CLASS FIELD THEORY GUIDANCE AND THREE FUNDAMENTAL DEVELOPMENTS IN ARITHMETIC OF ELLIPTIC CURVES

IVAN FESENKO

Class Field Theory (CFT). Is it a subject on which nothing final can be known?

Has it become something from a half-remembered dream? What have we forgotten?

Recall the opinion of Shafarevich: 'Weil was undoubtedly right when he asserted, in the preface to the Russian edition of his book on number theory, that since class field theory pertains to the foundation of mathematics, every mathematician should be as familiar with it as with Galois theory. Moreover, just like Galois theory before it, class field theory was reputed to be very complicated and accessible only to experts ... For class field theory, on the other hand, there is a wide range of essentially different expositions, so that it is not immediately obvious even whether the subject is the same'.[1] Weil's words are passé: nowadays even many number theorists are not well familiar with class field theory.

A rather complex and multi-vocal nature of CFT will be briefly presented below. It has been an enduring process of discovering new branches of CFT and its generalisations, among which many were quite complicated at their early stage and some were almost impossible to understand and accept for their contemporaries.

There are several generalisations of CFT. Three main generalisations will be discussed in this text. It is an open problem whether there might be a single unified generalisation of CFT.

The split of (one-dimensional) CFT into special CFT (SCFT) and general CFT (GCFT) is somehow repeated in two generalisations of CFT: the Langlands program (LP) which does not yet have any development parallel to GCFT, and higher CFT, almost all of which is parallel to GCFT. 50 years after the start of LP, in the sense of the lack of general theory we are in the 'pre-Takagi' stage of LP, similar to where CFT was 100 years ago. It is likely that some entirely new vision for the Langlands program is awaiting to be discovered.

It is natural to ask whether some further progress can be achieved in arithmetic LP with a new potential development there that is parallel to GCFT, or, at least some mixture of features of SCFT which works over all number fields. Such a mixture is a property of Mochizuki's IUT theory.

LP predicts important features of the L-functions of arithmetic schemes, but the L-functions are a more difficult, micro/quantum object to study in comparison to the zeta functions, a macro/abelian object. Efforts of many researchers have been concentrated on the study of the L-function rather than the study of the zeta functions of general arithmetic schemes (not just Shimura varieties). More work should be invested into the zeta functions. The study of the zeta functions of arithmetic surfaces using structures coming from 2dCFT and integration on non-locally compact abelian objects in 2dAAG has already brought various insights.

New perspectives for further developments of LP and higher CFT and related theories may become possible via the study of their interaction with the IUT theory that is based on anabelian geometry, the third generalisation of CFT.

Further interactions and links between the three generalisations of CFT can potentially be found if we go from them back to CFT, to trace their common roots.

[1] in Foreword to Local Fields and Their Extensions, by I.B. Fesenko and S.V. Vostokov, 2nd ed., AMS, 2002

This paper does not aim to include all results and theories in CFT and its generalisations. Sometimes, more space is given to relatively unknown theories.

This text was written on the occasion of delivering a talk at a meeting of Symmetries and Correspondences in Oxford on February 23 2018.

I am grateful to Shinichi Mochizuki and Laurent Lafforgue for many valuable discussions and remarks, and to Vladimir Drinfeld, Adam Topaz, Robert Langlands, Richard Taylor, Edward Frenkel and Alexander Beilinson for their comments.

1. KUMMER EXERCISE

For a field k its absolute Galois group is G_k and its maximal abelian quotient is $G_k^{\mathrm{ab}} = G_k/[G_k,G_k]$.

For a field k whose characteristic does not divide integer $m > 1$, linear algebra for the multiplicative group as Galois module, associated to the exact sequence of G_k-modules

$$1 \longrightarrow \mu_m \longrightarrow \mathrm{k}^\times \longrightarrow \mathrm{k}^\times \longrightarrow 1,$$

k is a separable closure of k, gives the Kummer map

$$k^\times/k^{\times m} \simeq H^1(G_k,\mu_m).$$

This is an H^1-theory.

If the μ_m is in k, then RHS $= \mathrm{Hom}(G_k,\mathbb{Z}/m)$, so get the Kummer pairing

$$k^\times/k^{\times m} \times G_k/([G_k,G_k]G_k^m) \longrightarrow \mathbb{Z}/m,$$

and finite abelian extensions of k of exponent m are in explicit one-to-one correspondence with subgroups B of $k^\times$ of exponent m: $B \longrightarrow k(\sqrt[m]{B})$.

This theory is not of much value for arithmetic fields, e.g. $\mathbb{Q}$ and $\mathbb{Q}_p$, since they contain too few roots, unlike function fields over algebraically closed fields. One can slightly extend, working with $\mathbb{Q}(\mu_m)$, for prime m, using the fact that $|\mathbb{Q}(\mu_m):\mathbb{Q}|$ is prime to m, but this is too little.

