ENTANGLEMENT SHARING: FROM QUBITS TO GAUSSIAN STATES

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Received 24 June 2005

It is a central trait of quantum information theory that there exist limitations to the free sharing of quantum correlations among multiple parties. Such monogamy constraints have been introduced in a landmark paper by Coffman, Kundu and Wootters, who derived a quantitative inequality expressing a trade-off between the couplewise and the genuine tripartite entanglement for states of three qubits. Since then, a lot of efforts have been devoted to the investigation of distributed entanglement in multipartite quantum systems. In this paper we report, in a unifying framework, a bird’s eye view of the most relevant results that have been established so far on entanglement sharing in quantum systems. We will take off from the domain of $N$ qubits, graze qudits, and finally land in the almost unexplored territory of multimode Gaussian states of continuous variable systems.

**Keywords:** Entanglement sharing; monogamy constraints; qubits and Gaussian states.

1. Coffman–Kundu–Wootters Inequality and Entanglement Sharing in Discrete-Variable Systems

The simplest conceivable quantum system in which multipartite entanglement can arise is a system of three two-level particles (qubits). Let two of these qubits, say A and B, be in a maximally entangled state (a Bell state). Then, no entanglement is possible between each of them and the third qubit C. In fact, entanglement between C and A (or B) would imply A and B being in a mixed state, which is impossible because they are sharing a pure Bell state. This simple observation embodies, in its sharpest version, the monogamy of quantum entanglement (see, for example, Ref. 1 and references therein), as opposed to classical correlations which can be freely shared.

We find it instructive to look at this feature as a simple consequence of the no-cloning theorem.$^{2,3}$ In fact, maximal couplewise entanglement in both
bipartitions AB and AC of a three-particle ABC system, would enable perfect $1 \rightarrow 2$ telecloning\textsuperscript{4} of an unknown input state, which is impossible due to the linearity of quantum mechanics. The monogamy constraints thus emerge as fundamental properties enjoyed by quantum systems involving more than two parties, and play a crucial role, e.g. in the security of quantum key distribution schemes based on entanglement,\textsuperscript{5} limiting the possibilities of the malicious eavesdropper. Just as in the context of cloning, where research is devoted to the problem of creating the best possible approximate copies of a quantum state, one can address the question of entanglement sharing in a weaker form. If the two qubits $A$ and $B$ are still entangled but not in a Bell state, one can then ask how much entanglement each of them is allowed to share with qubit $C$, and what is the maximum genuine tripartite entanglement that they may share all together. The answer is beautifully encoded in the Coffman–Kundu–Wootters (CKW) inequality\textsuperscript{6}

$$E^{A|\langle BC \rangle} \geq E^{A|B \oplus C} + E^{A|C \oplus B},$$

(1)

where $E^{A|\langle BC \rangle}$ denotes the entanglement between qubit $A$ and subsystem $(BC)$, globally in a state $\varrho$, while $E^{A|B \oplus C}$ denotes entanglement between $A$ and $B$ in the reduced state obtained tracing out qubit $C$ (and similarly for $E^{A|C \oplus B}$ exchanging the roles of $B$ and $C$). Inequality (1) states that the bipartite entanglement between one single qubit, say $A$, and all the others, is greater than the sum of all the possible couplewise entanglements between $A$ and each other qubit.

1.1. Which entanglement is shared?

While originally derived for system of three qubits, it is natural, due to the above considerations, to assume that inequality (1) be a general feature of any three-party quantum system in arbitrary (even infinite) dimensions. However, before proceeding, the careful reader should raise an important question, namely, how are we measuring the bipartite entanglement in the different bipartitions, and what the symbol $E$ stands for in inequality (1).

