

# Analysis on arithmetic schemes. I

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The functional equation of the twisted zeta function of algebraic number fields was first proved by Hecke (for a recent exposition see [N, Ch. VII]). Tate [T] and, for unramified characters without local theory, Iwasawa [I1–I2] lifted the zeta function to a zeta integral defined on an adelic space. Their method of proving the functional equation and deriving finiteness of several number theoretical objects is a generalization of one of the proofs of the functional equation by Riemann. The latter in the case of rational numbers uses an appropriate theta formula derived from a summation formula which itself follows from properties of Fourier transform. Hence the functional equation of the zeta integral reflects symmetries of the Fourier transform on the adelic object and its quotients, and the right mixture of their multiplicative and additive structures. With slight modification the approach of Tate and Iwasawa for characteristic zero can be extended to a uniform treatment of any characteristic, e.g. [W2]. Earlier, the functional field case was treated similarly to Hecke's method by Witt in 1936 (cf. [Rq, sect. 7.4]), and it was also established using essentially harmonic analysis on finite rings by Schmid and Teichmüller [ST] in 1943 (for a modern exposition see e.g. [M,3.5]); those works were not widely known.

Two components are important for a two dimensional generalization of the works of Iwasawa and Tate (both locally and globally). The first component is appropriate objects from the right type of higher class field theory. In the local case these are so called topological Milnor  $K^t$ -groups ([P1], [F1–F3], [IHLF]), for the local class field theory see [Ka1], [P1–P2], [F1–F3], [IHLF]. The second component is an appropriate theory of measure, integration and generalized Fourier transform on objects associated to arithmetic schemes of higher dimensions. In the local case such objects are a higher local field and its  $K^t$ -group. It is crucial that the additive and multiplicative groups of the field are not locally compact in dimension  $> 1$ . The bare minimum of the theory for two dimensional local fields is described in part 1 of this work. Unlike the dimension one case, this theory on the additive group of the field is not enough to immediately define a local zeta integral: for the zeta integral one uses the topological  $K^t_2$ -group of the field whose closed subgroups correspond (via class field theory) to abelian extensions of the field. A local zeta integral on them with its properties is introduced and discussed in part 2.

We concentrate on the dimension two case, but as usual, the two dimensional case should lead relatively straightforward to the general case.

We briefly sketch the contents of each part, see also their introductions. Of course, throughout the text we use ideas and constructions of the one dimensional theory, whose knowledge is assumed.

Two dimensional local fields are self dual objects in appropriate sense, c.f. section 3. In part 1, in the absence of integration theory on non locally compact abelian groups and harmonic analysis on them, we define a new translation invariant measure

$$\mu: \mathcal{A} \longrightarrow \mathbb{R}((X))$$

on appropriate ring  $\mathcal{A}$  of subsets of a two dimensional local field. This measure is not countably additive, but very close to such. The variable  $X$  can be viewed as an infinitesimally small positive

element of  $\mathbb{R}((X))$  with respect to the standard two dimensional topology, see section 1. The main motivation for this measure comes from nonstandard mathematics, but the exposition in this work does not use the latter. Elements of measure theory and integration theory are developed in sections 4–7 to the extent required for this work. We have basically all analogues of the one dimensional theory. Section 9 presents a generalization of the Fourier transform and a proof of a double transform formula (or transform inverse formula) for functions in a certain space. We comment on the case of formal power series over archimedean local fields in section 11.

In part 2, using a covering of a so called topological Milnor  $K_2^t$ -group of a two dimensional local field (which is the central object in explicit local class field theory) by the product of the group of units with itself we introduce integrals over the  $K_2^t$ -group. In this part we use three maps

$$\mathfrak{o}: T = \mathcal{O}^\times \times \mathcal{O}^\times \longrightarrow F \times F, \quad \mathfrak{r}, \mathfrak{t}: T \longrightarrow K_2^t(F),$$

which in some sense generalize the one dimensional map  $E \setminus \{0\} \longrightarrow E$  on the module, additive and multiplicative level structures respectively. In section 17 we define the main object – a local zeta integral  $\zeta(f, \chi)$  associated to a function  $f: F \times F \longrightarrow \mathbb{C}$  continuous on  $T$  and quasi-character  $\chi: K_2^t(F) \longrightarrow \mathbb{C}^\times$

$$\zeta(f, \chi) = \int_T f(\alpha) \chi_{\mathfrak{t}}(\alpha) d\mu_{F \times F}(\alpha),$$

for the notations see section 17. The definition of the zeta integral is the result of several trials which included global tests. Its several first properties in analogy with the one dimensional case are discussed and proved. A local functional equation for appropriate class of functions is proved in section 19. In the ramified case one meets new difficulties in comparison to the dimension one case, they are discussed in section 21. Zeta integrals for formal power series fields over archimedean local fields are introduced in section 23, their values are not really new in comparison to the dimension one case; this agrees with the well known fact that the class field theory for such fields degenerate and does not really use Milnor  $K_2$ .

2007 Update: For further papers on the measure and integration on two dimensional local fields see [F4], [KL2], [Mo1]–[Mo3], [KL1], [L], [HK1]–[HK2]. For an extension of this work to adelic and  $K$ -delic theory on arithmetic schemes and applications to zeta integral and zeta functions of arithmetic schemes see [F6] and [F5].

## 1. Measure and integration on higher local fields

We start with discussion of self duality of higher local fields which distinguishes them among other higher dimensional local objects. Then we introduce a nontrivial translation invariant measure on higher dimensional local fields, which may be viewed as a higher dimensional generalization of Haar measure on one dimensional fields. We define and study properties of integrals of a certain class of functions on the field. Self duality leads to a higher dimensional transform on appropriate space of functions. One of the key properties, a double transform formula for a class of functions in a space which generalizes familiar one dimensional spaces, is proved in section 9.

For various results about higher local fields see sections of [IHLF].

**1.** In higher dimensions sequential aspects of the topology are fundamental: for example, in dimensions greater than one the multiplication is not always continuous but is sequentially continuous. Certainly, in the one dimensional case one does not see the difference between the two.

In a two dimensional local field  $\mathbb{R}((X))$  a sequence of series  $(\sum a_{i,n} X^i)_n$  is a fundamental sequence if there is  $i_0$  such that for all  $n$  coefficients  $a_{i,n}$  are zero for all  $i < i_0$  and for every  $i$

the sequence  $(a_{i,n})_n$  is a fundamental sequence; similarly one defines convergence of sequences. Every fundamental sequence converges. Now consider on  $\mathbb{R}((X))$  the so called sequential saturation topology (sequential topology for short): a set is open if and only if every sequence which converges to any element of it has almost all its elements in the set.

The sequence  $(aX)^n$ ,  $n \in \mathbb{N}$ , tends to zero for every  $a \in \mathbb{R}$ . Hence, if a norm on  $\mathbb{R}((X))$  takes values in an extension of  $\mathbb{R}$  and is the usual module on the coefficients, then one deduces that  $|aX| < 1$  and so  $|X|$  is smaller than any positive real number. In other words, for two dimensional fields the local parameter  $X$  plays the role of an infinitesimally small element, and therefore it is very natural to use hyperconstructions of nonstandard mathematics. Nevertheless, to simplify the reading the following text does not contain anything nonstandard.

**2.** Let  $F$  be a two dimensional local field with local parameters  $t_1, t_2$  ( $t_2$  is a local parameter with respect to the discrete valuation of rank 1) with finite last residue field having  $q$  elements. Denote its ring of integers by  $O$  and the group of units by  $U$ ; denote by  $R$  the set of multiplicative representatives of the last finite residue field. The multiplicative group  $F^\times$  is the product of infinite cyclic groups generated by  $t_2$  and  $t_1$  and the group of units  $U$ .

Denote by  $\mathcal{O}$  the ring of integers of  $F$  with respect to the discrete valuation of rank one (so  $t_2$  generates the maximal ideal  $\mathcal{M}$  of  $\mathcal{O}$ ). There is a projection  $p: \mathcal{O} \rightarrow \mathcal{O}/\mathcal{M} = E$  where  $E$  is the (first) residue field of  $F$ .

Fractional ideals of  $F$  are of two types: principal  $t_2^i t_1^j O$  and nonprincipal  $\mathcal{M}^i$ .

For the definition of the topology on  $F$  see [sect. 1 part I of IHLF]. We shall view  $F$  endowed with the sequential saturation topology (see [sect. 6 part I of IHLF]), so sequentially continuous maps from  $F$  are the same as continuous maps. If the reader prefers to use the former topology on  $F$ , then the word ‘‘continuous’’ should everywhere below be replaced with ‘‘sequentially continuous’’.

**3.** For a field  $L((t))$  denote by  $\text{res}_i = \text{res}_{t^i}: L((t)) \rightarrow L$  the linear map  $\sum a_j t^j \rightarrow a_i$ . Similarly define  $\text{res} = \text{res}_{-1}: L\{\{t\}\} \rightarrow L$  in the case where  $L$  is a complete discrete valuation field of characteristic zero (for the definition of  $L\{\{t\}\}$  see [Z2]).