Anabelian geometry results show that every continuous automorphism of $G_\mathbb{Q}$ is inner. For non-archimedean completions k_v of number fields not every continuous automorphism of the absolute Galois group G_{k_v} is inner.We do know an algebraic description of the absolute Galois groups of k_v (Demushkin, Yakovlev, Jannsen–Winberg) but relations of this description with higher ramification filtration are in need of further work. It is known both for G_k and G_{k_v} that every nontrivial closed normal topologically finitely generated subgroup is open and every open subgroup has trivial centre. For number fields we still know little about G_k, but much more about G_k^{ab} due to CFT. There seems to be nothing done towards a solution of the Shafarevich conjecture: $[G_\mathbb{Q},G_\mathbb{Q}]$ is free in the category of profinite groups.

Problem: How to reach the maximal abelian extension and the maximal nilpotent extension of fixed nilpotent class and the separable closure of arithmetic fields and how to describe their finite subextensions in ways which allow a good range of applications? Which structures of arithmetic fields to use for such descriptions?

This problem for abelian extensions is solved by many types of CFT and higher CFT, for nilpotent extensions of local fields it is partially solved by arithmetic non-abelian local class field theory, whereas the Langlands program and anabelian geometry provide some partial answers in the general case.

2. CFT OF TWO TYPES

It is useful to recall two types of CFT:

$$\begin{tikzcd} & \mathrm{CFT} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{SCFT} & & \mathrm{GCFT} \end{tikzcd}$$

First type: SCFT = special CFT. These use special structures such torsion/division points or values of appropriate functions at torsion points and Galois action on them. The global number fields versions of these theories work over certain small fields only, the local and functional case theories work over any field. These theories are cohomology-free.

cyclotomic (historically, the first) Kronecker, Weber, Hilbert

using elliptic curves with CM (historically, the second) Kronecker, Weber, Takagi

using abelian varieties with CM, Shimura

positive characteristic, Hayes and Drinfeld (rank 1 Drinfeld modules). These theories work over all global fields of positive characteristic.

local theory that works over any local field with finite residue field by using Lubin–Tate formal groups.

In characteristic zero these theories are not extendable to arbitrary number fields.[2] SCFT are not easy. Staying too near[3], not able to see more general easier structures. Recall several Weber's erroneous published proofs of the first two CFTs, due to mistakes with the 2-primary part.[4] Hilbert Problem 12 was about extensions of SCFT to number fields, the best was achieved by Shimura, the general task is too optimistic and does not look to be realisable.

Second type: GCFT = general CFT. These theories follow very different conceptual patterns than SCFT. They works over any global or local field.

Hilbert, Furtwängler: some elements of such GCFT but for some fields only, without an associated local theory.

Takagi, the first work in GCFT, existence theorem and its applications.

Artin reciprocity map, whose full construction used a method from Chebotarev's work, missing in its first versions.

Hasse, the use of the Brauer group in CFT, the first local CFT, local-to-global.

Chevalley ideles, local-to-global, reciprocity law as the product of local reciprocity laws which on the diagonal image of global elements equals the identity automorphism.

classical approaches to CFT are presented, among many sources, in Hasse's Klassenkörperbericht, and in Weil's and Lang's books.

cohomological the Artin–Tate book and many other presentations.

All these class field theories are not really easy to embrace.[5]

positive characteristic: among various approaches, including some parallel to the number fields case, the easiest one is Kawada–Satake theory that uses Witt duality for the most difficult p-primary part. This approach does not have a number fields analogue; it is quite different from the SCFT approach of Hayes–Drinfeld.

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[2] It is incorrect that the cyclotomic theory and some general algebraic functoriality can produce general CFT for all arithmetic fields; number fields require more of individual attention.

[3] to an impressionist painting?

[4] More detail can be found in N. Schappacher's paper On the history of Hilbert's twelfth problem.

[5] E.g., I.R. Shafarevich in the 1950s was the only expert in CFT in fSU

explicit, post-cohomological and cohomologically-free local CFT: Tate–Dwork, Hazewinkel, Neukich, Fesenko, global CFT: Neukirch. These theories clarified and made explicit key structures of CFT. These are less dependent on torsion and they do not use the Brauer group: the Brauer group computation is not needed for CFT. They made CFT easy and in some sense for the first time explained CFT. It is somehow paradoxical that explicit GCFT, working over any global and local field, is easier than SCFT working over small number fields only. Note the very different type of explicitness in explicit GCFT in comparison to that in SCFT. The explicit GCFT was also highly useful in higher CFT and other CFT.

