcal{B}$. Then*

$$\omega(z, \mathcal{A}, D) \geq \frac{1}{2\pi} \exp\left\{-4 \int_a^z \frac{|d\zeta|}{\rho_{\mathcal{B}}(\zeta)}\right\},$$

where the integral is taken along Γ.

Proof of Lemma 5.20. The equality in the statement of the result follows from (5.5.3).

Let r_k be large, $\zeta \in S'_k$ and let w be a nearest point to ζ of $\mathcal{B} = \partial S_k \setminus A_k$. Then either $\arg w = 2\varepsilon_1$ or $\arg w = \pi - 2\varepsilon_1$. Using the fact that $\varepsilon - 2\varepsilon_1 > \varepsilon/2$, it follows that

$$\rho_{\mathcal{B}}(\zeta) = |\zeta - w| \geq |\zeta| \sin(\varepsilon/2).$$

For $z \in S'_k$, choose the path $\Gamma(t) = te^{i \arg z}$ for $t \in [r_k, r_k{}^2]$. Applying the previous lemma now yields

$$\omega(z, A_k, S_k) \geq \frac{1}{2\pi} \exp\left\{-4 \int_{r_k}^{|z|} \frac{dt}{t \sin(\varepsilon/2)}\right\} = \frac{1}{2\pi} \exp\left\{\frac{-4}{\sin(\varepsilon/2)} \log\left(\frac{|z|}{r_k}\right)\right\},$$

which gives the required result upon noting that $|z| < r_k{}^\sigma$. □

Using (5.5.2), (5.5.6) and Lemma 5.20, the Two Constants Theorem gives that, for $z \in S_k'$,

$$\log\left|\frac{g'(z)}{g(z)}\right| \leq \frac{-c_4 T(r_k, g)}{2\pi {r_k}^{\lambda/2}} + O(\log r_k), \qquad r_k \to \infty,$$

and in particular,

$$\left|\frac{g'(z)}{g(z)}\right| = o({r_k}^{-2}), \qquad r_k \to \infty. \tag{5.5.7}$$

Pick a point $z_k \in \Omega_k$ for each k. For large k, there are no zeroes or poles of g in S_k, and so for $z \in S_k'$ we can write

$$g(z) = g(z_k) \exp\left(\int_{z_k}^{z} \frac{g'(w)}{g(w)}\, dw\right) = 1 + o(1), \qquad k \to \infty,$$

using (5.5.4) and (5.5.7). By (5.5.5), if z is large and $\varepsilon < \arg z < \pi - \varepsilon$, then $z \in S_k'$ for some k, which tends to infinity with z. Hence, by the above,

$$g(z) \sim 1, \quad \text{as } z \to \infty,\ \varepsilon < \arg z < \pi - \varepsilon.$$

Since g is real this completes the proof.

Chapter 6

Non-real zeroes of derivatives of real entire functions

6.1 Introduction

6.1.1 Two conjectures of Pólya and Wiman

This chapter is motivated by the recent resolution of a long-standing conjecture attributed to Wiman. The conjecture dates back to around 1911 and involves the Laguerre-Pólya class LP. An entire function f belongs to the class LP if there exists a sequence of real polynomials with only real zeroes that converges locally uniformly to f. Such functions are necessarily real and have only real zeroes unless $f \equiv 0$. It is not difficult to show that LP is closed under differentiation; hence, all derivatives of a function in LP have only real zeroes. Pólya asked whether this last fact was enough to characterize the class LP, while Wiman's conjecture went a step further.

Former Conjecture (Pólya [52])**.** *If f is a real entire function such that $f^{(k)}$ has only real zeroes, for every $k \geq 0$, then $f \in LP$.*

Former Conjecture (Wiman [1, 2])**.** *If f is a real entire function such that ff'' has only real zeroes, then $f \in LP$.*

Wiman's conjecture therefore implies the following striking result: If the zeroes of a real entire function and its second derivative are real, then the zeroes of all its derivatives are confined to the real axis.

The first important steps towards a proof of Wiman's conjecture were made in 1960 by Levin and Ostrovskii [45] who introduced a factorisation of the logarithmic derivative that appears in almost all later work on this topic, see Section 6.3.2 for more details. Their second major contribution was the refinement of an analogue of the Nevanlinna characteristic for functions defined on a half-plane. This characteristic is described in

Section 6.1.3 below. Levin and Ostrovskii used this machinery to show that if a real entire function f is such that ff'' has only real zeroes, then its maximum modulus cannot grow too fast, in particular $\log\log M(r,f) = O(r \log r)$ as $r \to \infty$.

In 1977 Hellerstein and Williamson [26, 27] settled Pólya's conjecture by showing that a real entire function f must belong to LP if f, f' and f'' each have only real zeroes.

Sheil-Small proved Wiman's conjecture for functions of finite order in his 1989 Annals paper [55]. His main idea was to adopt a more geometric approach by studying how the logarithmic derivative f'/f and the Newton function $z - f/f'$ behave as mappings of the upper half-plane. Upon recalling that a function has finite order only when $\log\log M(r,f) = O(\log r)$ as $r \to \infty$, we see that there is a gap between Sheil-Small's result and the work of Levin and Ostrovskii. By bridging this gap, Bergweiler, Eremenko and Langley [7] finally completed the proof of Wiman's conjecture in 2002.

6.1.2 The classes U_{2p}^*

There are now many theorems related to the Pólya-Wiman conjectures, and the new results presented in Section 6.2 are best viewed in this context. Before proceeding we introduce a family of classes of real entire functions.

For each integer $p \geq 0$, the class V_{2p} consists of all functions

$$g(z) \exp\left(-a z^{2p+2}\right),$$

where $a \geq 0$ and g is a real entire function with real zeroes and genus at most $2p+1$; that is, g has a convergent representation

$$g(z) = A z^b e^{h(z)} \prod_k \left(1 - \frac{z}{a_k}\right) \exp\left(q_m(z/a_k)\right), \qquad q_m(z) = \sum_{n=1}^{m} \frac{z^n}{n},$$

where A and the a_k are real, b is a non-negative integer, $m \leq 2p+1$ and h is a real polynomial with degree at most $2p+1$. The classes U_{2p} are now defined by $U_0 = V_0$ and $U_{2p} = V_{2p} \setminus V_{2p-2}$ for $p \geq 1$. The connection with the Pólya-Wiman conjectures is made clear by the Laguerre-Pólya Theorem that $U_0 = LP$ [34, 51]. We denote by U_{2p}^* the class of real entire functions $f = Pf_0$, where $f_0 \in U_{2p}$ and P is a real polynomial. It follows that every real entire function of finite order with finitely many non-real zeroes belongs to exactly one of the classes U_{2p}^*.

The next result follows a convention that we shall adopt throughout this chapter: all counts of zeroes are made with regard to multiplicity unless explicitly stated otherwise. This result was first proved for $f \in U_{2p}$ and $k = 2$ in [55].

Theorem 6.1 ([12]). *Let f be a real entire function. If $f \in U_{2p}^*$, then $f^{(k)}$ has at least $2p$ non-real zeroes for all $k \geq 2$.*

Like the class LP, each of the classes U_{2p}^* is closed under differentiation [12, Corollary 2.12]. In particular, it suffices to prove Theorem 6.1 with $k = 2$. The corresponding infinite order result was proved for $k = 2$ in [7], and for $k \geq 3$ in [38].

Theorem 6.2 ([7, 38])**.** *Let f be a real entire function of infinite order. Then $ff^{(k)}$ has infinitely many non-real zeroes for all $k \geq 2$.*

One immediate corollary of these results is that if f is a real entire function and $ff^{(k)}$ has only real zeroes, for some $k \geq 2$, then $f \in LP$. This represents one natural generalisation of Wiman's conjecture. Two unpublished articles by Langley [41, 42] together give an excellent account of many of the key ideas used to prove Theorems 6.1 and 6.2 in the $k = 2$ cases, thereby establishing Wiman's conjecture.

6.1.3 The Tsuji half-plane characteristic

The characteristic for functions defined on a half-plane was first introduced by Tsuji [58] and was developed further by Levin and Ostrovskii [45]. We shall henceforth write $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ for the (open) upper half-plane and say that a function is *meromorphic on the closed upper half-plane* $\overline{H} \subseteq \mathbb{C}$ to mean that it is meromorphic on some domain containing $\overline{H}$. We describe how to define the Tsuji characteristic of a function f that is meromorphic on $\overline{H}$, and explore some of its basic properties.

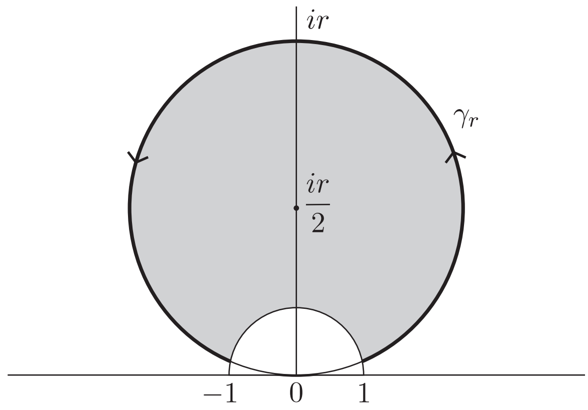

FIGURE 6.1: The sets used to define the Tsuji characteristic. A point on the bold arc γ_r with argument θ has modulus $r \sin\theta$.

For $r \geq 1$, let $\mathfrak{n}(r, f)$ denote the number of poles of f, counted with multiplicity, that lie in $\{z : |z - ir/2| \leq r/2,\ |z| \geq 1\}$. See Figure 6.1. The Tsuji integrated counting function is then given by

$$\mathfrak{N}(r, f) = \int_1^r \frac{\mathfrak{n}(t, f)}{t^2}\, dt \qquad (6.1.1)$$

for $r \geq 1$. Taking γ_r to be the arc of $S(ir/2, r/2)$ that lies outside the unit disc (the bold arc in Figure 6.1) traversed anticlockwise, we define the Tsuji proximity function

$$\mathfrak{m}(r,f) = \frac{1}{2\pi}\int_{\gamma_r} \frac{\log^+|f(z)|}{z^2}\,dz = \frac{1}{2\pi}\int_{\sin^{-1}(1/r)}^{\pi-\sin^{-1}(1/r)} \frac{\log^+|f(r\sin\theta e^{i\theta})|}{r\sin^2\theta}\,d\theta.$$

The Tsuji characteristic of f is then the sum

$$\mathfrak{T}(r,f) = \mathfrak{m}(r,f) + \mathfrak{N}(r,f).$$

Many results involving the Nevanlinna characteristic have analogues for the Tsuji characteristic. As in Nevanlinna theory, the inequalities

$$\mathfrak{T}(r,fg) \le \mathfrak{T}(r,f) + \mathfrak{T}(r,g), \qquad \mathfrak{T}(r,f+g) \le \mathfrak{T}(r,f) + \mathfrak{T}(r,g) + \log 2$$

follow from similar inequalities for the counting and proximity functions. For a non-constant f and $a \in \mathbb{C}$, the First Fundamental Theorem ([17, p.27], compare Theorem 1.1) states that

$$\mathfrak{T}(r,1/(f-a)) = \mathfrak{T}(r,f) + O(1) \quad \text{as } r \to \infty.$$

The Second Fundamental Theorem also holds [17, p.104–112] and leads to the following result. For distinct $a_j \in \mathbb{C}$,

$$\mathfrak{T}(r,f) \le \sum_{j=1}^{3} \mathfrak{N}(r,1/(f-a_j)) + O(\log r + \log^+ \mathfrak{T}(r,f)) \tag{6.1.2}$$

as $r \to \infty$ outside an exceptional set of finite measure. Despite these similarities between the Nevanlinna and half-plane characteristics, there are some important differences that should not be overlooked. A notable example is that $\mathfrak{T}(r,f) = O(\log r)$ does not imply that f is rational, and indeed $\mathfrak{T}(r,e^{-iz})$ is bounded as $r \to \infty$.

We shall say more about the Tsuji characteristic at the beginning of Section 6.3.

6.2 Statement of results

In the spirit of the Pólya-Wiman conjectures, the aim of this chapter is to seek out conditions under which a real entire function must belong to the class LP or to one of the more general classes U_{2p}^*. These conditions will typically involve the non-real zeroes of the function and its derivatives.

The first result below implies that a real entire function f belongs to LP if it has only real zeroes and all the non-real zeroes of f'' are critical points of f.

Theorem 6.3. *Let f be a real entire function with finitely many non-real zeroes. If $f \in U_{2p}^*$, then f'' has at least $2p$ non-real zeroes that are not critical points of f. If instead f is of infinite order, then f'' has infinitely many such zeroes.*

Theorem 6.3 is a minor strengthening of the $k = 2$ cases of Theorems 6.1 and 6.2. Our next result extends these cases in a different direction. It turns out that statements regarding the zeroes of ff'' can often be generalised to ones considering the zeroes of $ff'' - a(f')^2$ for certain values of a. To do this, we modify Sheil-Small's approach by using a 'relaxed' version of the Newton function.

The zeroes of the differential polynomial $ff'' - a(f')^2$ for a general meromorphic f have previously been studied in [5] and [36]. With all this in mind, we remark that if f is entire then a zero of $ff''/(f')^2 - a$ of multiplicity m is a zero of $ff'' - a(f')^2$ of multiplicity at least m.

Theorem 6.4. *Let $a < 1$ and let f be a real entire function with finitely many non-real zeroes. If $f \in U^*_{2p}$, then $ff''/(f')^2 - a$ has at least $2p$ non-real zeroes. If instead f is of infinite order, then $ff''/(f')^2 - a$ has infinitely many non-real zeroes.*

To see that we cannot take $a \geq 1$ in the above, let $f(z) = \exp(z^{2p})$ for $p \in \mathbb{N}$. Then $f \in U_{2p}$ and

$$\frac{ff''}{(f')^2} - a = \frac{2p - 1 - 2p(a-1)z^{2p}}{2pz^{2p}},$$

which has no zeroes if $a = 1$ and only $2p - 2$ non-real zeroes if $a > 1$.

The following corollary is proved in Section 6.5. Note that this time there are no assumptions about the zeroes of the function.

Corollary 6.5. *Let $a \leq \frac{1}{2}$ and let f be a real entire function such that f'/f is of finite lower order. If $ff''/(f')^2 - a$ has only finitely many non-real zeroes, then $f \in U^*_{2p}$ for some p. Moreover, if $ff''/(f')^2 \neq a$ on H, then $f \in LP$.*

Corollary 6.5 is new even for $a = 0$, in which case it shows that a real entire function f must belong to the class LP if f'/f has finite lower order and each non-real zero of ff'' is a critical point of f. The next result considers zeroes of higher derivatives and its proof is similar to that of Corollary 6.5. In fact, the $a = 0$ case of Corollary 6.5 implies the $k = 2$ case of Theorem 6.6.

Theorem 6.6. *Let $k \geq 2$ and let f be a real entire function such that $f^{(k-1)}/f^{(k-2)}$ is of finite lower order. Suppose that all (respectively, all but finitely many) of the non-real zeroes of $ff^{(k)}$ are also zeroes of $f^{(k-2)}$ and $f^{(k-1)}$. Then $f \in LP$ (respectively, $f \in U^*_{2p}$ for some p).*

The hypothesis that $f^{(k-1)}/f^{(k-2)}$ is of finite lower order is certainly satisfied if either f or f'/f is of finite order. See Lemma 6.34 for a proof of the latter fact.