For a one dimensional local field  $L$  denote by  $\psi_L$  a character of the additive group  $L$  with conductor  $O_L$ . Introduce

$$\psi' = \psi_L \circ \text{res}_0: L((t)) \rightarrow \mathbb{C}^\times, \quad \psi' = \psi_L \circ (\pi_L^{-1} \text{res}): L\{\{t\}\} \rightarrow \mathbb{C}^\times,$$

where  $\pi_L$  is a prime element of  $L$ . The conductor of  $\psi'$  is the ring  $O$  of the corresponding field: i.e.  $\psi'(O) = 1 \neq \psi'(t_1^{-1}O)$ . An arbitrary two dimensional local field  $F$  of mixed characteristic has a finite extension of type  $L\{\{t\}\}$  which is the compositum of  $F$  and a finite extension of  $\mathbb{Q}_p$  [sect. 3 of Z2], so the restriction of  $\psi'$  on  $F$  has conductor  $O_F$ .

Thus, in each case we have a character

$$\psi_0: F \rightarrow \mathbb{C}_1^\times < \mathbb{C}^\times$$

with conductor  $O_F$ , where  $\mathbb{C}_1^\times$  are complex numbers of module 1.

The additive group  $F$  is algebraically self-dual:

**LEMMA.** *The group of characters on  $F$  consists of characters of the form  $\alpha \mapsto \psi(a\alpha)$  where  $a$  runs through all elements of  $F$ .*

This assertion is easy, at least for the usual topology on  $F$  in the positive characteristic case where  $F$  is dual to appropriately topologized  $\Omega_{F/\mathbb{F}_q}^2$  (c.f. [Y]) which itself is noncanonically dual to  $F$ . Since we are working with a stronger, sequential saturation topology, we indicate a short proof of duality.

*Proof.* Given a character  $\psi'$ , there are  $i, j$  such that  $\psi'(t_2^i t_1^j O) = 1$ , and so we may assume that the conductor of  $\psi'$  is  $O$ . In the equal characteristic case the restriction of  $\psi'$  on  $\mathcal{O}$  induces a character on  $E$  which by the one dimensional theory is a “shift” of  $\psi_E$ , hence there is  $a_0 \in \mathcal{O}$  such that  $\psi_1(\alpha) = \psi'(\alpha)\psi(a_0\alpha)^{-1}$  is trivial on  $\mathcal{O}$ . Similarly by induction, there is  $a_i \in t_2^i \mathcal{O}$  such that  $\psi_{i+1}(\alpha) = \psi_i(\alpha)\psi(a_i\alpha)^{-1}$  is trivial on  $t_2^{-i} \mathcal{O}$ . Then  $\psi'(\alpha) = \psi(a\alpha)$  with  $a = \sum_0^{+\infty} a_i$ .

In the mixed characteristic case it suffices to consider the case of  $\mathbb{Q}_p\{\{t\}\}$ . The restriction of  $\psi'$  on  $t^i \mathbb{Q}_p$  is of the form  $\alpha \mapsto \psi(a_i \alpha)$  with  $a_i \in t^{-i-1} p\mathbb{Z}_p$ , and  $a_i \rightarrow 0$  when  $i \rightarrow +\infty$ ; hence  $\psi'(\alpha) = \psi(a\alpha)$  for  $a = \sum_{-\infty}^{+\infty} a_i$ .

REMARK. Equip the group of characters of  $F$  with the topology of uniform convergence on compact subsets of  $F$ . It is easy to check that for every open subgroup  $U$  and a compact subset  $C$  there is an open subgroup  $V$  such that  $VC \subset U$ , and for every open subgroup  $V$  there is a compact subset  $C$  and an open subgroup  $U$  such that  $\{\alpha : \alpha C \subset U\} \subset V$ . This implies that the map  $a \mapsto (\alpha \mapsto \psi(a\alpha))$  is a homeomorphism between the additive group of  $F$  and the group of its characters.\*

4. To introduce a measure on  $F$  we first specify a nice class of measurable sets.

DEFINITION. A distinguished subset is empty or of the form  $\alpha + t_2^i t_1^j O$ . Denote by  $\mathcal{A}$  the minimal ring in the sense of [H] containing all distinguished sets.

It is easy to see that if the intersection of two distinguished sets is nonempty, then it equals to one of them.

Every element of  $\mathcal{A}$  is a disjoint (maybe infinite countable) union of some distinguished subsets. Distinguished sets are closed but not open. For example, sets  $O \setminus t_2 \mathcal{O} = \bigcup_{j \geq 0} t_1^j U$  and  $\mathcal{O}^\times = \bigcup_{j \in \mathbb{Z}} t_1^j U$  do not belong to  $\mathcal{A}$ .

Alternatively,  $\mathcal{A}$  is the minimal ring which contains sets  $\alpha + t_2^i p^{-1}(S)$ ,  $i \in \mathbb{Z}$ , where  $S$  is a compact open subset of  $E$ .

Every nonempty element of  $\mathcal{A}$  can be written as a disjoint union of sets  $A_n$  each of which is the difference between a distinguished set and a finite disjoint union of distinguished sets. Every such  $A_n$  can be written as  $A'_n \setminus A''_n$  with  $A'_n \supset A''_n$ , where  $A'_n$  is a disjoint union of distinguished sets  $B_m$  and  $A''_n$  is a disjoint union of distinguished sets  $C_l$  which satisfy the property: there is  $i$  such that  $X^{-i} \mu(B_m)$  is real for all  $m$  and  $\mu(C_l) \in X^{i+1} \mathbb{R}[[X]]$  for all  $l$ , i.e. the sets  $C_l$  are at higher levels than the sets  $B_m$  with respect to  $t_2^i \mathcal{O}$ -filtration.

DEFINITION–LEMMA. There is a unique measure  $\mu$  on  $F$  with values in  $\mathbb{R}((X))$  which is a translation invariant finitely additive measure on  $\mathcal{A}$  such that  $\mu(\emptyset) = 0$ ,

$$\mu(t_2^i t_1^j O) = q^{-j} X^i.$$

The proof immediately follows from the properties of the distinguished sets. The measure  $\mu$  depends on the choice of  $t_2$  but not on the choice of  $t_1$ .

We get  $\mu(t_2^i p^{-1}(S)) = X^i \mu_E(S)$  where  $\mu_E$  is the normalized Haar measure on  $E$  such that  $\mu_E(O_E) = 1$ .

\* An addition property is this: choose a one-dimensional local field  $k$  which is a subfield of  $F$  with the induced structure. See [C, sect.6] for the proof that as locally convex  $k$ -spaces,  $F$  is homeomorphic to the space of continuous  $k$ -linear maps from  $F$  to  $k$  endowed with the topology of uniform convergence on compactoid  $o_k$ -submodules.

REMARKS.

1. This measure can be viewed as induced (in appropriate sense) from a measure which takes values in hyperreals  ${}^*\mathbb{R}$ . A field of hyperreal numbers  ${}^*\mathbb{R}$  introduced by Robinson [Ro1] (for a modern exposition see e.g. [Go1]) is a minimal field extension of  $\mathbb{R}$  which contains infinitesimally small elements and on which one has all reasonable analogues of analytic constructions. If one fixes a positive infinitesimal  $\omega^{-1} \in {}^*\mathbb{R}$ , then a surjective homomorphism from the fraction field of approachable polynomials  $\mathbb{R}[X]^{ap}$  to  $\mathbb{R}((X))$ , and  $\omega^{-1} \mapsto X$  determines an isomorphism of a subquotient of  ${}^*\mathbb{R}$  onto  $\mathbb{R}((X))$ .

The first variant of this work employed hyperobjects, and its conceptual value ought to be emphasized.

2. The measure  $\mu$  takes values in  $\mathbb{R}((X))$  (for its topology see section 1) which has the total ordering:  $\sum_{n \geq n_0} a_n X^n > 0$ ,  $a_{n_0} \in \mathbb{R}^\times$ , iff  $a_{n_0} > 0$ . Notice two unusual properties:  $a_n = 1 - q^{-n}$  is smaller than  $1 - X$ , but the limit of  $(a_n)$  is 1; not every subset bounded from below has an infimum. Thus, the standard concepts in real valued (or Banach spaces valued) measure theory, e.g. the outer measure, do not seem to be useful here. In particular, the integral to be defined below will possess some unusual properties like those in section 8.

3. The set  $O \in \mathcal{A}$  of measure 1 is the disjoint union of  $t_2 O \in \mathcal{A}$  of measure  $X$ ,  $t_2 t_1^{-j} O \setminus t_2 t_1^{-j+1} O$  of measure  $q^j(1 - q^{-1})X$  for  $j > 0$ , and  $t_1^l O \setminus t_1^{l+1} O$  of measure  $q^{-l}(1 - q^{-1})$  for  $l \geq 0$ . Since  $\sum_{j>0} q^j$  diverges, the measure  $\mu$  is not  $\sigma$ -additive.

The measure is  $\sigma$ -additive for those sets of countably many disjoint sets  $A_n$  in the ring of sets  $\mathcal{A}$  which "don't accumulate at break points" from one horizontal line  $t_2^m O$  to  $t_2^{m+1} O$ , i.e. for which the series  $\sum \mu(A_n)$  absolutely converges (see section 6).

5. For  $A \in \mathcal{A}$ ,  $\alpha \in F^\times$  one has  $\alpha A \in \mathcal{A}$  and  $\mu(\alpha A) = |\alpha| \mu(A)$ , where  $|\cdot|$  is a two dimensional module:  $|0| = 0$ ,  $|t_2^i t_1^j u| = q^{-j} X^i$  for  $u \in U$ . The module is a generalization of the usual module on locally compact fields. For example, every  $\alpha \in F$  can be written as a convergent series  $\sum \alpha_{i,j}$  with  $\alpha_{i,j} \in t_2^i t_1^j O$  and  $|\alpha_{i,j}| \rightarrow 0$ . This simplifies convergence conditions in previous use, e.g. [Z2].