There are many fundamental differences between these classes of GCFT.[6]

3. Four parts of CFT

I Functorial reciprocity map

$$\Psi_k\colon C_k \longrightarrow G_k^{\mathrm{ab}}$$

almost a topological isomorphism, from the topological abelian group C_k (the multiplicative group of a local field or the idele class group of a global field) associated to the ground field k, inducing the isomorphism $C_k/N_{l/k}C_l \simeq G(l/k)^{\mathrm{ab}}$ for finite Galois extensions.

II Existence theorem, describing finite abelian extensions of the ground field k in terms of open subgroups of finite index of C_k. Important for many applications, e.g. the proof that the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ of $\mathbb{Q}$ is its maximal cyclotomic extension $\mathbb{Q}(\text{roots})$.

III Explicit formulas for the (wild) Hilbert pairing, already asked for in Hilbert Problem 9. Important for numerous applications and computations. The nature of explicitness here differs from that in explicit CFT.

IV Compatibility with ramification theory is an additional arithmetic flavour of CFT.

I Hundreds of papers in GCFT. The list of names of Japanese contributors fills in a whole page.

Cohomological approach: there are several. One of them uses the Kummer map, locally

$$\mathrm{Hom}(G_k,\mathbb{Z}/m)\times k^\times/k^{\times m} \longrightarrow H^1(G_k,\mathbb{Z}/m)\times H^1(G_k,\mu_m) \longrightarrow H^2(G_k,\mu_m) = {}_m\mathrm{Br}(k) \simeq \mathbb{Z}/m,$$

which gives the map $k^\times \longrightarrow \varprojlim k^\times/k^{\times m} \longrightarrow G_k^{\mathrm{ab}}$.

Not far from the Kummer pairing, but so different.

The map $H^1\times H^1 \to H^2$ involves the cup product. Note: the cup product is quite non-explicit, which implies the non-explicit nature of cohomological class field theory.

Note: knowing the Brauer group helps to get the reciprocity map, but not the existence theorem and not explicit formulas. In other words, CFT is not reduced to the computation of the Brauer group. Moreover, the Brauer group is not needed in explicit CFT.

Cohomological CFT is an H^2-theory. It seems to be fair to remark that generations of number theorists went too fast through the cohomological approach and not learning the substance of CFT well. 50 years ago Hasse wrote, 'For the sharply profiled lines and individual features of this magnificent edifice seem to me to have lost somewhat of their original splendour and plasticity by the penetration of class field theory with cohomological

[6] see the quote of I.R. Shafarevich at the beginning of the paper

concepts and methods'.[7] Linear algebraic (cohomological) methods miss various aspects of profound arithmetic issues of CFT.

Class formations approach was a long search for clarification of CFT. Deduce as much as possible in CFT from as few axioms as possible. Typically, the axioms are about H^0 (index of the norm group) and H^1 (Hilbert Theorem 90) for cyclic extensions.

Explicit CFT give more information about the image in $C_k/N_{l/k}C_l$ of a Galois automorphism σ of finite Galois extension l/k, which can be schematically represented by $\sigma \mapsto N_{\Sigma/k}\pi_\Sigma$. In the Neukirch approach one works with appropriate free abelian pro-p-extensions of rank 1: the maximal unramified for local fields and the only $\hat{\mathbb{Z}}$-subextension $f/\mathbb{Q}$ of $\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}$ for global fields. In the local theory one uses the facts that for unramified extensions each prime element is sent to the Frobenius automorphism and for every automorphism σ of finite Galois totally ramified extension l/k there is a finite extension e/k such that le/e is unramified and σ is in the image of $G(le/e) \to G(l/k)$.

In characteristic p one can use Artin–Schreier–Witt theory to explicitly get the p-primary part of CFT.

II Existence theorem related with the name 'class field': open subgroups of finite index in C_k are in one-to-one correspondence with finite abelian extensions l/k: $l/k \longrightarrow N_{l/k}C_l$.

III Explicit formulas for the (local) Hilbert pairing:

Let the μ_m be in k. The Hilbert pairing

$$(\ ,\ )_m\colon k^\times/k^{\times m} \times k^\times/k^{\times m} \longrightarrow \mu_m$$

is defined as $(a,b)_m = \sigma(c)/c$ where $c^m = b$ and σ is the restriction on $k(\sqrt[m]{b})/k$ of the image of a with respect to the local reciprocity map. If one uses the Kummer map, then $(\ ,\ )_m\colon k^\times/k^{\times m} \times k^\times/k^{\times m} \longrightarrow H^1(G_k,\mu_m) \times H^1(G_k,\mu_m) \longrightarrow H^2(G_k,\mathbb{Z}/m) = {}_m\mathrm{Br}(k) \simeq \mu_m$ after appropriate choices are made.