The results stated above all require that the function under consideration either has only finitely many non-real zeroes or satisfies an order condition. We now seek results that are free of these particular restrictions. Instead, we take integers $M \geq k \geq 2$ and define the following hypotheses for an analytic function f:

(I) all the non-real zeroes of $ff^{(k)}$ are zeroes of f with multiplicity at least k but at most M;

(I′) all but finitely many of the non-real zeroes of $ff^{(k)}$ are zeroes of f with multiplicity at least k but at most M;

(II) $ff'' - a(f')^2$ has no non-real zeroes, for some $a \in \mathbb{C} \setminus \{\frac{1}{2}, 1\}$;

(II′) $ff'' - a(f')^2$ has finitely many non-real zeroes, for some $a \in \mathbb{C} \setminus \{\frac{1}{2}, 1\}$.

Under these hypotheses, the next result provides a bound on the Tsuji characteristic as defined in Section 6.1.3.

Theorem 6.7. *If f is analytic on $\overline{H}$ and satisfies either* (I′) *or* (II′) *then, for all $j \geq 0$,*

$$\mathfrak{N}(r, 1/f) = O(\log r) \quad \textit{and} \quad \mathfrak{T}\left(r, \frac{f^{(j+1)}}{f^{(j)}}\right) = O(\log r) \quad \textit{as} \quad r \to \infty. \tag{6.2.1}$$

In Section 6.6 we will apply Theorem 6.7 to obtain the following three results.

Theorem 6.8. *Let f be a real entire function and take real $a < \frac{1}{2}$ and $M \geq k \geq 2$. Suppose that either*

(i) all (respectively, all but finitely many) of the non-real zeroes of $ff^{(k-1)}f^{(k)}$ are zeroes of f with multiplicity at least k but at most M; or

(ii) $ff'' - a(f')^2$ has no (respectively, finitely many) non-real zeroes and f' has finitely many non-real zeroes.

Then $f \in LP$ (respectively, $f \in U_{2p}^*$ for some p).

We need the following definition which makes exact the notion of points not occurring too frequently. Let $a_1, a_2, \ldots$ be a sequence of complex numbers, this sequence either being finite or tending to infinity. Writing $n(r)$ for the number of a_j lying in $\{z : |z| \leq r\}$, and setting

$$N(r) = \int_1^r \frac{n(r)}{t}\, dt,$$

we say that the sequence a_j has *finite exponent of convergence* if and only if

$$\limsup_{r \to \infty} \frac{\log N(r)}{\log r} < \infty.$$

For a meromorphic function g, we shall say that "the zeroes of g have finite exponent of convergence" to mean that the sequence of zeroes repeated according to multiplicity has finite exponent of convergence. We comment that it follows from the First Fundamental Theorem (Theorem 1.1) that the zeroes of a finite order function must have finite exponent of convergence.

Theorem 6.9. *Let f be a real entire function.*

(i) If (I′) *holds and the zeroes of $f^{(j)}$ have finite exponent of convergence for some $0 \leq j \leq k-1$, then $f \in U_{2p}^*$ for some p. If in addition* (I) *holds, then $f \in LP$.*

(ii) If (II′) *holds and the zeroes of f or f' have finite exponent of convergence, then f'/f has finite order. Moreover, if $a < \frac{1}{2}$ then we have $f \in U_{2p}^*$ for some p, and in fact $f \in LP$ if* (II) *also holds.*

The final two results only place a 'finite exponent of convergence' condition on certain *non-real* zeroes. That is, we simply consider the sequence of non-real zeroes repeated according to multiplicity. There is no restriction on the frequency of the real zeroes.

Theorem 6.10. *Let f be an entire function satisfying either* (I′) *or* (II′)*. Suppose that the non-real zeroes of $f^{(j)}$ have finite exponent of convergence for some $j \geq 0$. Then* $\log\log M(r,f) = O(r \log r)$ *as $r \to \infty$.*

The particular estimate for the rate of growth found in Theorem 6.10 has a long history in this area. We have already mentioned that Levin and Ostrovskii [45] established this bound for a real entire function f such that ff'' has only real zeroes. It is through Shen's generalisation [56] of one of Levin and Ostrovskii's results that Theorem 6.10 does not require a real function.

Our last theorem extends the theme of Theorem 6.8(i) and Theorem 6.9(i).

Theorem 6.11. *Let $1 \leq j < k < M < \infty$ and let f be a real entire function such that all (respectively, all but finitely many) of the non-real zeroes of $ff^{(j)}f^{(k)}$ are zeroes of f with multiplicity at least k but at most M. Assume further that these non-real zeroes have finite exponent of convergence. Then $f \in LP$ (respectively, $f \in U_{2p}^*$ for some p).*

6.3 Preliminaries

We begin with two established lemmas involving the Tsuji characteristic. The first is a version of Hayman's Alternative that goes back essentially to Levin and Ostrovskii [45].

Lemma 6.12. *Let g be meromorphic on $\overline{H}$. If*

$$\mathfrak{N}(r, 1/g) = O(\log r) \quad \textit{and} \quad \mathfrak{N}\left(r, \frac{1}{g'-1}\right) = O(\log r), \quad r \to \infty,$$

then $\mathfrak{T}(r,g) = O(\log r)$.

The proof of Lemma 6.12 is obtained from the proof of Hayman's Alternative [20, p.60] by replacing the Nevanlinna characteristic with the Tsuji half-plane characteristic and using the fact that the lemma of the logarithmic derivative continues to hold in the Tsuji case [45, p.332] (see also [17, p.108]). We remark that, by (6.1.1), a function g satisfies the hypothesis of Lemma 6.12 if g and $g' - 1$ both have finitely many zeroes in the upper half-plane.

The next result will be used to provide a connection between the Nevanlinna and Tsuji proximity functions. We define

$$m_{0\pi}(r, g) = \frac{1}{2\pi} \int_0^{\pi} \log^+ |g(re^{i\theta})| \, d\theta, \tag{6.3.1}$$

and note that for a real meromorphic function on the plane $m(r, g) = 2m_{0\pi}(r, g)$.

Lemma 6.13 ([45]). *If g is meromorphic on $\overline{H}$ and $\mathfrak{m}(r, g) = O(\log r)$ as $r \to \infty$, then*

$$\int_R^{\infty} \frac{m_{0\pi}(r, g)}{r^3} \, dr = O\left(\frac{\log R}{R}\right), \quad R \to \infty.$$

The following lemma concerns subharmonic functions as defined in Section 1.2.

Lemma 6.14 ([59]). *Let u be a non-constant continuous subharmonic function on the plane. For $r > 0$, let $\theta^*(r)$ be the angular measure of that subset of $S(0, r)$ on which $u(z) > 0$, except that $\theta^*(r) = \infty$ if $u(z) > 0$ on the whole circle $S(0, r)$. Then, for $r > 0$,*

$$B(r, u) = \max\{u(z) : |z| = r\} \leq \frac{3}{2\pi} \int_0^{2\pi} \max\{u(2re^{it}), 0\} \, dt$$

and, if $r \leq R/4$ and r is sufficiently large,

$$B(r, u) \leq 9\sqrt{2} B(R, u) \exp\left(-\pi \int_{2r}^{R/2} \frac{ds}{s\theta^*(s)}\right).$$

6.3.1 Transcendental singularities of the inverse function

Recall the discussion of the singularities of the inverse function in Section 1.4. The asymptotic values of a transcendental meromorphic function g are called the *transcendental singularities of g^{-1}*. These are further classified as direct or indirect as follows. Suppose that $g(z)$ tends to $\alpha \in \mathbb{C}$ as z goes to infinity along a path γ. For each $\varepsilon > 0$, let $C(\varepsilon)$ denote that component of the set $\{z : |g(z) - \alpha| < \varepsilon\}$ which contains an unbounded subpath of γ. Two different asymptotic paths on which $g \to \alpha$ are considered to determine separate transcendental singularities if and only if the corresponding components $C(\varepsilon)$ are distinct for some $\varepsilon > 0$. The path γ determines an *indirect* transcendental singularity over α if $C(\varepsilon)$ contains infinitely many α-points of g for every $\varepsilon > 0$. Otherwise, the singularity is called *direct* and $C(\varepsilon)$, for all sufficiently small ε, contains no α-points. Transcendental singularities over ∞ are defined and classified by considering $1/g$. A transcendental singularity will be referred to as "lying in a domain D" if $C(\varepsilon) \subseteq D$ for small ε.

The Denjoy-Carleman-Ahlfors Theorem [47, §XI.4] places a bound on the number of direct transcendental singularities. In particular, we have the following result.

Lemma 6.15 ([47, §XI.4]). *A meromorphic function of finite lower order has finitely many direct transcendental singularities.*

In subsequent sections we shall often want to limit the number of singularities of an inverse function found in the upper half-plane. Lemmas 6.15, 6.16 and 6.17 will be used several times for this purpose.

Lemma 6.16 ([40]). *Let g be a meromorphic function such that $\mathfrak{T}(r,g) = O(\log r)$ as $r \to \infty$. Then there is at most one direct singularity of g^{-1} lying in H.*

The Bergweiler-Eremenko Theorem is an important result about indirect transcendental singularities. We state Hinchliffe's extension of it to include functions of finite lower order.

Lemma 6.17 ([6, 30]). *Let g be a meromorphic function of finite lower order. Then any indirect transcendental singularity of g^{-1} must be a limit point of critical values. In particular, if g has finitely many critical values, then g^{-1} has no indirect transcendental singularities.*

The next result is standard; a proof is included for completeness.

Lemma 6.18. *Let D be an unbounded simply-connected domain whose boundary consists of two simple curves γ_1 and γ_2, both tending to infinity and disjoint apart from their common starting point. Let g be analytic on a domain containing the closure $\overline{D}$. If $g(z) \to \alpha_j$ as $z \to \infty$ on γ_j, for $j = 1, 2$, where $\alpha_1, \alpha_2 \in \mathbb{C}$ are distinct, then there is a direct transcendental singularity over ∞ lying in D.*

Proof. Since $\alpha_1 \neq \alpha_2$, an application of a strong form of the Phragmén-Lindelöf principle [59, p.308] gives that g is unbounded in D. Therefore, the sets $D_n = \{z \in D : |g(z)| > n\}$ are non-empty and $\overline{D_n} \subseteq D$ for $n \geq n_0$. Let C_{n_0} be a component of D_{n_0}. We inductively choose a sequence of nested components $C_n \subseteq D_n$ for $n > n_0$. To do this, first assume that C_n has been chosen appropriately and define $v_n(z) = |g(z)|$ for $z \in C_n$, and $v_n(z) = n$ for $z \in \mathbb{C} \backslash C_n$. Then the real-valued function v_n is continuous and subharmonic in the plane, see Section 1.2. The Liouville Theorem for subharmonic functions [53, p.31] states that a bounded subharmonic function on $\mathbb{C}$ is constant. Since $|g|$ is non-constant on any domain in D, it follows that v_n is non-constant and hence unbounded. Therefore, g is unbounded on C_n and we can choose C_{n+1} to be a component of D_{n+1} lying in C_n.

The proof is now completed by simply choosing a path $\gamma : [n_0, \infty) \to D$ such that $\gamma(t) \in C_n$ for $t \geq n$. □

6.3.2 The Levin-Ostrovskii factorisation

Nearly half a century after the Pólya-Wiman conjectures were posed, the first significant progress was made by Levin and Ostrovskii [45]. They wrote the logarithmic derivative as the product of two functions, one having few poles and one mapping the upper half-plane into itself. Variations of this technique are central to the proofs of Theorems 6.1 and 6.2.

Lemma 6.19 ([7, 40])**.** *Let f be a real entire function with finitely many non-real zeroes. Then the logarithmic derivative has a factorisation*

$$L = \frac{f'}{f} = \phi\psi \tag{6.3.2}$$

in which ϕ and ψ are real meromorphic functions satisfying the following:

(i) either $\psi \equiv 1$ or $\psi(H) \subseteq H$;

(ii) ψ has a simple pole at each real zero of f, and no other poles;

(iii) ϕ has finitely many poles, none of them real;

(iv) on each component of $\mathbb{R} \setminus f^{-1}(\{0\})$ the number of zeroes of ϕ is either infinite or even;

(v) if $f \in U^*_{2p}$, then ϕ is a rational function, and if in addition f has at least one real zero, then the degree at infinity of ϕ is even and satisfies

$$\deg_\infty(\phi) = \lim_{z\to\infty} \frac{\log|\phi(z)|}{\log|z|} \geq 2p; \tag{6.3.3}$$

(vi) if f has infinite order, then ϕ is transcendental.

Parts (i)–(v) are proved in [40, Lemma 4.2], as the cited lemma applies to any real entire f with finitely many non-real zeroes. Part (vi) is [7, Lemma 5.1]. We briefly elucidate the construction of ψ in the case where the set of real zeroes a_k of f is unbounded above and below. Assume that $a_k < a_{k+1}$ and for simplicity that $a_0 = 0$. Since L has a positive residue at each zero of f, there exists a zero b_k of L in (a_k, a_{k+1}). We take ψ to be the product of the terms $p_k(z)$, where

$$p_0(z) = \frac{b_0 - z}{a_0 - z}, \qquad p_k(z) = \frac{1 - z/b_k}{1 - z/a_k},\ k \neq 0;$$

this product converging by the alternating series test. For $z \in H$, we observe that $\arg p_k(z)$ is the angle between the lines from z to a_k and b_k respectively, so that $\arg\psi(z) = \sum \arg p_k(z) \in (0, \pi)$ and thus $\psi(z) \in H$.

The next result is the Carathéodory inequality [44, Ch. I.6, Theorem 8′], which is essentially the Schwarz lemma on a half-plane.

Lemma 6.20 ([44]). *Let $\psi : H \to H$ be analytic. Then*

$$\frac{|\psi(i)|\sin\theta}{5r} < |\psi(re^{i\theta})| < \frac{5r|\psi(i)|}{\sin\theta} \qquad \text{for} \qquad r \geq 1, \quad \theta \in (0,\pi).$$

This shows that away from the real axis ψ is neither too large nor too small, so that in (6.3.2) the growth of f'/f is dominated by that of ϕ.

6.4 Proof of Theorems 6.3 and 6.4

Theorems 6.3 and 6.4 are proved by making a number of small alterations to the proofs of Theorems 6.1 and 6.2. The main difference is that we shall consider a 'relaxed' Newton function $z - hf/f'$, where the constant h is no longer always taken to be 1.

We shall first prove both theorems in the infinite order case, as these results can be quickly deduced from a theorem of Bergweiler, Eremenko and Langley [7]. We then tackle the remaining finite order case, where we base our arguments on existing proofs, but cannot so easily quote suitable results from the literature. Some of the original papers on this subject can be difficult to follow, hence the reader is directed to [42] which gives a unified presentation of the proof of Theorems 6.1 and 6.2, and upon which this section draws heavily.

6.4.1 Infinite order case

The following is the theorem of Bergweiler, Eremenko and Langley mentioned above.

