Iteration of quasiregular analogues of trigonometric functions

Dan Nicks

University of Nottingham

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Joint work with Alastair Fletcher
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• Dynamics of a quasiregular version of tangent
Quasiregular mappings

Quasiregular functions on $\mathbb{R}^n$ generalize analytic functions on $\mathbb{C}$.

**Definition**

- A continuous function $f : U \rightarrow \mathbb{R}^n$ on a domain $U \subseteq \mathbb{R}^n$ is called quasiregular if $f \in W^{1}_{n,\text{loc}}(U)$ and there exists $K \geq 1$ such that
  
  $$\|Df(x)\|^n \leq KJ_f(x) \quad \text{a.e. in } U.$$ 

- More generally, a continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n \cup \{\infty\}$ is called quasiregular (or quasimeromorphomorphic) if the set of poles $f^{-1}(\infty)$ is discrete and if $f$ is quasiregular on $\mathbb{R}^n \setminus f^{-1}(\infty)$. 
The Zorich mapping

The Zorich map $Z : \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$ is a quasiregular analogue of the exponential function. It can be defined as follows:

1. Choose a bi-Lipschitz map

$$h : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^2 \to \{(x, y, z) : x^2 + y^2 + z^2 = 1, \ z \geq 0\}.$$

2. Define $Z : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^2 \times \mathbb{R} \to \{(x, y, z) : z \geq 0\}$ by

$$Z(x, y, z) = e^zh(x, y).$$

3. Extend $Z$ to all of $\mathbb{R}^3$ by repeatedly reflecting in planes.

The Zorich map is quasiregular on $\mathbb{R}^3$ and doubly-periodic with periods $(2\pi, 0, 0)$ and $(0, 2\pi, 0)$. 
Trigonometric analogues

- Quasiregular maps of $\mathbb{R}^n$ which generalize the sine and cosine functions have been constructed by Drasin, by Mayer and by Bergweiler and Eremenko.

- Constructed by mapping a half-infinite beam to a half-space, then reflecting in planes.

- By iterating their map $S$, Bergweiler and Eremenko obtained a seemingly paradoxical decomposition of $\mathbb{R}^n$.

- They also showed that the escaping set

  $$I(S) = \{ x \in \mathbb{R}^n : S^k(x) \to \infty \text{ as } k \to \infty \}$$

  is dense in $\mathbb{R}^n$. 
Dynamics of the qr sine analogue $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$

We say $x$ is a periodic point of $S$ if $S^p(x) = x$ for some $p$.

**Theorem**

The periodic points of $S$ are dense in $\mathbb{R}^n$.

**Corollary**

$\partial I(S) = \mathbb{R}^n$.

**Theorem**

$S$ has the blowing-up property everywhere in $\mathbb{R}^n$; that is,

$$\bigcup_{k=0}^{\infty} S^k(U) = \mathbb{R}^n,$$

for any non-empty open $U \subseteq \mathbb{R}^n$.

...so the “Julia set” of $S$ is equal to $\mathbb{R}^n$. 
In the rest of this talk we will

• construct a 3-dimensional quasimeromorphic analogue $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of the meromorphic tangent function

• compare the dynamics of $\lambda T$ and $\lambda \tan z$ for $\lambda > 0$. 
Construction of a generalized tangent mapping

Observe that the complex function
\[ \tan \zeta = \frac{i(1 - e^{2i\zeta})}{1 + e^{2i\zeta}} \]
is the composition of a Möbius map and the exponential function.

Define a sense-preserving Möbius map \( A : \mathbb{R}^3 \to \mathbb{R}^3 \cup \{\infty\} \) by
\[ A(x, y, z) = (0, 0, 1) + \frac{(2x, 2y, -2(z + 1))}{x^2 + y^2 + (z + 1)^2}. \]

We then define our 3-dimensional analogue of tangent by
\[ T(x) = (A \circ Z)(2x). \]
Expressions for $T$

$T$ contains embedded copies of the usual (complex) tangent function:

- $T(0, y, z) = (0, \Re(\tan(y + iz)), \Im(\tan(y + iz)))$,

- $T(x, 0, z) = (\Re(\tan(x + iz)), 0, \Im(\tan(x + iz)))$.

If $M(x, y) = \max\{|x|, |y|\} \leq \pi/4$ and we write $\zeta = M(x, y) + iz$, then

$$T(x, y, z) = \left( \frac{x}{\sqrt{x^2 + y^2}} \Re(\tan \zeta), \frac{y}{\sqrt{x^2 + y^2}} \Re(\tan \zeta), \Im(\tan \zeta) \right).$$
Geometric properties of $T$

Comparing $T$ with tan, the $z$-axis plays the role of the imaginary axis, while the $xy$-plane plays the role of the real axis.

