ABELIAN EXTENSIONS OF COMPLETE DISCRETE VALUATION FIELDS

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Introduction

In 1968 Y. Ihara [Ih] proposed to study class field theory of the p-adically complete field $\widehat{\mathbb{Q}(t)}_p$ that is the quotient field of $\varprojlim \mathbb{Z}[t]/p^i\mathbb{Z}[t]$. This field in modern terminology is a two-dimensional local-global field. Ihara considered cyclic extensions of degree p of this field. He suggested that its class field theory could "explain arithmetically" the map $j \to \Pi_j$ which associates to each $j \in {\mathbb{F}_p}^{\mathrm{sep}}, j \neq 0; 1$ $(p \neq 2; 3)$ the subgroup $\Pi_j \subset \mathbb{Q}_p^\times$ generated by α'/α where $(1-\alpha u)(1-\alpha' u)$ is the L-function, the numerator of the zeta function $Z_E(u) = \zeta_E(q^{-s})$, $u = q^{-s}$, of E with E running all elliptic curves defined over finite extensions of $\mathbb{F}_p(j)$ with modulus j.

That work of Ihara had stimulated two completely different series of works on abelian extensions of complete discrete valuation fields with very general (imperfect) residue field by H. Miki ([M2], 1977, also [M1]) and by K. Kato ([Ka1–Ka7], 1977–1982). The first direction essentially describes some classes of abelian extensions of a complete discrete valuation field with imperfect residue field via study of the group of principal units without using cohomological methods. In the second direction abelian extensions of an n-dimensional local field are described in terms of the Milnor K_n-groups. Kato's theory uses advanced Galois cohomology groups calculations. Independently, A. N. Parshin proposed higher local theory in positive characteristic by using quotients of Milnor K-groups endowed with certain topology, and then its extension to a description of abelian coverings of two-dimensional arithmetic schemes ([P1–P5], 1975–78, 85, 90).

The aim of this work is to sketch the present–day scenery (as it was in 1994) of local class field theories including [Kur], [Ko1–Ko2], [Sp], [F1–F5].

Higher local theories

First we introduce main objects describing abelian extensions of multidimensional local fields: Milnor K-groups and topological K-groups (1° – 6°). We follow [P1–P5], [Ka3–Ka4], [F1,F2,F4]. Then we consider higher local class field theories (7° – 10°).

1. *Multidimensional fields.*

Given a two-dimensional smooth projective over a finite field of characteristic p scheme X one can attach to a point $x \in X$ and a curve $y \subset X$ passing smoothly through x the quotient field of the completion $(\widehat{\widehat{\mathcal{O}_{X,x}}})_y$ of the localization at y of the completion $\widehat{\mathcal{O}_{X,x}}$ of the localization at x. This is a two-dimensional local field over a finite field (which is itself considered as a 0-dimensional local field). More generally, an n-dimensional local field F is a complete discrete valuation field with residue field being $(n-1)$-dimensional. Due to classical structure theorems F is noncanonically isomorphic to $k_m((t_{m+1}))\dots((t_n))$ where k_m is a coefficient field corresponding to the $(n-m)$th residue field of F and either $m = 0$ or $\mathrm{char}(k_m) = 0, \mathrm{char}(k_{m-1}) = p$. Note that the group of principal units of F with respect to the discrete valuation of rank $n-m$ is divisible, and so is not of interest for class field theory. The field k_m for $m \neq 0$ is called a mixed characteristic field, it is a natural higher analog of a $\mathfrak{p}$-adic fields. Lifting prime elements from F and residue fields $k_{n-1}, \dots, k_1$ to the field F one obtains an ordered system of local parameters $t_n, \dots, t_1$ (t_n is a prime element of F).

published in Number Theory Seminars Paris 1993/94, Cambridge Univ. Press 1996, 47–74

If a field M has mixed characteristic and $\mathcal{O}_M$ is its ring of integers with respect to the discrete structure of rank m and t_m is a main local parameter, then the quotient field $M\{\{t\}\}$ of $\varprojlim \mathcal{O}_M[[t]][t^{-1}]/t_m^i\mathcal{O}_M[[t]][t^{-1}]$ is an $(m+1)$-dimensional mixed characteristic field. In general a mixed characteristic field is a finite extension of a field like $\mathbb{Q}_p\{\{t_1\}\}\dots\{\{t_{m-1}\}\}$ ([FV, sect. 5 Ch. II], for more details see [Zh1]).

