Analysis on arithmetic schemes. III

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This work further extends the *programme of adelic analysis and geometry on surfaces*, with its applications to key properties of the zeta functions via two-dimensional zeta integrals [F1–F4].

In this paper we study the zeta function of elliptic surfaces at the central point, thus developing an approach in this direction which was schematically presented in section 9 of [F3] and in section 58 of [F4].

Already in geometric dimension two, generalisations of one-dimensional structures branch out into several different structures. For example, there are *two distinct adelic structures on relative surfaces*: one is *geometric adeles* [F4 sect. 28], [F5], which essentially takes into account geometry and divisors, another is *analytic adeles* which is best suited to the study of closed points on surfaces and their zeta functions via the two-dimensional translation invariant integration and zeta integrals [F3–F4]. The method of [F3], [F4] and of this paper uses the two adelic structures on surfaces, geometric and analytic dualities and global geometry of surfaces.

Interactions between the multiplicative groups of the two adelic structures on elliptic surfaces (a regular proper flat model of an elliptic curve over a global field) is crucial for understanding fundamental properties of the zeta functions. Among several applications, these interactions are very useful for a proper understanding of the equality of the geometric and analytic ranks of the elliptic surface, which is closely related to the original conjecture of Birch and Swinnerton–Dyer on the analytic and arithmetic ranks of the elliptic curve. The new method to deal with the BSD conjecture differs substantially, from the point of view of its origin and conceptual position, from all the previous methods. In particular, it does not aim and does not need to produce rational points on elliptic curves.

Both in the study of special values of zeta and L-functions and in the study of aspects of the Langlands correspondences, "Relations between the analytic theory and the arithmetical have often been uneasy" [Ls, p. 458]. To a certain degree, we address this issue in the case of an elliptic surface, by using adelic methods and two adelic structures associated to the surface. The geometric and analytic ranks of an elliptic surface are numbers which are associated to the two adelic structures. Towards a proof of the numerical relation, i.e. the equality of the geometric and analytic ranks, we study nontrivial relations between these two adelic structures using certain natural constructions from *explicit two-dimensional class field theory*.

Briefly speaking, for the geometric rank of the elliptic surface one uses the multiplicative up of the two-dimensional geometric adeles and its double subquotient (which generalises

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the familiar quotient of the one-dimensional ideles) isomorphic to the (generalised) Neron–Severi and Picard groups of the surface. This is a two-dimensional version of the standard adelic description of the class number of a global field. For the analytic rank we use the theory of two-dimensional translation invariant measure, integration and harmonic analysis on the analytic adeles, analytic duality (two-dimensional theta-formula) and the two-dimensional Iwasawa–Tate theory of [F1–F4]. The latter allows us to reduce the analytic part of the problem to a computation of the order of the pole at the central point of a certain boundary term studied in [F3–F4], which itself is an integral over a subquotient of the two-dimensional analytic adeles. To compute the order of the pole at the central point we need to get more information on the structure of the adelic subquotient. We achieve this by relating the analytic adeles with the geometric adeles. This relation comes from an intrinsic pairing between the multiplicative groups of the two adelic structures, which has already played a key role in explicit two-dimensional class field theory. This allows to restate the rank part of the BSD conjecture as a property of a certain adelic integral. See Theorem **6**.

In positive characteristic things are easier by several reasons. In particular, here we can also use the recent work [F5] which directly connects geometric adeles on algebraic surfaces with their algebraic geometry. Geometric adelic duality underlies Serre's duality and properties of geometric adelic objects give a simple proof of the Riemann–Roch theorem. These arguments involve a proof of the *discreteness of the global functions in the full geometric adelic space*. Using this and the theory mentioned in the previous paragraphs we establish a new inequality between the geometric and analytic ranks in **7**. This inequality together with well known previous results, obtained by completely different methods several decades ago, imply the equality of the ranks, see section **8**. It is expected that one can establish the full rank equality using the higher adelic method of this paper only.

In characteristic zero, as far as the rank part of the BSD conjecture is concerned, there is one missing important ingredient, namely, a two-dimensional adelic reinterpretation of the Arakelov intersection pairing and of the Faltings–Riemann–Roch theorem for regular models of elliptic curves and a related proof of the discreteness of the rational functions in the geometric adelic ring, in other words an arithmetic version of the geometric case established in [F5]. The study of this ingredient will be the subject of another paper.

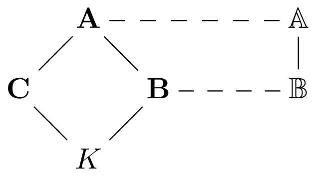
There are two-dimensional tools which can be used to provide more information on the special value of the zeta function and compute its residue at the central point. Work in this direction is started in Part II of this paper.

Now we provide a more technical introduction of the method of the paper. The reader not familiar with the theory of [F1–F4] is recommended to read first [F6] for an introduction into some of the key features of higher adelic analysis and geometry and their applications, [F3] for the two-dimensional adelic analysis programme, and also the introduction of [F4].

Let E be an elliptic curve over a global field k. Let B be the spectrum of the ring of integers of k in characteristic zero, or a proper, smooth, connected curve over a finite field with function field k. Let $\pi\colon \mathcal{E} \longrightarrow B$ be a proper regular model of the elliptic curve E such that every singular point of every fibre of $\mathcal{E} \longrightarrow B$ is split ordinary double. Let

$\mathcal{E}$ be geometrically irreducible. Assume in addition that the reduction of E in residual characteristic 2 and 3 is good or multiplicative and E is not defined over a finite field. Let $K = \mathrm{k}(E)$ be the function field of $\mathcal{E}$.

There are six two-dimensional adelic objects on $\mathcal{E}$:



where the punctured lines stand for subsets $\mathbb{A} \subset \mathbf{A}$ with adelic structure different from the induced one. The four adelic objects on the left hand side are of geometric origin and their modification is defined in sect. 28 of [F4], see also [F5]. The two adelic objects on the right hand side are of analytic and arithmetic origin and defined in sect. 29 of [F4] for a fixed set $S_\prime$ of finitely many horizontal irreducible smooth curves and all fibres on $\mathcal{E}$. The geometric adelic objects are defined on the whole set of points on curves on $\mathcal{E}$, and we can also consider their $S_\prime$-part.

Class field theory involves K_2 or K_2^t functors applied to objects in the two top levels of the diagram. The additive groups of geometric adelic spaces and analytic adelic spaces are very far away from each other, but their multiplicative groups intertwine via explicit two-dimensional class field theory and Milnor K_2^t-groups, see **3**.