Lemma 6.21 ([7]). *Let $\tilde{L}$ be a real meromorphic function such that all but finitely many poles of $\tilde{L}$ are real and simple and have positive residues. Suppose that $\tilde{L} = \tilde{\phi}\psi$, where $\tilde{\phi}$ and ψ are real meromorphic functions such that: either $\psi \equiv 1$ or $\psi(H) \subseteq H$; every pole of ψ is real and simple and is a simple pole of $\tilde{L}$; and $\tilde{\phi}$ is transcendental with finitely many poles. Then $\tilde{L} + \tilde{L}'/\tilde{L}$ has infinitely many non-real zeroes.*

Let f be a real entire function of infinite order with only finitely many non-real zeroes. By Lemma 6.19, we have the Levin-Ostrovskii factorisation $L = f'/f = \phi\psi$. For $a < 1$, let

$$\tilde{\phi} = (1-a)\phi \quad \text{and} \quad \tilde{L} = (1-a)L = \tilde{\phi}\psi.$$

Then $\tilde{L}$, $\tilde{\phi}$ and ψ satisfy the hypothesis of Lemma 6.21 by Lemma 6.19(i)–(iii) and (vi). Therefore, Lemma 6.21 gives that

$$\tilde{M} = \tilde{L} + \frac{\tilde{L}'}{\tilde{L}} = (1-a)\frac{f'}{f} + \frac{f}{f'}\left(\frac{ff'' - (f')^2}{f^2}\right) = \frac{f'}{f}\left(\frac{ff''}{(f')^2} - a\right)$$

has infinitely many non-real zeroes. Since $\tilde{M}$ does not vanish at a zero of L, this establishes the infinite order case of Theorem 6.4. Setting $a = 0$ gives the infinite order case of Theorem 6.3.

6.4.2 Finite order case

We assume that $f \in U_{2p}^*$ for some $p \geq 1$ and that $ff''/(f')^2 - a$ has finitely many non-real zeroes, where $a < 1$. To prove Theorem 6.3, we make these assumptions with $a = 0$. Let $L = f'/f$.

Lemma 6.22. *The Tsuji characteristic of the logarithmic derivative satisfies*

$$\mathfrak{T}(r, L) = O(\log r), \qquad r \to \infty. \tag{6.4.1}$$

Proof. Note that poles of L are zeroes of f, and so only finitely many of them can be non-real. Write $g = 1/L$; then g has finitely many zeroes in H, and $g' = 1 - ff''/(f')^2$ takes the value $1 - a$ only finitely often in H. An application of Lemma 6.12 now gives that $\mathfrak{T}(r, L) = \mathfrak{T}(r, g/(1-a)) + O(1) = O(\log r)$ as $r \to \infty$. □

We make the following definitions.

$$h = \frac{1}{1-a} > 0,$$

$$G(z) = z - h\frac{f(z)}{f'(z)} = z - \frac{h}{L(z)}, \qquad G' = h\left(\frac{ff''}{(f')^2} - a\right), \tag{6.4.2}$$

$$W = \{z \in H : G(z) \in H\}, \qquad Y = \{z \in H : L(z) \in H\}. \tag{6.4.3}$$

Observe that $Y \subseteq W$, since h is positive. For $h = 1$, this key observation is due to Sheil-Small [55], who was the first to consider these sets. It is through a detailed study of how G maps components of W into H that Theorems 6.3 and 6.4 will be proved.

If $h = 1$, then G is the Newton function for f, otherwise it is called a relaxed Newton function. These functions arise when using the (relaxed) Newton method [4, §6] to find the zeroes of f by iterating G (in this context, usually $|h - 1| < 1$).

Lemma 6.23. *The closure of Y contains no real zeroes of f.*

Proof. This is from [55, p.181]. If x is a real zero of f, then it is a simple pole of L with positive residue. Then since L is univalent near x and real on the real axis, we see that $\operatorname{Im} L(z) < 0$ for points in H near x. Thus x does not lie in the closure of Y. □

We continue to follow [42]. Our next result (cf [40, §5]) deals with transcendental singularities as discussed in Section 6.3.1.

Lemma 6.24. *The function G has no asymptotic values in $\mathbb{C} \setminus \mathbb{R}$, while the function L has only finitely many.*

Proof. Suppose that $\alpha \in \mathbb{C} \setminus \mathbb{R}$ is an asymptotic value of G. Since G has finitely many non-real critical values by (6.4.2), the Bergweiler-Eremenko Theorem (Lemma 6.17) shows that α must be a direct transcendental singularity of G^{-1}. Therefore, there exist $\varepsilon \in (0,1)$ and a component D of the set $\{z \in \mathbb{C} : |G(z) - \alpha| < \varepsilon\}$ such that $G(z) \neq \alpha$ on D. Since G is real meromorphic, we may assume that $D \subseteq H$. We define a continuous subharmonic function on the plane by

$$v(z) = \begin{cases} \log \dfrac{\varepsilon}{|G(z) - \alpha|}, & z \in D \\ 0, & z \in \mathbb{C} \setminus D. \end{cases}$$

Lemma 6.14 gives that

$$B(r/2, v) \leq \frac{3}{2\pi} \int_0^\pi \log^+ \frac{\varepsilon}{|G(re^{it}) - \alpha|} \, dt \leq 3 m_{0\pi}\left(r, \frac{1}{G - \alpha}\right).$$

By (6.4.1) and (6.4.2), we have $\mathfrak{T}(r, 1/(G - \alpha)) = O(\log r)$ as $r \to \infty$. Using this and the above, together with Lemma 6.13 and the fact that $B(r, v)$ is increasing [53, §2.3], now leads to

$$\frac{B(R/2, v)}{2R^2} \leq \int_R^\infty \frac{B(r/2, v)}{r^3} \, dr \leq 3 \int_R^\infty \frac{m_{0\pi}(r, 1/(G - \alpha))}{r^3} \, dr = O\left(\frac{\log R}{R}\right).$$

Hence,

$$B(R, v) = O(R \log R) \quad \text{as } R \to \infty. \tag{6.4.4}$$

We now let δ be small and positive, and claim that

$$G(z) \to \infty \quad \text{as } z \to \infty, \quad \delta < \arg z < \pi - \delta. \tag{6.4.5}$$

Let $L = f'/f = \phi\psi$ be the Levin-Ostrovskii factorisation described in Lemma 6.19. If f has at least one real zero, then $\deg_\infty(\phi) \geq 2$ by (6.3.3), and then (6.4.5) follows from Lemma 6.20 and (6.4.2). Otherwise, f has no real zeroes, and so there exist real polynomials P and Q such that

$$f = Pe^Q, \quad L = P'/P + Q', \quad \deg Q \geq 2p, \tag{6.4.6}$$

using the fact that $f \in U_{2p}^*$ (see Section 6.1.2). Hence L is a rational function with a pole at infinity, and again (6.4.5) follows from (6.4.2).

By (6.4.5), the angular measure of $D \cap S(0, r)$ is at most 2δ, for all $r \geq r_0$. Thus Lemma 6.14 gives, as $R \to \infty$,

$$B(R, v) \geq \frac{B(r_0, v)}{9\sqrt{2}} \exp\left(\pi \int_{2r_0}^{R/2} \frac{ds}{2\delta s}\right) = cR^{\pi/2\delta}$$

for some positive constant c. As δ is arbitrarily small, this contradicts (6.4.4), showing that G cannot have an asymptotic value in $\mathbb{C} \setminus \mathbb{R}$.

Suppose now that L has infinitely many non-real asymptotic values. Since L has finitely many non-real poles, we see from Lemma 6.18 that L^{-1} must have at least two direct transcendental singularities over ∞ lying in H. By (6.4.1), this stands in contradiction to Lemma 6.16. $\square$

Therefore, G has no asymptotic values in H by the previous lemma, and finitely many critical values in H by (6.4.2). We use these facts to obtain the next result, which is Lemma 7.1 of [42].

Lemma 6.25. *For each component A of W there is a positive integer k_A such that G maps A onto H with valency k_A; that is, each value $w \in H$ is taken k_A times in A. Furthermore, G' has at least $k_A - 1$ zeroes in A.*

Lemma 6.25 is proved by the following standard argument (see [7, p.987–988] or [38, §11]). Let $\gamma \subseteq \overline{H}$ be a bounded simple curve such that $H^* = H \setminus \gamma$ is simply-connected and contains no singular values of G^{-1}. Then each component of $G^{-1}(H^*)$ is mapped univalently onto H^* by G, and G maps every component of W onto H with finite valency. The final assertion is proved by an application of the Riemann-Hurwitz formula.

We introduce some more notation before stating our next lemma. Denote by $2q$ the number of distinct non-real zeroes of f and define

$$D(\lambda) = \{z \in H : |z| < \lambda\}, \qquad E(\Lambda) = \{z \in H : |z| > \Lambda\}.$$

The next result is Lemma 6.1 of [42].

Lemma 6.26. *For sufficiently small positive λ, and sufficiently large positive Λ, there are at least $p + q$ pairs of bounded components $K_j \subseteq V_j \subseteq H$ such that the following conditions are satisfied:*

(i) K_j is a component of the set $L^{-1}(D(\lambda))$, mapped univalently onto $D(\lambda)$ by L;

(ii) V_j is a component of the set $G^{-1}(E(\Lambda))$, mapped univalently onto $E(\Lambda)$ by G;

(iii) the V_j are pairwise disjoint;

(iv) $\partial K_j \cap \partial V_j$ contains one zero of L.

Proof. Let Z be a finite set of zeroes of L and let λ and $1/\Lambda$ be small. The proof of [42, Lemma 6.1], which is essentially the argument of [38, p.383–385] and [40, Lemma 8.1], contains an elementary analysis of the behaviour of L near its zeroes which shows that each $\zeta \in Z$ gives rise to pairs $\{K_j, V_j\}$ as in the statement of the lemma as follows:

- If $\zeta \in Z \cap H$ is a zero of L of multiplicity m, then there exist m such pairs $\{K_j, V_j\}$ with $\zeta \in \partial K_j \cap \partial V_j$.
- If $\zeta \in Z \cap \mathbb{R}$ is a zero of even multiplicity m, then there exist $m/2$ such pairs $\{K_j, V_j\}$ with $\zeta \in \partial K_j \cap \partial V_j$. In this case, the sign of $L(x)$ does not change as real x passes through ζ from left to right.
- Now suppose that $\zeta \in Z \cap \mathbb{R}$ is a zero of L of odd multiplicity m. If $L^{(m)}(\zeta) > 0$, then there exist $(m+1)/2$ pairs $\{K_j, V_j\}$ and $L(x)$ has a positive sign change at ζ; that is, $L(x)$ changes from negative to positive as x passes through ζ from left to right. If instead $L^{(m)}(\zeta) < 0$, then ζ gives rise to $(m-1)/2$ pairs $\{K_j, V_j\}$ and $L(x)$ has a negative sign change at ζ. In either case, $\zeta \in \partial K_j \cap \partial V_j$ for each pair.

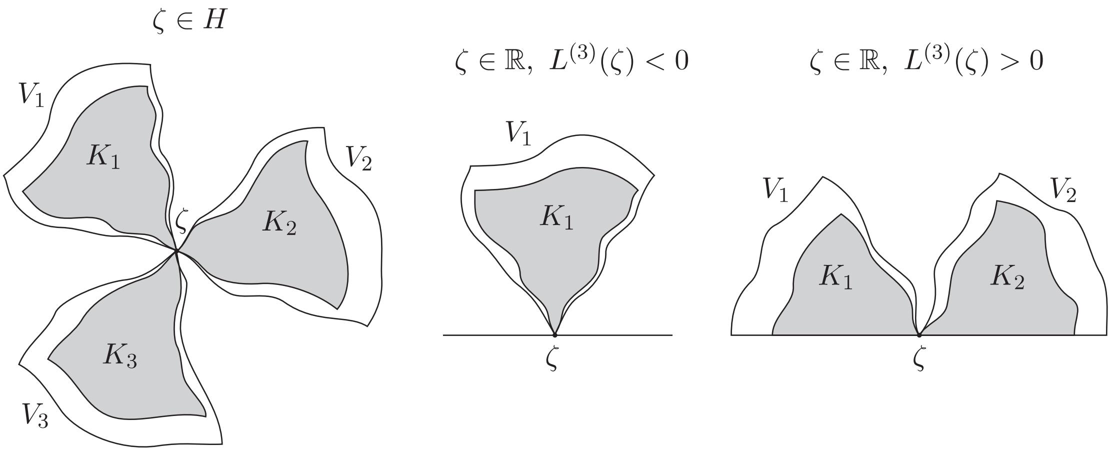

Figure 6.2: The three cases for pairs $\{K_j, V_j\}$ when L has a triple zero at ζ.

Provided that λ and $1/\Lambda$ are chosen sufficiently small, the components arising from distinct zeroes are disjoint. It remains to show that we can find at least $p+q$ components K_j. To this end, we again make use of the factorisation $L = \phi\psi$ from Lemma 6.19, where ϕ is rational by Lemma 6.19(v). Let I be a component of $\mathbb{R} \setminus f^{-1}(\{0\})$ containing $\mu_I > 0$ zeroes of ϕ and m_I zeroes of L, not forgetting our convention that zeroes are counted with regard to multiplicity. Then $m_I \geq \mu_I$, and μ_I is even by Lemma 6.19(ii) and (iv). Hence, by the statements above, the interval I gives rise to

$$n_I = \frac{m_I + s_I}{2} \geq \frac{\mu_I + s_I}{2} \tag{6.4.7}$$

components K_j, where s_I is the number of positive sign changes minus the number of negative sign changes undergone by $L(x)$ on I. Since $s_I \geq -1$ and μ_I is even, (6.4.7) implies that $n_I \geq \mu_I/2$.

Denote by $2r$ and $2t$ respectively the number of real and non-real zeroes of ϕ, and recall that the non-real zeroes of L are precisely the non-real zeroes of ϕ. By summing over all components of $\mathbb{R} \setminus f^{-1}(\{0\})$ that contain real zeroes of L, the arguments above show that there are at least $r + t$ components K_j that satisfy the conditions of the lemma.

The function ϕ has only simple poles in the plane, and these occur precisely at the $2q$ distinct non-real zeroes of f. Hence, equating the number of zeroes and poles of ϕ in $\mathbb{C} \cup \{\infty\}$ leads to

$$2r + 2t = 2q + \deg_\infty(\phi). \tag{6.4.8}$$

Thus the conclusion of the lemma follows at once from (6.3.3), except in the case where f has no real zeroes. In this last case, however, we must once more have (6.4.6). If $\deg Q \geq 2p+1$, then $\deg_\infty(\phi) \geq 2p-1$ by Lemma 6.20, and again the result follows from (6.4.8). Suppose finally that $\deg Q = 2p$. Then since $f \in U^*_{2p}$, the leading coefficient c of Q is positive and $L(z) \sim 2pcz^{2p-1}$ as $z \to \infty$ by (6.4.6). Here there is one component $I = \mathbb{R}$ and $s_I = 1$. As L has simple poles at the distinct zeroes of f, equating the number of zeroes and poles of L in $\mathbb{C} \cup \{\infty\}$ gives

$$m_I + 2t = 2q + \deg_\infty(L) = 2q + 2p - 1. \tag{6.4.9}$$

Therefore, in this case we have at least p^* components K_j, where, using (6.4.7) and (6.4.9),

$$p^* \geq n_I + t = \frac{m_I + 1}{2} + t = p + q. \qquad \square$$

We are now ready to complete the proof as in [42]. Choose $\theta \in (\pi/4, 3\pi/4)$ such that the ray $\gamma(s) = se^{i\theta}$, $s \in (0, \infty)$, contains no singular values of L^{-1}. This is possible because L has countably many critical values and, by Lemma 6.24, finitely many asymptotic values in H. For each K_j, choose $z_j \in K_j$ with $L(z_j) \in \gamma$, and continue L^{-1} along γ in the direction of infinity. Let Γ_j be the image of this continuation starting at z_j. Then Γ_j is a path in Y on which $L(z) \to \infty$, where Y is defined by (6.4.3). Hence, Γ_j tends either to infinity or to a pole of L, which must be a zero of f in H by Lemma 6.23. Since $K_j \subseteq Y \subseteq W$, each K_j lies in some component A of W. A component A_ν of W will be called type (α) if there exists $K_j \subseteq A_\nu$ such that Γ_j tends to infinity, and type (β) otherwise.