Even if the system of three qubits is globally in a pure state, its reductions will obviously be mixed. In fact, the various physical processes responsible for the interpretation of the entropy of entanglement as the unique proper entanglement measure for pure states, cease to be equivalent for mixed states. To give a typical example, one must spend more to produce a mixed state $\varrho$ out of an ensemble of pure entangled states, than what one earns back by distilling entanglement from $\varrho$ to a set of singlets via local operations and classical communication (LOCC). Formally, the entanglement of formation\textsuperscript{7} $E_F$ is greater than the distillable entanglement $E_D$ of generic mixed states, while both reduce to the entropy of entanglement on pure states. More generally, there are several, inequivalent measures of entanglement for mixed states, leading to different orderings on the set of entangled states living in a specified Hilbert space.\textsuperscript{8} Thus, a mixed state $\varrho_A$ can be more entangled than another state $\varrho_B$ with respect to a given measure, but less entangled than $\varrho_B$ with respect to another measure.
In this piebald scenario (which we cannot further explore in this paper), we should convince ourselves that different measures of entanglement must be chosen, depending on the problem one needs to address, and/or on the desired use of the entangled resources. This picture is consistent, provided that each required measure is selected out of the cauldron of bona fide entanglement measures, at least positive on inseparable states and monotone under LOCC. Here, we are addressing the problem of entanglement sharing: one should not be so surprised to discover that not all entanglement measures satisfy inequality (1). In particular, the entanglement of formation fails to fulfill the task, and this fact led CKW to define, for qubit systems, a new measure of bipartite entanglement consistent with the quantitative monogamy constraint expressed by inequality (1).

1.2. Entanglement of two qubits

In discrete-variable systems, separability of a mixed state \( \varrho \) of two qubits (and of a system of one qubit and one qutrit) is equivalent to the positivity of the partial transpose\(^8,9\) (PPT) \( \hat{\varrho} \) of \( \varrho \), defined as the result of transposition performed on only one of the two subsystems in some given basis. From a quantitative point of view, a proper measure of entanglement is provided by the entanglement of formation

\[
E_F(\varrho) \equiv \min_{\{p_i, \psi_i\}} \sum_i p_i E_o(|\psi_i\rangle\langle \psi_i|),
\]

where the minimization is taken over those probabilities \( \{p_i\} \) and pure states \( \{\psi_i\} \) that realize the density matrix \( \varrho = \sum_i p_i |\psi_i\rangle\langle \psi_i| \), and \( E_o \) is the entropy of entanglement of \( |\psi_i\rangle \). The latter is the von Neumann entropy \( \text{Tr}_A \hat{\varrho}_A^i \log \hat{\varrho}_A^i \) of the reduced density matrix \( \hat{\varrho}_A^i = \text{Tr}_B |\psi_i\rangle\langle \psi_i| \) obtained from the state \( |\psi_i\rangle \) of the bipartite system AB, by tracing out the degrees of freedom of B. For two qubits, the entanglement of formation has been computed by Wootters\(^11\) and reads

\[
E_F(\varrho) = f[C(\varrho)],
\]

with \( f(x) = H[(1 + \sqrt{1-x^2})/2] \) and \( H(x) = -x \log_2 x - (1-x) \log_2 (1-x) \). The quantity \( C(\varrho) \) is called the concurrence\(^12\) of the state \( \varrho \) and is defined as

\[
C(\varrho) \equiv \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\},
\]

where the \( \lambda_i \)'s are the eigenvalues of the matrix \( \varrho (\sigma_y \otimes \sigma_y) \varrho^* (\sigma_y \otimes \sigma_y) \) in decreasing order, \( \sigma_y \) is the Pauli spin matrix and the star denotes complex conjugation in the computational basis \( \{|ij\} \equiv |i\rangle \otimes |j\rangle, \ i, j = 0, 1 \). Because \( f(x) \) is a monotonic convex function of \( x \in [0, 1] \), the concurrence \( C(\varrho) \) and its square, the tangle\(^9\) \( \tau(\varrho) \equiv C^2(\varrho) \), are proper entanglement monotones as well. On pure states, they are monotonically increasing functions of the entropy of entanglement. The concurrence coincides (for pure states) with another entanglement monotone, the negativity\(^13\), defined in general as

\[
\mathcal{N}(\varrho) = (||\hat{\varrho}||_1 - 1)/2,
\]

where \( ||\hat{\varrho}||_1 = \text{Tr} |\hat{\varrho}| \) stands for the trace norm of the Hermitian operator \( \hat{\varrho} \). The quantity \( \mathcal{N}(\varrho) \) is equal to \( \sum_i |\lambda_i| \), the modulus of the sum of the negative eigenvalues of \( \hat{\varrho} \), quantifying the extent to which the PPT criterion is violated. On
the other hand, the tangle is equal (for pure states $|\psi\rangle$) to the linear entropy of entanglement $E_L$, defined as the linear entropy $S_L(\rho_A) \equiv 1 - \text{Tr}_A \rho_A^2$ of the reduced state $\rho_A = \text{Tr}_B |\psi\rangle\langle \psi|$ of one party.