It occurs that the Milnor K-groups are not the most suitable objects to be related with abelian extensions of an n-dimensional local field F. It is more convenient to work with quotients $K_m(F)/\Lambda_m(F)$ of Milnor K-groups endowed with a special topology ($\Lambda_m(F)$ is the intersection of all neighbourhoods of zero). Arithmetical homomorphisms from Milnor K-groups (like a reciprocity map) usually factorize through such quotients.

2. *Topologies on the multiplicative group.*

It is natural to expect compatibility of theories of a field and its residue field (lifting of extensions of the residue field as unramified extensions of the field and the border homomorphism in K-theory). Thus, a genuine topology on a multidimensional field has to take into account topologies of its residue fields. For class field theory sequential continuity seems to be more important than continuity. This is a hidden phenomenon in dimension 1 and 2, where continuity is the same as sequential continuity.

Denote by $\mathcal{O}_0$ the field in F corresponding to the last finite residue field k_0 when $\mathrm{char}(F)=p$ and the image in F of the ring of Witt vectors of k_0 corresponding to the homomorphism $k_0\to k_{m-1}$ (which is uniquely determined, see [FV, sect. 5 Ch. II]) when $\mathrm{char}(F)=0$. The ring $\mathcal{O}_0$ contains the set of canonical liftings $\mathcal{R}$ from k_0, so called multiplicative representatives. Denote by $\mathcal{O}_1$ the integral closure of $\mathcal{O}_0$.

In general, define a topology λ by induction on dimension. Let F be an n-dimensional local field with $\mathrm{char}(k_{n-1})=p$. Define a lifting (and thus a set of representatives S) of k_{n-1} in F: elements of k_0 are mapped to their multiplicative representatives in $\mathcal{O}_0$, for a system of local parameters t_i their residues $\bar t_i\in k_{n-1}$, $1\leqslant i\leqslant n-1$, are mapped to t_i in F. Given the topology on the additive group k_{n-1}, introduce the following topology on the additive group F. First, an element $\alpha\in F$ is said to be a limit of a sequence of elements $\alpha_v\in F$, $v\to+\infty$, iff given any expression $\alpha_v=\sum_i\theta_{v,i}t_n^i$, $\alpha=\sum_i\theta_i t_n^i$ with $\theta_*\in S$, for every set $\{U_i,\quad -\infty<i<+\infty\}$ of neighbourhoods of zero in k_{n-1} and every i_0 for almost all v the residue of $\theta_{v,i}-\theta_i$ belongs to U_i for all $i<i_0$. Second, a subset U in F is called open iff for every $\alpha\in U$ and every sequence $\alpha_v\in F$ having α as a limit almost all α_v belong to U. This determines a topology λ on F. Then α is a limit of α_v iff the sequence α_v converges to α with respect to the topology λ.

By induction on dimension one verifies that a limit is uniquely determined, each Cauchy sequence with respect to the topology λ converges in F, the limit of the sum of two convergent sequences is the sum of their limits.

If a subgroup $A=\{\alpha=\sum\theta_i t_n^i:\alpha\in F,\theta_i\in S_i\subset S\}$ is open, then all sets of residues $\overline{S_i}$ are open subgroups in k_{n-1}.

Note that the topology λ on the additive group is different from that introduced in [P4] for $n\geqslant 2$: for example, the set $W=F\setminus\{t_2^a t_1^{-c}+t_2^{-a}t_1^c: a,c\geqslant 1\}$ in $F=\mathbb{F}_p((t_1))((t_2))$ is open in the just defined topology, ie for each convergent sequence $x_v\to x\in W$ almost all x_v belong to W. If for some open subgroups U_i in the additive group of $\mathbb{F}_p((t_1))$ such that $U_i=\mathbb{F}_p((t_1))$ for $i\geqslant a$ the group $\{x=\sum a_i t_2^i: x\in F, a_i\in U_i\}$ were contained in W, then for any positive c such that $t_1^c\in U_{-a}$ we would have $t_2^a t_1^{-c}+t_2^{-a}t_1^c\in W$, a contradiction. However, a sequence of elements in F converges to $x\in F$ with respect to λ iff it converges with respect to the topology introduced by Parshin. The group F is not a topological group for $n\geqslant 2$ with respect to λ, for example, if $W'+W'\subset W$, then W' is not open with respect to λ. The topology λ is the sequentially saturated topology: the finest topology in which the set of convergent sequences is the same as in the topology introduced by Parshin.