6. We introduce a space  $R_F$  of complex valued functions on  $F$  and their integrals against the measure  $\mu$ .

DEFINITION. Call a series  $\sum c_n$ ,  $c_n \in \mathbb{C}((X))$ , absolutely convergent if it converges and if  $\sum |\text{res}_{X^i}(c_n)|$  converges for every  $i$ .

For an absolutely convergent series  $\sum c_n$  and every subsequence  $n_m$  the series  $\sum c_{n_m}$  absolutely converges and the limit does not depend on the terms order.

LEMMA. Suppose that a function  $f: F \rightarrow \mathbb{C}$  can be written as  $\sum c_n \text{char}_{A_n}$  with countably many disjoint sets  $A_n$ ,  $c_n \in \mathbb{C}$ , where each  $A_n$  is the difference of a distinguished set and a finite disjoint union of distinguished sets, and  $\text{char}_C$  is the characteristic function of  $C$ . Suppose that the series  $\sum c_n \mu(A_n)$  absolutely converges. If  $f$  has a second presentation of the same type  $\sum d_m \text{char}_{B_m}$ , then  $\sum c_n \mu(A_n) = \sum d_m \mu(B_m)$ .

*Proof.* The proof is reduced to the case of distinguished sets  $A_n, B_m$ . Notice then that if  $\bigcup A_n = \bigcup B_m$  for distinguished disjoint sets, then for every  $n$  either  $A_n$  equals to the union of some of  $B_m$ , or the union of  $A_n$  and possibly several other  $A$ 's equals to one of  $B_m$ . It remains to use the following property: if a distinguished set  $C$  is the disjoint union of distinguished sets  $C_n$  and

$\sum \mu(C_n)$  absolutely converges, then for every  $a \in F$  and integer  $i$  condition  $C \supset a + t_2^i \mathcal{O}$  implies for all  $n$   $C_n \supset a_n + t_2^i \mathcal{O}$  with some  $a_n$ ; therefore  $\mu(C) = \sum \mu(C_n)$ .

DEFINITION. Define  $R_F$  as the vector space generated by functions  $f$  as in the previous lemma and by functions which are zero outside finitely many points. For an  $f$  as in the previous lemma define its integral

$$\int f d\mu = \sum c_n \mu(A_n),$$

and for  $f$  which are zero outside finitely many points define its integral as zero.

The previous definition implies that the integral is an additive function on  $R_F$ .

EXAMPLE. Let a function  $f$  satisfy the following conditions:

there is  $i_0$  such that  $f(\beta) = 0$  for all  $\beta \in t_2^i t_1^j U$ , all  $i \neq i_0$ ;

there is  $k(j)$  such that for every  $\alpha \in t_2^{i_0} t_1^j U$   $f(\alpha) = f(\alpha + \beta)$  for all  $\beta \in t_2^{i_0} t_1^{k(j)} \mathcal{O}$ .

For every  $\theta \in R^\times$  write the set  $\theta t_2^{i_0} t_1^j + t_2^{i_0} t_1^{j+1} \mathcal{O}$  as a disjoint union of finitely many  $c_{\theta,j,l} + t_2^{i_0} t_1^{k(j)} \mathcal{O}$ ,  $l \in L_{\theta,j}$ ; then

$$\int f d\mu = \left( \sum_{j \in \mathbb{Z}} \sum_{\theta \in R^\times} \sum_{l \in L_{\theta,j}} f(c_{\theta,j,l}) q^{-k(j)} \right) X^{i_0}.$$

If the series in the brackets absolutely converges, then  $f \in R_F$ .

Redenote  $f$  as  $f_{i_0}$ . Then  $\sum_{i \geq i_1} f_i$  also belongs to  $R_F$ .

For another example see section 8.

7. Some important for harmonic analysis functions do not belong to  $R_F$ , for example, the function  $\alpha \mapsto \psi(a\alpha) \text{char}_A(\alpha)$  for, say,  $A = \mathcal{O} \in \mathcal{A}$ ,  $a \notin \mathcal{O}$ . In the one dimensional case all such functions do belong to the analogue of  $R_F$ .

Recall that in the one dimensional case the integral over an open compact subgroup of a nontrivial character is zero. This leads to the following natural

DEFINITION. Denote  $\psi(C) = 0$  if  $\psi$  takes more than one value on a distinguished set  $C$  and = the value of  $\psi$  if it is constant on  $C$ . For a distinguished set  $A$  and  $a \in F^\times$  define

$$\int \psi(a\alpha) \text{char}_A(\alpha) d\mu(\alpha) = \mu(A) \psi(aA).$$

Extend the definition of the integral of  $\psi(a\alpha) \text{char}_A(\alpha)$  by linearity to  $A \in \mathcal{A}$ .

If  $A \subset \mathcal{O}$  and is the preimage of a shift of a compact open subgroup of the residue field  $E$  with respect to  $p$ , and if  $a \in \mathcal{O} \setminus t_2 \mathcal{O}$ , then the previous definition agrees with the property of a character on  $E$ , mentioned above.

Correctness of the definition follows from the following

LEMMA. For a function  $f = \sum c_n \psi(a_n \alpha) \text{char}_{A_n}(\alpha)$  with finite set  $\{a_n\}$  and with countably many disjoint sets  $A_n$  such that each  $A_n$  is the difference of a distinguished set and a finite disjoint union of distinguished sets and such that the series  $\sum c_n \mu(A_n)$  absolutely converges, the sum  $\sum c_n \int \psi(a_n \alpha) \text{char}_{A_n}(\alpha) d\mu(\alpha)$  does not depend on the choice of  $c_n, a_n, A_n$ .

*Proof.* To show correctness, one can reduce to sets on which  $|f|$  is constant, then use a simple fact that if a distinguished set  $C$  is the disjoint union of distinguished sets  $C_n$ , and  $\psi(a\alpha) \text{char}_C(\alpha) = \sum d_n \psi(b_n \alpha) \text{char}_{C_n}(\alpha)$  with  $|d_n| = 1$ , absolutely convergent series  $\sum d_n \mu(C_n)$  and finitely many distinct  $b_n$ , then  $\psi(aC) = \sum d_n \psi(b_n C_n) \mu(C_n)$ .

DEFINITION. Put

$$\int f d\mu = \sum c_n \int \psi(a_n \alpha) \text{char}_{A_n}(\alpha) d\mu(\alpha).$$

Denote by  $R'_F$  the space generated by functions  $f(\alpha) \psi(a\alpha)$  with  $f \in R_F$ ,  $a \in F$ , and functions which are zero outside a point; all of them are integrable. This space will be enough for the purposes of this work.

REMARK. Here is a slightly different approach to the extension of  $R_F$ : Suppose that a function  $f: F \rightarrow \mathbb{C}$  is zero outside a distinguished subgroup  $A$  of  $F$ . Suppose that there are finitely many  $a_1, \dots, a_m \in F$  such that the function  $g(x) = \sum_i f(a_i + x)$  belongs to  $R_F$ . Then *define*

$$\int f = \frac{1}{m} \int g d\mu.$$

First of all, this is well defined: if  $h(x) = \sum_j f(b_j + x)$  belongs to  $R_F$  for  $b_1, \dots, b_n \in A$ , then  $\sum_{i=1}^m h(a_i + x) = \sum_{j=1}^n g(b_j + x)$ , and from  $h(a_i + x) \in R_F$  and the translation invariant property, and the similar property for shifts of  $g$  one gets  $m \int h d\mu = n \int g d\mu$ .

Second, for two functions  $f_1, f_2: A \rightarrow \mathbb{C}$  if  $\sum_i f_1(a_i + x), \sum_j f_2(b_j + x) \in R_F$ , then  $\sum_{i,j} f_l(a_i + b_j + x) \in R_F$  for  $l = 1, 2$  and so  $\int f_1 + f_2 = \int f_1 + \int f_2$ . Denote by  $R''_F$  the space of all such  $f$ .

Third, this definition is compatible with all the properties of  $R_F$  in the previous section: if  $f$  already belongs to  $R_F$  then  $\int f = \int f d\mu$ . Also,  $\int f(x) = \int f(a + x)$  for  $a \in A$ .

Finally, this definition is compatible with the preceding definitions of this section: of course, for nontrivial characters on a distinguished subgroup one has  $g = 0$ . The space  $R'_F$  is a subspace of  $R''_F$ : for a function  $f = \sum c_n \psi(a\alpha) \text{char}_{A_n}(\alpha)$ , such that  $\psi(aA_n) = 0$ , i.e.  $A_n + a^{-1}t_1^{-1}O = A_n$ , for every  $n$ , choose  $a_i$  in  $a^{-1}t_1^{-1}O$ .

For  $f \in R'_F$  we have

$$\int f(\alpha + a) d\mu(\alpha) = \int f(\alpha) d\mu(\alpha)$$

and using section 5

$$\int f(\alpha) d\mu(\alpha) = |a| \int f(a\alpha) d\mu(\alpha).$$

For a subset  $S$  of  $F$  put  $\int_S f d\mu = \int f \text{char}_S d\mu$ .