1.3. The residual tangle: A measure of tripartite entanglement

After this survey, we can now recall the crucial result that, for three qubits, the desired measure $E$ such that the CKW inequality (1) is satisfied is exactly the tangle $\tau^6$. The general definition of the tangle, needed, e.g. to compute the leftmost term in inequality (1) for mixed states, involves a convex roof analogous to that defined in Eq. (2), namely,

$$\tau (\rho) \equiv \min_{\{p_i, \psi_i\}} \sum_i p_i \tau (|\psi_i\rangle\langle \psi_i|).$$

(5)

With this general definition, which implies that the tangle is a convex measure on the set of density matrices, it was sufficient for CKW to prove inequality (1) only for pure states of three qubits, to have it satisfied for free by mixed states as well.$^6$

Once one has established a monogamy inequality like inequality (1), the following natural step is to study the difference between the leftmost quantity and the rightmost one, and to interpret this difference as the residual entanglement, not stored in couplewise correlations, that thus quantifies the genuine tripartite entanglement shared by the three qubits. The emerging measure $\tau^3_A|B|C \equiv \tau^A(BC) - \tau^A|B'C - \tau^A|C'B$, known as the three-way tangle,$^6$ has indeed some nice features. For pure states, it is invariant under permutations of any two qubits, and more remarkably it has been proven to be a tripartite entanglement monotone under LOCC.$^{14}$ However, no operational interpretation for the three-tangle, possibly relating it to the optimal distillation rate of some canonical “multipartiy singlet,” is currently known. The reason lies probably in the fact that the notion of a well-defined maximally entangled state becomes fuzzier when one moves to the multipartite setting. In this context, it has been shown that there exist two classes of three-party fully inseparable pure states of three qubits, inequivalent under stochastic LOCC operations, namely the Greenberger–Horne–Zeilinger (GHZ) state$^{15}$ $|\psi_{GHZ}\rangle = (1/\sqrt{2}) \left(|000\rangle + |111\rangle\right)$, and the $W$ state$^{14}$ $|\psi_W\rangle = (1/\sqrt{3}) \left(|001\rangle + |010\rangle + |100\rangle\right)$. From the point of view of entanglement, the big difference between them is that the GHZ state has maximum residual three-party tangle $[\tau^3(\psi_{GHZ}) = 1]$ with zero couplewise quantum correlations in any two-qubit reduction, while the $W$ state contains maximum two-party entanglement between any couple of qubits in the reduced states and it consequently saturates inequality (1) $[\tau^3(\psi_W) = 0]$.

1.4. Monogamy inequality for $N$ parties

So far, we have recalled the known results on the problem of entanglement sharing in systems of three parties, leading to the definition of the residual tangle as a proper measure of genuine tripartite entanglement for three qubits. However, if the
monogamy of entanglement is really a universal property of quantum systems, one
should aim at finding more general results.

There are two axes along which one can move, pictorially, in this respect. One
direction concerns the investigation on distributed entanglement in systems of more
than three parties, starting with the simplest case of $N \geq 4$ qubits (thus moving
along the horizontal axis of increasing number of parties). On the other hand,
one should analyze the sharing structure of multipartite entanglement in higher-
dimensional systems, like qudits, moving, in the end, towards continuous variable
(CV) systems (thus going along the vertical axis of increasing Hilbert space dimen-
sions). The final goal would be to cover the entire square spanned by these two
axes, in order to establish a really complete theory of entanglement sharing.