If $\mathrm{char}(k_{n-1})=p$, then define the topology λ on $F^\times$ as the product of the induced from F topology on the group of principal units $V_F=1+(t_n,\dots,t_1)\mathcal{O}_F$, the discrete topologies on the cyclic groups generated by t_i and the cyclic group of non-zero multiplicative representatives of k_0 in F.

Each Cauchy sequence with respect to the topology λ converges in F, the limit of the sum of two convergent sequences is the sum of their limits. The multiplication in $F^{\times}$ is sequentially continuous.

By induction on dimension one can check that the topological spaces V_F, $F^{\times}/F^{\times p^s}$ satisfy the following completeness property: the intersection of a decreasing sequence of closed subsets is nonempty.

A general principle on higher dimensions states that there are two essential changes in objects and methods that are to be involved: when one goes from dimension 1 to 2 and from dimension 2 to >2. For a 2-dimensional local field its multiplicative group $F^{\times}$ is a topological group and it has a countable base of open subgroups. In the case of at least three dimensional field both assertions don't hold. For example, if $W' \ni 0$ is a subset of F and $W' \cdot W' \subset 1 + Wt_3 + \mathcal{O}_L t_3^2$ for $L = F((t_3))$ with W as above, then W' is not open in $1 + \mathcal{O}_L t_3$.

Call a subset X of elements of $(\mathbb{Z})^n$ greater than 0 admissible if for every $1 < m \leqslant n$ and every $(i_m, \dots, i_n)$ there is $j(i_m, \dots, i_n)$ such that $(i_1, \dots, i_m, \dots, i_n) \in X$ implies $i_{m-1} \geqslant j(i_m, \dots, i_n)$ and there is j such that all $(i_1, \dots, i_m, \dots, i_n) \in X$ implies $i_n \geqslant j$.

If $\operatorname{char}(k_{n-1}) = p$, then every element $\alpha \in F^{\times}$ can be expanded into a convergent product:

$$\alpha = t_n^{a_n} \dots t_1^{a_1} \theta \prod (1 + \theta_{i_n, \dots, i_1} t_n^{i_n} \dots t_1^{i_1})$$

with $\{(i_1, \dots, i_n) : \theta_{i_n, \dots, i_1} \neq 0\}$ being an admissible set (see [MZh]).

Include a principal unit $\varepsilon \in V_F$ into an arbitrary topological basis ε_α of V_F and consider the subgroup topologically generated by $t_n^{a_n}, \dots, t_1^{a_1}$, principal units $\varepsilon_\alpha \neq \varepsilon$ and ε^{p^s}. It is an open subgroup of finite index of $F^{\times}$. Denote the shift-invariant topology τ on $F^{\times}$ which has these open subgroups as the base of neighbourhoods of 1. A sequence of elements in $F^{\times}$ converges to 1 with respect to λ if and only if it converges with respect to τ. In what follows "open" and "closed" means with respect to λ.

The intersection of all open subgroups of finite index containing a closed subgroup H coincides with H. Subgroups $V_F^{p^s}$ are closed in V_F. The product of a closed subgroup in V_F and $V_F^{p^s}$ is closed.

If $\operatorname{char}(F) = \operatorname{char}(k_m) = 0$, $\operatorname{char}(k_{m-1}) = p$, then define the topology λ (τ resp.) on $F^{\times}$ as the product of the trivial topology on the divisible part of $F^{\times}$, the discrete topology on the cyclic groups generated by t_i with $i > m$ and the topology λ (τ resp.) on $k_m^{\times}$.

For a subgroup H of $F^{\times}$ a sequence of elements of $F^{\times}/H$ converges with respect to the quotient topology of τ if and only if it converges with respect to the quotient topology of λ. Hence, if two sequences converge in $F^{\times}/H$, then their sum converges to the sum of their limits.

The topology λ on the multiplicative group is different from that introduced by Parshin in [P4]. For example, for $n \geqslant 3$ each open subgroup A in $F^{\times}$ with respect to the topology introduced in [P4] possesses the property: $1 + t_n^2 \mathcal{O}_F \subset (1 + t_n^3 \mathcal{O}_F)A$. However, the subgroup in $1 + t_n \mathcal{O}_F$ topologically generated by $1 + \theta t_n^{i_n} \dots t_1^{i_1}$ with $(i_n, \dots, i_1) \neq (2, 1, \dots, 0)$, $i_n \geqslant 1$ (ie the sequential closure of the subgroup generated by these elements), is open in λ and doesn't satisfy the above-mentioned property.