Lemma 6.27. *Let n_ν denote the number of K_j contained in a component A_ν of W.*

- *If A_ν is type (α), then n_ν is at most the number of zeroes of G' in A_ν.*
- *If A_ν is type (β), then n_ν is at most the number of distinct zeroes of f in A_ν.*

Proof. First suppose that A_ν is type (α). By Lemma 6.25, it will suffice to show that $n_\nu \leq k_{A_\nu} - 1$. But the fact that the valency k_{A_ν} of G on A_ν exceeds the number n_ν is made clear by the following observation: each of the n_ν sets $K_j \subseteq A_\nu$ corresponds to a bounded component $V_j \subseteq A_\nu$ which is mapped onto $E(\Lambda)$ by G, while we also have a path tending to infinity in A_ν on which $L(z) \to \infty$ and consequently $G(z) \to \infty$, by (6.4.2).

Now suppose instead that A_ν is type (β). For each K_j contained in A_ν, the path Γ_j must tend to a zero w_j of f in H. Since L has a simple pole at w_j, it is univalent near w_j, and there cannot be two different paths Γ_j, $\Gamma_{j'}$ near w_j that are both mapped onto γ by L. Therefore, the w_j corresponding to different K_j must be distinct. Moreover, using (6.4.2) gives that $G(w_j) = w_j \in H$, so that $w_j \in A_\nu$. $\square$

This completes the proof of Theorem 6.4, since Lemma 6.26 gives $p+q$ components K_j, but by (6.4.2) and Lemma 6.27, the number of K_j does not exceed q plus the number of zeroes of $ff''/(f')^2 - a$ in H.

To prove Theorem 6.3, we put $a = 0$ and note that G' does not vanish at any zero of f' by (6.4.2). Hence, using (6.4.2) again, the number of zeroes of G' in A_ν is at most the number of distinct zeroes of f in A_ν plus the number of zeroes of f'' in A_ν that are not zeroes of f'. Lemma 6.26 still provides $p+q$ components K_j, but now Lemma 6.27 shows that this cannot exceed q plus the number of zeroes of f'' in H that are not critical points of f.

6.5 An iteration argument

The field of complex dynamics studies the behaviour of the iterates of analytic and meromorphic functions on the complex plane, see for example [4, 46]. This is a very active area of research and has enjoyed many successes in recent years. We will use some of the well-known elements of iteration theory to establish a useful lemma.

We shall write F^n for the nth iterate of the function F; that is, $F^0(z) = z$ and $F^n(z) = F(F^{n-1}(z))$ for $n \geq 1$. If F is a rational function, then the iterates F^n are also rational and so are defined at all points $z \in \mathbb{C} \cup \{\infty\}$. On the other hand, a transcendental F cannot sensibly be defined at infinity, so that F^n is only defined at points $z \in \mathbb{C}$ that are not poles of $F, F^2, \ldots, F^{n-1}$.

The Fatou and Julia sets are central to complex dynamics. Qualitatively, the Fatou set of a function is that part of $\mathbb{C}\cup\{\infty\}$ on which the function's iterates behave smoothly, while the Julia set is the region where they behave chaotically. To give a formal definition, let F be meromorphic and let $\mathcal{F} = \{F^n : n \in \mathbb{N}\}$ be the family of iterates of F. A point $z \in \mathbb{C} \cup \{\infty\}$ belongs to the Fatou set of F if and only if the family $\mathcal{F}$ is defined and normal on some neighbourhood of z. The Julia set is then defined to be the complement of the Fatou set in $\mathbb{C} \cup \{\infty\}$. It follows that both the Fatou set and the

Julia set are invariant under F, in the sense that both sets are mapped into themselves by F. Moreover, the Fatou and Julia sets for any iterate F^n are the same as those for F.

A point z_0 is called a *fixed point* of F if $F(z_0) = z_0$. Such a fixed point is said to be *attracting* if $|F'(z_0)| < 1$, and *superattracting* if $F'(z_0) = 0$. Lemma 6.28 below is based upon the fact that, as a function is iterated, each attracting fixed point draws in a singularity of the inverse function (see Section 1.4). For a rational function F, we denote by $\text{sing}(F^{-1})$ the set of critical values of F, including ∞ if F has any multiple poles. For a transcendental function, $\text{sing}(F^{-1})$ consists of these critical values together with any finite asymptotic values of F. We now define the sets

$$\mathcal{A}(F) = \{z \in \mathbb{C} \setminus \mathbb{R} : F(z) = z \text{ and either } 0 < |F'(z)| < 1 \text{ or } F'(z) = -1\} \qquad (6.5.1)$$

and

$$\mathcal{C}(F) = \{z \in \mathbb{C} \setminus \mathbb{R} : z \in \text{sing}(F^{-1}),\ |F(z) - z| + |F'(z)| > 0\}, \qquad (6.5.2)$$

so that $\mathcal{C}(F)$ contains the non-real singularities of the inverse function F^{-1} that are not superattracting fixed points of F. We can now state the aforementioned useful lemma.

Lemma 6.28. *Let F be a real meromorphic function on the plane. If $\mathcal{C}(F)$ is finite, then so is $\mathcal{A}(F)$ and $|\mathcal{A}(F)| \leq |\mathcal{C}(F)|$.*

Proof. Let $z_j \in \mathcal{A}(F)$. We suppose first that $|F'(z_j)| < 1$. It then follows that z_j lies in a component C_j of the Fatou set of F. This component is called the *immediate attracting basin* of the attracting fixed point z_j, and we have

$$F^n(z) \to z_j \quad \text{as} \quad n \to \infty, \qquad z \in C_j. \qquad (6.5.3)$$

If we suppose instead that $F'(z_j) = -1$, then there must exist at least two components of the Fatou set on which $F^n(z) \to z_j$ and which include z_j in their boundary. These components are called *Leau domains* [46, §10]. In this case, we let C_j be the union of all these Leau domains, so that we again have (6.5.3).

It follows from (6.5.3) that distinct points $z_j \in \mathcal{A}(F)$ give rise to disjoint subsets C_j of the Fatou set. Since F is real, we see also that no C_j can meet the real axis. Moreover, $\infty \notin C_j$ because if F is a real rational function, then $F^n(\infty) \in \mathbb{R} \cup \{\infty\}$.

It is well known [4, §4.3] that each set C_j must contain a point of $\text{sing}(F^{-1})$, say w_j. By the previous paragraph, $w_j \in \mathbb{C} \setminus \mathbb{R}$. If w_j is a fixed point of F, then $w_j = z_j$ by (6.5.3), in which case $|F'(w_j)| = |F'(z_j)| > 0$ since $z_j \in \mathcal{A}(F)$. Hence, $w_j \in \mathcal{C}(F)$ and the result follows. □

6.5.1 Proof of Corollary 6.5

As in the statement of the corollary, we take $a \leq \frac{1}{2}$ and let f be a real entire function such that f'/f is of finite lower order. Suppose that $ff''/(f')^2 - a$ has only finitely many non-real zeroes.

We aim to show that f has only finitely many non-real zeroes. Let G be defined by (6.4.2), where $h = (1-a)^{-1}$ and so $0 < h \leq 2$. By (6.4.2) and our hypotheses on f, we see that G has finite lower order and G' has finitely many non-real zeroes. Lemmas 6.15 and 6.17 now show that G^{-1} has finitely many direct, and no indirect, transcendental singularities over $\mathbb{C} \setminus \mathbb{R}$. Thus the set $\mathcal{C}(G)$ is finite (this is trivial if G is a rational function), and Lemma 6.28 implies that $\mathcal{A}(G)$ is also finite.

If $\zeta \in \mathbb{C} \setminus \mathbb{R}$ is a zero of f of multiplicity m, then $G(\zeta) = \zeta$ and

$$G'(\zeta) = 1 - h \left(\frac{f}{f'} \right)' (\zeta) = 1 - \frac{h}{m} \in [-1, 1).$$

Hence, assuming that ζ is not one of the finitely many non-real zeroes of G', we have that $\zeta \in \mathcal{A}(G)$. We therefore deduce that f has a finite number of non-real zeroes. Theorem 6.4 now gives that $f \in U^*_{2p}$ for some p.

Now suppose that $ff''/(f')^2 \neq a$ on H. Then by (6.4.2), the finite critical values of G are all real. Since $f \in U^*_{2p}$, Lemma 6.24 applies and shows that G has no asymptotic values in $\mathbb{C} \setminus \mathbb{R}$. Thus $\mathcal{C}(G)$ is empty. Therefore, Lemma 6.28 shows that $\mathcal{A}(G)$ is also empty, and so f cannot have any zeroes $\zeta \in \mathbb{C} \setminus \mathbb{R}$. Hence, f must belong to the class LP by Theorem 6.4 (recall from Section 6.1.2 that $U_0 = LP$).

6.5.2 Proof of Theorem 6.6

Let $k \geq 2$ and let f be a real entire function such that $f^{(k-1)}/f^{(k-2)}$ is of finite lower order (we exclude the case where $f^{(k-2)} \equiv 0$). Assume that all but finitely many of the non-real zeroes of $ff^{(k)}$ are also zeroes of $f^{(k-2)}$ and $f^{(k-1)}$. Write

$$F_k(z) = z - \frac{f^{(k-2)}(z)}{f^{(k-1)}(z)}, \qquad F_k' = -\frac{f^{(k-2)} f^{(k)}}{(f^{(k-1)})^2}. \tag{6.5.4}$$

Then F_k is the Newton function of $f^{(k-2)}$ and is, in particular, a real meromorphic function of finite lower order.

Lemma 6.29. *If ζ is a non-real zero of $f^{(k-2)}$ of multiplicity $m \geq 2$, then $\zeta \in \mathcal{A}(F_k)$, where $\mathcal{A}(F_k)$ is defined by* (6.5.1) *and* (6.5.4).

Proof. Observe that $F_k(\zeta) = \zeta$ and calculate

$$F_k'(\zeta) = 1 - \left(\frac{f^{(k-2)}}{f^{(k-1)}} \right)' (\zeta) = 1 - \frac{1}{m}.$$

Hence, $\frac{1}{2} \leq F_k'(\zeta) < 1$ and thus $\zeta \in \mathcal{A}(F_k)$. □

By Lemma 6.29, all but finitely many of the non-real zeroes of $ff^{(k)}$ are members of $\mathcal{A}(F_k)$, because of our assumption about these zeroes.

Our next task is to prove that $\mathcal{C}(F_k)$ is finite. The main observation here is that by (6.5.4), all but finitely many of the non-real critical points of F_k are also fixed points of F_k. Using (6.5.2), this immediately implies that $\mathcal{C}(F_k)$ contains only a finite number of critical values of F_k. A second consequence of our observation is that the set of critical values of F_k can have no limit points in $\mathbb{C} \setminus \mathbb{R}$. Therefore, by Lemma 6.17 there are no indirect transcendental singularities of F_k^{-1} lying in $\mathbb{C} \setminus \mathbb{R}$. Hence, using Lemma 6.15 we see that F_k has only finitely many non-real asymptotic values, and so $\mathcal{C}(F_k)$ is indeed finite.

An application of Lemma 6.28 now shows that $ff^{(k)}$ has finitely many non-real zeroes. Theorem 6.2 then implies that f has finite order, and hence $f \in U_{2p}^*$ for some p.

We prove next that $f \in LP$ if all the non-real zeroes of $ff^{(k)}$ are zeroes of $f^{(k-2)}$ and $f^{(k-1)}$. For such a function f, Lemma 6.29 shows that all the non-real zeroes of $ff^{(k)}$ lie in $\mathcal{A}(F_k)$. Therefore, by Theorem 6.1 and Lemma 6.28, it will suffice to show that $\mathcal{C}(F_k) = \emptyset$ in this case. By (6.5.4), a non-real zero of F_k' is now necessarily a zero of $f^{(k-2)}$, and so a fixed point of F_k. Using (6.5.2), it follows that no critical values of F_k belong to $\mathcal{C}(F_k)$, and it just remains to show that F_k has no non-real asymptotic values. Since the class U_{2p}^* is closed under differentiation [12, Corollary 2.12], we have that $f^{(k-2)} \in U_{2p}^*$. Therefore all the statements made when proving Theorems 6.3 and 6.4 in Section 6.4 remain valid with $f^{(k-2)}$ in place of f. In particular, if we replace f with $f^{(k-2)}$ and set $a = 0$, then the function G defined in (6.4.2) becomes F_k. The result we require is then provided by Lemma 6.24.

6.6 Theorem 6.7 and applications

In this section, we will establish Theorem 6.7 and then apply it to prove Theorems 6.8, 6.9 and 6.10.

6.6.1 Proof of Theorem 6.7

We shall first obtain a normal families result for functions satisfying (I$'$) or (II$'$). This leads to a lower bound for the distance between the distinct zeroes of such functions, from which a careful counting argument gives the first estimate of (6.2.1). The half-plane versions of some standard value distribution results then complete the proof of Theorem 6.7.

We begin with the following theorem of Bergweiler and Langley [8], where $\mathrm{Res}(F, w)$

denotes the residue of F at w, and the differential operators Ψ_k are defined by

$$\Psi_1(y) = y, \qquad \Psi_{k+1}(y) = y\Psi_k(y) + \frac{d}{dz}(\Psi_k(y)).$$

Lemma 6.30 ([8])**.** *Let $k \geq 2$ and let $\mathcal{F}_0$ be a family of functions meromorphic on a domain Ω. Then $\mathcal{F}_0$ is a normal family on Ω if there exists $\delta \in (0,1]$ such that the following conditions hold for all $F \in \mathcal{F}_0$.*

- *$\Psi_k(F)$ has no zeroes.*
- *If w is a simple pole of F, then $|\operatorname{Res}(F,w) - j| \geq \delta$ for all $j \in \{0, 1, \ldots, k-1\}$.*
- *For all discs $D(c,R) \subseteq \Omega$ such that $D(c,\delta R)$ contains two poles of F counting multiplicities, but $D(c,R) \setminus D(c,\delta R)$ contains none, we have*

$$\left| \sum_{w \in D(c,\delta R)} \operatorname{Res}(F,w) - (k-1) \right| \geq \delta.$$

We repeat the observation made in [8] that an easy proof by induction yields

$$\Psi_k(g'/g) = g^{(k)}/g. \tag{6.6.1}$$

Lemma 6.31. *Let $k \geq 2$, let $a \in \mathbb{C} \setminus \{\frac{1}{2}, 1\}$ and let $\mathcal{G}$ be a family of functions analytic on a domain Ω. Suppose that for each $g \in \mathcal{G}$, either*

(i) every zero of $gg^{(k)}$ in Ω is a zero of g with multiplicity at least k; or

(ii) $gg'' - a(g')^2$ has no zeroes in Ω.

Then $\mathcal{F} = \{g/g' : g \in \mathcal{G}\}$ is a normal family on Ω.

Remark. In fact, Lemma 6.31 holds for a family of meromorphic functions provided that every member satisfies condition (ii) and $\frac{1}{a-1} \notin \mathbb{N}$ as well as $a \neq \frac{1}{2}, 1$. The proof needs only minor modification, but we will not need this result.

Proof of Lemma 6.31. We may assume that either every $g \in \mathcal{G}$ satisfies (i) or that every $g \in \mathcal{G}$ satisfies (ii). Suppose first that each $g \in \mathcal{G}$ satisfies condition (i) and let $G = g'/g$. Then using (6.6.1), we see that $\Psi_k(G) = g^{(k)}/g$ does not vanish in Ω. Moreover, the poles of G are simple and have integer residues not less than k. Therefore, the family $\mathcal{F}_0 = \{g'/g : g \in \mathcal{G}\}$ satisfies the hypotheses of Lemma 6.30, and hence both $\mathcal{F}_0$ and $\mathcal{F}$ are normal on Ω.