EXAMPLE. We have  $\int_{t_2^i t_1^j O} \psi(a\alpha) d\mu(\alpha) = q^{-j} X^i$  if  $a \in t_2^{-i} t_1^{-j} O$  and  $= 0$  otherwise (since then  $\psi(a\alpha)$  is a nontrivial character on  $t_2^i t_1^j O$ ). Hence

$$\int_{t_2^i t_1^j U} \psi(a\alpha) d\mu(\alpha) = \begin{cases} 0 & \text{if } a \notin t_2^{-i} t_1^{-j-1} O, \\ -q^{-1-j} X^i & \text{if } a \in t_2^{-i} t_1^{-j-1} U, \\ q^{-j} (1 - q^{-1}) X^i & \text{if } a \in t_2^{-i} t_1^{-j} O. \end{cases}$$

**8. EXAMPLES.** First,  $\int \text{char}_{t_2^m \mathcal{O}} d\mu = 0$  for every  $m$ . Indeed, as in Lemma 6, write the function  $\text{char}_{t_2^m \mathcal{O}}$  as  $\text{char}_{t_2^{m-1} \mathcal{O}} - \sum_{n \geq 0} \text{char}_{t_2^{m-1} t_1^n \mathcal{O}^\times}$ , with  $\sum_{n \geq 0} \mu(t_2^{m-1} t_1^n \mathcal{O}^\times)$  absolutely convergent and equal to  $X^{m-1}$ . Then  $\int \text{char}_{t_2^m \mathcal{O}} d\mu = X^{m-1} - X^{m-1} = 0$ .

Second, if two functions  $f, h: \mathcal{O} \rightarrow \mathbb{C}$  are constant on  $t_2 \mathcal{O} \setminus \{0\}$  and their restriction to  $\mathcal{O} \setminus t_2 \mathcal{O}$  coincide, then

$$\int_{\mathcal{O}} f d\mu = \int_{\mathcal{O}} h d\mu.$$

Indeed, let  $(f - h)|_{t_2 \mathcal{O} \setminus \{0\}} = c$ . Then

$$\int_{\mathcal{O}} (f - h) d\mu = \int_{t_2 \mathcal{O}} c d\mu + \int_{\mathcal{O} \setminus t_2 \mathcal{O}} (f - h) d\mu = 0.$$

Note that the sets  $\mathcal{O}, \mathcal{O} \setminus t_2 \mathcal{O}$  are not in  $\mathcal{A}$ .

REMARK. One has  $\int_{Z_l} d\mu = \sum_{-l \leq j \leq l} q^{-j}$  where  $Z_l = \bigcup_{-l \leq j \leq l} t_1^j U$ , and  $\mathcal{O} \setminus t_2 \mathcal{O}$  is the “limit” of  $Z_l$  when  $l \rightarrow \infty$ . Compare with “equality”  $\sum_{n \in \mathbb{Z}} z^n = 0$  used by L. Euler. One of interpretations of it is to view  $z$  as a complex variable, then for every integer  $m$  the sum of analytic continuations of two rational functions  $\sum_{n \in \mathbb{Z}, n < m} z^n$  and  $\sum_{n \in \mathbb{Z}, n \geq m} z^n$  is zero. Euler’s equality can be applied to show equivalence of the Riemann–Roch theorem and the functional equation of the zeta function of a one dimensional global field of positive characteristic (cf. [Rq, sect. 4.3.3]).

From the definitions we immediately get

LEMMA. Suppose that  $g: E \rightarrow \mathbb{C}$  is (absolutely) integrable over  $E$  with respect to the normalized Haar measure  $\mu_E$  as in section 4. Then the function  $g \circ p$  extended by zero outside  $\mathcal{O}$  is in  $R_F$  and

$$\int_{\mathcal{O}} g \circ p d\mu = \int_E g d\mu_E.$$

One can say that the Haar measure  $\mu_E$  equals  $p_*(\mu')$  where the measure  $\mu'$  on  $\mathcal{O}$  is induced by  $\mu$  by extending functions on  $\mathcal{O}$  by zero to  $F$ .

**9. DEFINITION.** For  $f \in R_F$  introduce the transform function

$$\mathcal{F}(f)(\beta) = \int_F f(\alpha) \psi(\alpha\beta) d\mu(\alpha).$$

In particular,  $\mathcal{F}(\text{char}_{\mathcal{O}}) = \text{char}_{\mathcal{O}}$ .

DEFINITION. Denote by  $Q_F$  the subspace of  $R_F$  consisting of functions  $f$  with support in  $\mathcal{O}$  and such that  $f|_{\mathcal{O}} = g \circ p|_{\mathcal{O}}$  for a function  $g: E \rightarrow \mathbb{C}$  which belongs to the Bruhat space generated by characteristic functions of shifts of proper fractional ideals of  $E$ .

THEOREM. Given  $f \in Q_F$ , the function  $\mathcal{F}(f)$  belongs to  $Q_F$  and we have a double transform formula

$$\mathcal{F}^2(f)(\alpha) = f(-\alpha).$$

*Proof.* First note that  $\psi(\alpha) = \psi_E \circ p(\alpha)$  for all  $\alpha \in \mathcal{O}$  where  $\psi_E$  is an appropriate character on  $E$  with conductor  $O_E$ .

Using sections 7 and 8 we shall verify that



if  $\beta \notin \mathcal{O}$  then  $\mathcal{F}(f)(\beta) = 0$ ;

if  $\beta \in \mathcal{O}$  then  $\mathcal{F}(f)(\beta) = \mathcal{F}(g) \circ p(\beta)$  (where  $\mathcal{F}(g)$  denotes the Fourier transform of  $g$  with respect to  $\psi_E$  and  $\mu_E$ ).

For  $\beta \notin \mathcal{O}$  the definitions imply

$$\mathcal{F}(f)(\beta) = \int_{\mathcal{O}} f(\alpha) \psi(\alpha\beta) d\mu(\alpha) = 0.$$

For  $\beta \in \mathcal{O} \setminus \mathcal{O}^\times$

$$\begin{aligned} \mathcal{F}(f)(\beta) &= \int_{\mathcal{O}} f(\alpha) d\mu(\alpha) = \int_{\mathcal{O}} g \circ p(\alpha) d\mu(\alpha) \\ &= \int_E g(\bar{\alpha}) d\mu_E(\bar{\alpha}) = \mathcal{F}(g)(0) = \mathcal{F}(f)(0). \end{aligned}$$

For  $\beta \in \mathcal{O}^\times$

$$\begin{aligned} \mathcal{F}(f)(\beta) &= \int_F f(\alpha) \psi(\alpha\beta) d\mu(\alpha) = \int_{\mathcal{O}} f(\alpha) \psi(\alpha\beta) d\mu(\alpha) \\ &= \int_{\mathcal{O}} g \circ p(\alpha) \psi_E \circ p(\alpha\beta) d\mu(\alpha) = \int_E g(\bar{\alpha}) \psi_E(\bar{\alpha}\bar{\beta}) d\mu_E(\bar{\alpha}) = \mathcal{F}(g)(p(\beta)). \end{aligned}$$

It remains to use the one dimensional double transform formula.

As a corollary, we deduce that if a function  $f: F \rightarrow \mathbb{C}$  with support in  $\mathcal{O}^\times$  coincides on  $\mathcal{O}^\times$  with a function  $h \in Q_F$ , then

$$\mathcal{F}^2(f)(\alpha) = \begin{cases} f(-\alpha) & \text{if } \alpha \in \mathcal{O}^\times \cup (F \setminus \mathcal{O}), \\ h(0) & \text{if } \alpha \in \mathcal{O} \setminus \mathcal{O}^\times. \end{cases}$$

REMARK. It is more natural to integrate  $\mathbb{C}((X))$ -valued functions on  $F$ : for a function

$$f = \sum_{i \geq i_0} f_i X^i: F \rightarrow \mathbb{C}((X)), \quad f_i \in R'_F$$

define

$$\int_F f d\mu = \sum X^i \int_F f_i d\mu.$$

Similarly to the previous text one checks the correctness of the definition and properties of the integral. In particular, the previous theorem remains true for functions  $f = \sum_{i \geq i_0} f_i X^i$ ,  $f_i$  in the space  $Q'_F = \{\alpha \mapsto \sum_{n \geq n_0} g_n(t_2^{-n} \alpha)\}$  where  $g_n$  belong to  $Q_F$  (or even just "lifts" of functions on  $E$  for which the double transform formula holds).

**10.** Introduce the product measure  $\mu_{F \times F} = \mu_F \otimes \mu_F$  on  $F \times F$  and define spaces  $R_{F \times F} = R_F \otimes R_F$ ,  $R'_{F \times F} = R'_F \otimes R'_F$  and  $Q_{F \times F} = Q_F \otimes Q_F$ .

For a function  $f \in Q_{F \times F}$  define its transform

$$\mathcal{F}(f)(\beta_1, \beta_2) = \int_{F \times F} f(\alpha_1, \alpha_2) \psi(\alpha_1\beta_1 + \alpha_2\beta_2) d\mu(\alpha_1) d\mu(\alpha_2).$$

From section 9 we deduce  $\mathcal{F}^2(f)(\alpha_1, \alpha_2) = f(-\alpha_1, -\alpha_2)$  for  $f \in Q_{F \times F}$ . In the case of a character  $\psi$  with conductor different from  $\mathcal{O}$  to get the double transform formula as above we have to choose a self-dual measure  $\mu$  on  $\mathcal{O}$ , see 20.