Let us start moving to the right. It is quite natural to expect that, in an $N$-party
system, the entanglement between qubit $p_i$ and the rest should be greater than the
total two-party entanglement between qubit $p_i$ and each of the other $N-1$ qubits.
So, the generalized version of inequality (1) reads

$$E_{p_i | P_i} \geq \sum_{j \neq i} E_{p_i | p_j},$$

(7)

with $P_i \equiv (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_N)$. Proving inequality (7) for any quantum sys-
tem in arbitrary dimension, would definitely fill the square; it appears though as a
formidable task. However, partial, encouraging results have recently been obtained.

Osborne and Verstraete have shown that the generalized monogamy inequality
(7) holds true for any (pure or mixed) state of a system of $N$ qubits,\textsuperscript{16} proving a
longstanding conjecture due to CKW themselves.\textsuperscript{6} Again, the entanglement has to
be measured by the tangle $\tau$. This is an important result; nevertheless, one must
admit that, if more than three parties are concerned, it is not so obvious why all
the bipartite entanglements should be decomposed only with respect to a single
elementary subsystem. One has in fact an exponentially increasing number of ways
to arrange blocks of subsystems and to construct multiple splittings of the whole set
of parties, across which the bipartite (or, even more intriguingly, the multipartite)
entanglements can be compared. This may be viewed as a third, multifolded axis
in our “geometrical” description of the possible generalizations of inequality (1).
Leaving aside in the present paper this intricated plethora of additional situations,
we stick to the monogamy constraint of inequality (7), obtained decomposing the
bipartite entanglements with respect to a single particle, while keeping in mind that
for more than three particles the residual entanglement emerging from inequality (7)
is not necessarily the measure of multipartite entanglement. Rather, it properly
quantifies the entanglement not stored in couplewise correlations, and thus finds
interesting applications, for instance, in the study of quantum phase transitions
and criticality in spin systems.\textsuperscript{17–19}

1.5. Entanglement sharing among qudits

The first problem one is faced with when trying to investigate the sharing of
quantum correlations in higher-dimensional systems is to find the correct measure
for the quantification of bipartite entanglement. Several approaches to generalize Wootters’ concurrence and/or tangle have been developed.\textsuperscript{21–23} In the present context, maybe the most relevant result has been recently obtained by Yu and Song,\textsuperscript{24} who established the general monogamy inequality (7) for an arbitrary number of qudits (i.e. \( d \)-dimensional quantum systems), for any finite \( d \). They used a generalization of the tangle \( \tau \), defined for mixed states as the convex-roof extension Eq. (5) of the linear entropy of entanglement \( E_L \) for pure states. Moreover, the authors claim that the corresponding residual tangle is a proper measure of multipartite entanglement. Let us remark however that, at the present stage in the theory of entanglement sharing, trying to make sense of a heavy mathematical framework (within which, moreover, a proof of monotonicity of the \( N \)-way tangle under LOCC has not been established yet for \( N > 3 \), not even for qubits) with little, if any, physical insight, is likely not worth trying. Probably the CKW inequality is interesting not because of the multipartite measure it implies, but because it embodies a quantifiable trade-off between the distribution of bipartite entanglement.

In this respect, it seems relevant to address the following question, raised by Dennison and Wootters.\textsuperscript{25} One is interested in computing the maximum possible bipartite entanglement between \textit{any} couple of parties, in a system of three or more qudits, and in comparing it with the entanglement capacity \( \log_2 d \) of the system. Their ratio \( \varepsilon \) would provide an immediate quantitative bound on the shareable entanglement, stored in couplewise correlations. Results obtained for \( d = 2, 3 \) and \( 7 \) (using the entanglement of formation) suggest for three qudits a general trend of increasing \( \varepsilon \) with increasing \( d \).\textsuperscript{25} While this is only a preliminary analysis, it raises intriguing questions, pushing the interest in entanglement sharing towards infinite-dimensional systems. In fact, if \( \varepsilon \) saturated to 1 for \( d \to \infty \), this would entail the really counterintuitive result that entanglement could be freely shared in this limit! We notice that, being the entanglement capacity infinite for \( d \to \infty \), \( \varepsilon \) vanishes if the maximum couplewise entanglement is not infinite. And this is the case, because again an infinite shared entanglement between two two-party reductions would allow perfect \( 1 \to 2 \) telecloning exploiting Einstein–Podolski–Rosen (EPR)\textsuperscript{26} correlations, but this is forbidden by quantum mechanics. Nevertheless, the study of entanglement sharing in CV systems yields surprising consequences.