The theory developed in [F1–F4] studies the zeta function $\zeta_{\mathcal{E}}(s)$ of $\mathcal{E}$ via a zeta integral of $\mathcal{E}$, as a two dimensional generalisation of the Iwasawa and Tate theories, [I], [T1].

The surface zeta integral defined and studied in [F4]

$$\zeta(f, |\ |^s) = \int_{\mathfrak{T}} f \, \|\ \|^s$$

lives on a rescaled object $\mathfrak{T}$ of $T = (K_1 \times K_1)(\mathbb{A})$. Here $\|\ \|$ is a twisted module on $\mathfrak{T}$ and the integration is taken with respect to a new translation invariant measure on two-dimensional analytic addles, see sect. 36 and 39 of [F4]. The function f is chosen to be almost an eigenfunction of the two-dimensional Fourier transform, see sect. 40 of [F4]. It is proved in [F4] that the zeta integral $\zeta(f, |\ |_2^{s})$ equals the product of an exponential factor $\mathfrak{c}_{\mathcal{E}}^{1-s}$, the square of the zeta function $\zeta_{\mathcal{E}}(s)$, and the product of the squares of the one-dimensional zeta integrals at $s/2$ of the horizontal curves in $S_\prime$, each of which satisfies a functional equation with respect to $s \to 2 - s$. A two-dimensional theta-formula for $\mathcal{E}$ is established in sect. 44 of [F4], its proof relies on two-dimensional harmonic analysis on analytic adeles. In sect. 45 of [F4] it is proved that

$$\zeta(f, |\ |_2^{s}) = \xi(|\ |_2^{s}) + \xi(|\ |_2^{2-s}) + \omega(|\ |_2^{s})$$

on the half-plane $\Re(s) > 2$, where the function $\xi(|\ |_2^{s})$ is an absolutely and uniformly convergent integral on the complex plane, and hence it extends to an entire function on the complex plane and where the boundary term $\omega(|\ |_2^{s})$ has an explicit adelic integral representation.

The function $\omega(|\ |_2^s)$ can be written, up to an entire function, as a triple integral whose internal integral is an integral over the weak boundary ∂T_0 of $T_0 = (K_1 \times K_1)(\mathbb{B})$ of a function which is essentially the difference between $\mathcal{F}(f_\alpha)$ and $f_\alpha\colon \gamma \mapsto f(\alpha\gamma)$.

Due to the theory of [F3–F4], the study of such fundamental problems as the analytic continuation, functional equation, generalised Riemann hypothesis and the BSD conjecture are reduced to the study of appropriate properties of the boundary integral $\omega(|\ |_2^s)$.

In this paper we study the boundary integral $\omega(|\ |_2^s)$ at the central point $s = 1$, assuming the meromorphic continuation and functional equation of the zeta function where necessary. We get more information on related analytic adelic subquotients using geometric adelic structures, thus relating the analytic and geometric invariants of $\mathcal{E}$. We will use explicit two-dimensional class field theory and the diagram in **3**, originating in two-dimensional explicit class field theory, to relate geometric adeles $\mathbf{B}^\times$ with T_0 and to study the behaviour of the function $\omega(|\ |_2^s)$ at $s = 1$.

The paper contains two parts.

The first part studies the rank part of the BSD conjecture, following sect. 9 of [F3] and sect. 58 of [F4]. We reduce the rank part of the BSD conjecture to a certain adelic property $\circledcirc$ in **6**. In section **7** we study this property in positive characteristic using, in particular, the discreteness of K in the geometric adelic space $\mathbf{A}$ established in [F5]. This discreteness is related with geometric adelic duality and relations between adelic geometry and algebraic geometry on algebraic surfaces over perfect fields. In section **8** we derive applications in the positive characteristic case. This part will be further extended to include the case of elliptic curves over number fields.

The second part, in preparation, studies the residue at the central point. There we use additional two-dimensional adelic structures and objects related to two-dimensional class field theory.

Needless to say, the method of [F4] and this work is entirely different from the previous methods applied to the study of the BSD and Tate conjectures.

We keep the notation of [F4], see its sect. 24 and its last pages for the index of notation.

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1. The rank part of the BSD from higher adelic point of view

To study the boundary term $\omega(|\ |_2^s)$ we can use explicit two-dimensional class field theory and the commutative diagram in Lemma 41(c) of [F4] reproduced below in **3**. In particular, as far as the special value at $s = 1$ is concerned, [F4] relates the analytic rank of $\zeta_\mathcal{E}$ to the order of the pole of the boundary term and hence the structure of the internal integral over the weak boundary ∂T_0. On the other hand, the arithmetic–geometric rank of $\mathcal{E}$ is related to the discrete part of the Picard group of $\mathcal{E}$ and hence to a quotient of $K_1(\mathbf{B})$, which participates in the commutative diagram reproduced below in **3**. Then the vertical map in that diagram provides a relation between geometry and analysis and helps to study the structure of ∂T_0. We will use this relation to study the equality of the arithmetic and analytic ranks.

The main results are Theorems **6**, **7**, **8**.

For the reader's convenience we repeat an appropriate part of sect. 58 of [F4].

1. Recall that using the formulas in Proposition 45 of [F4], we get

$$\omega(|\ |_2^s) = \int_{M^-} \omega_m(|\ |_2^s)\, d\mu_{M^-}(m), \qquad \omega_m(|\ |_2^s) = \omega_m^{(1)}(|\ |_2^s) + \omega_m^{(2)}(|\ |_2^s),$$

where

$$\int_{M^-} \omega_m^{(1)}(|\ |_2^s)\, d\mu_{M^-}(m) = \int_{M^-} |m|^{s-2} \int_{\mathfrak{T}_1} f(\mathfrak{m}^{-1}\alpha^{-1})\big(|\alpha|^{-1} - 1\big)\, d\mu(\alpha)\, d\mu_{M^-}(m)$$

extends to an entire function on the complex plane, see Remark 1 of 45 of [F4], and

$$\begin{aligned}&\int_{M^-} \omega_m^{(2)}(|\ |_2^s)\, d\mu_{M^-}(m)\\ &= \int_{\mathfrak{T}^-/T_0} \int_{\partial T_0} ||\alpha||^s \big(|\alpha|^{-1}\, \mathcal{F}(f)(\alpha^{-1}\beta) - f(\alpha\beta)\big)\, d\mu(\beta)\, d\mu(\alpha)\\ &= \int_{M^-} |m|^s \int_{\mathfrak{T}_1/T_0} \int_{\partial T_0} \big(|\mathfrak{m}\gamma|^{-1} f(\mathfrak{m}^{-1}\nu^{-1}\gamma^{-1}\beta) - f(\mathfrak{m}\gamma\beta)\big)\, d\mu(\beta)\, d\mu(\gamma)\, d\mu_{M^-}(m)\end{aligned}$$