**11.** In the case of two dimensional local fields of type  $F = K((t))$ , where  $K$  is an archimedean local field,  $O, U$  are not defined. Define a character  $\psi: K((t)) \rightarrow \mathbb{C}^\times$  to be the composite of the res $_0: K((t)) \rightarrow K$  and the archimedean character  $\psi_K(\alpha) = \exp(2\pi i \operatorname{Tr}_{K/\mathbb{R}}(\alpha))$  on  $K$  (for the character  $\psi_K$  it is classical to take  $\exp(-2\pi i \operatorname{Tr}_{K/\mathbb{R}}(\alpha))$  which we will do in the adelic theory). The role of distinguished sets is played by  $B = a + t^i D + t^{i+1} K[[t]]$  where  $D$  is an open ball in  $K$ . In this case if the intersection of two distinguished sets is nonempty then it equals either to one of them, or to a smaller distinguished set. The measure is a translation invariant additive measure  $\mu$  on the ring generated by distinguished sets and such that  $\mu(B) = \mu_K(D)X^i$ . Here, as in the one dimensional theory [T1],  $\mu_K$  is the ordinary Lebesgue measure on  $K$  if  $K$  is real, and is twice the ordinary Lebesgue measure on  $K$  if  $K$  is complex. The measure  $\mu$  can be extended via  $\mu(a + t^i C + t^{i+1} K[[t]]) = \mu_K(C)X^i$ , where  $C \subset K$  is a Lebesgue measurable set. In particular,  $\mu(t^{i+1} K[[t]]) = 0$ .

The module is  $|\sum_{i \geq i_0} a_i t^i| = |a_{i_0}|_K X^{i_0}$ , where the module on the real  $K$  is the absolute value and on the complex  $K$  is the square of the absolute value.

In the global two dimensional theory it will be sufficient to use the one dimensional theory for archimedean local fields. However, we briefly sketch the analogues of the previous constructions as part of the two dimensional local theory.

The definitions of spaces  $R_F$  and  $R'_F$  follow the general pattern of sections 6 and 7: for disjoint sets  $A_n$  where each  $A_n$  is the difference of a distinguished set and a disjoint union of distinguished sets and such that  $\sum c_n \mu(A_n)$  absolutely converges put

$$\int \sum c_n \operatorname{char}_{A_n} d\mu = \sum c_n \mu(A_n).$$

For a function  $f = \sum c_n \psi(a_n \alpha) \operatorname{char}_{A_n}(\alpha)$  with finite set  $\{a_n\}$  and with countably many disjoint sets  $A_n$  each of which is the difference of a distinguished set and a finite disjoint union of distinguished sets and such that the series  $\sum c_n \mu(A_n)$  absolutely converges put

$$\int f d\mu = \sum c_n \int_{A_n} \psi(a_n \alpha) d\mu(\alpha),$$

where

$$\int_B \psi(c\alpha) d\mu(\alpha) = \psi(ca) X^i \int_D \psi_K(c_{-i}\beta) d\mu_K(\beta), \quad c_{-i} = \operatorname{res}_{t^{-i}} c,$$

$B, a, D, i$  are defined at the beginning of this section. Then we extend this definition to sets of type  $A$ .

The analogues of the definitions and results of sections 8 and 9 hold. The transform of a function  $f \in R_F$  is defined by the same formula

$$\mathcal{F}(f)(\beta) = \int_F f(\alpha) \psi(\alpha\beta) d\mu(\alpha).$$

The space  $Q_F$  consists of functions  $f \in R_F$  such that  $f = g \circ p$  for a function  $g$  in the Schwartz space on  $K$ , where  $p = \operatorname{res}_{t^0}$ . The double transform formula is  $\mathcal{F}^2(f)(\alpha) = f(-\alpha)$  for  $f \in Q_F$ .

**REMARK.** It is interesting to investigate if further extensions of this sort of measure theory and harmonic analysis (they seem to be more powerful than applications of Wiener's measure) would be useful for mathematical description of Feynman integrals and integration on loop spaces.

**12.** In general, for the global theory, we need to work with normalized measures corresponding to shifted characters. Similar to the one dimensional case, if we start with a character  $\psi$  with conductor  $aO$  then the dual to  $\mu$  measure  $\mu'$  on  $F$  is normalized such that  $|a|\mu(O)\mu'(O) = 1$ . The double transform formula of section 9 holds.

For example, if  $\mu' = \mu$  then  $\mu(O) = |a|^{-1/2}$ . If  $aO = t^iO$  then in the last formula of section 8 one should insert the coefficient  $X^{-i/2}$  on the right hand side. If  $f_0(\alpha) = f(a\alpha)$  belongs to  $Q_F$  or  $Q'_F$  then  $\mathcal{F}(f) = |a|^{1/2} \mathcal{F}_0(f_0)$  where  $\mathcal{F}_0$  is the transform with respect to character  $\psi_0(\alpha) = \psi(a\alpha)$  of conductor  $O$  and measure  $\mu_0$  such that  $\mu_0(O) = 1$ . In particular,  $\mathcal{F}(\text{char}_O) = |a|^{-1/2} \text{char}_{aO}$ .

**13.** Generally, given an integral domain  $A$  with principal ideal  $P = tA$  and projection  $A \rightarrow A/P = B$  and an  $\mathbb{R}$ -valued translation invariant measure on  $B$ , similar to the previous one defines a measure and integration on  $A$ . For example, the analogue of the ring  $\mathcal{A}$  is the minimal ring which contains sets  $\alpha + t^i p^{-1}(S)$ , where  $S$  is from a class of measurable subsets of  $B$ , its measure is by definition  $X^i \mu_B(S)$ ; the space of integrable functions is generated by functions  $\alpha \rightarrow g \circ p(t^{-i} \alpha)$  extended by zero outside  $t^i A$ , where  $g$  is an integrable function on  $B$ . Such a  $\mathbb{R}((X))$ -valued measure and integration on  $A$  is natural to call a *lift* of the corresponding measure and integration on the base  $B$ .

In particular, if  $B = B_1 \times B_2$  and the translation invariant measures are chosen such that  $\mu_B = \mu_{B_1} \otimes \mu_{B_2}$ , then for their lifts  $\mu, \mu_1, \mu_2$  we get  $\mu_1(p^{-1}(X_1 \times X_2)) = \mu_1(p^{-1}(X_1)) \mu_2(p^{-1}(X_2))$ , where  $X_1, X_2$  are measurable sets with respect to  $\mu_1, \mu_2$ .

REMARKS.

1. Satoh [S] and Kapranov [Kp] suggested two other very different approaches to define a measure on two dimensional fields (notice that Satoh's work was an attempt to use Wiener's measure). The work [Kp] does not introduce a measure, and deal with distributions and functions; local zeta integrals are not discussed.

2. In this work we do not need a general theory of analysis on non locally compact abelian groups, since the case of higher dimensional local fields and natural adelic objects composed of them is enough for applications in number theory. Undoubtedly, there is a more general theory of harmonic analysis on non locally compact groups than presented above, see also the next remark.

3. The existence of Haar measure on locally compact abelian groups can be proved in the shortest and most elegant conceptual way by viewing it as induced from the counting measure on a covering hyperfinite abelian group (e.g. [Gor1]); the same is also true for the Fourier transform [Gor2].

Conceptually, the integration theory of this work is supposed to be induced by integration on hyper locally compact abelian groups (or hyper hyper finite groups). This would also give an extension of this theory to a more general class of non locally compact groups than those of this part.

## 2. Dimension two local zeta integral

Using the theory of the previous part we explain how to integrate over topological  $K$ -groups of higher local fields, define a local zeta integral and discuss its properties and new phenomena of the dimension two situation. Using a covering of a so called topological Milnor  $K_2^t$ -group of a two dimensional local field (which is the central object in explicit local class field theory) by the product of the group of units with itself we introduce integrals over the  $K_2^t$ -group. For this we use additional maps  $\tau, \mathfrak{t}, \mathfrak{o}$  whose meaning can only be finally clarified in constructions of the global part [F5]. In section 17 we define the main object – a higher dimensional local zeta integral associated to a

function on the field and a character of the  $K_2^t$ -group, followed by concrete examples. Several first properties of the zeta integral, in analogy with the one dimensional case, are discussed and proved. In sections 22 and 23 we consider the case of two dimensional local fields whose residue field is an archimedean local field.

**14.** On  $F^\times$  one has the induced from  $F \oplus F$  topology with respect to

$$F^\times \longrightarrow F \oplus F, \quad \alpha \mapsto (\alpha, \alpha^{-1}),$$

and the translation invariant measure

$$\mu_{F^\times} = (1 - q^{-1})^{-1} \mu / | \cdot |.$$

Explicit higher class field theory describes abelian extensions of  $F$  via closed subgroups in the topological Milnor  $K$ -group  $K_2^t(F) = K_2(F)/\Lambda_2(F)$  where  $\Lambda_2(F)$  is the intersection of all open neighbourhoods of zero in the strongest topology on  $K_2(F)$  in which the subtraction in  $K_2(F)$  and the map

$$F^\times \times F^\times \longrightarrow K_2(F), \quad (a, b) \mapsto \{a, b\}$$

are sequentially continuous. For more details see [IHLF, sect. 6 part I] and [F3]. Denote by  $UK_2^t(F)$  the subgroup generated by symbols  $\{u, \alpha\}$ ,  $u \in U, \alpha \in F^\times$ .