2. Entanglement Sharing in Continuous Variable Systems

The first study of entanglement sharing in CV systems was performed in Ref. 27, focusing on the physically relevant class of Gaussian states.

2.1. Entanglement of Gaussian states

In a CV system consisting of \( N \) canonical modes, associated to an infinite-dimensional Hilbert space, and described by the vector \( \hat{X} \) of the field quadrature operators, Gaussian states (such as squeezed, coherent and thermal states) are those states characterized by first and second moments of the canonical operators.
When addressing physical properties invariant under local unitary operations, like entanglement, first moments can be neglected and Gaussian states can then be fully described by the $2N \times 2N$ real covariance matrix (CM) $\sigma$, whose entries are $\sigma_{ij} = 1/2\langle\{\hat{X}_i, \hat{X}_j\}\rangle$. A physical CM $\sigma$ must fulfill the uncertainty relation $\sigma + i\Omega \geq 0$, with the symplectic form $\Omega = \oplus_{i=1}^{2N} \omega$ and $\omega = \delta_{ij-1} - \delta_{ij+1}$, $i, j = 1, 2$.

In phase space, any $N$-mode Gaussian state can be written as $\sigma = S^T \nu S$, with $\nu = \text{diag}\{n_1, n_1, n_2, n_2, \ldots, n_N, n_N\}$ and $S$ a symplectic operation. The set $\Sigma = \{n_i\}$ constitutes the symplectic spectrum of $\sigma$ and its elements must fulfill the conditions $n_i \geq 1$, ensuring positivity of the density matrix $\rho$ associated to $\sigma$. The degree of purity $\mu = \text{Tr} \rho^2$ of a Gaussian state with CM $\sigma$ is simply $\mu = 1/\sqrt{\det \sigma}$.

Concerning the entanglement, the PPT criterion is again a necessary and sufficient condition for separability of $(N+1)$-mode Gaussian states of $(1 \times N)$-mode bipartitions$^{28,29}$ and of $(M+N)$-mode bisymmetric Gaussian states of $(M \times N)$-mode bipartitions.$^{30}$ In phase space, partial transposition with respect to a $(1 \times N)$-mode bipartition amounts to a mirror reflection of one quadrature associated to the single-mode party. If $\{\tilde{n}_i\}$ is the symplectic spectrum of the partially transposed CM $\tilde{\sigma}$, then a $(N+1)$-mode Gaussian state with CM $\sigma$ is separable if and only if $\tilde{n}_i \geq 1 \ \forall\ i$. This implies that bona fide measures of CV entanglement are the negativity $N$ Eq. (4) and, more properly, the logarithmic negativity$^{31-33}$

$$E_N \equiv \log \|\tilde{\rho}\|_1,$$

which is readily computed in terms of the symplectic spectrum $\tilde{n}_i$ of $\tilde{\sigma}$ as $E_N = -\sum_{i: \tilde{n}_i < 1} \log \tilde{n}_i$. The logarithmic negativity is additive on tensor product states and constitutes an upper bound on the distillable entanglement. For two-mode symmetric Gaussian states only, the entanglement of formation Eq. (2) has been computed$^{34}$ and it is completely equivalent to $E_N$ in that subcase.

### 2.2. The continuous variable tangle

After this brief introduction to Gaussian states (see Ref. 35 for a recent review), let us now look for the proper measure of bipartite entanglement, which would be the CV analogue of the tangle. While a formal, mathematically justified definition of this new measure was given in Ref. 27, we believe it is more instructive to follow, here, a simple trial-and-error strategy to arrive at the correct result.