where $\mathfrak{T}^- = \mathfrak{M}^-\mathfrak{T}_1$, $\mathcal{F}(f)(\alpha) = f(\nu^{-1}\alpha)$ as in 40, and we use $|\gamma| = ||\gamma||_-$ for $\gamma \in \mathfrak{T}_1$. Recall that the zeta integral is the integral over $\mathfrak{T}$, which is a rescaled version of T, so in particular $\mathfrak{T}_1$ is different from T_1, and horizontal curves zeta integrals are essentially one-dimensional zeta integrals at $s/2$.

2. Let r be the arithmetic rank of E over k, i.e. the rank of the free part of E(k).

DEFINITION. Choose horizontal curves y_i, $i \in I$, $|I| = r + 1$, the images of sections of $\pi\colon \mathcal{E} \longrightarrow B$ which include the image of the zero section and the curves on the surface, corresponding to a choice of free generators of the group E(k).

Denote $S_- = \{y_i : i \in I\}$ and let $S_\prime$ be the union of it and all the fibres, $T = T_{S_\prime}$, $T_0 = T_{0,S_\prime}$.

For every singular fibre $\mathcal{E}_b$ take all the ($k(b)$-rational) components of its reduced part except one which intersects the zero section and denote them by y_j, $1 \leqslant j \leqslant n_b$, where n_b is as in 48, i.e. the number of components of the special fibre $\mathcal{E}_b$ with the component intersecting the zero section excluded. In addition, choose one nonsingular fibre y_*, and if K is of positive characteristic add it to the above curves. Denote the whole collection of curves in this paragraph by y_j, $j \in J$; then $|J| = \sum n_b$ in characteristic zero and $|J| = \sum n_b + 1$ in positive characteristic.

Put

$$S_+ = \cup_{i\in I} y_i \cup \cup_{j\in J} y_j .$$

Consider

$$\psi: K^\times \longrightarrow \mathbf{B}_{\mathcal{E}}^\times/\big(\mathbf{B}_{\mathcal{E}}^\times \cap \mathbf{VA}_{\mathcal{E}}^\times\big),$$

see sect. 28, 36 of [F4] for the notation, and denote its cokernel by $F_{\mathcal{E}}$.

Note that $\mathbf{B}_{\mathcal{E}}^\times/\big(\mathbf{B}_{\mathcal{E}}^\times \cap \mathbf{VA}_{\mathcal{E}}^\times\big)$ is isomorphic to the group of divisors on $\mathcal{E}$, similarly to the one dimensional classical case. and the group Div($\mathcal{E}$) is the direct sum of vertical divisors and horizontal divisors, the latter correspond to divisors on its generic fibre. Thus, $F_{\mathcal{E}}$ is isomorphic to the Picard group of $\mathcal{E}$, whose structure is well known. In positive characteristic the free part of NS($\mathcal{E}$) is generated by classes of y_i, y_j, $i \in I, j \in J$, without $y_\star$, thus its rank is $r + 1 + \sum n_b$, [Sh], [T2], [Go]. The group $\pi^*\mathrm{Pic}^0(B)$ is of finite index in the subgroup of divisors numerically equivalent to 0, and the kernel of the natural surjective map from NS($\mathcal{E}$) modulo its torsion to Num($\mathcal{E}$) is an isomorphism, e.g. [Ma]. Thus, in positive characteristic the group $F_{\mathcal{E}}$ has rank $r + 2 + \sum n_b$ and its subgroup generated by classes of $\mathbf{B}_{y_i}^\times, \mathbf{B}_{y_j}^\times$, $i \in I, j \in J$, is of finite index.

In characteristic zero, extending the argument in the positive characteristic, we have a similar description of $F_{\mathcal{E}}$, for example using the intersection pairing. In particular, similar to Lemma 1.4 of [Sh] a divisor D on $\mathcal{E}$ which does not meet the generic fibre is uniquely expressed modulo the subgroup generated by fibres as a linear combination $(y_j)A^{-1}((D, y_j))^T$ where A is a block matrix consisting of blocks, for each b in the ramification locus of π, whose entries are formed by the intersection indices (y_j, y_l) of the subset of $\{y_j\}$ associated to b. See also Ch.III of [Lg], which does not use Arakelov theory, and in the finite part of Thm. 2.3 of Ch. IV of [Lg]. Thus, the subgroup generated by classes of $\mathbf{B}_{y_i}^\times, \mathbf{B}_{y_j}^\times$, $i \in I, j \in J$, is of finite index in $F_{\mathcal{E}}$ and its rank is $r + 1 + \sum n_b$. For a very recent presentation of the rank of Pic($\mathcal{E}$) in relation to the rank of E(k) in characteristic zero see [Ta].

Put

$$K_{S_\prime}^\times = \{\alpha \in K^\times : \alpha \in \mathcal{O}_y^\times \quad \text{for all } y \notin S_\prime\},$$

i.e. the support of the principal divisor for α lies inside $S_\prime$.

Then by the previous material

$$\mathbf{B}_{S_\prime}^\times/\big(\big(\mathbf{B}_{S_\prime}^\times \cap \mathbf{VA}_{S_\prime}^\times\big)K_{S_\prime}^\times\big) \simeq F_{\mathcal{E}}.$$

REMARK. The group $K^\times \cap V\mathbb{A}^\times$ of ψ is of rank 0 in positive characteristic and is of rank $r_1 + r_2 - 1$ (the rank of the group of units of the ring of integers of k) in characteristic zero.

3. Lemma 36 of [F4] induces the commutative diagram

$$\begin{array}{ccccc} & & \mathbb{A}^\times \otimes \mathbf{A}^\times_{S,}/\mathbf{V}\mathbf{A}^\times_{S,} & & \\ & & \downarrow & \searrow & \\ T & \longrightarrow & \mathbb{A}^\times/V\mathbb{A}^\times \times \mathbb{A}^\times/V\mathbb{A}^\times & \longrightarrow & J_{S,}/(J_{S,} \cap VJ_{\mathcal{E}}), \end{array}$$

the tensor product is the adelic tensor product, see sect. 36 of [F4].