**15.** Introduce a module on  $K_2^t(F)$ :

$$| \cdot |_2: K_2^t(F) \longrightarrow \mathbb{R}_+^\times, \quad \alpha \mapsto q^{-v(\alpha)},$$

where  $v: K_2^t(F) \longrightarrow K_0(\mathbb{F}_q) = \mathbb{Z}$  is the composite of two boundary homomorphisms.

DEFINITION. Introduce a subgroup

$$T = \mathcal{O}^\times \times \mathcal{O}^\times = \{(t_1^j u_1, t_1^l u_2) : j, l \in \mathbb{Z}, u_i \in U\}$$

of  $F^\times \times F^\times$ . The closure of  $T$  in  $F \times F$  equals  $\mathcal{O} \times \mathcal{O}$ .

DEFINITION. Introduce a map (morphism of multiplicative structures)

$$\sigma': T \longrightarrow F \times F, \quad (t_1^j u_1, t_1^l u_2) \mapsto (t_1^{2j} u_1, t_1^{2l} u_2)$$

and denote by  $\sigma$  the inverse bijection  $\sigma'(T) \longrightarrow T$ .

For a complex valued continuous function  $f \in R_F$  whose domain includes  $T$  and is a subset of  $F \times F$  form the composite  $f \circ \sigma: \sigma'(T) \longrightarrow \mathbb{C}$  and extend it by zero outside  $\sigma'(T)$ , denote the result by  $f_\sigma: F \times F \rightarrow \mathbb{C}$ .

To be able to apply the transform  $\mathcal{F}$  introduce an extension  $e(g)$  of a function  $g = f_\sigma: F \times F \longrightarrow \mathbb{C}$  as the continuous extension on  $F \times F$  of the function

$$e(g)(\alpha_1, \alpha_2) = g(\alpha_1, \alpha_2) + \sum_{1 \leq i \leq 3} g((\alpha_1, \alpha_2) \nu_i), \quad (\alpha_1, \alpha_2) \in F^\times \times F^\times,$$

where  $\nu_1 = (t_1^{-1}, t_1^{-1})$ ,  $\nu_2 = (t_1^{-1}, 1)$ ,  $\nu_3 = (1, t_1^{-1})$ .

For example, if  $f = \text{char}_{(t_1^j \mathcal{O}, t_1^l \mathcal{O})}$  then  $f_\sigma(\alpha_1, \alpha_2) = 1$  for  $(\alpha_1, \alpha_2) \in F \times F$  only if  $\alpha_1 \in t_1^{2k} U, \alpha_2 \in t_1^{2m} U$  for  $k \geq j, m \geq l$ . In particular,  $f_\sigma$  is not a continuous function. However  $e(f_\sigma) = \text{char}_{(t_1^{2j} \mathcal{O}, t_1^{2l} \mathcal{O})}$  is a continuous function.

DEFINITION. For a function  $f: T \rightarrow \mathbb{C}$  such that  $e(f_\circ) \in R_{F \times F}$  define its transform

$$\widehat{f} = \mathcal{F}(e(f_\circ)) \circ \circ': T \rightarrow \mathbb{C}.$$

16. Due to the well known useful equality

$$\{1 - \alpha, 1 - \beta\} = \{\alpha, 1 + \alpha\beta/(1 - \alpha)\} + \{1 - \beta, 1 + \alpha\beta/(1 - \alpha)\},$$

and a topological argument we have surjective maps ([IHLF, sect. 6 part I]):

$$\begin{aligned} \mathfrak{r}: T &\rightarrow K_2^t(F), & \mathfrak{r}((t_1^j, t_1^l)(u_1, u_2)) &= \min(j, l) \{t_1, t_2\} + \{t_1, u_1\} + \{u_2, t_2\}, \\ \mathfrak{t}: T &\rightarrow K_2^t(F), & \mathfrak{t}((t_1^j, t_1^l)(u_1, u_2)) &= (j + l) \{t_1, t_2\} + \{t_1, u_1\} + \{u_2, t_2\}. \end{aligned}$$

For  $(\alpha_1, \alpha_2) \in T$  we have

$$|\mathfrak{t}(\alpha_1, \alpha_2)|_2 = |\alpha_1| |\alpha_2| = |(\alpha_1, \alpha_2)|.$$

The homomorphism  $\mathfrak{t}$  plays an important role in the study of topological Milnor  $K$ -groups of higher local fields ([P3], [F3]). If we denote  $H = \{(\alpha, \beta) : \alpha, \beta \in \langle t_1 \rangle U, \alpha\beta^{-1} \in U\}$ , then  $\mathfrak{r}(H) = K_2^t(F)$ . Note that if  $K_2^t(F)$  is replaced by the usual  $K_2(F)$  then none of the previous maps is surjective. The topology of  $K_2^t(F)$  is the quotient topology of the sequential saturation of the multiplicative topology on  $H$  (cf. [F3], [IHLF, sect. 6 part I]).

The maps  $\mathfrak{r}, \mathfrak{t}$  depend on the choice of  $t_1, t_2$  but the induced map to  $K_2^t(F)/\ker(\partial)$  (see section 18 for the definition) does not depend on the choice of  $t_2$ . Define a map  $\mathcal{O}^\times \times F^\times / \mathcal{O}^\times \rightarrow \mathcal{O}^\times \times \mathcal{O}^\times / \mathcal{O}^\times$  as the identity map on the first component and  $t_2^i \rightarrow t_1^i$  on the second. Then the composition of this map with the map  $\mathcal{O}^\times \times \mathcal{O}^\times / \mathcal{O}^\times \rightarrow K^2(F)/\ker(\partial)$  induced by  $\mathfrak{t}$  equals the map  $\mathcal{O}^\times \times F^\times / \mathcal{O}^\times \rightarrow K^2(F)/\ker(\partial)$  induced by the symbol map.

The homomorphism  $\mathfrak{t}$  (which depends on the choice of  $t_1, t_2$ ) is much more convenient to use for integration on  $K_2^t(F)$  than the canonically defined map  $F^\times \times F^\times \rightarrow K_2^t(F)$ .

For a function  $h: K_2^t(F) \rightarrow \mathbb{C}$  denote by  $h_\tau: F \times F \rightarrow \mathbb{C}$  ( $h_t: F \times F \rightarrow \mathbb{C}$ ) the composite  $h \circ \mathfrak{r}$  (resp. the composite  $h \circ \mathfrak{t}$ ) extended by zero outside  $T$ .

17. Let  $\chi: K_2^t(F) \rightarrow \mathbb{C}^\times$  be a continuous homomorphism. Similarly to the one dimensional case one easily proves that  $\chi = \chi_0 | \cdot |_2^s$  ( $s \in \mathbb{C}$ ), where  $\chi_0$  is a lift of a character of  $UK_2^t(F)$  such that  $\chi_0(\{t_1, t_2\}) = 1$ . The group  $V = 1 + t_1\mathcal{O}$  of principal units has the property: every open subgroup contains  $V^{p^n}$  for sufficiently large  $n$ , c.f. [Z1, Lemma 1.6]. This property and the definitions imply that  $|\chi_0(K_2^t(F))| = 1$ . Put  $s = s(\chi)$ .

DEFINITION. For a function  $f$  such that  $\text{char}_{\circ'(T)} f_\circ \in R_{F \times F}$  define

$$\int_T f d\mu_T := (1 - q^{-1})^{-2} \int_{\circ'(T)} f_\circ d\mu_{F \times F} = (1 - q^{-1})^{-2} \int_{\circ'(T)} e(f_\circ) d\mu_{F \times F}.$$

The definitions imply that

$$\int_T f d\mu_T = (1 - q^{-1})^{-2} \int_{F \times F} \text{char}_T f | \cdot | d\mu_{F \times F} = \int_T f | \cdot |^2 d\mu_{F^\times \times F^\times}.$$

For a function  $f: F \times F \rightarrow \mathbb{C}$  continuous on  $T$  and a quasi-character  $\chi: K_2^t(F) \rightarrow \mathbb{C}^\times$  introduce a zeta integral

$$\zeta(f, \chi) = \int_T f(\alpha) \chi_{\mathfrak{t}}(\alpha) |\mathfrak{t}(\alpha)|_2^{-2} d\mu_T(\alpha) = \int_{\circ'(T)} f_\circ \chi_{\mathfrak{t}_\circ} d\mu_{F^\times \times F^\times} = \int_T f \chi_{\mathfrak{t}} d\mu_{F^\times \times F^\times}$$

as an element of  $\mathbb{C}((X))$  if the integral converges.

From the definitions we obtain

$$\zeta(f, \chi) = (1 - q^{-1})^{-2} \sum_{j, l \in \mathbb{Z}} (q^{-s})^{j+l} \int_{U \times U} f(t_1^j u_1, t_1^l u_2) \chi_0(\mathfrak{t}(u_1, u_2)) d\mu(u_1) d\mu(u_2).$$

If for fixed  $j, l$  the value  $f(t_1^j u_1, t_1^l u_2)$  is constant  $= f_0(j, l)$ , then we get

$$\zeta(f, \chi) = (1 - q^{-1})^{-2} \sum_{j, l \in \mathbb{Z}} (q^{-s})^{j+l} f_0(j, l) \int_{U \times U} \chi_0(\mathfrak{t}(u_1, u_2)) d\mu(u_1) d\mu(u_2).$$

EXAMPLES.