We can reasonably assume that inequality (1) is true for three-mode Gaussian states, like it should be for any three-party quantum system. The problem is to find the proper measure $E$. The first attempt is naturally to use the entanglement of formation (when computable) or the negativities. Immediate inspection reveals that they actually fail, even in the simplest instance of pure, fully symmetric, three-mode Gaussian states.$^{27}$ So, next trial. Let us construct a generalization of the tangle via the convex roof like in Eq. (5), where for pure states the tangle is defined as the linear entropy of entanglement, just like in the case of qubits and qudits. The corresponding tangle for CV systems would range from 0 to 1, which is uncommon
when dealing with states whose entanglement can be infinite; and, in fact, this
candidate, which works fine for the quantification of entanglement sharing in any
finite dimension $2 \leq d < \infty$, fails for $d = \infty$, for instance, on pure, bisymmetric $^{30}$
three-mode Gaussian states. This could prima facie discourage us from trying defining
a CV tangle; indeed, there is another chance left, thanks to the following crucial
observation. For mixed states of two qubits, the tangle can be viewed equivalently
as the convex-roof extension of the squared negativity (the latter coinciding with
the concurrence for pure states). This fact then suggests defining a CV tangle via
the negativity or, better, via the logarithmic negativity. In fact, if the monogamy
inequality is satisfied using a measure $E$ of bipartite entanglement, it will hold as
well using any other increasing and convex function of $E$. This is exactly the case
for negativities, because $\mathcal{N}$ is a convex function of $E_\mathcal{N}$.

From the above considerations, it follows that a privileged candidate to com-
ply with the CV versions of the monogamy inequalities (1) and (7) is thus the
continuous-variable $^{27}$ tangle, or, in short, the contangle $^{27} E_\tau$. For a generic pure state $|\psi\rangle$ of a $(1 + N)$-mode CV system, we can formally define the tangle as

$$ E_\tau(\psi) \equiv \log^2 \|\tilde{\varrho}\|_1, \quad \varrho = |\psi\rangle\langle\psi|. \quad (9) $$

$E_\tau(\psi)$ is a proper measure of bipartite entanglement, being a convex, increasing
function of the logarithmic negativity $E_\mathcal{N}$, which is equivalent to the entropy of
entanglement for arbitrary pure states. In the case of a pure Gaussian state $|\psi\rangle$
with CM $\sigma^p$, $E_\tau(\sigma^p) = \log^2(1/\mu_1 - \sqrt{1/\mu_1^2 - 1})$, where $\mu_1 = 1/\sqrt{\text{Det} \sigma_1}$ is the
local purity of the reduced state of mode 1, described by a CM $\sigma_1$ (we are dealing
with a $1 \times N$ bipartition). Definition (9) is naturally extended to generic mixed
states $\varrho$ of $(N + 1)$-mode CV systems through the convex-roof formalism, namely,

$$ E_\tau(\varrho) \equiv \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E_\tau(\psi_i), \quad (10) $$

where the infimum is taken over the decompositions of $\varrho$ in terms of pure states
$\{|\psi_i\rangle\}$. Dealing with infinite Hilbert spaces the index $i$ is continuous, so the sum
in Eq. (10) should be replaced by an integral, and the probabilities $\{p_i\}$ by a
distribution $\pi(\psi)$. Let us recall that any multimode mixed Gaussian state with CM
$\sigma$, admits a decomposition in terms of an ensemble of pure Gaussian states. The
infimum of the average contangle, taken over all pure Gaussian decompositions only,
defines the Gaussian contangle $G_\tau$, which is an upper bound to the true contangle $E_\tau$, and an entanglement monotone under Gaussian local operations and classical
communications (GLOCC). $^{36,37}$ The Gaussian contangle, similarly to the Gaussian
entanglement of formation, $^{36,37}$ acquires the simple form $G_\tau(\sigma) \equiv \inf_{\sigma^p \leq \sigma} E_\tau(\sigma^p)$,
where the infimum runs over all pure Gaussian states with CM $\sigma^p \leq \sigma$.