Denote by IT the subgroup of T of elements (α, α^{-1}), $\alpha \in \mathbb{A}^\times$.

The following is immediate from the definitions.

LEMMA 1. *The group T is generated by IT and the preimage in T of the image of $\mathbb{A}^\times \otimes \mathbf{A}^\times_{S,}/\mathbf{V}\mathbf{A}^\times_{S,}$ in $\mathbb{A}^\times/V\mathbb{A}^\times \times \mathbb{A}^\times/V\mathbb{A}^\times$.*

The diagram in Lemma 41 (c) of [F4] induces a commutative diagram

$$\begin{array}{ccccc} & & \mathbb{B}^\times \otimes \mathbf{B}^\times_{S,}/(\mathbf{B}^\times_{S,} \cap \mathbf{V}\mathbf{A}^\times_{S,}) & & \\ & & \downarrow & \searrow & \\ T_0 & \longrightarrow & \mathbb{B}^\times/(\mathbb{B}^\times \cap V\mathbb{A}^\times) \times \mathbb{B}^\times/(\mathbb{B}^\times \cap V\mathbb{A}^\times) & \longrightarrow & P_{S,}/(P_{S,} \cap VJ_{\mathcal{E}}), \end{array}$$

$P_{S,} = P \cap J_{S,}$.

REMARK. Surjective homomorphisms $\mathfrak{t}_{x,z}\colon \mathcal{O}^\times_{x,z} \times \mathcal{O}^\times_{x,z} \longrightarrow K^t_2(K_{x,z})$ defined in sect. 16 of [F1] and used in sect. 41 of [F4] can be included in a commutative diagram of continuous maps

$$\begin{array}{ccc} \mathcal{O}^\times_{x,z} \times K^\times_{x,z} & & \\ \downarrow & \searrow & \\ \mathcal{O}^\times_{x,z} \times \mathcal{O}^\times_{x,z} & \longrightarrow & K^t_2(K_{x,z}) \end{array}$$

where the diagonal map is the symbol map and the existence of the vertical map is proved in [K, §7] and in the first paragraph of the proof of [F7, Thm 4.6].

Their adelic version is the following commutative diagram of continuous maps (see sect. 33 of [F4])

$$\begin{array}{ccc} \mathbb{A}^\times \times \mathbf{A}^\times_{S,} & & \\ \downarrow & \searrow & \\ \mathbb{A}^\times \times \mathbb{A}^\times & \longrightarrow & J_{S,}. \end{array}$$

Previous A- and B-diagrams are induced by this commutative diagram.

The previous Lemma implies

LEMMA 2. *The group T_0 is generated by the subgroup $IT\cap T_0$ and the preimage in T_0 of the image of $\mathbb{B}^\times\otimes\mathbf{B}_{S'}^\times/(\mathbf{B}_{S'}^\times\cap\mathbf{VA}_{S'}^\times)$ in $\mathbb{B}^\times/(\mathbb{B}^\times\cap V\mathbb{A}^\times)\times\mathbb{B}^\times/(\mathbb{B}^\times\cap V\mathbb{A}^\times)$.*

DEFINITION. Denote by $\mathcal{K}$ the product of K = the image of $\mathbb{B}^\times\otimes K_{S'}^\times$ in $\mathbb{B}^\times\times\mathbb{B}^\times$, $(\beta_y)\otimes\kappa\mapsto((\beta_y^{v_y(\kappa)}),1)$, and $\mathcal{U}$ = the product of $IT\cap T_0$ and $VT\cap T_0$ where $VT = V\mathbb{A}^\times\times V\mathbb{A}^\times$.

We have a commutative diagram

$$\begin{array}{ccccc}
\mathbb{B}^\times\otimes K_{S'}^\times & \xrightarrow{\text{symb}} & K_2(\mathbf{B}_{S'}) & \longrightarrow & K_2(\mathbf{A}_{S'})\\
\big\downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{div}} & & \big\downarrow{\scriptstyle\partial} & & \big\downarrow{\scriptstyle\partial}\\
\oplus_{y\in S'}K_1(\mathbb{B}_y) & \xrightarrow{p} & \oplus_{y\in S'}K_1(k(y)) & \longrightarrow & \oplus_{y\in S'}K_1(\mathbb{A}_{k(y)})\\
 & \searrow & \big\downarrow{\scriptstyle\oplus\mathrm{div}_y} & & \big\downarrow\\
 & & \oplus_{y\in S'}\mathrm{PDiv}(\tilde y) & \longrightarrow & \oplus_{y\in S'}\mathrm{Div}(\tilde y)
\end{array}$$

where $\tilde y$ is the normalisation of y.

Then K is the image of the image of the top left object in $\oplus_{y\in S'}K_1(\mathbb{B}_y)$ in $\mathbb{B}^\times\times\mathbb{B}^\times$ with respect to $\oplus_{y\in S'}K_1(\mathbb{B}_y)\to\mathbb{B}^\times\to\mathbb{B}^\times\times\mathbb{B}^\times$, where the last map is $\beta\mapsto(\beta,1)$.

The image of $\mathcal{K}$ in $P_{S'}/(P_{S'}\cap VJ_{\mathcal{E}})$ coincides with the image of $\mathbb{B}^\times\otimes K_{S'}^\times/(K_{S'}^\times\cap\mathbf{VA}_{S'}^\times)$ in $P_{S'}/(P_{S'}\cap VJ_{\mathcal{E}})$, i.e. the subgroup generated by $\{\mathbb{B}^\times, K_{S'}^\times\}$ modulo units.

Using **2** we obtain

LEMMA 3. *The subgroup generated by $\mathcal{K}$ and T_{0,S_+} is of finite index in T_0.*

4. The weak topology on $\mathbb{A}_{S_o}$ for a finite subset S_o of S' was defined in sect. 44 of [F4].

The weak topology on $\mathbb{A}_y$ induces the weak topology on $\mathbb{B}_y$. It is easy to see from the definitions that in characteristic 0 it is the lift of the topology on $k(y)$ which is induced from the topology on $k(y)\otimes_{\mathbb{Q}}\mathbb{R}$ of the vector space over the complete field $\mathbb{R}$. In positive characteristic the weak topology on $\mathbb{F}_q(t)$ is such that a sequence of functions f_n tends to 0 iff for every positive integer r there is n_0 such that for every $n>n_0$ and every monic irreducible polynomial g of degree prime to p the value $v_g(f_n)\in(-\infty,-r)\cup\{0\}$.