If m is prime to the residue characteristic p, then the Hilbert pairing is a power of the tame symbol with its explicit linear algebra formula.

When $m = p^r$ explicit formulas for the wild Hilbert symbol and its generalisations are far from linear. Such formulas are heart of arithmetic.

Two types of explicit formulas (for the wild Hilbert pairing and its generalisation to formal groups):

Partial/special explicit formulas: Artin–Hasse, Iwasawa, Coates–Wiles, Fontaine, Perrin-Riou, Kato–Kurihara–Tsuji, Benois (related to the p-adic Hodge theory)[8] and others.

Full explicit formulas (for full values of the argument, so these formulas define a symbol pairing independently of CFT): Shafarevich, Vostokov, Henniart, Fesenko, Kato.

IV The local reciprocity map is (a) compatible with higher ramification groups filtration on the abelian part of the absolute Galois group. This filtration is due to (b) Herbrand's theorem and satisfies (c) Hasse–Arf theorem: higher ramification jumps of abelian extensions are integers.

The encore. CFT is more than abelian, it includes some information about abelian by finite extensions, the Shafarevich–Weil theorem: for an arithmetic field k and finite $l/f/k$, l/f abelian, f/k normal, the class of the

[7] quoting from H. Hasse's History of class field theory, in LMS proc. 1965 Brighton conf., Academic Press 1967. Compare with P. Cartier's words 'New tools appeared: sheaf theory and homological algebra (invented by Jean Leray in the first case, and by Henri Cartan and Samuel Eilenberg in the second: their treatise Homological Algebra was published in 1956). These were admirable in their generality and flexibility'in his Inference paper.

[8] part of it, the theory of φ-Γ-modules can be viewed as a weak version of the earlier Tate–Dwork's theorem in explicit CFT

group extension in $H^2(G(f/k), C_f/N_{l/f}C_l)$ of the exact sequence $1 \longrightarrow C_f/N_{l/f}C_l \longrightarrow G(l/k) \longrightarrow G(f/k) \longrightarrow 1$ equals the image of the canonical class in $H^2(G(f/k), C_f)$. Koch–de Shalit's metabelian local class field theory, of SCFT type, may be viewed as one of developments related to this theorem. Another development is general arithmetic non-abelian CFT of GCFT type, it includes local theory for arithmetically profinite extensions, with its existence theorem and compatibility with ramification theory (Fesenko, Ikeda–Serbest), and some global theory (Ikeda).

4. From CFT to Algebra: 'Higher Kummer theory' for Milnor K_r

The Hilbert pairing satisfies the norm property: $(a,b)_m = 1$ iff $b \in N_{k(\sqrt[m]{a})/k} k(\sqrt[m]{a})^\times$.

This implies the Steinberg property $(a, 1-a)_m = 1$ for $a \neq 1$.

So the Hilbert pairing is a symbol map from Milnor $K_2(k)$ to μ_m. In fact, it is a universal continuous map from $K_2(k)$ to finite abelian groups.

For any field k of characteristic not dividing m, using the Kummer map and the cup-product, one gets the 2-Kummer map:

$$k^\times/k^{\times m} \times k^\times/k^{\times m} \longrightarrow H^1(G_k, \mu_m) \times H^1(G_k, \mu_m) \longrightarrow H^2(G_k, \mu_m \otimes \mu_m).$$

If k is local, this is closely related to the Hilbert pairing. The 2-Kummer map satisfies the Steinberg property, hence induces the norm residue symbol

$$K_2(k)/m \longrightarrow H^2(G_k, \mu_m \otimes \mu_m).$$

Tate: the norm residue symbol is an isomorphism for local and global fields, using CFT.

Merkuriev–Suslin: the norm residue symbol is an isomorphism for all fields.

Voevodsky (using results of Suslin, Morel and Rost): the generalisation of the norm residue symbol to Milnor K_r is an isomorphism for all r.[9]

5. Three generalisations of CFT

Each of the following three generalisations was started some 40–50 years ago. There are some other generalisations of CFT, not discussed here.

$$\begin{tikzcd}
A\text{: the Langlands program} \arrow[dr, dash] & B\text{: higher CFT} \arrow[d, dash] & C\text{: anabelian geometry} \arrow[dl, dash] \\
 & \text{CFT} &
\end{tikzcd}$$

A: the Langlands program, a well known approach which uses linear representation theory and in the arithmetic case a lot of analytic considerations. Its main conjectures are stated over all global and local fields. They do not have analogues of parts II, III, IV of CFT. The Langlands program aims to classify (irreducible) linear representations of the Galois group of a global field. Therefore it misses various important features of the full absolute Galois group that are not of linear representation type.[10] [11] It is important to note that 'the methods