Equipped with these properties and definitions, one can prove a series of
results. $^{37}$ In particular, inequality (7) is satisfied by all pure three-mode and all pure symmetric $N$-mode Gaussian states, using either $E_\tau$ or $G_\tau$ to quantify bipartite
entanglement, and by all the corresponding mixed states using $G_\tau$. Furthermore, there is numerical evidence supporting the conjecture that the general CKW inequality (7) should hold for all nonsymmetric $N$-mode Gaussian states as well.

The sharing constraint (1) leads to the definition of the residual contangle as a tripartite entanglement quantifier. However, for generic three-mode Gaussian states the residual contangle is partition-dependent. In this respect, a proper quantification of tripartite entanglement is provided by the minimum residual contangle

$$E_{\tau}^{i|j|k} \equiv \min_{(i,j,k)} [E_{\tau}^{i(jk)} - E_{\tau}^{ij} - E_{\tau}^{ik}],$$

where $(i, j, k)$ denotes all the permutations of the three mode indexes. This definition ensures that $E_{\tau}^{i(jk)}$ is invariant under mode permutations and is thus a genuine three-way property of any three-mode Gaussian state. We can adopt an analogous definition for the minimum residual Gaussian contangle $G_{\tau}^{i|j|k}$. One finds that the latter is a proper measure of genuine tripartite CV entanglement, because it can be proven to be an entanglement monotone under tripartite GLOCC for pure three-mode Gaussian states.$^{27}$

2.3. Promiscuous sharing of continuous variable entanglement

Let us now analyze the sharing structure of CV entanglement by taking the residual contangle as a measure of tripartite entanglement, by analogy with the study done for three qubits.$^{14}$ Namely, we pose the problem of identifying the three-mode analogues of the two fully inseparable and symmetric three-qubit pure states, the GHZ state$^{15}$ and the $W$ state,$^{14}$ discussed in Sec. 1.3. Surprisingly enough, in symmetric three-mode Gaussian states, if one aims at maximizing (at given single-mode squeezing) either the two-mode contangle $E_{\tau}^{i|l}$ in any reduced state (i.e. aiming at the CV $W$-like state), or the genuine tripartite contangle (i.e. aiming at the CV GHZ-like state), one finds the same, unique family of pure symmetric three-mode squeezed states. These states, previously named “GHZ-type” states,$^{35}$ can be defined for generic $N$-mode systems, and their multimode entanglement scaling can be studied.$^{30,38}$ The peculiar nature of entanglement sharing in this class of CV GHZ/$W$ states is further confirmed noting that if one requires $E_{\tau}^{i|j|k}$ to be maximum under the constraint of separability of all two-mode reductions, one finds states whose residual contangle is strictly smaller than the one of the GHZ/$W$ states, at fixed squeezing.

Therefore, in symmetric three-mode Gaussian states, when there is no two-mode entanglement, the three-party one is not enhanced, but frustrated. These results, unveiling a major difference between discrete-variable and CV systems, establish the promiscuous nature of entanglement sharing in symmetric Gaussian states. Being associated with degrees of freedom with continuous spectra, states of CV systems need not saturate the sharing inequality to achieve maximum couplewise correlations. In fact, without violating the monogamy constraint (1), pure symmetric three-mode Gaussian states are maximally three-way entangled and, at the same
time, maximally robust against the loss of one of the modes due, for instance, to decoherence.

Finally, the residual contangle Eq. (11) in this class of GHZ/W states acquires a clear operative meaning in terms of the optimal fidelity in a three-party CV teleportation network. This result enforces the interpretation of the contangle $E_\tau$ as a bona fide measure of tripartite entanglement for Gaussian states, because it appears as the most natural infinite-dimensional extension of the tangle $\tau$ which quantifies entanglement sharing among qubits and qudits.

Acknowledgments

Financial support from MIUR, INFN and INFM is acknowledged.

Note added in proof. Recently, Hiroshima, Adesso and Illuminati have proven inequality (17) for all Gaussian states of $N$-mode CV systems, by using the Gaussian tangle defined as the (convex-roof extended) squared negativity. This fundamental result extends the findings of Ref. 27 and establishes the general monogamy of distributed Gaussian entanglement. The conjecture raised in Sec. 2.2 is indeed true.

References