Recall that $\mathbf{A}$ is endowed with the topology of inductive limit of $\mathbf{A}=\cup_{S_o\subset S'}\mathbf{A}^{S_o}$ for finite subsets S_o of the set of curves, where $\mathbf{A}^{S_o}=\mathbf{A}_{S_o}\times\prod_{y\notin S_o}\mathbf{OA}_y$ is endowed with the product topology of appropriately defined topologies on the factors, [F5].

Define the weak topology on $\mathbb{A}$ is the inductive topology with respect to $\mathbb{A}=\cup_{S_o\subset S'}\mathbb{A}^{S_o}$ for finite subsets S_o of S' containing all horizontal curves, where $\mathbb{A}^{S_o}=\mathbb{A}_{S_o}\times\prod_{y\notin S_o}O\mathbb{A}_y$ with the product topology of the weak topology on $\mathbb{A}_{S_o}$ and the lift of the discrete topology on $\prod_{y\notin S_o}p(O\mathbb{A}_y)$. Similarly define the weak topology on $\mathbb{A}\times\mathbb{A}$.

Then the weak boundary

$$\partial T_0=\bigcup_{S_o\subset S'}\partial T_{0,S_o}\times T_{0,S'\setminus S_o},\qquad \partial T_{0,S_o}=\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}\setminus(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times,$$

where S_o runs through all finite subsets of $S_{\prime}$, is the boundary with respect to the weak topology, the difference of the weak closure $\overline{T_0}$ of T_0 in $\mathbb{A} \times \mathbb{A}$ and of T_0. Indeed,

$$p(T_0) = \cup_{S_o \subset S_{\prime}} p(T_0 \cap \mathbb{A}^{S_o} \times \mathbb{A}^{S_o}) = \cup_{S_o \subset S_{\prime}} K_1 \times K_1 \big(\prod_{y \in S_o} k(y) \times \prod_{y \notin S_o} \mathfrak{f}(k(y)) \big)$$

where $\mathfrak{f}(k(y))$ is the maximal finite subfield of $k(y)$ and S_o runs through finite subsets of $S_{\prime}$ which contain all horizontal curves. For a vertical curve y the set $\mathbb{B}_y^{\times} \cap O\mathbb{A}_y = p^{-1}(\mathfrak{f}(k(y))^{\times})$ is weakly closed.

The weak closure of the finite direct product of sets is direct product of the weak closures of sets.

5. *From now on we assume that the boundary term* $\omega(|\ |_2^s)$ (*and hence* $\zeta_{\mathcal{E}}(s)$ *and* $L_E(s)$) *extend meromorphically to the complex plane.* This is closely related to mean-periodicity, in appropriate functional space, of the boundary function h defined in sect. 46 of [F4], see sect. 47–48 of [F4] and [FRS]. In positive characteristic we know the meromorphic continuation, of course.

Modify, if necessary, the horizontal part of the function f from sect. 40 of [F4], so that the zeta integral along the zero section y_0, and hence the zeta integral along every horizontal curve y_i, still has the same poles as the square of the completed zeta function of k at $s/2$ and does not vanish at 1 (there are global fields in characteristic zero and positive characteristic whose zeta function vanishes at 1/2).

We are interested to compute the order of the pole at $s = 1$ *of the function* (*meromorphically extended to the complex plane*)

$$\mathfrak{I}_{S_+,R} = \int_{\mathfrak{T}^-_{S_+}/T_{0,S_+}} \int_R ||\alpha||^s \big(|\alpha|^{-1}\, \mathcal{F}(f)(\alpha^{-1}\beta) - f(\alpha\beta)\big)\, d\mu(\beta)\, d\mu(\alpha),$$

where $\mathfrak{T}^-_{S_+} = \{\alpha \in \mathfrak{T}_{S_+} : ||\alpha|| \in N^-\}$, $R = \prod_{*\in S_+} Q_*$ *and* Q_* *is* $\partial T_{0,*}$, $T_{0,*}$, *or* $\partial T_{0,*} \cup T_{0,*} = \mathbb{B}_* \times \mathbb{B}_*$.

We will check that it equals $2(|I| + |J|) = 2|S_+|$ *for an appropriate choice of* R.

We can rewrite as

$$\mathfrak{I}_{S_+,R} = \int_{\mathfrak{T}^-_{S_+}/T_{0,S_+}} \int_R ||\alpha||^s |\alpha|^{-1}\, \mathcal{F}(f)(\alpha^{-1}\beta)\, d\mu(\beta)\, d\mu(\alpha)$$
$$- \int_{\mathfrak{T}^-_{S_+}/T_{0,S_+}} \int_R ||\alpha||^s f(\alpha\beta)\, d\mu(\beta)\, d\mu(\alpha).$$

REMARK. The proof of Theorem 40 of [F4] includes special contribution from singular points of fibres as the square of the Euler factor times and exponential factor, but this does not affect the value of the zeta integral along an irreducible component of the fibre at $s = 1$.

For a vertical curve the integral

$$\mathfrak{J}^2_*(\gamma_*) = \int_{M^-_*} |m|^s \int_{T_{1,*}/T_{0,*}} \int_{Q_*} f(m\gamma_*\gamma\beta)\, d\mu(\beta)\, d\mu(\gamma)\, d\mu(m),$$

where $M_* = |T_*|$, $\gamma_* \in T_\star$, has pole of order 2 at $s = 1$ if $Q_* = T_{0,*}$ or $\partial T_{0,*}$ and does not have pole if $Q_* = \partial T_{0,*} \cup T_{0,*}$.

Comparing with the one-dimensional zeta integral and taking into account $||\alpha||^s = |\alpha|^{s/2}$ on horizontal curves, we see that the order of the pole at $s = 1$ of

$$\mathcal{I}^2_{S_+,R} = \int_{\mathfrak{T}^-_{S_+}/T_{0,S_+}} \int_R ||\alpha||^s f(\alpha\beta)\, d\mu(\beta)\, d\mu(\alpha)$$

does not exceed the maximal contribution from vertical curves in $S_\prime$, i.e. $2|J|$.

THEOREM. *The order of the pole at* $s = 1$ *of*

$$\mathcal{I}^1_{S_+,R} = \int_{\mathfrak{T}^-_{S_+}/T_{0,S_+}} \int_R ||\alpha||^s |\alpha|^{-1} \mathcal{F}(f)(\alpha^{-1}\beta)\, d\mu(\beta)\, d\mu(\alpha)$$

does not exceed $2|S_+|$ *and equals it for a special choice of* R.