1. Let  $f = \text{char}_{OK_2^t(F)_\mathfrak{v}}$  where  $OK_2^t(F) = v^{-1}(\{0, 1, \dots\})$ . So  $f_0(j, l) = 1$  if  $j, l \geq 0$  and  $f_0(j, l) = 0$  otherwise. Then

$$\zeta(f, \chi | \mathbb{Z}_2^s) = \left( \frac{1}{1 - q^{-s}} \right)^2$$

which converges for  $\text{Re}(s) > 0$ , and has a single valued meromorphic continuation to the whole complex plane. Note that for this specific choice of  $f$  the local zeta integral *does not involve*  $X$ .

2. More generally, for a function  $f \in Q_{F \times F}$  the local zeta integral  $\zeta(f, \chi)$  converges for  $\text{Re}(s) > 0$  to a rational function in  $q^s$  and therefore has a meromorphic continuation to the whole complex plane.

To show this, one can assume that  $f(\alpha_1, \alpha_2) = \text{char}_{A_1}(\alpha_1) \text{char}_{A_2}(\alpha_2)$  with distinguished sets  $A_1, A_2$ , each of which is either of type  $t_1^l(a + t_1^r O)$ ,  $a \in U$ ,  $r > 0$ , or of type  $t_1^m O$ . For example, if  $A_1$  of the first type and  $A_2$  of the second type, then

$$\zeta(f, \chi) = (1 - q^{-1})^{-2} \sum_{j \geq m} (q^{-s})^{j+l} \int_{(a+t_1^r O) \times U} \chi_0(\mathfrak{t}(u_1, u_2)) d\mu(u_1) d\mu(u_2),$$

which for  $\text{Re}(s) > 0$  converges to  $(1 - q^{-1})^{-2} (q^{-s})^{m+l} (1 - q^{-s})^{-1} c$  where

$c = \int_{(a+t_1^r O) \times U} \chi_0(\mathfrak{t}(u_1, u_2)) d\mu(u_1) d\mu(u_2)$ , and therefore equals to this rational function of  $q^s$  on the whole plane.

3. For  $f \in Q_{F \times F}$  and  $(\alpha_1, \alpha_2) \in T$  define, in analogy with A. Weil's definition in [W1]

$$W(f)(\alpha_1, \alpha_2) = f(\alpha_1, \alpha_2) - f(t_1^{-1} \alpha_1, \alpha_2) - f(\alpha_1, t_1^{-1} \alpha_2) + f(t_1^{-1} \alpha_1, t_1^{-1} \alpha_2).$$

Then from the definitions and the properties of  $\mu$  we get

$$\zeta(W(f), \chi | \mathbb{Z}_2^s) = (1 - q^{-s})^2 \zeta(f, \chi | \mathbb{Z}_2^s).$$

In particular, let  $g = W(f)$ ,  $f|_T = (\text{char}_{OK_2^t(F)})_\mathfrak{v}$ . Then

$$\zeta(g, \chi | \mathbb{Z}_2^s) = 1.$$

**18.** Let  $\partial: K_2(F) \rightarrow K_1(E)$  be the boundary homomorphism. Note that  $\partial(\Lambda_2(F)) = 1$ . Define

$$\lambda: K_1(E) \rightarrow K_2^t(F)/\ker(\partial), \quad \alpha \mapsto \{\tilde{\alpha}, t_2\}$$

where  $\tilde{\alpha} \in \mathcal{O}$  is a lifting of  $\alpha \in E^\times$ . Then  $\partial(\lambda(\alpha)) = \alpha$ .

Let  $f: K_2^t(F) \rightarrow \mathbb{C}$  be a continuous function which factorizes through  $K_2^t(F)/\ker(\partial)$ . Let  $\chi: K_2^t(F) \rightarrow \mathbb{C}^\times$  be a weakly ramified continuous quasi-character, i.e.  $\chi_0(\ker(\partial)) = 1$ , so  $\chi$  is

induced by a (not necessarily unramified) quasi-character of the first residue field  $E$  of  $F$ . Then using Lemma in section 8 one immediately deduces that

$$\zeta(f_\tau, \chi) = \frac{1 + q^{-s}}{1 - q^{-s}} \zeta_E(f \circ \lambda, | \cdot |_1^{2s} \chi_0 \circ \lambda)$$

where  $\zeta_E$  is the one dimensional zeta integral on  $E$  which corresponds to the normalized Haar measure on  $E$ .

Similarly, if  $F$  is a mixed characteristic field, and  $K$  is the algebraic closure of  $\mathbb{Q}_p$  in  $F$ , then one has a residue map  $\mathfrak{d}: K_2^t(F) \rightarrow K_1(K)$ , see e.g. [Ka2, sect. 2], the map  $\mathfrak{d}$  is  $-res$  defined there.

Assume that a prime element of  $K$  is a  $t_2$ -local parameter of  $F$ , then we can identify  $F = K\{\{t_1\}\}$ . Define  $\mathfrak{l}: K_1(K) \rightarrow K_2^t(F)$ ,  $\alpha \mapsto \{t_1, \alpha\}$ . Then  $\mathfrak{d}(\mathfrak{l}(\alpha)) = \alpha$ .

Let  $f: K_2^t(F) \rightarrow \mathbb{C}$  and a quasi-character  $\chi$  have analogous to the above properties but with respect to  $\mathfrak{d}$ . Then similarly one has

$$\zeta(f_\tau, \chi) = \frac{1 + q^{-s}}{1 - q^{-s}} \zeta_K(f \circ \mathfrak{l}, | \cdot |_1^{2s} \chi_0 \circ \mathfrak{l}).$$

Thus, the two dimensional zeta integral for  $F$  links zeta integrals for both  $E$  and  $K$ , local fields of characteristic  $p$  and  $0$ .

**19.** Put  $\widehat{\chi} = \chi^{-1} | \cdot |_2^2$ .

PROPOSITION. Let  $g, h \in Q_{F \times F}$  be continuous functions, and let  $\chi: K_2^t(F) \rightarrow \mathbb{C}^\times$  be a quasi-character. For  $0 < \operatorname{Re} s(\chi) < 2$  (and therefore for all  $s$ ) one has a local functional equation

$$\zeta(g, \chi) \zeta(\widehat{h}, \widehat{\chi}) = \zeta(\widehat{g}, \widehat{\chi}) \zeta(h, \chi).$$

*Proof.* One proof is similar to Example 2 in the previous section and reduces the verification to the case where both  $g$  and  $h$  are of the simple form  $\operatorname{char}_{A_1} \operatorname{char}_{A_2}$ . We indicate another, similar to dimension one, with one new feature – a factor  $k(\beta)$  which corresponds to the transform  $\widehat{f}$  involving the map  $\sigma$ .

For  $\alpha = (\alpha_1, \alpha_2)$  denote  $\alpha^{-1} = (\alpha_1^{-1}, \alpha_2^{-1})$ ,  $\alpha\beta = (\alpha_1\beta_1, \alpha_2\beta_2)$ ,  $|\alpha| = |\alpha_1||\alpha_2|$ ,  $\mu(\alpha) = \mu_{F \times F}(\alpha_1, \alpha_2)$ ,  $\psi(\alpha) = \psi(\alpha_1)\psi(\alpha_2)$ . Put

$$k(\beta) = \psi(\beta) + q^{-2}\psi(\nu_1^{-1}\beta) + q^{-1}\psi(\nu_2^{-1}\beta) + q^{-1}\psi(\nu_3^{-1}\beta),$$

$\nu_i$  defined in section 15. We will use  $|\operatorname{to}(\alpha)|_2^2 = |\alpha|$  for  $\alpha \in \mathfrak{o}'(T)$ .

We have

$$\begin{aligned} \zeta(g, \chi) \zeta(\widehat{h}, \widehat{\chi}) &= \iint_{T \times T} g(\alpha) \widehat{h}(\beta) \chi_t(\alpha\beta^{-1}) |\beta|^2 |\alpha|^{-2} |\beta|^{-2} d\mu_T(\alpha) d\mu_T(\beta) \\ &= \int_{F \times F} \int_{F \times F} g_\sigma(\alpha) \mathcal{F}(\mathbf{e}(h_\sigma))(\beta) \chi_{\operatorname{to}}(\alpha\beta^{-1}) |\alpha|^{-1} d\mu(\alpha) d\mu(\beta) \\ &= \iiint_{F^6} g_\sigma(\alpha) \mathbf{e}(h_\sigma)(\gamma) \psi(\beta\gamma) \chi_{\operatorname{to}}(\alpha\beta^{-1}) |\alpha|^{-1} d\mu(\alpha) d\mu(\beta) d\mu(\gamma) \end{aligned}$$

$$\begin{aligned}
&= \iiint_{F^6} g_{\mathfrak{o}}(\alpha) h_{\mathfrak{o}}(\gamma) \psi(\beta\gamma) \chi_{\mathfrak{to}}(\alpha\beta^{-1}) |\alpha|^{-1} d\mu(\alpha) d\mu(\beta) d\mu(\gamma) \\
&+ \sum_{1 \leq i \leq 3} \iiint_{F^6} g_{\mathfrak{o}}(\alpha) h_{\mathfrak{o}}(\gamma) \psi(\beta\nu_i^{-1}\gamma) \chi_{\mathfrak{to}}(\alpha\beta^{-1}) |\alpha|^{-1} d\mu(\alpha) d\mu(\beta) d\mu(\nu_i^{-1}\gamma) \\
&= \iiint_{F^6} g_{\mathfrak{o}}(\alpha) h_{\mathfrak{o}}(\gamma) k(\beta\gamma) \chi_{\mathfrak{to}}(\alpha\beta^{-1}) |\alpha|^{-1} d\mu(\alpha) d\mu(\beta) d\mu(\gamma) \\
&\quad (\beta \rightarrow \gamma^{-1}\beta) \\
&= \iiint_{F^6} g_{\mathfrak{o}}(\alpha) h_{\mathfrak{o}}(\gamma) \chi_{\mathfrak{to}}(\alpha\gamma) |\alpha\gamma|^{-1} d\mu(\alpha) d\mu(\gamma) k(\beta) \chi_{\mathfrak{to}}(\beta^{-1}) d\mu(\beta)
\end{aligned}$$

(due to to the integrals are actually taken over  $\mathfrak{o}'(T)$  where one can apply the Fubini property). The symmetry in  $\alpha, \gamma$  implies the local functional equation.