[9] see Haesemeyer–Weibel's book, to be published soon, for the first full and relatively compact presentation

[10] 'In the case of the absolute Galois group of a field, set-theoretic actions of this group correspond to separable extensions of the field. The world of its linear representations is so different that it is usually extremely difficult and deep to derive concrete consequences for separable extensions of results obtained for linear representations of this group', from L. Lafforgue's email message of Feb 21 2018

[11] see also the end of the introduction of Sh. Mochizuki's IUT-I paper

used to construct Langlands' correspondence (for function fields) or pieces of this correspondence (for number fields) are highly non-explicit, as they consist in considering and studying l-adic cohomology spaces of sophisticated algebraic varieties. Galois representations constructed as components of cohomology spaces are not at all objects one can describe.'[12]

A conceptually different from the Langlands program approach to the description of certain noncommutative Galois extensions of function fields of curves over finite fields was developed by Ihara and his students 50–45 years ago. For example, this theory describes the maximal unramified cover of the projective line over $\mathbb{F}_{p^2}$ minus three points which is at most tamely ramified at the three points and in which some special finite set of points decomposes completely, in terms of subgroups of finite index of the quotient of the subgroup of $SL_2(\mathbb{Z}[1/p])$ congruent to the identity mod 2 by its cyclic subgroup of order 2. There are interesting arithmetic phenomena in this theory which deserve further development.[13]

The Langlands program is non-abelian but not utmost non-abelian. In some sense it may be viewed as a linear theory over abelian CFT. Its main arithmetic advances are of special type only: over $\mathbb{Q}$ and small number fields (totally real, etc.), due to Wiles, with a very substantial non-linear arithmetic part in his work, and Taylor, and generalisations of the Wiles method, and other results of Taylor and his collaborators; there are also Harris–Taylor and Henniart's results about the local case for the general linear group using global methods. Recent work by L. Lafforgue sheds new light on functoriality.

In characteristic p the GL_r case was established by Drinfeld for $r = 2$, and by L. Lafforgue for all r, using Drinfeld modules and FH-sheaves[14], and with further important work for arbitrary reductive algebraic groups by V. Lafforgue.

The proof of the fundamental lemma by Ngô, using Hitchin systems, is one of recent highlights. Using it, Arthur estabilised his trace formulas and various arithmetic applications.

The geometric Langlands program, further away from arithmetical aspects of CFT, was initiated by Deligne and Laumon and developed by Beilinson–Drinfeld, Gaitsgory, Frenkel and others, it uses Hitchin systems; there are various open problems including the ramified case. Recent work of Langlands on geometric theory sheds a new, more analytic, light.[15]

Still, most fundamental problems in the arithmetic Langlands program remain open.[16] Even the $GL_2(\mathbb{Q})$ case is still open, 'We still don't know how to associate an Artin representation to a Galois Maass form'and 'the problem of Galois Maass forms is just the tip of an iceberg'[17]. It looks that some fundamental arithmetic insight is missing. Linear algebraic and geometric methods and vision cannot fully cover or substitute various profound arithmetic issues, in particular those revealed by CFT and its other generalisations.

Note: apparently, no significant development within A is parallel to (or can be viewed as a generalisation of) GCFT. Thus, in the Langlands program we are still in the 'pre-Takagi' stage, similar to where we were 100 years ago in CFT.

[12] from L. Lafforgue's email message of Feb 21 2018

[13] Y. Ihara, On congruence monodromy problems, MSJ Memoirs 18(2008)

[14] nicely named so by G. Harder and D. Kazhdan in their 1979 report paper, F for Frobenius, H for Hecke, instead of the original name штуки (pl.), one of the main translations of штука (sing.) is whatsit, thingummy, gizmo; compare with this paper second line that was included much earlier than this footnote

[15] R. Langlands, Об аналитическом виде геометрической теории автоморфных форм, An analytic approach to the geometrical theory of automorphic forms, in Russian, April 2018

[16] talks of R. Langlands at S&C conference `https://www.maths.nottingham.ac.uk/personal/ibf/files/sc3.html`

[17] from R. Taylor's email messages of Mar 4 and Mar 13 2018

Problems: find an approach to Langlands type theory which is parallel to some of GCFTs. Find an application of a generalisation of the Neukirch method to the Langlands program. Find an approach to Langlands type theory which is parallel to some of post-cohomological CFT, thus circumventing the problem of using non-abelian cohomology.

It is quite likely that better knowledge and use of GCFT can be useful for further progress. Langlands emphasises the importance of Hasse's Klassenkörperbericht.[18]

B: 2d (and higher) CFT typically but not always (Lang, Wiesend) use Milnor K_2. Most if not all of them are of GCFT type. Almost all main theorems are established.