- $T$ is doubly-periodic with periods $(\pi, 0, 0)$ and $(0, \pi, 0)$.
- $T$ omits the values $(0, 0, \pm 1)$. These are asymptotic values of $T$:

$$\lim_{z \to \pm \infty} T(x, y, z) = (0, 0, \pm 1).$$

- $T : \{xy\text{-plane}\} \rightarrow \{xy\text{-plane}\} \cup \{\infty\}$.
- The $\{z > 0\}$ and $\{z < 0\}$ half-spaces are completely invariant under $T$.

$T$ is highly symmetric: If $R$ is a reflection in a co-ordinate plane then

$$T(R(x)) = R(T(x)).$$
Iteration of tangent maps on $\mathbb{C}$

For a parameter $\lambda > 0$, Devaney and Keen described the dynamics of the meromorphic tangent family $\tau_\lambda(\zeta) = \lambda \tan \zeta$.

**Theorem (Devaney and Keen)**

- If $0 < \lambda < 1$, then $\mathcal{J}(\tau_\lambda) \subseteq \mathbb{R}$ is locally a Cantor set. Attracting fixed point at origin.

- If $\lambda = 1$, then $\mathcal{J}(\tau_\lambda) = \mathbb{R}$. Parabolic fixed point at origin.

- If $\lambda > 1$, then $\mathcal{J}(\tau_\lambda) = \mathbb{R}$. Attracting fixed points at $\pm i \xi_0$, where $\xi_0 > 0$ solves $\xi_0 = \lambda \tanh \xi_0$. 
Dynamics of $\lambda T$

For $\lambda > 0$ we put

$$T_\lambda(x) = \lambda T(x).$$

We iterate $T_\lambda$ and aim to establish an analogue of the $\lambda \tan \zeta$ results.

First, we describe the behaviour on the upper and lower half-spaces.

**Theorem**

- If $0 < \lambda < 1$, then $T_\lambda$ has an attracting fixed point at the origin.

- If $0 < \lambda \leq 1$, then $T^k_\lambda(x) \to 0$ on $\{(x, y, z) : z \neq 0\}$, as $k \to \infty$.

- If $\lambda > 1$, then $T_\lambda$ has attracting fixed points at $(0, 0, \pm \xi_0)$, where $\xi_0 = \lambda \tanh \xi_0$, and

  $$T^k_\lambda(x) \to (0, 0, \pm \xi_0) \text{ on } \{(x, y, z) : \pm z > 0\}. $$
What’s a Julia set?

For a meromorphic function $f$ with poles, the Julia set $J(f)$ satisfies

$$J(f) = \overline{O_f^{-}(\infty)} = \partial I(f),$$

where $I(f) = \{\zeta : f^k(\zeta) \to \infty \text{ as } k \to \infty\}$.

**Theorem**

For all $\lambda > 0$,

$$\overline{O_{T_{\lambda}}^{-}(\infty)} = \partial I(T_{\lambda}) = \overline{I(T_{\lambda})}.$$  

Call this set $J$. Then $J$ is an uncountable perfect set. If $U$ is an open set that meets $J$ then, for some $m > 0$,

$$T_{\lambda}^m(U) = (\mathbb{R}^3 \cup \{\infty\}) \setminus \{(0, 0, \pm \lambda)\}.$$  

$J$ is contained in the closure of the set of periodic points of $T_{\lambda}$. 
What does $J$ look like?

$$J = \overline{O_{T_\lambda}(\infty)} = \partial I(T_\lambda) \subseteq \{xy\text{-plane}\}$$

**Theorem**

If $\lambda \geq 1$ then $J$ is connected. If $0 < \lambda < 1$ then $J$ is not connected.

Open questions: When $0 < \lambda < 1$, is $J$ locally Cantor? Does $J$ equal $\{x : T^k_\lambda(x) \nrightarrow 0\}$?

**Theorem**

If $\lambda > \sqrt{2}$ then $J = \{xy\text{-plane}\}$.

The constant $\sqrt{2}$ here cannot be replaced by any smaller value.

When $\lambda < \sqrt{2}$, a (relatively) open subset of the $xy$-plane lies in the attracting basin of $0 \ldots$
Each square is the subset \([-\frac{\pi}{4}, \frac{\pi}{4}]^2\) of the xy-plane. The shaded points lie in the basin of attraction of 0.
A numerical plot for $\lambda = 0.9$. Blue points $\to 0$ fast, red points $\to 0$ slow.
A numerical plot for $\lambda = 1$. Blue points $\to 0$ fast, red points $\to 0$ slow.
Around a pole for $\lambda = 0.7$. Thanks to Dan Goodman for code.