The order of the pole at $s = 1$ *of* $\mathcal{I}_{S_+}$ *is* $2|S_+|$.

Proof. Note that the order of the pole of a curve zeta integral at $s = 1$ does not exceed 2, since all the associated curve zeta integrals are related to squares of one-dimensional zeta integrals which have order of pole at $s = 1$ not greater than 1.

Recall that $|\gamma| = ||\gamma||_-$ for $\gamma \in \mathfrak{T}_1$. Then the $*$-component of $||\alpha||^s|\alpha|^{-1}$ is $|m|^{s-1}$ for vertical curves and $|m|^{s-2}||\gamma||_*^{-1}$ for horizontal curves.

To make the function $\mathcal{I}^1_{S_+,R}$ holomorphic in the half-place $\Re(s) > 1$ we let $Q_* = T_{0,*}$ for horizontal curves $*$, since the only pole in that half-plane is at $s = 2$ and arises from $\partial T_{0,*}$.

For a vertical curve $* = y_j$, $j \in J$, consider

$$\mathcal{I}^1_*(\gamma_*) = \int_{M_*^-} |m|^s \int_{T_{1,*}/T_{0,*}} \int_{Q_*} |m|^{-1} \mathcal{F}(f)(m^{-1}\gamma^{-1}\gamma_*\beta)\, d\mu(\beta)\, d\mu(\gamma)\, d\mu(m)$$

where $\gamma_* \in T_*$, $M_* = q_j^{\mathbb{Z}}$, q_j is the cardinality of the finite constant field of y_j.

The classical one-dimensional theory implies that for $Q_* = T_{0,*}$ and M^- there is no pole at $s = 1$. For $Q_* = \partial T_{0,*}$ the order of the pole at $s = 1$ is 2. For Q_* equal to $T_{0,*} \cup \partial T_{0,*}$ the order of the pole is 2.

If $Q_* = \partial T_{0,*}$ for vertical curves then there is no restriction on vertical γs and hence on horizontal γs. For a horizontal curve $* = y_i$, $i \in I$, we then need to look at the integral

$$\int_{M_*^-} ||\mathfrak{m}||^{s-2} \int_{\mathfrak{T}_*/T_{0,*}} \int_{Q_*} ||\gamma||^{-1} \mathcal{F}(f)(\mathfrak{m}^{-1}\gamma^{-1}\beta)\, d\mu(\beta)\, d\mu(\gamma)\, d\mu(m)$$

where $M_* = |T_*|$. The integral

$$\int_{M_*^-} ||\mathfrak{m}||^{s-2} \int_{\gamma \in \mathfrak{m}^{-1}\mathfrak{M}_*^+} ||\gamma||^{-1}\, d\mu(\gamma)\, d\mu(m)$$

has pole of order 1 at $s = 1$. It is then easy to deduce that, both in characteristic zero and positive characteristic, for Q_* equal to $T_{0,*}$the order of the pole of $\mathcal{I}^1_*$ at $s = 1$ is 2.

Thus, if we take $Q_* = \partial T_{0,*}$ for vertical curves in S_+ and $Q_* = T_{0,*}$ for horizontal curves in S_+ the integral $\mathcal{I}^1_{S_+,R}$ has pole of order $2|S_+|$ at $s = 1$, for all other choices of R the order of the pole is less than $2|S_+|$.

6. DEFINITION. Denote by $\circledcirc$ the following (auxiliary) condition:
when the internal integral in $\int_{M^-} \omega_m^{(2)}(|\ |_2^s)\, d\mu_{M^-}(m)$ *is taken over* $\mathcal{K}$ *and its (weak) closure, in each case the order of zero at* $s = 1$ *of the full integral equals* $2\,\mathrm{rk}(K^\times \cap V\mathbb{A}^\times)$*, see* Remark 2. *In other words, the full integral* $\int_{M^-} \omega_m^{(2)}(|\ |_2^s)\, d\mu_{M^-}(m)$ *has no zero and no pole at* $s = 1$ *in positive characteristic and has zero of order* $2(r_1 + r_2 - 1)$ *at* $s = 1$ *in characteristic zero.*

In particular, in positive characteristic this condition $\circledcirc$ tells that the analytic contribution to the boundary term of the low level object in the three level adelic picture in dimension two

$$\begin{array}{c}\mathbf{A}^\times\\ |\\ \mathbf{B}^\times\\ |\\ K^\times\end{array}$$

is non-essential at $s = 1$.

Now, using the previous theory of [F4] and this paper, we obtain the following result.

THEOREM. *Let* E *and* $\mathcal{E}$ *be as in the beginning of this paper.*

Property $\circledcirc$ *holds if and only if the order of the pole at* $s = 1$ *of the boundary term* $\omega(|\ |_2^s)$ *is* $2(|I| + |J| - r_1 - r_2 + 1)$*, and if and only if the order of the zero of* L_E *at* $s = 1$ *equals the rank of* $E(\mathrm{k})$.

Proof. Recall that the order of the pole of the zeta integral $\zeta(f, |\ |_2^s)$ at $s = 1$ is twice the order of the pole of the zeta function of $\mathcal{E}$ at $s = 1$. The latter is the order of the zero of $L_E(s)$ at $s = 1$ plus $|J| + 1$ minus $r_1 + r_2 - 1$, see sect. 40, 48 of [F4].

Compute the order of the pole of the zeta integral at $s = 1$ using Lemma 2 and 3 in **3**, Proposition **5** and the following observation: for two curves $y \neq y'$ the (y, y')-part of $\mathcal{I}$, with the y-part of $\mathfrak{T}/T_0$ and y'-part of ∂T_0, does not have a pole or zero at $s = 1$.

7. On property $\circledcirc$ in positive characteristic. Let K be of positive characteristic. It is proven in [F5], as part of study of two-dimensional adelic geometry, that K is a discrete subspace of $\mathbf{A}$. The discreteness of K in $\mathbf{A}$ is closely related to the two-dimensional adelic form of the Arakelov intersection pairing in [F5].

Since the topology on $\mathbf{A}^\times$ is defined by the embedding $\mathbf{A}^\times \longrightarrow \mathbf{A} \times \mathbf{A}$, $\alpha \mapsto (\alpha, \alpha^{-1})$, $K^\times$ is a discrete subset of $\mathbf{A}^\times$. Hence $K^\times$ is a discrete subset of $\mathbf{B}^\times$ and $K^\times_{S,}$ is a discrete subset of $\mathbf{B}^\times_{S,}$. Using this one deduces that K, defined in **3**, is weakly closed (note that unlike dimension one case, 0 does not belong to ∂T_0). On the other hand, it is easy to see that the units $\mathcal{U}$ are weakly closed. From **4** we deduce that $\mathcal{K}$ coincides with its weak closure.