60 years old Lang's geometric CFT for function fields of varieties over finite fields used commutative algebraic groups over finite fields and their isogenies. It influenced Drinfeld's theory. It can be deduced from higher CFT.

Cohomological theory developed by Kato, Bloch, Kato–Saito, Spieß, Jannsen–Saito. It generalises one of cohomological approaches to CFT. The local part of Kato's theory works with $H^3(G_k, \mu_m \otimes \mu_m)$ instead of the m-torsion part of the Brauer group. The cohomological 2dCFT is an H^3-theory. This theory is not easy. Other 1d cohomological approaches cannot be (easily) generalised. In particular, for a finite Galois extension L/F the homomorphism $K_2(F) \longrightarrow K_2(L)^{G(L/F)}$ is neither injective nor surjective in general, so the 1d class formations approach does not generalise directly.

Parshin's approach to higher CFT in positive characteristic was an extension[19] of classical Kawada–Satake GCFT, by also involving some topological considerations. Fesenko's explicit higher CFT in any characteristic generalised Weil–Chevalley's method and Neukirch's method. It works with several topologies on Milnor K-groups of higher fields and their separated quotient K_2^t, and made use of higher Vostokov explicit pairing that exists independently of CFT.

All these 2dCFT are GCFT, they do have analogues of II, III, but not of IV due to lack of satisfactory 2d ramification theory satisfying analogues of all (a), (b), (c).

Problem: develop satisfactory 2d ramification theory compatible with CFT.

More recently, a π_1-approach, cohomologically free and without using K-groups, to parts of higher global CFT in any characteristic was developed by Wiesend. It analyses which finite compatible covers of closed points and curves on an arithmetic scheme correspond to a finite Galois cover of the scheme, using the Hilbert irreducibility theorem, Grothendieck's specialisation of the fundamental group and finiteness of topological generators of the tame fundamental group of smooth curves over separably closed fields. Restricting to abelian covers and using 1d global CFT leads to a simplified approach to parts of global higher CFT. Unusually for CFT, this approach does not have local and local-to-global parts. Some corrections, extensions and applications of Wiesend's approach are due to Schmidt and Kerz.

Note: unlike the classical one-dimensional case, 2dCFT is somehow separated from the usual algebraic geometry working with 1-cocycles.

[18] in two email messages of Feb 9 2016, Feb 24 2018

[19] substantial mistakes and gaps in the original published papers were eventually fixed, see Fesenko's generalised Kawada-Satake theory below

$$\begin{tikzcd}[column sep=small] & \mathbf{A} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbf{B} \arrow[dr, dash] & & \mathbf{C} \arrow[dl, dash] \\ & K & \end{tikzcd}$$

The distance between full geometric adeles $\mathbf{A}$ and the function field K is two-step. 2dCFT operates with K_2 of the top two levels, i.e. $K_2(\mathbf{A})$ and $K_2(\mathbf{B})+K_2(\mathbf{C})$, the quotient knows about the Chow group of zero-cycles, while divisors and the Picard group come from $K_1(\mathbf{B})$ and the quotient $K_1(\mathbf{B})/K_1(K)$, i.e. the bottom two levels.

Note: apparently, no significant development within B is parallel to (or can be viewed as a generalisation of) SCFT!

Problem: would a special higher CFT which uses torsion structures provide new insights into 2dCFT?

Thus, we observe that the split of CFT into SCFT and GCFT is currently somehow reproduced in the split of generalisations of CFT into A: the Langlands program and B: higher CFT.

Note: for the p-primary part of CFT in characteristic p one cannot use Kummer theory. Kawada–Satake method and its generalisations, using Witt theory, makes the p-primary part the easiest part of CFT in characteristic p. The nature of the existence theorem is clarified: it corresponds to topological reflexivity of (generally non-locally compact) groups with respect to a generalised explicit pairing.[20]

C: anabelian geometry Neukirch, Iwasawa, Ikeda, Uchida (for 1d fields, it uses the computation of the Brauer group of arithmetic fields); Pop (finitely generated fields); Nakamura, Tamagawa, Mochizuki (curves over fields), Stix (more on curves over fields of positive characteristic).[21] There is no analogue of II, III in C, but there might be some analogue of IV.

Anabelian geometry is a sort of utmost non-abelian and non-linear, working with full topological groups such as the absolute Galois groups and fundamental groups of hyperbolic curves, using 2d structures. Main theorems of 2d anabelian geometry (i.e. for hyperbolic curves over various fields, including global fields) were established by Mochizuki. Typically, proofs in anabelian geometry reduce to abelian by finite groups. Mono-anabelian geometry further extends anabelian geometry.