Next suppose instead that each $g \in \mathcal{G}$ satisfies condition (ii). We may assume that a is non-zero, otherwise every $g \in \mathcal{G}$ satisfies (i) with $k = 2$. This time we set $G = (1-a)g'/g$ and again appeal to Lemma 6.30. We see that

$$\Psi_2(G) = G^2 + G' = (1-a)\left((1-a)\left(\frac{g'}{g}\right)^2 + \frac{gg'' - (g')^2}{g^2}\right) = \frac{1-a}{g^2}(gg'' - a(g')^2),$$

and so $\Psi_2(G)$ has no zeroes in Ω. Condition (ii) implies that g has only simple zeroes, so that G has only simple poles, each with residue $1-a$. Since $1-a$ is neither zero nor one, and $2(1-a) \neq 1$, we find that the family $\mathcal{F}_1 = \{(1-a)g'/g : g \in \mathcal{G}\}$ satisfies the hypotheses of Lemma 6.30 with $k = 2$. Therefore, $\mathcal{F}_1$ is normal on Ω by Lemma 6.30, and the result follows. $\square$

The next lemma is essentially contained in [15, Lemma 2.1]; its proof is reproduced here for completeness. Recall the notation

$$E(R) = \{z \in H : |z| > R\}. \tag{6.6.2}$$

Lemma 6.32 ([15]). *Let $R \geq 0$, $d > 0$ and $0 < c < 1$. Suppose that u is meromorphic on $\overline{H}$ such that the family*

$$\left\{ \frac{u(z_0 + (c \operatorname{Im} z_0)z)}{c \operatorname{Im} z_0} : z_0 \in E(R) \right\}$$

is normal on the unit disc, and $|u'(\zeta)| \geq d$ whenever $u(\zeta) = 0$ with $\zeta \in E(R)$.

Then there exists $b > 0$ with the following property: any pair $z_1, z_2 \in H$ of distinct zeroes of u satisfies $|z_1 - z_2| \geq b \operatorname{Im} z_1$.

Proof. Let $z_1 \in H$ be a zero of u. Since u has only a finite number of zeroes lying in $\{z \in H : |z| \leq R\}$, there is no loss of generality in assuming that $z_1 \in E(R)$. By equicontinuity, there exists a positive constant δ, independent of the choice of z_1, such that

$$\left| \frac{u(z_1 + (c \operatorname{Im} z_1)z)}{c \operatorname{Im} z_1} \right| \leq 1 \quad \text{for} \quad z \in B(0, 2\delta);$$

equivalently, $|u(z)| \leq c \operatorname{Im} z_1$ for $z \in B(z_1, 2\delta c \operatorname{Im} z_1)$. Now assume that z_2 is a zero of u with $0 < |z_1 - z_2| \leq \delta c \operatorname{Im} z_1$. The function

$$h(z) = \frac{u(z)}{(z - z_1)(z - z_2)}$$

is analytic on $B(z_1, 2\delta c \operatorname{Im} z_1)$, and satisfies

$$|h(z)| \leq \frac{c \operatorname{Im} z_1}{(2\delta c \operatorname{Im} z_1)(\delta c \operatorname{Im} z_1)}$$

on the boundary of $B(z_1, 2\delta c \operatorname{Im} z_1)$, and so on the whole disc by the Maximum Principle. Therefore,

$$d \leq |u'(z_1)| = |(z_1 - z_2)h(z_1)| \leq \frac{|z_1 - z_2|}{2\delta^2 c \operatorname{Im} z_1},$$

which gives the required lower bound for $|z_1 - z_2|$. $\square$

Lemma 6.33. *Let $b > 0$ and suppose that u is meromorphic on $\overline{H}$ such that any pair $z_1, z_2 \in H$ of distinct zeroes of u satisfies $|z_1 - z_2| \geq b \operatorname{Im} z_1$. If the zeroes of u have bounded multiplicities, then $\mathfrak{N}(r, 1/u) = O(\log r)$ as $r \to \infty$.*

Proof. We begin by claiming that, for $r > 1$,

$$\left\{z : |z| \geq 1,\ \left|z - \frac{ir}{2}\right| \leq \frac{r}{2}\right\} \subseteq D_r = \left\{x + iy : \frac{1}{r} \leq y \leq r,\ |x| \leq \sqrt{ry}\right\}. \tag{6.6.3}$$

See Figure 6.3. To prove this claim, suppose that $x + iy$ lies in the set on the left-hand side of (6.6.3). By calculating that $S(0, 1)$ intersects $S(ir/2, r/2)$ at points with imaginary part $1/r$, we get that $1/r \leq y \leq r$. Then

$$\begin{aligned} \left|x + iy - \frac{ir}{2}\right| \leq \frac{r}{2} \quad &\Rightarrow \quad x^2 + \left(y - \frac{r}{2}\right)^2 \leq \frac{r^2}{4} \\ &\Rightarrow \quad |x| \leq \left|ry - y^2\right|^{1/2} \leq \sqrt{ry} \end{aligned}$$

and hence $x + iy \in D_r$.

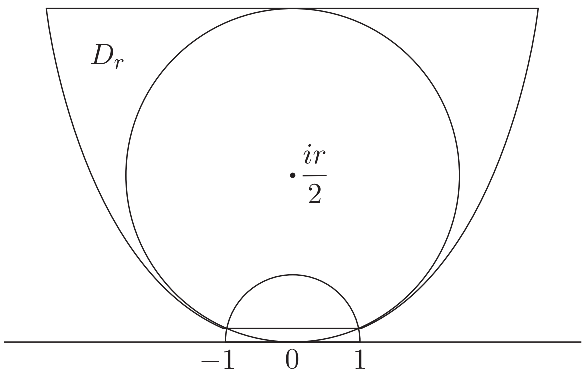

Figure 6.3: The truncated parabola D_r.

Cover the upper half-plane H with squares

$$A_{p,q} = \left\{z : 2^{p-1} \leq \operatorname{Im} z \leq 2^p,\ |\operatorname{Re} z - 2^{p-1}q| \leq 2^{p-2}\right\}, \quad p, q \in \mathbb{Z},$$

as shown in Figure 6.4. Observe that each square $A_{p,q}$ contains at most N zeroes of u, where N is independent of p and q. This is because the distinct zeroes in $A_{p,q}$ are separated by a distance of at least $2^{p-1}b$ and have bounded multiplicities. It now follows from (6.6.3) that $\mathfrak{n}(r, 1/u)$ is at most N times the number of squares $A_{p,q}$ that meet D_r. To count these squares, first note that row p meets D_r if and only if $2^p \geq 1/r$ and $2^{p-1} \leq r$; or equivalently, $-L \leq p \leq L+1$, where L is the greatest integer not exceeding $\log_2 r$. When row p meets D_r, the square $A_{p,q}$ does so if and only if

$$2^{p-1}\left(|q| - \frac{1}{2}\right) \leq \sqrt{r2^p},$$

and there can be at most $4\left(2^{-p/2}\sqrt{r}\right) + 2$ such integers q. Therefore, the number of squares $A_{p,q}$ that intersect D_r does not exceed

$$\begin{aligned} \sum_{p=-L}^{L+1} \left(4\left(2^{-p/2}\sqrt{r}\right) + 2\right) \quad &\leq \quad 4\sqrt{r}\frac{2^{L/2}}{1 - 2^{-1/2}} + 4L + 4 \\ &\leq \quad \frac{4r}{1 - 2^{-1/2}} + 4\log_2 r + 4. \end{aligned}$$

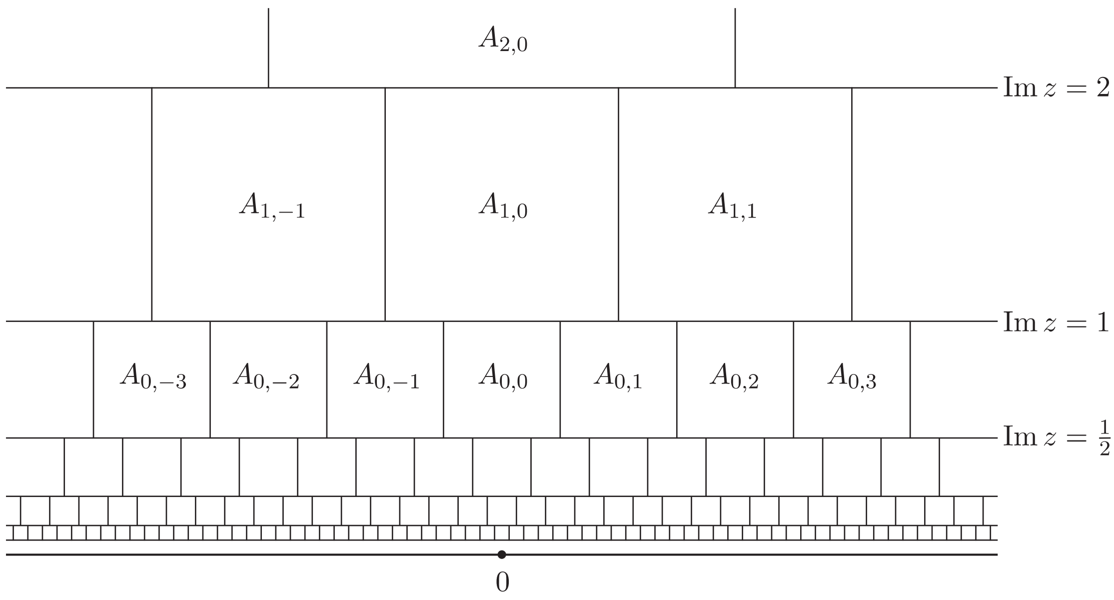

FIGURE 6.4: Each square $A_{p,q}$ has side length 2^{p-1}.

Hence, $\mathfrak{n}(r, 1/u) = O(r)$ as $r \to \infty$. Recalling definition (6.1.1) now completes the proof. $\square$

We are now able to apply the preceding sequence of lemmas to establish Theorem 6.7. To this end, let f be analytic on $\overline{H}$ and satisfy either (I′) or (II′). Fix $c \in (0, 1)$. Then for a sufficiently large choice of R, the family

$$\mathcal{G} = \{f(z_0 + (c \operatorname{Im} z_0)z) : z_0 \in E(R)\} \tag{6.6.4}$$

of analytic functions on the unit disc satisfies the hypothesis of Lemma 6.31. Hence,

$$\mathcal{F} = \left\{ \frac{f(z_0 + (c \operatorname{Im} z_0)z)}{(c \operatorname{Im} z_0) f'(z_0 + (c \operatorname{Im} z_0)z)} : z_0 \in E(R) \right\} \tag{6.6.5}$$

is a normal family on the unit disc by Lemma 6.31.

We note that the multiplicities of the non-real zeroes of f are bounded above by some constant M_0. In case (II′), this follows from the fact that f has only finitely many non-real multiple zeroes. We now write $u = f/f'$. If ζ is a non-real zero of u, then ζ must also be a zero of f, say of multiplicity m, and so $u'(\zeta) = 1/m \geq 1/M_0$. Therefore Lemma 6.32 applies to u with $d = 1/M_0$, since we have shown that (6.6.5) is normal on the unit disc. Upon combining the conclusion of Lemma 6.32 with the observation that u has only simple zeroes, we obtain from Lemma 6.33 that

$$\mathfrak{N}(r, 1/f) \leq M_0 \mathfrak{N}(r, 1/u) = O(\log r), \quad r \to \infty. \tag{6.6.6}$$

This establishes the first estimate of (6.2.1).

We now assert that

$$\mathfrak{T}(r, f'/f) = O(\log r), \quad r \to \infty. \tag{6.6.7}$$

In the case that f satisfies (II′), we can use Hayman's Alternative to deduce (6.6.7) as follows. Since $\mathfrak{N}(r, 1/u) = O(\log r)$ by (6.6.6), and

$$u' - 1 + a = -\frac{ff'' - a(f')^2}{(f')^2}$$

has finitely many non-real zeroes by (II′), Hayman's Alternative (Lemma 6.12) gives that $\mathfrak{T}(r, u) = O(\log r)$ as $r \to \infty$.

Now suppose instead that f satisfies (I′). Observe that it will suffice to show that (6.6.7) holds as $r \to \infty$ outside a set of finite measure, because the Tsuji characteristic differs from a non-decreasing continuous function by a bounded additive term [17, p.27]. Hence, if (6.6.7) fails to hold, then there must exist a set J of infinite measure such that $\log r = o(\mathfrak{T}(r, f'/f))$ as $r \to \infty$ through values in J. Since f is analytic on $\overline{H}$ and satisfies (I′), we get from (6.6.6) that

$$\mathfrak{N}(r, 1/f) + \mathfrak{N}(r, 1/f^{(k)}) + \mathfrak{N}(r, f) = O(\log r) = o(\mathfrak{T}(r, f'/f)) \quad \text{as } r \to \infty \text{ on } J.$$

Since the lemma of the logarithmic derivative holds for the Tsuji characteristic (see [17, p.108]), we can now apply the standard Tumura-Clunie argument [20, Thm 3.10, p.74] on J to obtain a contradiction. Here we use the fact that all the exceptional sets arising in the proof have finite measure, and that the exceptional cases encountered all imply (6.6.7) anyway. See also Lemma 1 of [28] and the remark of [28, p.476].

Write

$$L_j = \frac{f^{(j+1)}}{f^{(j)}},$$

so that $\mathfrak{T}(r, L_0) = O(\log r)$ as $r \to \infty$, by (6.6.7). Assume the inductive hypothesis that $\mathfrak{T}(r, L_j) = O(\log r)$ as $r \to \infty$, for some $j \geq 0$. As L_j'/L_j only has (simple) poles at the zeroes and poles of L_j, we know that

$$\mathfrak{N}(r, L_j'/L_j) = O(\mathfrak{T}(r, L_j)) = O(\log r), \quad r \to \infty.$$

Moreover, the lemma of the logarithmic derivative on a half-plane [17, p.108] gives that $\mathfrak{m}(r, L_j'/L_j) = O(\log r)$. Thus, using the relation

$$L_{j+1} = L_j + \frac{L_j'}{L_j} \tag{6.6.8}$$

and a standard inequality from Section 6.1.3 shows that $\mathfrak{T}(r, L_{j+1}) = O(\log r)$ as $r \to \infty$. The second estimate of (6.2.1) now follows by induction.

Remark. Theorem 6.7 states that f has very few zeroes from the viewpoint of the Tsuji characteristic. However, f could have many non-real zeroes in the Nevanlinna sense; in fact, these zeroes could have infinite exponent of convergence. This difference can arise when the zeroes are concentrated near the real axis, as suggested by Figure 6.4. We remark, however, that the condition on the separation of the zeroes in Lemma 6.33 is not strong enough to conclude that the zeroes form an A-set as studied, for example, by Shen in [56].

6.6.2 Proof of Theorem 6.8

The proofs of parts (i) and (ii) are similar but for clarity they are presented separately.

Part (i). Suppose that all but finitely many of the non-real zeroes of $ff^{(k-1)}f^{(k)}$ are zeroes of f with multiplicity at least k but at most M. To show that $f \in U_{2p}^*$ for some p, it will suffice by Theorem 6.2 to show that $ff^{(k)}$ has only finitely many non-real zeroes. Define F_k by (6.5.4), and note that all but finitely many of the non-real zeroes of $ff^{(k)}$ belong to $\mathcal{A}(F_k)$ by Lemma 6.29. Hence, by Lemma 6.28 it will suffice to prove that $\mathcal{C}(F_k)$ is finite. As in Section 6.5.2, the hypothesis on f and (6.5.4) imply that all but a finite number of the non-real critical points of F_k are fixed points of F_k, so that $\mathcal{C}(F_k)$ contains only finitely many critical values of F_k by (6.5.2). It remains to show that F_k does not have infinitely many non-real asymptotic values.