THEOREM. *For a finite subset $S_o \supset S_+$ of $S_\prime$ define*

$$\mathcal{J}^1_{S_o} = \int_{\mathfrak{T}^-_{S_o}/T_{0,S_o}} \int_{\mathcal{K}_{S_o}} ||\alpha||^s|\alpha|^{-1}\, \mathcal{F}(f)(\alpha^{-1}\beta)\, d\mu(\beta)\, d\mu(\alpha),$$

$$\mathcal{J}^2_{S_o} = \int_{\mathfrak{T}^-_{S_o}/T_{0,S_o}} \int_{\mathcal{K}_{S_o}} ||\alpha||^s f(\alpha\beta)\, d\mu(\beta)\, d\mu(\alpha)$$

where $\mathcal{K}_{S_o}$ is the S_o-projection of $\mathcal{K}(\mathbb{B}_{S_+}\times\mathbb{B}_{S_+})\cap\partial T_{0,S_o}(VT_{S_\prime\setminus S_o}\cap T_{0,S_\prime\setminus S_o})$.

Let K be of positive characteristic. Then the order of the pole of $\mathcal{J}^1_{S_o}$ at $s=1$ does not increase when one replaces S_o with $S_o\cup y$ where y is an element of $S_\prime\setminus S_o$, hence necessarily a vertical irreducible curve. The order of the pole of of $\mathcal{J}^2_{S_o}$ at $s=1$ does not exceed the order of the pole of of $\mathcal{J}^2_{S_+}$ at $s=1$ plus 2. The residues at $s=1$ are uniformly bounded with respect to S_o.

Proof. For an element $\beta\in\mathcal{K}_{S_o}$ write $\beta=\beta_1\beta_2\beta_3$ where $\beta_1\in\mathbf{K}$, $\beta_2\in\mathcal{U}$, $\beta_3\in\mathbb{B}_{S_+}\times\mathbb{B}_{S_+}$. Then the y-component of β_1 is 1 for $y\notin S_o$. Denote $S_p=S_0\cup\{y\}$.

The boundary $\partial T_{0,S_p}$ is the disjoint union of $\partial T_{0,S_o}\times\mathbb{B}_y\times\mathbb{B}_y$ and $T_{0,S_o}\times\partial T_{0,y}$. Since $\mathcal{K}\cap\partial T_0=\emptyset$, we get $\mathcal{K}_{S_p}\subset\mathcal{K}_{S_o}\times T_{0,y}$.

In the definition of integration over T_0 in sect. 43 of [F4] we use the measure $\mu'_{(\mathbb{B}\times\mathbb{B})^\times}$. Denote by Y the support of the function $\beta\mapsto f(\alpha\beta)$, $\beta\in\mathbb{B}\times\mathbb{B}$, $\alpha\in T$. It is easy to see $\mu'_{(\mathbb{B}\times\mathbb{B})^\times}(Y\cap\mathcal{U})$ is finite positive uniformly for $\alpha\in\mathfrak{T}^-/T_0$. Hence when the internal integral in $\int_{M^-}\omega^{(2)}_m(|\ |^{\frac{s}{2}})\,d\mu_{M^-}(m)$ is taken over $\mathcal{U}$, its value is in $\mathbb{C}$ at $s=1$.

The group $K^\times$ is discrete in $\mathbf{A}^\times$, hence the image of $\mathbb{A}^\times\times K^\times$ with respect to the vertical map of the commutative digram in Remark **3** is discrete in $\mathbb{A}^\times\times\mathbb{A}^\times$. Hence the integrals of the type $\mathcal{J}^1$ and $\mathcal{J}^2$ over $\mathcal{K}$ do not contribute poles at $s=1$.

Due to the computation of $\mathcal{J}^1_*$ in **5** and the preceding formula, the integral $\mathcal{J}^1_{S_p}$ does not have a higher order pole at $s=1$ than $\mathcal{J}^1_{S_o}$ and the residue at $s=1$ is uniformly bounded with respect to S_o.

Due to the properties of the integral $\mathcal{J}^2_*$ in **5**, the order of the pole of the integral $\mathcal{J}^2_{S_p}$ at $s=1$ is not more than 2 plus the order of the pole of $\mathcal{J}^2_{S_+}$ and the residue at $s=1$ is uniformly bounded with respect to S_o.

COROLLARY. *Let K be of positive characteristic. The order of the pole of the boundary term and of the zeta integral at $s=1$ does not exceed $2|S_+|=2(|I|+|J|)$, i.e. the analytic rank of E does not exceed the arithmetic rank of E.*

Proof. Using Lemma 3 we deduce that the order of the pole at $s=1$ of $\mathcal{J}_{S_\prime}$ does not change when one replaces ∂T_0 by $\mathcal{K}(\mathbb{B}_{S_+}\times\mathbb{B}_{S_+})\cap\partial T_0$. Letting $S_o\to S_\prime$ in the previous theorem gives the latter integral.

REMARK. It is very natural to expect that a similar argument, which incorporates an adelic approach to Arakelov geometry on $\mathcal{E}$, will be applicable in characteristic zero to deduce the relation between the analytic and arithmetic ranks. The results of this ongoing work will be added to this text in due course, to extend the results of sections **7** and **8** of this text.

8. The Birch–Swinnerton-Dyer and Tate conjectures in positive characteristic. In positive characteristic it is well known that

(a) the analytic rank of E is not smaller than the arithmetic rank of E,

(b) if the BSD conjecture holds for an elliptic curve $E \times_{\mathrm{k}} \mathrm{k}'$ over a finite extension k' of k then it holds for E over k,

(c) the BSD conjecture holds for isotrivial elliptic curves,

(d) the rank part of the BSD conjecture implies the full conjecture,

see [T2], [M1], [Go].

From (a), Corollary in the previous section, (b) and (c) we immediately get the rank part of the following theorem, and then (d) implies the rest.

THEOREM. *Let E be an elliptic curve over a global field* k *of positive characteristic, then the order of the zero of L_E at $s = 1$ equals the rank of the free part of of E*(k) *and the conjectured formula for the residue of L_E at $s = 1$ holds.*

It is well known that several other important properties such as the Artin–Tate conjecture follow from the BSD conjecture in positive characteristic, [T2], [M1], [Go], [Sa1], one can also consult [KT].