Grothendieck's theory of the Teichmüller tower already had implication for $G_{\mathbb{Q}}$, as well as Belyi's theorem and its consequences. The Grothendieck–Teichmüller group (GT) defined by Drinfeld was used to provide a conjectural concrete description of $G_{\mathbb{Q}}$. Recent work of Mochizuki–Hoshi–Minamide shows that GT is, up to S_{n+3}, just the group of outomorphisms of the étale fundamental group of the n-dimensional configuration space, $n>1$, associated to a hyperbolic curve of genus 0 with 3 punctures over an algebraically closed field of characteristic zero. This result suggests strongly that the natural inclusion of $G_{\mathbb{Q}}$ into GT is not surjective.

Note: the Neukirch's explicit approach to CFT was partially motivated by his experience in 1d anabelian geometry.

Note: mono-anabelian geometry provides a new insight into class formations in CFT: separation of ring structure dependent part (checking the class formations axioms validity using the ring structure of arithmetic fields) from the group structure only dependent part (CFT mechanism to deduce the reciprocity map from the axioms).

[20] another forgotten theory, Inaba's work on matrix Artin–Schreier theory, which describes all finite Galois extensions of degree a power of p of fields of characteristic p, may lead to a simplified p-primary theory of the Langlands correspondence in characteristic p

[21] and Bogomolov, Bogomolov–Tschinkel, Pop, Topaz for function fields of varieties of dimension >2 over $\mathbb{C}$

6. Three generalisations of CFT and arithmetic of elliptic curves

Example: elliptic curves E over number fields k. Choose a proper regular model $\mathscr{E}$ flat over the ring of integers of k. Denote $K = k(E) = k(\mathscr{E})$.

A: in this case it can also be called the Taniyama–Shimara conjecture (still not solved over arbitrary number field). It uses representations of G_k on torsion points of $E_k(\mathrm{k})$, k is a separable closure of k. Does not use 2d objects, works with the generic fibre of $\mathscr{E}$ (hence with horizontal curves on $\mathscr{E}$) or with special fibres of $\mathscr{E}$ but not with the full arithmetic surface $\mathscr{E}$.

B: 2d reciprocity map from K_2^t-idele classes $K_2^t(\mathbf{A})/(K_2^t(\mathbf{B})+K_2^t(\mathbf{C}))$ of $\mathscr{E}$ to G_K^{ab}, 2d local reciprocity map from $K_2^t(F)$ to G_F^{ab} where F is a 2d local field associated to a point on a curve on $\mathscr{E}$.

C: utmost non-abelian and non-linear, for a sufficiently large class of Es, it works with hyperbolic curves associated to E, e.g. $X = E \setminus \{0\}$ over k, and with the homomorphism $\pi_1(X) \to G_k$, using 2d structures. Mochizuki proved that from the surjective homomorphism $\pi_1(X) \to G_k$ of topological groups one can get the topological groups G_K, G_K^{ab} and restore the fields k and completions k_v of k. It is well known that from G_{k_v} one cannot restore k_v in general.

C led to Mochizuki's IUT = inter-universal Teichmüller theory = arithmetic deformation theory and applications to the abc, Szpiro and Vojta conjectures. IUT uses generalised Kummer theory and the computation of the local Brauer group. It does not use anything else from CFT. It works with values of certain non-archimedean functions (etalé theta functions) at torsion points, in this respect it is nearer to SCFT; on the other hand, it works over any number field and in this respect it is nearer to GCFT.[22]

Problem: can the use of non-archimedean 'modular/automorphic' functions in the Langlands program lead to a theory which works over any number field?

Recall that for 50 years or so global CFT used a computation with the zeta function to prove one of key inequalities.

B led to 2dAAG = 2d adelic analysis and geometry[23], to study meromorphic properties of the zeta function of $\mathscr{E}$, its generalised Riemann hypothesis and the Birch–Swinnerton-Dyer conjecture in Tate's form (BSDT). 2dAAG uses objects at all three levels of the previous diagram. There is an informal and mysterious link between structures of CFTs and their generalisations and structures used in the study of zeta functions of arithmetic schemes and their L-factors. However, CFT and B are not used in such study of zeta functions.

Fundamental analytic objects corresponding to CFT, A, B and 2dAAG, C and IUT:

CFT: Dedekind zeta function and its twists by characters, 1d Iwasawa–Tate zeta integrals theory

A: L-functions (micro objects, generally hard to study, if not known to be automorphic)

B and 2dAAG: 2d zeta integral theory to study zeta functions of arithmetic surfaces (macro objects, easier to study, zeros of L of the generic fibre generally becomes poles of zeta), mean-periodicity correspondence (a weaker version of the Langlands correspondence) and GRH.