The function f satisfies (I′), so Theorem 6.7 and (6.5.4) give that

$$\mathfrak{T}(r, F_k) = \mathfrak{T}(r, f^{(k-1)}/f^{(k-2)}) + O(1) = O(\log r) \quad \text{as } r \to \infty.$$

Lemma 6.16 now shows that there is at most one direct transcendental singularity of F_k^{-1} lying in H. Observe that our hypothesis on f implies that F_k has a finite number of poles in H. It follows that F_k has at most two asymptotic values in H, since any pair of indirect transcendental singularities requires a direct singularity over ∞ lying between them by Lemma 6.18. Therefore, F_k has at most four non-real asymptotic values. This completes the proof that $f \in U_{2p}^*$.

Now assume that all of the non-real zeroes of $ff^{(k-1)}f^{(k)}$ are zeroes of f with multiplicity at least k but at most M. We have already shown that $f \in U_{2p}^*$, so f has finite order and, in particular, $f^{(k-1)}/f^{(k-2)}$ must have finite lower order. We conclude that $f \in LP$ by Theorem 6.6.

Part (ii). Suppose that f' and $ff''-a(f')^2$ both have only finitely many non-real zeroes. We aim to show that the zeroes of f are real with finitely many exceptions, so that

$f \in U_{2p}^*$ for some p by Theorem 6.4. We define G by (6.4.2) with $h = (1-a)^{-1}$, so that $h \in (0,2)$. Then G has finitely many non-real critical points by (6.4.2) and our assumptions on f. Note that if $\zeta \in \mathbb{C} \setminus \mathbb{R}$ is a zero of f, but is not one of the finitely many non-real zeroes of G' or f', then by (6.4.2),

$$G(\zeta) = \zeta \quad \text{and} \quad G'(\zeta) = 1 - h\left(\frac{f}{f'}\right)'(\zeta) = 1 - h \in (-1,1),$$

and so $\zeta \in \mathcal{A}(G)$. Therefore, to show that f has finitely many non-real zeroes, it will again suffice by Lemma 6.28 to show that $\mathcal{C}(G)$ is finite. Since G has a finite number of non-real critical values, we only need to limit the number of non-real asymptotic values.

Using the fact that f satisfies condition (II′), we deduce from Theorem 6.7 that $\mathfrak{T}(r,G) = O(\log r)$ as $r \to \infty$. The proof that G has at most four non-real asymptotic values is now exactly as in part (i), using the fact that non-real poles of G can only occur at the finitely many non-real zeroes of f'. This completes the proof that $f \in U_{2p}^*$, and we note that this certainly implies that f'/f is of finite lower order. Under the stronger assumption that $ff'' - a(f')^2$ has no non-real zeroes, Corollary 6.5 immediately gives that $f \in LP$ (see also the sentence preceding Theorem 6.4).

6.6.3 Proof of Theorem 6.9

We will use the following simple lemma.

Lemma 6.34. *Let g be a meromorphic function and let $L_j = g^{(j+1)}/g^{(j)}$. Then the orders satisfy $\rho(L_{j+1}) \le \rho(L_j)$.*

Proof. Assume that L_j has finite order, otherwise there is nothing to prove. Equation (6.6.8) holds for the L_j, and so

$$T(r, L_{j+1}) \le T(r, L_j) + T(r, L_j'/L_j) + O(1) \le 4T(r, L_j) + O(\log r), \quad r \to \infty,$$

using the lemma of the logarithmic derivative. □

To prove Theorem 6.9, suppose that f is a real entire function such that either

(i) condition (I′) holds and the zeroes of $f^{(j)}$ have finite exponent of convergence for some $0 \le j \le k-1$; or

(ii) condition (II′) holds and the zeroes of $f^{(j)}$ have finite exponent of convergence for $j = 0$ or 1.

In either case, let $L^* = f'/f$ if $j = 0$, and let $L^* = f^{(j-1)}/f^{(j)}$ if $j > 0$. Then the poles of L^* have finite exponent of convergence, and so there exists $K \geq 3$ such that

$$I_1 = \int_1^\infty \frac{N(t, L^*)}{t^K}\, dt < \infty.$$

Theorem 6.7 gives that $\mathfrak{T}(r, L^*) = O(\log r)$ as $r \to \infty$. Hence, by Lemma 6.13 and the sentence preceding it, we have

$$I_2 = \int_1^\infty \frac{m(t, L^*)}{t^3}\, dt < \infty.$$

Since $T(r, L^*)$ is an increasing function of r, we see that for $r \geq 1$,

$$\frac{T(r, L^*)}{(2r)^K} r \leq \int_r^{2r} \frac{T(t, L^*)}{t^K}\, dt \leq I_1 + I_2,$$

from which we deduce that L^* has finite order.

In case (ii), the function L^* is either f'/f or f/f', and so f'/f must have finite order. In this case, the proof is now completed by applying Corollary 6.5.

To conclude the proof in case (i), we first appeal to Lemma 6.34 to show that $\rho(f^{(k-1)}/f^{(k-2)}) \leq \rho(L^*)$. Then $f^{(k-1)}/f^{(k-2)}$ certainly has finite lower order, and the required results follow from Theorem 6.6.

6.6.4 Proof of Theorem 6.10

As in the statement of the theorem, suppose that f is an entire function satisfying either (I′) or (II′), and assume that the non-real zeroes of $f^{(j)}$ have finite exponent of convergence for some $j \geq 0$.

There exists an entire function Π whose zeroes are precisely the non-real zeroes of $f^{(j)}$, and whose order is equal to the exponent of convergence of these zeroes and so is finite. (Here Π may be formed as a Weierstrass product, see [20, p.24–30].) Pick three distinct values $a_1, a_2, a_3 \in \mathbb{C}$. Checking a straightforward set inclusion shows that

$$\mathfrak{n}(r, 1/(\Pi - a_\nu)) \leq n(r, 1/(\Pi - a_\nu))$$

and since Π has finite order, it follows that there exists $K > 0$ such that

$$\mathfrak{N}(r, 1/(\Pi - a_\nu)) \leq N(r, 1/(\Pi - a_\nu)) = O(r^K).$$

The Second Fundamental Theorem for the Tsuji characteristic (6.1.2) now gives

$$\mathfrak{T}(r, \Pi) \leq \sum_{\nu=1}^{3} \mathfrak{N}(r, 1/(\Pi - a_\nu)) + O(\log r + \log \mathfrak{T}(r, \Pi)) = O(r^K)$$

as $r \to \infty$ outside a set of finite measure. It follows that in fact $\mathfrak{T}(r, \Pi) = O(r^K)$ as $r \to \infty$ without an exceptional set, because $\mathfrak{T}(r, \Pi)$ differs from a non-decreasing continuous function by a bounded additive term [17, p.27]. Using this, the lemma of the logarithmic derivative [17, p.108] gives that $\mathfrak{m}(r, \Pi'/\Pi) = O(\log r)$ as $r \to \infty$.

Define the entire function g by $f^{(j)} = \Pi g$, so that g has only real zeroes. Then

$$\mathfrak{m}\left(r, \frac{g'}{g}\right) \leq \mathfrak{m}\left(r, \frac{f^{(j+1)}}{f^{(j)}}\right) + \mathfrak{m}\left(r, \frac{\Pi'}{\Pi}\right) + O(1) = O(\log r), \tag{6.6.9}$$

as $r \to \infty$, by using the above and Theorem 6.7. Since g satisfies (6.6.9) and has only real zeroes, Theorem 1A of [56] states that $\log\log M(r, g) = O(r \log r)$ as $r \to \infty$. As Π has finite order, it follows that

$$\log\log M(r, f^{(j)}) = O(r \log r), \quad r \to \infty. \tag{6.6.10}$$

By integrating $f^{(j)}$ a total of j times, it is easy to see that

$$M(r, f) \leq r^j M(r, f^{(j)}) + O(r^{j-1}), \quad r \to \infty,$$

so that (6.6.10) leads to the required estimate,

$$\log\log M(r, f) = O(r \log r), \quad r \to \infty.$$

6.7 Proof of Theorem 6.11

For functions of finite order, Theorem 6.11 follows immediately from Theorem 6.6. Therefore to prove Theorem 6.11 in full, it will suffice to show that any function satisfying the more general hypotheses has finite order. Note further that the $j = k - 1$ case of Theorem 6.11 is contained in Theorem 6.8(i).

Henceforth, we shall assume that f is an infinite order function that satisfies the more general hypotheses of Theorem 6.11 with $j \leq k - 2$. We aim to demonstrate a contradiction with Theorem 6.2 by showing that $f f^{(k)}$ has only finitely many non-real zeroes. The proof will then be complete.

We will again study a suitable Newton function. Let

$$F(z) = z - \frac{f^{(k-2)}(z)}{f^{(k-1)}(z)}, \qquad F' = \frac{f^{(k)} f^{(k-2)}}{(f^{(k-1)})^2}. \tag{6.7.1}$$

The next result is absolutely central to Theorem 6.11, but we postpone its proof to Section 6.7.2. Instead, we first describe how we may obtain the desired contradiction from it by applying the ideas of Section 6.5.

Proposition 6.35. *F^{-1} has no indirect transcendental singularities over $\mathbb{C} \setminus \mathbb{R}$.*

In fact, once Proposition 6.35 is established, it is easy to show that F has only a finite number of non-real asymptotic values. To do this, observe that f satisfies condition (I$'$)

of page 77, so that $\mathfrak{T}(r, F) = O(\log r)$ by Theorem 6.7. Then Lemma 6.16 gives that F^{-1} has at most two direct transcendental singularities over $\mathbb{C} \setminus \mathbb{R}$.

Using (6.7.1) and the hypotheses on f, we see that all but finitely many of the non-real critical points of F are also fixed points. Hence, the set $\mathcal{C}(F)$ defined by (6.5.2) is finite. Lemma 6.28 now gives that the set $\mathcal{A}(F)$ of (6.5.1) must also be finite. Since Lemma 6.29 applies to zeroes of f with multiplicity at least k, we find that the non-real zeroes of $ff^{(k)}$ belong to $\mathcal{A}(F)$ with only finitely many exceptions. This leads us to deduce that $ff^{(k)}$ has only finitely many non-real zeroes. As indicated earlier, this contradiction with Theorem 6.2 is enough to complete the proof of Theorem 6.11.

6.7.1 An estimate required for Proposition 6.35

Write, for $m = 0, 1, \ldots, k$,

$$L_m = \frac{f^{(m+1)}}{f^{(m)}}.$$

Then because f satisfies condition (I′) of page 77, we get from Theorem 6.7 that

$$\mathfrak{T}(r, L_m) = O(\log r), \quad \text{as } r \to \infty. \tag{6.7.2}$$

This section is devoted to proving the following result, which will later be used in the proof of Proposition 6.35. Both these proofs will use many ideas from [38], where Theorem 6.2 was proved for $k \geq 3$.

Proposition 6.36. *Let $\delta > 0$ and $P > 0$. Then on a set of r of logarithmic density 1 we have*

$$|L_m(z)| = \left| \frac{f^{(m+1)}(z)}{f^{(m)}(z)} \right| > |z|^P, \qquad |z| = r,\ \delta \leq \arg z \leq \pi - \delta \tag{6.7.3}$$

for $0 \leq m \leq k$.

In fact, Proposition 6.36 holds for any real entire function f of infinite order that satisfies (I′) and has non-real zeroes with finite exponent of convergence.

We use a Levin-Ostrovskii factorisation

$$L_m = \phi_m \psi_m \tag{6.7.4}$$

similar to that discussed for L_0 in Section 6.3.2. See also [38, §4].

Lemma 6.37. *For $0 \leq m \leq k$, there exist real meromorphic functions ϕ_m and ψ_m satisfying* (6.7.4) *such that*

(i) either $\psi_m \equiv 1$ or $\psi_m(H) \subseteq H$;

(ii) ϕ_m has only simple poles, all of which are zeroes of $f^{(m)}$ and only finitely many of which are real; and

(iii) ϕ_m has finite order.

Proof. If $f^{(m)}$ has a finite number of real zeroes, then we set $\psi_m \equiv 1$. Otherwise, $f^{(m)}$ has infinitely many real zeroes a_n. The a_n are simple poles of L_m and we may assume that $a_n < a_{n+1}$. By Rolle's Theorem, there exists a zero b_n of $f^{(m+1)}$, and hence of L_m, in (a_n, a_{n+1}). For $|n|$ at least some large n_0, the numbers a_n and a_{n+1} have the same sign. We now take ψ_m to be the product of the terms

$$p_n(z) = \frac{1 - z/b_n}{1 - z/a_n}, \quad |n| \geq n_0,$$

this product converging by the alternating series test. For $z \in H$, we observe that $\arg p_n(z)$ is the angle between the lines from z to a_n and b_n respectively, so that $\arg \psi_m(z) = \sum_{|n| \geq n_0} \arg p_n(z) \in (0, \pi)$, and thus $\psi_m(z) \in H$. Hence, ψ_m and ϕ_m satisfy (i) and (ii) by construction.

From part (i) and Lemma 6.20, we get that $m_{0\pi}(r, 1/\psi_m) = O(\log r)$ as $r \to \infty$, where $m_{0\pi}(r, 1/\psi_m)$ is defined by (6.3.1). Using this, (6.7.2) and (6.7.4), and applying Lemma 6.13, gives that

$$\int_1^\infty \frac{m_{0\pi}(r, \phi_m)}{r^3}\, dr \leq \int_1^\infty \frac{m_{0\pi}(r, L_m) + m_{0\pi}(r, 1/\psi_m)}{r^3}\, dr < \infty. \tag{6.7.5}$$

Following [38, Lemma 4.1], we now claim that there exists $q \geq 1$ such that, for $0 \leq m \leq k$,

$$n(r, \phi_m) \leq \sum_{0 \leq \mu < m} n(r, 1/\phi_\mu) + O(r^q) \quad \text{as } r \to \infty. \tag{6.7.6}$$

To prove this we need only consider the non-real poles of ϕ_m, since ϕ_m has only finitely many real poles by part (ii). When $m = 0$, the estimate (6.7.6) follows from noting that the (simple) non-real poles of ϕ_0 are non-real zeroes of f, and so have finite exponent of convergence. Now suppose that $m \geq 1$ and z_0 is a non-real pole of ϕ_m. Then z_0 is a simple pole of ϕ_m and a zero of $f^{(m)}$. Let $0 \leq p \leq m$ be the least integer such that $f^{(p)}(z_0) = 0$. Then either $p \geq 1$ and $\phi_{p-1}(z_0) = 0$; or else z_0 is a non-real zero of f, and these have finite exponent of convergence. This completes the proof of (6.7.6), as claimed.

We now prove part (iii) of the lemma by induction on m. Suppose that $\rho(\phi_\nu) < \infty$ for $0 \leq \nu \leq m-1$ (we assume nothing when $m = 0$). Then from (6.7.6) we have that, for some $q_m \geq 1$,

$$N(r, \phi_m) = O(r^{q_m}) \quad \text{as } r \to \infty.$$

Hence, using (6.7.5) and the fact that ϕ_m is a real function,

$$\int_1^\infty \frac{T(r, \phi_m)}{r^{q_m+2}}\, dr \leq \int_1^\infty \frac{2m_{0\pi}(r, \phi_m)}{r^3}\, dr + \int_1^\infty \frac{N(r, \phi_m)}{r^{q_m+2}}\, dr < \infty.$$

Since $T(r, \phi_m)$ is increasing, it follows that ϕ_m is of finite order. □

The next lemma provides a pointwise estimate for the logarithmic derivative of a finite order function. It is a special case of Corollary 2 of [19].