REMARK. Undoubtedly, one can extend this proposition to a larger class of continuous functions  $g, h: F \times F \rightarrow \mathbb{C}$ .

**20.** Similar to section 12, we need to take care of zeta integrals which involve normalized measures corresponding to shifted characters. The zeta integral should be more correctly written as  $\zeta(f, \chi | \frac{s}{2}, \mu)$  since it depends on the normalization of the measure  $\mu$ . Above we used a normalized measure  $\mu$  such that  $\mu(O) = 1$  (which corresponds to the conductor of  $\psi$  being  $O$ ).

More generally, let the conductor of a character  $\psi$  be  $t_1^d O$ . Use  $\psi$  and  $\psi(\alpha_1, \alpha_2)$  for the transform  $\mathcal{F}$  of functions in  $Q_F$  and  $Q_{F \times F}$  as in sections 9–10. Then self dual measures  $\mu$  on  $F$  and  $F \times F$  with respect to  $\psi$  and Fourier transform  $\mathcal{F}$  satisfy  $\mu(O) = q^{d/2}$ ,  $\mu(O, O) = q^d$ . Indeed, let  $f = \text{char}_{(t_1^c O, t_1^{c'} O)}$ . Then  $\mathcal{F}$  with respect to  $\psi$  is  $\mathcal{F}(f) = q^{d-c-c'} \text{char}_{(t_1^{d-c} O, t_1^{d-c'} O)}$  and  $\mathcal{F}^2(f) = f$ , as required.

We get

$$\zeta(f, | \frac{s}{2}, \mu) = q^{d-(c+c')s} \left( \frac{1}{1-q^{-s}} \right)^2.$$

Now assume that  $d$  is even. Then  $\widehat{f} = q^{d-2c-2c'} \text{char}_{(t_1^{d/2-s} O, t_1^{d/2-c'} O)}$  and

$$\zeta(\widehat{f}, | \frac{2-s}{2}, \mu) = q^{s(d-c-c')} \left( \frac{1}{1-q^{s-2}} \right)^2,$$

so the local constant, the quotient  $\zeta(f, | \frac{s}{2}, \mu) \zeta(\widehat{f}, | \frac{2-s}{2}, \mu)^{-1}$  is  $q^{d(1-s)}$ .

**21.** The previous constructions of the maps  $\mathfrak{o}'$ ,  $\mathfrak{r}$ ,  $\mathfrak{t}$ ,  $\mathfrak{e}$  are suitable for unramified characters, but are not good enough for ramified characters. In addition to section 18 here is another example.

Let a weakly ramified character  $\chi$  of  $K_2^t(F)$  be induced by a character  $\bar{\chi}$  of the field  $E$  with conductor  $1 + \bar{t}_1^r O_E$ ,  $r > 0$ . By analogy with the one dimensional case it seems reasonable to calculate the zeta integral  $\zeta(f, \chi)$  for  $f: T \rightarrow \mathbb{C}$  which is defined by  $(t_1^j u_1, t_1^l u_2) \mapsto 1$  if and only if  $j \geq 0, l = 0, u_2 \in 1 + t_1^r O$ . Using the dimension one calculation in [T] one easily obtains

$$\begin{aligned}
\zeta(f, \chi) &= \frac{1}{1-q^{-s}} (1-q^{-1})^{-1} q^{-r}, \\
\zeta(\widehat{f}, \bar{\chi}) &= \frac{1}{1-q^{s-2}} (1-q^{-1})^{-1} q^{-(s+1)r/2-\delta} \rho_0(\bar{\chi}),
\end{aligned}$$

where  $\rho_0(\bar{\chi})$  is the root number as in [T, 2.5],  $\delta = s/2$  if  $r$  is odd and  $= 0$  if  $r$  is even.



Notice that unlike the case of unramified characters in section 20,  $\widehat{f}$  differs from  $f$  on  $T$  due to the difference between  $e(\widehat{f}_\mathfrak{o})$  and  $\mathcal{F}(e(f_\mathfrak{o}))$ ; so for the current choice of  $\mathfrak{o}$  and  $e$  as above there is no canonical local constant associated to  $\chi$ .

All this may be related to the well known difficulty of constructing a general higher ramification theory which is still not available. It is expected that one can refine  $\mathfrak{o}$ ,  $e$ ,  $\tau$ ,  $t$  to get canonical local constants for ramified characters as well.

**22.** We describe elements of the theory for  $K_2^t(K((t)))$  where  $K$  is an archimedean local field (two  $t$  have different meanings). Class field theory for such fields is described in [KS]. In this case class field theory does not really match nicely the structure of the  $K_2^t$ -group of the field, and this is reflected in the constructions of this section.

Introduce the sequential topology on  $L = K((t))$  as in section 1. Define

$$K_2^t(L) = K_2(L)/\Lambda_2(L), \quad \Lambda_2(L) = \Lambda_2'(L) + \{K^\times, 1 + tK[[t]]\} + \{t, 1 + tK[[t]]\},$$

$\Lambda_2'(L)$  is the intersection of all neighbourhoods of zero in the strongest topology on  $K_2(L)$  in which the subtraction in  $K_2(L)$  and the map

$$L^\times \times L^\times \longrightarrow K_2(L), \quad (a, b) \mapsto \{a, b\}$$

are sequentially continuous. Then  $K_2^t(L)$  is the direct sum of the group, generated by  $\{a, t\}$ ,  $a \in K^\times$  and isomorphic to  $K^\times$ , and of the group generated by  $\{-1, -1\}$  (which is zero if  $\sqrt{-1} \in K$ ). We have  $\partial(\Lambda_2(L)) = 1$  where  $\partial$  is the boundary map. It induces a homomorphism  $K_2^t(L) \longrightarrow K_1(K)$ .

Introduce a module on  $K_2^t(L)$ :  $|\alpha|_2 = |\partial(\alpha)|_K$ .

A continuous homomorphism  $\chi: K_2^t(L) \longrightarrow \mathbb{C}^\times$  can be written as  $\chi = \chi_0 | \cdot |_2^s$  ( $s \in \mathbb{C}$ ), where  $\chi_0(\{K^{\times 2}, t_2\}) = 1$  and  $|\chi_0(K_2^t(L))| = 1$ ; so  $\chi_0$  is determined by its value on  $\{-1, -1\}$  and  $\{-1, t\}$ . The former symbol corresponds via class field theory to the totally ramified extension  $K((t^{1/2}))/K((t))$ , the latter – to  $K(\sqrt{-1})(t)/K((t))$ .

**23.** Let

$$\begin{aligned} \mathfrak{v}: K^\times &\longrightarrow K_2^t(L), & \alpha &\mapsto \{\alpha, t\} \\ \mathfrak{t}: T = K^\times \times K^\times &\longrightarrow K_2^t(L), & (\alpha, \beta) &\mapsto \{\alpha\beta, t\}. \end{aligned}$$

For a function  $f: L \times L \longrightarrow \mathbb{C}$  continuous on  $T$  and rapidly decaying at  $(\pm\infty, \pm\infty)$  and a quasi-character  $\chi: K_2^t(L) \longrightarrow \mathbb{C}^\times$  preliminary introduce a zeta integral

$$\zeta(f, \chi) = \int_{K \times K} f(\alpha, \beta) \chi_t(\alpha, \beta) \frac{d\mu_{K \times K}(\alpha, \beta)}{|\alpha|_K |\beta|_K}.$$

Put  $\lambda = \mathfrak{v}$ . Let  $f: K_2^t(L) \longrightarrow \mathbb{C}$  factorize through  $K_2^t(L)/\ker(\partial)$ . Let  $\chi: K_2^t(L) \longrightarrow \mathbb{C}^\times$  be a continuous quasi-character such that  $\chi_0(\ker(\partial)) = 1$  (i.e.  $\chi$  is an unramified character with respect to  $L/K$ ). Then

$$\zeta(g, \chi) = \zeta_1(f \circ \lambda, \chi \circ \lambda)^2,$$

where  $g(\alpha, \beta) = f_\tau(\alpha) f_\tau(\beta)$ , and  $\zeta_1$  is the one dimensional zeta integral.

REMARK. The well known technique in dimension one which extends the local integrals associated to representations of the multiplicative group to representations of algebraic groups is likely to give a similar extension of this work.

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