C and IUT: related analytic objects are not currently known. IUT somehow replaces analytic structures (at archimedean places) with non-archimedean ones, via the product formula.

Problem: find (archimedean) analytic functions related to IUT.

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[22] see also Remark 2.3.3 of the fourth IUT paper

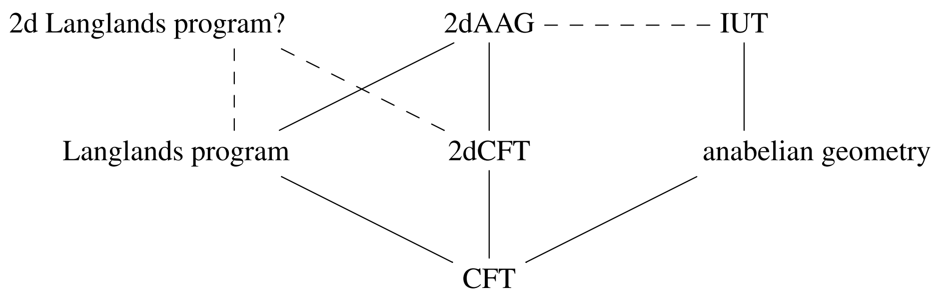
[23] sect. I,J,K of `https://www.maths.nottingham.ac.uk/personal/ibf/mp.html`

The L-functions of arithmetic schemes are like micro/quantum objects, requiring the use of representation theory, once developed for the needs fo quantum mechanics and then finding applications in various parts of mathematics including number theory. In comparison, the zeta functions of arithmetic schemes are like macro objects, they can be studied without using representation theory, using abelian (but higher dimensional methods).

For example, the zeta function of $\mathscr{E}$ is $\zeta_{\mathscr{E}}(s) = \prod_x \big(1 - |k(x)|^{-s}\big)^{-1}$, where x runs through all closed points of $\mathscr{E}$. Its conjectural functional equation $c_{\mathscr{E}}^{2-s}\zeta_{\mathscr{E}}(2-s)^2 = c_{\mathscr{E}}^{s}\zeta_{\mathscr{E}}(s)^2$ with some positive $c_{\mathscr{E}}$ not depending on the archimedean data; note the absence of the Γ-functions. Dependent on the generic fibre of $\mathscr{E}$ only, the (Hasse–Weil) zeta function of E satisfies the equation $\zeta_{\mathscr{E}}(s) = n_{\mathscr{E}}(s)\,\zeta_E(s)$ where the factor $n_{\mathscr{E}}(s)$ is the product of finitely many zeta functions of affine lines over finite fields, corresponding to what happens at singular fibres of $\mathscr{E}$. If $\mathscr{E}$ is the global minimal Weierstraß model of E, then $n_{\mathscr{E}}(s) = 1$. The L-factor is the denominator of ζ_E: $\zeta_E(s) = \zeta_{\mathbb{P}^1(k)}(s)/L_E(s)$. Thus, the conjectural functional equation of L_E is more complicated than that of $\zeta_{\mathscr{E}}$, and its Gamma-factor has its origin due to the 1d Gamma-factor in the functional equation of $\zeta_k(s)\,\zeta_k(s-1)$.

It is one of paradoxes, which may discussed by future historians of number theory, that by far most of the efforts of many previous researchers were concentrated for so long on the study of the L-factors of the zeta functions as opposite to the study of the zeta functions of arithmetic schemes.

Here are some relations between the three generalisations of CFT and their further developments:



Relations between 2dAAG and IUT include the following analogy. IUT uses (after fixing a prime number l) two symmetries: additive geometric and multiplicative arithmetic. 2dAAG uses adelic structures on arithmetic surfaces: geometric adeles (their additive structure is related to Zariski cohomology and the intersection pairing, their K_2-structure is important for 2dCFT) and multiplicative arithmetic adeles $\mathbb{A}^\times$ (on which 2d zeta integral is defined, zero cycles). In particular, an interaction between the multiplicative groups of the two adelic structures $K_1(\mathbf{A}) \times K_1(\mathbb{A}) \to K_2(\mathbf{A})$, originating from 2dCFT, specialises to the relation between the analytic and geometric ranks as predicted by the BSDT conjecture. This is the only general approach to BSD known to me; all other known approaches, including the use of specific Euler systems, are of special type.

Problem: find direct relations between A and C and IUT. Undoubtedly, uncovering many of such relations requires going back to and via CFT. Such relations when found may lead to new approaches to the Langlands program which are parallel to general CFT, or, at least have some mixture of features of special CFT but work over all number fields, one of characteristics of IUT.