Lemma 6.38 ([19])**.** *If h is a meromorphic function of finite order, then*

$$\log^+\left|\frac{h'(z)}{h(z)}\right| = O(\log r)$$

as $|z| = r \to \infty$ outside a set of finite logarithmic measure.

By Lemma 6.37, each of the functions ϕ_m has finite order. We can therefore apply Lemma 6.38 to show that, for $0 \le m \le k$,

$$\log^+\left|\frac{\phi_m'(z)}{\phi_m(z)}\right| = O(\log r) \tag{6.7.7}$$

as $|z| = r \to \infty$ outside a set of finite logarithmic measure.

Lemma 6.37(i) states that if $\psi_m \not\equiv 1$, then $\psi_m(H) \subseteq H$. In this case, an analytic branch of $\log\psi_m$ may be defined on H. By Bloch's Theorem, the image of $B\big(z, \frac{\operatorname{Im} z}{2}\big)$ under $\log\psi_m$ must contain a disc of radius at least $C_B|(\log\psi_m)'(z)|\frac{\operatorname{Im} z}{2}$, where C_B is Bloch's Constant. As this image is contained in $\log H$, the radius of such a disc cannot exceed $\pi/2$ and therefore

$$\left|\frac{\psi_m'(z)}{\psi_m(z)}\right| \le \frac{\pi}{C_B \operatorname{Im} z}. \tag{6.7.8}$$

Using (6.7.4) and the definition of the L_m, we obtain the identity

$$L_m = L_{m-1} + \frac{L_{m-1}'}{L_{m-1}} = L_{m-1} + \frac{\phi_{m-1}'}{\phi_{m-1}} + \frac{\psi_{m-1}'}{\psi_{m-1}},$$

which immediately leads to

$$\log^+|L_m(z)| \ge \log^+|L_{m-1}(z)| - \log^+\left|\frac{\phi_{m-1}'(z)}{\phi_{m-1}(z)}\right| - \log^+\left|\frac{\psi_{m-1}'(z)}{\psi_{m-1}(z)}\right| - \log 3. \tag{6.7.9}$$

If we now take z with $|z| = r$ and $\delta \le \arg z \le \pi - \delta$, and repeatedly use (6.7.9) together with (6.7.7) and (6.7.8), then we conclude that

$$\log^+|L_m(z)| \ge \log^+|L_0(z)| - O(\log r) \tag{6.7.10}$$

as $r \to \infty$ outside a set of finite logarithmic measure. As a result of (6.7.10), we see that it will suffice to prove Proposition 6.36 with $m = 0$. We shall now concentrate on this particular case.

Let Π be a real entire function of finite order whose zeroes are precisely the non-real zeroes of f. For example, Π may be formed as a Weierstrass product [20, p.24–30] because the non-real zeroes of f are assumed to have finite exponent of convergence. Define g by

$$f = \Pi g;$$

then g is real entire and has only real zeroes. We take the Levin-Ostrovskii factorisation $g'/g = \phi\psi$ as described in Lemma 6.19. The function ϕ is entire by Lemma 6.19(i) and (iii), as g has no non-real zeroes. Moreover, ϕ is transcendental by Lemma 6.19(vi) because f, and hence also g, are of infinite order. Observe that

$$L_0 = \frac{f'}{f} = \frac{\Pi'}{\Pi} + \frac{g'}{g} = \frac{\Pi'}{\Pi} + \phi\psi. \tag{6.7.11}$$

We show next that the order of ϕ does not exceed 1. The characteristic $T(r, \phi)$ of the real entire function ϕ is equal to $2m_{0\pi}(r, \phi)$, and because this is increasing we immediately obtain the inequality

$$\frac{T(R, \phi)}{(2R)^3} R \leq \int_R^{2R} \frac{2m_{0\pi}(r, \phi)}{r^3} \, dr. \tag{6.7.12}$$

From the fact that Π has finite order, we can use the Tsuji half-plane versions of the Second Fundamental Theorem and the lemma of the logarithmic derivative to show that $\mathfrak{m}(r, \Pi'/\Pi) = O(\log r)$, as in Section 6.6.4. Together with (6.7.2) and (6.7.11), this gives that $\mathfrak{m}(r, g'/g) = O(\log r)$. We see from Lemma 6.19(i) and Lemma 6.20 that $m_{0\pi}(r, 1/\psi) = O(\log r)$. Using these estimates and applying Lemma 6.13 to g'/g, we deduce that

$$\int_R^\infty \frac{m_{0\pi}(r, \phi)}{r^3} \, dr \leq \int_R^\infty \frac{m_{0\pi}(r, g'/g) + m_{0\pi}(r, 1/\psi)}{r^3} \, dr = O\left(\frac{\log R}{R}\right)$$

as $R \to \infty$. The first inequality here just uses the fact that $\phi = (g'/g)/\psi$. Comparing this estimate with (6.7.12) reveals that $T(R, \phi) = O(R \log R)$, so that the order of ϕ is indeed no greater than 1.

By combining the next lemma with the fact that ϕ is transcendental, we are able to find points of large modulus that satisfy the inequality in Proposition 6.36 when $m = 0$.

Lemma 6.39. *Given $\varepsilon > 0$ and $\delta > 0$, we can find $\sigma \in (0, \delta]$ and a set $E_1 \subseteq [1, \infty)$ of upper logarithmic density at most ε with the following property. For each $r \notin E_1$, there exists $\theta = \theta(r) \in (\sigma, \pi - \sigma)$ such that*

$$\log |L_0(re^{i\theta})| > \frac{T(r, \phi)}{2} - O(\log r) \qquad \text{as } r \to \infty.$$

Proof. We begin by calling again upon two standard growth estimates that both hold outside small exceptional sets. As the function Π has finite order, Lemma 6.38 tells us that

$$\log^+ \left| \frac{\Pi'(z)}{\Pi(z)} \right| = O(\log r) \tag{6.7.13}$$

as $|z| = r \to \infty$ outside a set of finite logarithmic measure. Meanwhile, the order of ϕ does not exceed 1, and so we learn from Lemma 3.6 that, provided $C > 1$,

$$T(2r, \phi) \leq CT(r, \phi) \tag{6.7.14}$$

outside a set of upper logarithmic density at most $\log 2/\log C$. We now set $C = 2^{1/\varepsilon}$ and let E_1 be the union of the above two exceptional sets. Then $\overline{\text{logdens}}\, E_1 \leq \varepsilon$.

As ϕ is entire, Lemma 1.3 and (6.7.14) lead to

$$\log M(r,\phi) \leq 3T(2r,\phi) \leq 3CT(r,\phi), \quad r \notin E_1. \tag{6.7.15}$$

We now take $\sigma = \min\left\{\frac{\pi}{24C}, \delta\right\}$ and claim that, for each $r \notin E_1$, we can pick $\theta \in (\sigma, \pi-\sigma)$ such that

$$\log|\phi(re^{i\theta})| > \frac{T(r,\phi)}{2}. \tag{6.7.16}$$

Otherwise, if no such θ exists, then we could obtain a contradiction as follows, by using (6.7.15) and the fact that ϕ is a real function:

$$\begin{aligned} T(r,\phi) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\phi\left(re^{it}\right)\right| dt \\ &\leq \frac{4\sigma}{2\pi} 3CT(r,\phi) + \frac{T(r,\phi)}{2} \leq \frac{3T(r,\phi)}{4}. \end{aligned}$$

We can now complete the proof of the lemma by using (6.7.11), (6.7.13), (6.7.16) and Lemma 6.20,

$$\begin{aligned} \log|L_0(re^{i\theta})| &\geq \log|\phi(re^{i\theta})| + \log|\psi(re^{i\theta})| - \log^+\left|\frac{\Pi'(re^{i\theta})}{\Pi(re^{i\theta})}\right| - \log 2 \\ &> \frac{T(r,\phi)}{2} - O(\log r) \end{aligned}$$

as $r \to \infty$ outside E_1. □

Lemma 6.40. *Given $\varepsilon > 0$ and $\sigma > 0$, we can find $\lambda > 1$ such that $ff^{(k)}$ has no zeroes in*

$$A(r) = \{z : r/\lambda < |z| < \lambda r,\ \sigma/2 < \arg z < \pi - \sigma/2\}$$

for all r outside a set E_2 of upper logarithmic density at most ε.

Proof. Fix $c \in (0,1)$ and let $\mathcal{G}$ and $\mathcal{F}$ be the families of functions on the unit disc given by (6.6.4) and (6.6.5) respectively, where $E(R)$ is as in (6.6.2). As f satisfies condition (I′), a sufficiently large choice of R ensures that each member of $\mathcal{G}$ satisfies hypothesis (i) of Lemma 6.31, and so we deduce that $\mathcal{F}$ is normal on the unit disc. We now write $u = f/f'$. The argument following (6.6.5) shows that u satisfies the hypothesis of Lemma 6.32.

Denote by $z_1, z_2, \ldots$ those distinct zeroes of $ff^{(k)}$ that lie in

$$\{z : \sigma/2 < \arg z < \pi - \sigma/2\}.$$

Applying Lemma 6.32 to u gives $b > 0$ such that, if z_p, z_q are distinct zeroes of f, then

$$|z_p - z_q| \geq b\,\text{Im}\, z_p \geq b\sin(\sigma/2)|z_p|.$$

Since all but finitely many of the z_n are zeroes of f, we may assume that the above inequality holds for all distinct pairs z_p, z_q by reducing b if necessary. It follows that the number of the z_n that lie in any annulus $\{z : r < |z| < 2r\}$ has an upper bound independent of r. Therefore, we can find a constant B such that

$$\#\{z_n : |z_n| < r\} \leq B \log r, \qquad r \geq 2.$$

We now take $\lambda = \exp(\varepsilon/2B)$ and

$$E_2 = \bigcup_{n=1}^{\infty} \left[\frac{|z_n|}{\lambda}, \lambda|z_n| \right].$$

Then

$$\begin{aligned}
\overline{\text{logdens}}\, E_2 &= \limsup_{r\to\infty} \frac{1}{\log r} \int_{E_2 \cap [1,r]} \frac{dt}{t} \\
&\leq \limsup_{r\to\infty} \frac{1}{\log r} \sum_{|z_n|<\lambda r} \int_{|z_n|/\lambda}^{\lambda|z_n|} \frac{dt}{t} \\
&\leq \limsup_{r\to\infty} \frac{B\log \lambda r}{\log r} 2\log\lambda = \varepsilon.
\end{aligned}$$

It just remains to note that if $w \in A(r)$ and $ff^{(k)}(w) = 0$, then $w = z_n$ for some n. In this case, $r/\lambda < |z_n| < \lambda r$ and hence $r \in E_2$. □

Lemma 6.41 ([38, Lemma 2.4]). *Let $s > 0$ and let h be analytic on $B(0, 2s)$ with $h(z)h^{(k)}(z) \neq 0$ there. Then $G = h'/h$ satisfies*

$$\log M(s, G) \leq c_0(1 + \log^+|G(0)|),$$

in which $c_0 > 0$ depends only on s.

The estimate for L_0 provided by Lemma 6.39 is valid at only one point for each value of the modulus r. We now aim to use Lemmas 6.40 and 6.41 to extend this estimate to a large arc of the circle $|z| = r$.

Choose $\varepsilon > 0$ small, let σ and E_1 be as in Lemma 6.39, and let λ and E_2 be as in Lemma 6.40. Let $r \geq 1$ with $r \notin E_1 \cup E_2$, and take $\theta = \theta(r)$ as given by Lemma 6.39. Define the scaled functions

$$f_r(z) = f(rz), \qquad G_r(z) = \frac{f_r'(z)}{f_r(z)} = rL_0(rz). \tag{6.7.17}$$

Lemma 6.40 gives that $ff^{(k)}$ has no zeroes in $A(r)$, and so it follows that $f_r f_r^{(k)}$ is non-zero on $A(1)$. Therefore, repeated application of Lemma 6.41 gives a constant c_1, depending only on λ and σ, such that

$$\log^+|G_r(e^{i\theta})| \leq c_1(1 + \log^+|G_r(e^{it})|)$$

for all $t \in [\delta, \pi - \delta]$. It is clear from (6.7.17) that

$$|L_0(rz)| \leq |G_r(z)| \leq r|L_0(rz)|,$$

and so we can re-write the above as

$$\log^+|L_0(re^{i\theta})| \leq c_1(1 + \log r + \log^+|L_0(re^{it})|), \qquad t \in [\delta, \pi - \delta].$$

Combining this with the result of Lemma 6.39 gives that

$$\log^+|L_0(re^{it})| \geq c_2 T(r, \phi) - O(\log r), \qquad t \in [\delta, \pi - \delta], \tag{6.7.18}$$

as $r \to \infty$ outside $E_1 \cup E_2$, and where the constant c_2 is independent of r and t.

By recalling (6.7.10) and the fact that ϕ is transcendental, the estimate (6.7.18) shows that (6.7.3) holds for r outside an exceptional set with upper logarithmic density at most 2ε. Since ε may be chosen arbitrarily small, this completes the proof of Proposition 6.36.

6.7.2 Proof of Proposition 6.35

Assume that F^{-1} has an indirect transcendental singularity over some $\alpha \in H$. Our strategy for demonstrating a contradiction is based upon [38, §10] and will be as follows. First, we find a whole sequence of asymptotic values β_n such that $F(z) \to \beta_n$ as z tends to infinity on a path Γ_n. From (6.7.1), we have that

$$L_{k-2}(z) = \frac{f^{(k-1)}(z)}{f^{(k-2)}(z)} = \frac{1}{z - F(z)}. \tag{6.7.19}$$

Hence, Proposition 6.36 shows that $F(z) \approx z$ in most of the plane. It follows that the region where F is near β_n must be narrow, and this fact can be used to deduce that $F \to \beta_n$ quickly on Γ_n. Via (6.7.19), this leads to a good description of how L_{k-2} behaves like $(z - \beta_n)^{-1}$ on Γ_n. By integrating this, we discover the asymptotics of $f^{(k-2)}$ on Γ_n, and then also of $f^{(j)}$ and $f^{(j-1)}$ by further integration. The hypothesis on the zeroes of $f^{(j)}$ implies that $1/L_{j-1} = f^{(j-1)}/f^{(j)}$ has only finitely many non-real poles. This lack of poles, together with our asymptotic knowledge of this function, allows us to show that $1/L_{j-1}$ grows rapidly in the upper half-plane. The contradiction between this fast rate of growth and the estimate of (6.7.2) will ultimately establish Proposition 6.35.

Following the above outline, the details of the proof will now be presented under the assumption that F^{-1} has an indirect transcendental singularity over $\alpha \in H$. We are guided by [38, §10] throughout.