2. The residue of $\zeta_{\mathcal{E}}(s)$ at $s = 1$

In this chapter we compute the residue of the zeta integral at $s = 1$ using the boundary term $\omega(|\ |^s)$. One of its factors is the order of the Brauer group of $\mathcal{E}$ which is a much more suitable object of study from the point of view of two-dimensional adelic analysis than the Ш. Except the order of Brauer group, all other factors in the conjectural description of the zeta function at $s = 1$ are known to have adelic description, see e.g. [Bl]; for the Brauer group we will use the previous section.

9. The groups $Ш(E)$ and $\mathrm{Br}(\mathcal{E})$. Here we collect several known results and their extensions which will be useful in an extension of this section. They show how closely related are aspects of two-dimensional class field K_2-theory and the Brauer group and K_1-theory. Results cited from [Sa1] and [Sa2] were proven there under the assumption k does not have real places, we include below a general version whose proof follows the same argument.

The $\mathrm{Br}(\mathcal{E})$ sits in the exact sequence

$$0 \longrightarrow \mathrm{Br}(\mathcal{E}) \longrightarrow \mathrm{Br}(K) \longrightarrow \oplus_{y \in S_1} \mathrm{Br}(K_y)/\mathrm{Br}(\mathcal{O}_y)$$

(see (7-3) of [Sa1] or Prop. (6-8) of [Sa2]). By p. 582 of [Sa1] the $\mathrm{Br}(\mathcal{E})$ coincides with the kernel of

$$\mathrm{Br}(K) \longrightarrow \oplus_{y\in S_1} \oplus_{x\in y} \mathrm{Br}(K_{x,z})/\mathrm{Br}(\mathcal{O}_{x,z}).$$

The $\mathrm{Br}(\mathcal{E})$ sits also in the exact sequence

$$0 \longrightarrow \mathrm{Br}(\mathcal{E}) \longrightarrow \mathrm{Br}(E) \longrightarrow \oplus_{\text{na } v} \mathrm{Br}(E_v)$$

where v runs through all nonarchimedean places of k, i.e. closed points of B, $E_v = \mathcal{E}\times_B k_v$, k_v is the completion of k with respect to v; see e.g. Lemma 2.6 of [M2].

Artin's theorem tells that the Shafarevich–Tate group of E fits into the exact sequence

$$0 \longrightarrow Ш(E) \longrightarrow \mathrm{Br}(\mathcal{E}) \to \oplus_{\text{a } v} \mathrm{Br}(E_v),$$

where v runs through archimedean places of k see sect. 3 of [T2] and Br III of [Gr], and also Prop. 2.7 of [M2]. For other related results see also [LLR] and [Ge].

Thus we get an exact sequence

$$0 \longrightarrow Ш(E) \longrightarrow \mathrm{Br}(E) \longrightarrow \oplus_{\text{all } v} \mathrm{Br}(E_v).$$

Recall the pairings induced by $K_1 \longrightarrow H^1$,

$$X(K)\times K_2(K_{x,z}) \longrightarrow X(K_{x,z})\times K_2(K_{x,z}) \longrightarrow H^3(K_{x,z}) \simeq \mathbb{Q}/\mathbb{Z},$$
$$\mathrm{Br}(K)\times K_1(K_{x,z}) \longrightarrow \mathrm{Br}(K_{x,z})\times K_1(K_{x,z}) \longrightarrow H^3(K_{x,z}) \simeq \mathbb{Q}/\mathbb{Z}$$

and the adelic gluing of these

$$X(K)\times K_2(\mathbf{A}_{\mathcal{E}}) \longrightarrow \mathbb{Q}/\mathbb{Z}, \quad \mathrm{Br}(K)\times K_1(\mathbf{A}_{\mathcal{E}}) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

It is known that this induces pairings

$$X(K)\times\ K_2(\mathbf{A}_{\mathcal{E}})/\big(\mathrm{im}(K_2(\mathbf{B}_{\mathcal{E}})) + \mathrm{im}(K_2(\mathbf{C}_{\mathcal{E}}))\big) \longrightarrow \mathbb{Q}/\mathbb{Z},$$
$$\mathrm{Br}(K)\times\ K_1(\mathbf{A}_{\mathcal{E}})/\big(K_1(\mathbf{B}_{\mathcal{E}}) + K_1(\mathbf{C}_{\mathcal{E}})\big) \longrightarrow \mathbb{Q}/\mathbb{Z},$$

see 28 for the definitions.

These two pairings induce the reciprocity map

$$\mathbf{\Phi}_K\colon K_2(\mathbf{A}_{\mathcal{E}})/\big(\mathrm{im}(K_2(\mathbf{B}_{\mathcal{E}})) + \mathrm{im}(K_2(\mathbf{C}_{\mathcal{E}}))\big) \longrightarrow G_K^{\mathrm{ab}},$$

see 34, and a homomorphism

$$\mathbf{\Psi}_K\colon \mathrm{Br}(K) \longrightarrow \mathrm{Hom}_c\big(\mathbf{A}_{\mathcal{E}}^\times/\mathbf{B}_{\mathcal{E}}^\times, \mathbb{Q}/\mathbb{Z}\big) \longrightarrow \mathrm{Hom}_c\big(\mathbf{A}_{\mathcal{E}}^\times/(\mathbf{B}_{\mathcal{E}}^\times\mathbf{C}_{\mathcal{E}}^\times), \mathbb{Q}/\mathbb{Z}\big),$$

Hom_c stands for continuous homomorphisms of finite order.

By (7-5) of [Sa1] the l-primary part of $\mathbf{\Psi}_K$ is surjective iff the l-primary part of $\mathrm{Br}(\mathcal{E})$ is finite.

The $\mathrm{Br}(\mathcal{E})$ coincides with the kernel of

$$\mathrm{Br}(K) \longrightarrow \mathrm{Hom}_c\big(\mathbf{V}\mathbf{A}_{\mathcal{E}}^\times/(\mathbf{B}_{\mathcal{E}}^\times \cap \mathbf{V}\mathbf{A}_{\mathcal{E}}^\times), \mathbb{Q}/\mathbb{Z}\big).$$

To show this, use p. 582 of [Sa1] (extended to the general case without the restriction on the absence of real places) and the adelic form for $D_I(\mathcal{E})$ of (6-17) of [Sa1], available from [KS2].

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