Recall that the non-real critical values of F form a discrete set because, by (6.7.1), all but finitely many of the non-real critical points are fixed points. The proof of [38, Lemma 10.3] uses this fact to show that, for $n = 0, 1, 2, \ldots$, there exist pairwise distinct $\beta_n \in H$ and pairwise disjoint simple paths to infinity $\Gamma_n \subseteq H$ such that

$$F(z) \to \beta_n \quad \text{as} \quad z \to \infty \quad \text{on} \quad \Gamma_n.$$

We now appeal to the argument of Lemmas 10.4, 10.5 and 10.6 of [38] — these rely on [38, Lemma 9.2], the conclusion of which is provided in our case by Proposition 6.36 and (6.7.19). By doing so, we are able to find constants $A_n \in \mathbb{C} \setminus \{0\}$ and error functions τ_n such that

$$f^{(k-2)}(z) = A_n(z - \beta_n) + \tau_n(z), \qquad \tau_n(z) = O(|z|^{-1}), \tag{6.7.20}$$

as $z \to \infty$ on Γ_n (this is Lemma 10.4 and (42) of [38]). Furthermore, for any $K \in \mathbb{N}$,

$$\int_{\Gamma_n} |u^K \tau_n(u)||du| < \infty. \tag{6.7.21}$$

This assertion is part of [38, Lemma 10.6] and means that the error term τ_n decays quickly on Γ_n. The next lemma is essentially Lemma 10.7 of [38].

Lemma 6.42. *Let $0 \le m \le k-2$. Then, as $z \to \infty$ on Γ_n,*

$$f^{(m)}(z) = \frac{A_n(z - \beta_n)^{k-m-1}}{(k-m-1)!} + O(|z|^{k-m-3}).$$

Proof. If $m = k-2$, then the result is an immediate consequence of (6.7.20). Now assume that $m \le k-3$. Fix $z_0 \in \Gamma_n$ and write

$$h(z) = f^{(m)}(z) - \frac{A_n(z - \beta_n)^{k-m-1}}{(k-m-1)!}.$$

Then (6.7.20) gives that $h^{(k-m-2)}(z) = \tau_n(z)$. Taylor's formula with the integral form of the remainder gives a polynomial Q of degree at most $k-m-3$ such that

$$h(z) = Q(z) + \int_{z_0}^{z} \frac{(z-u)^{k-m-3}}{(k-m-3)!} \tau_n(u)\, du.$$

Using (6.7.21) now shows that $h(z) = O(|z|^{k-m-3})$ as $z \to \infty$ on Γ_n, as required. □

Recalling our assumption that $1 \le j \le k-2$, we apply Lemma 6.42 with $m = j-1$ and $m = j$ to show that, as $z \to \infty$ on Γ_n,

$$\frac{f^{(j-1)}(z)}{f^{(j)}(z)} = \frac{(z-\beta_n)^{k-j} + O(|z|^{k-j-2})}{(k-j)(z-\beta_n)^{k-j-1} + O(|z|^{k-j-3})} = \frac{z-\beta_n}{k-j} + O(|z|^{-1}). \tag{6.7.22}$$

By the hypothesis on the non-real zeroes of $f^{(j)}$, there exists a large r_1 such that $E(r_1) = \{z \in H : |z| > r_1\}$ contains no poles of $f^{(j-1)}/f^{(j)}$. We can now choose simple paths Γ_n^* in $E(r_1)$, each tending to infinity and pairwise disjoint apart from a common starting point, such that (6.7.22) holds as $z \to \infty$ on Γ_n^*. Relabelling if necessary, we obtain pairwise disjoint simply-connected subdomains $D_1, D_2, \ldots$ of $E(r_1)$, with D_n bounded by Γ^*_{n-1} and Γ^*_n. Set

$$H_n(z) = \frac{f^{(j-1)}(z)}{f^{(j)}(z)} - \frac{z - \beta_n}{k - j}. \tag{6.7.23}$$

The construction of the D_n shows that H_n is analytic on the closure $\overline{D_n}$. Furthermore, by considering (6.7.22), we see that H_n tends to zero as $z \to \infty$ on Γ^*_n, while H_n tends to the non-zero value $\frac{\beta_n - \beta_{n-1}}{k-j}$ as $z \to \infty$ on Γ^*_{n-1}. Therefore, H_n must be unbounded on D_n by the Phragmén-Lindelöf principle [59, p.308] (see also Lemma 6.18).

Let N be a large integer. Take $c^* > 0$ large, and for $n = 1, \ldots, N$ define

$$u_n(z) = \begin{cases} \log^+ \left| \dfrac{H_n(z)}{c^*} \right|, & z \in D_n \\ 0, & z \in \mathbb{C} \setminus D_n. \end{cases}$$

Then each u_n is a continuous subharmonic function on the plane that is both non-negative and non-constant. Let $\theta_n(s)$ be the angular measure of the intersection of D_n with the circle $|z| = s$. Applying Lemma 6.14 to u_n, with r_2 large and $1 \leq n \leq N$, gives

$$\begin{aligned} \int_{r_2}^{r} \frac{\pi \, ds}{s\theta_n(s)} \leq \log B(2r, u_n) + O(1) &\leq \log \left(\frac{1}{2\pi} \int_0^{\pi} u_n(4re^{it}) \, dt \right) + O(1) \\ &\leq \log \left(m_{0\pi}(4r, H_n) \right) + O(1) \\ &\leq \log^+ \left(m_{0\pi} \left(4r, \frac{f^{(j-1)}}{f^{(j)}} \right) \right) + o(\log r) \end{aligned}$$

as $r \to \infty$, using (6.7.23). Summing this over n, and combining with the the Cauchy-Schwarz inequality

$$N^2 \leq \sum_{n=1}^{N} \theta_n(s) \sum_{n=1}^{N} \frac{1}{\theta_n(s)} \leq \sum_{n=1}^{N} \frac{\pi}{\theta_n(s)},$$

yields

$$N^2 \log r \leq N \log^+ \left(m_{0\pi} \left(4r, \frac{f^{(j-1)}}{f^{(j)}} \right) \right) + o(\log r), \qquad r \to \infty.$$

Since $f^{(j)}/f^{(j-1)} = L_{j-1}$, this implies that

$$(N - o(1)) \log r \leq \log^+ (m_{0\pi}(4r, 1/L_{j-1})), \qquad r \to \infty,$$

and so, for all large r,

$$m_{0\pi}(r, 1/L_{j-1}) \geq r^{N-1}. \tag{6.7.24}$$

However, (6.7.2) gives that $\mathfrak{T}(r, 1/L_{j-1}) = O(\log r)$ as $r \to \infty$. Therefore, by Lemma 6.13 the integral

$$\int_1^{\infty} \frac{m_{0\pi}(r, 1/L_{j-1})}{r^3} \, dr$$

converges. As N is large, this clear contradiction with (6.7.24) is enough to complete the proof of Proposition 6.35.

References

[1] M. Ålander, Sur les zéros extraordinaires des dérivées des fonctions entières réelles, *Ark. för Mat., Astron. och Fys.* **11** No. 15 (1916), 1-18.

[2] M. Ålander, Sur les zéros complexes des dérivées des fonctions entières réelles, *Ark. för Mat., Astron. och Fys.* **16** No. 10 (1922), 1-19.

[3] P. D. Barry, The minimum modulus of small integral and subharmonic functions, *Proc. London Math. Soc.* (3) **12** (1962), 445-495.

[4] W. Bergweiler, Iteration of meromorphic functions, *Bull. Amer. Math. Soc.* **29** (1993), 151-188.

[5] W. Bergweiler, On the zeros of certain homogeneous differential polynomials, *Arch. Math.* **64** (1995), 199-202.

[6] W. Bergweiler and A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, *Rev. Mat. Iberoamericana* **11** (1995), 355-373.

[7] W. Bergweiler, A. Eremenko and J. K. Langley, Real entire functions of infinite order and a conjecture of Wiman, *Geom. Funct. Anal.* **13** (2003), 975-991.

[8] W. Bergweiler and J. K. Langley, Nonvanishing derivatives and normal families, *J. Anal. Math.* **91** (2003), 353-367.

[9] A. Edrei, Meromorphic functions with three radially distributed values, *Trans. Amer. Math. Soc.* **78** (1955), 276-293.

[10] A. Edrei and W. H. J. Fuchs, On meromorphic functions with regions free of poles and zeros, *Acta Math.* **108** (1962), 113-145.

[11] A. Edrei and W. H. J. Fuchs, Bounds for the number of deficient values of certain classes of meromorphic functions, *Proc. London Math. Soc.* (3) **12** (1962), 315-344.

[12] S. Edwards and S. Hellerstein, Non-real zeros of derivatives of real entire functions and the Pólya-Wiman conjectures, *Complex Var. Theory Appl.* **47** (2002), 25-57.

[13] A. Eremenko, J. K. Langley and J. Rossi, On the zeros of meromorphic functions of the form $f(z) = \sum_{k=1}^{\infty} a_k/(z - z_k)$, *J. Anal. Math.* **62** (1994), 271-286.

[14] A. Eremenko and M. Yu. Lyubich, Dynamical properties of some classes of entire functions, *Ann. Inst. Fourier Grenoble* **42** (1992), 989-1020.

[15] A. Fletcher, J. K. Langley and J. Meyer, Nonvanishing derivatives and the MacLane class $\mathcal{A}$, *Illinois J. Math.* **53** (2009), 379-390.

[16] W. H. J. Fuchs, Proof of a conjecture of G. Pólya concerning gap series, *Illinois J. Math.* **7** (1963), 661-667.

[17] A. A. Gol'dberg and I. V. Ostrovskii, *Value distribution of meromorphic functions*, Transl. Math. Monogr. **236**, Amer. Math. Soc., Providence RI, 2008. Translated from the 1970 Russian original, NAUKA, Moscow.

[18] A. A. Gol'dberg and O. P. Sokolovskaya, Some relations for meromorphic functions of order or lower order less than one, *Izv. Vyssh. Uchebn. Zaved. Mat.* **31** (1987), 26-31 (translation: *Soviet Math. (Izv. VUZ)* **31** (1987), 29-35).

[19] G. G. Gundersen, Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates, *J. London Math. Soc.* (2) **37** (1988), 88-104.

[20] W. K. Hayman, *Meromorphic functions*, Oxford at the Clarendon Press, 1964.

[21] W. K. Hayman, On the characteristic of functions meromorphic in the plane and of their integrals, *Proc. London Math. Soc.* (3) **14A** (1965), 93-128.

[22] W. K. Hayman, The local growth of power series: a survey of the Wiman-Valiron method, *Canad. Math. Bull.* **17** (1974), 317-358.

[23] W. K. Hayman, *Subharmonic Functions Vol. 2*, Academic Press, London, 1989.

[24] W. K. Hayman, *Multivalent Functions*, 2nd edition, Cambridge Tracts in Mathematics 110, Cambridge University Press, Cambridge 1994.

[25] S. Hellerstein, L. C. Shen and J. Williamson, Reality of the zeros of an entire function and its derivatives, *Trans. Amer. Math. Soc.* **275** (1983), 319-331.

[26] S. Hellerstein and J. Williamson, Derivatives of entire functions and a question of Pólya, *Trans. Amer. Math. Soc.* **227** (1977), 227-249.

[27] S. Hellerstein and J. Williamson, Derivatives of entire functions and a question of Pólya, II, *Trans. Amer. Math. Soc.* **234** (1977), 497-503.

[28] S. Hellerstein and C. C. Yang, Half-plane Tumura-Clunie theorems and the real zeros of successive derivatives, *J. London Math. Soc.* (2) **4** (1972), 469-481.

[29] E. Hille, *Ordinary differential equations in the complex domain*, Wiley, New York, 1976.

[30] J. D. Hinchliffe, The Bergweiler-Eremenko theorem for finite lower order, *Result. Math.* **43** (2003), 121-128.

[31] A. Hinkkanen and J. Rossi, On a problem of Hellerstein, Shen and Williamson, *Proc. Amer. Math. Soc.* **92** (1984), 72-74.

[32] G. Jank and L. Volkmann, *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen*, Birkhäuser, Basel, 1985.

[33] W. P. Kohs and J. Williamson, Derivatives of meromorphic functions of finite order, *Trans. Amer. Math. Soc.* **306** (1988), 765-772.

[34] E. Laguerre, Sur les fonctions du genre zéro et du genre un, *C. R. Acad. Sci. Paris* **95** (1882); *Oeuvres* **1** 174-177.

[35] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin-New York, 1993.

[36] J. K. Langley, A lower bound for the number of zeros of a meromorphic function and its second derivative, *Proc. Edinburgh Math. Soc.* **39** (1996), 171-185.

[37] J. K. Langley, Deficient values of derivatives of meromorphic functions in the class S, *Comput. Methods Funct. Theory* **4** (2004), 237-247.

[38] J. K. Langley, Non-real zeros of higher derivatives of real entire functions of infinite order, *J. Anal. Math.* **97** (2005), 357-396.

[39] J. K. Langley, Meromorphic functions in the class S and the zeros of the second derivative, *Comput. Methods Funct. Theory* **8** (2008), 73-84.

[40] J. K. Langley, Non-real zeros of linear differential polynomials, *J. Anal. Math.* **107** (2009), 107-140.

[41] J. K. Langley, *The Wiman conjecture*, 2007, available online at http://www.maths.nottingham.ac.uk/personal/jkl/wimanconjecture.pdf.

[42] J. K. Langley, *The Wiman conjecture: a unified approach*, 2007, available online at http://www.maths.nottingham.ac.uk/personal/jkl/allwiman.pdf.

[43] J. K. Langley and J. H. Zheng, On the fixpoints, multipliers and value distribution of certain classes of meromorphic functions, *Ann. Acad. Sci. Fenn.* **23** (1998), 133-150

[44] B. Ja. Levin, *Distribution of zeros of entire functions*, GITTL, Moscow, 1956. 2nd English transl., Amer. Math. Soc., Providence RI, 1980.

[45] B. Ja. Levin and I. V. Ostrovskii, The dependence of the growth of an entire function on the distribution of zeros of its derivatives, *Sibirsk. Mat. Zh.* **1** (1960), 427-455. English transl., *Amer. Math. Soc. Transl.* (2) **32** (1963), 323-357.

[46] J. Milnor, *Dynamics in one complex variable*, 3rd edition, Ann. Math. Stud. **160**, Princeton University Press, Princeton, 2006.

[47] R. Nevanlinna, *Eindeutige analytische Funktionen*, 2nd edition, Springer-Verlag, Berlin, 1953.

[48] D. A. Nicks, Deficiencies of certain classes of meromorphic functions, *Ann. Acad. Sci. Fenn. Math.* **34** (2009), 157-171.

[49] D. A. Nicks, Rational deficient functions of derivatives of mappings in the classes S and B, *Comput. Methods Funct. Theory* **9** (2009), 239-253.

[50] D. A. Nicks, Real meromorphic functions and a result of Hinkkanen and Rossi, *Illinois J. Math.* **53** (2009), 605-622.

[51] G. Pólya, Über Annäherung durch Polynome mit lauter reellen Wurzeln, *Rend. Circ. Mat. Palermo* **36** (1913), 279-295.

[52] G. Pólya, Sur une question concernant les fonctions entières, *C. R. Acad. Sci. Paris* **158** (1914), 330-333.

[53] T. Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts 28, Cambridge University Press, Cambridge, 1995.

[54] P. J. Rippon and G. M. Stallard, Iteration of a class of hyperbolic meromorphic functions, *Proc. Amer. Math. Soc.* **127** (1999), 3251-3258.

[55] T. Sheil-Small, On the zeros of the derivatives of real entire functions and Wiman's conjecture, *Annals of Math.* **129** (1989), 179-193.

[56] L.-C. Shen, Influence of the distribution of the zeros of an entire function and its second derivative on the growth of the function, *J. London Math. Soc.* (2) **31** (1985), 305-320.

[57] E. C. Titchmarsh, *The theory of functions*, 2nd edition, Oxford University Press, 1939.

[58] M. Tsuji, On Borel's directions of meromorphic functions of finite order, I, *Tôhoku Math. J.* **2** (1950), 97-112.

[59] M. Tsuji, *Potential theory in modern function theory*, 2nd edition, Chelsea Publishing Co., New York, 1975.