For other definitions and details see [Ka3], [F1,F2,F4], [MZh].

3. Pairings of K-groups of higher local fields.

Let F be an n-dimensional local field of characteristic p. For $\alpha_1, \dots, \alpha_n \in F^{\times}$, and a Witt vector $(\beta_0, \dots, \beta_r) \in W_r(F)$ put

$$(\alpha_1, \dots, \alpha_n, (\beta_0, \dots, \beta_r)]_r = \operatorname{Tr}_{k_0/\mathbb{F}_p}(\gamma_0, \dots, \gamma_r)$$

where k_0 is the last finite residue field of F and the ith ghost component $\gamma^{(i)}$ of $(\gamma_0,\dots,\gamma_r)$ is defined as $\operatorname{res}_{k_0}(\beta^{(i)}\alpha_1^{-1}d\alpha_1\wedge\dots\wedge\alpha_n^{-1}d\alpha_n)$.

This is a sequentially continuous symbolic in the first n coordinates map. It defines the *Artin–Schreier–Witt–Parshin pairing*:

$$K_n^{\mathrm{M}}(F)/p^r \times W_r(F)/(\mathrm{Frob}-1)W_r(F) \to W_r(\mathbb{F}_p)$$

where Frob is the Frobenius map.

Let F be an n-dimensional mixed characteristic local field and let a primitive p^rth root of unity ζ be contained in F. Let $X_1,\dots,X_n$ be independent indeterminates over the quotient field of $\mathcal{O}_0$. For an element

$$\alpha = t_n^{a_n}\dots t_1^{a_1}\theta\prod(1+\theta_{i_n,\dots,i_1}t_n^{i_n}\dots t_1^{i_1})$$

of $F^\times$, with $\theta\neq 0, \theta_{i_n,\dots,i_1}\in\mathcal{R}$ put

$$\alpha(X) = X_n^{a_n}\dots X_1^{a_1}\theta\prod(1+\theta_{i_n,\dots,i_1}X_n^{i_n}\dots X_1^{i_1}).$$

The formal power series $\alpha(X)\in\mathcal{O}_0((X_1))\dots((X_n))$ depends on the choice of local parameters and the choice of the decomposition of α. Denote $z(X)=\zeta(X)$, $s(X)=z(X)^{p^r}-1$. Define the action of the operator Δ on θ's and on X_i as raising to the pth power. For $\alpha\in F^\times$ put $l(\alpha)=p^{-1}\log\alpha(X)^{p-\Delta}$.

Now for elements $\alpha_1,\dots,\alpha_{n+1}\in F^\times$ define $\Phi(\alpha_1,\dots,\alpha_{n+1})$ as

$$\sum_{i=1}^{n+1}(-1)^{n+1-i}l(\alpha_i)\left(\frac{\mathrm{d}\,\alpha_1}{\alpha_1}\wedge\dots\wedge\frac{\mathrm{d}\,\alpha_{i-1}}{\alpha_{i-1}}\wedge p^{-1}\frac{\mathrm{d}\,\alpha_{i+1}^{\Delta}}{\alpha_{i+1}^{\Delta}}\wedge\dots\wedge p^{-1}\frac{\mathrm{d}\,\alpha_{n+1}^{\Delta}}{\alpha_{n+1}^{\Delta}}\right).$$

Let μ_{p^r} denote the cyclic group generated by ζ. Define a map

$$V_r\colon (F^\times)^{\oplus n+1}\to\mu_{p^r}$$

as

$$V_r(\alpha_1,\dots,\alpha_{n+1})=\zeta^\gamma,\qquad \gamma=\mathrm{Tr}_{\mathcal{O}_0/\mathbb{Z}_p}\operatorname{res}\Phi(\alpha_1,\dots,\alpha_{n+1})/s(X).$$

It is a very deep result [V2] that V_r doesn't depend on the attaching formal power series to elements of F (for another proof involving syntomic cohomologies of Fontaine–Messing see [Ka8]). The map V_r is sequentially continuous and symbolic. It defines the *Vostokov pairing*:

$$K_n^{\mathrm{M}}(F)/p^r \times F^\times/p^r\to\mu_{p^r}.$$

Let F be an n-dimensional local field and let the cardinality of the group μ of all roots of unity of order prime to p be $q-1$ (in other words, $k_0=\mathbb{F}_q$). For an element $\alpha\in F^\times$ and its expression as above put $v^{(j)}(\alpha)=a_j$ for $1\leqslant j\leqslant n$. For elements $\alpha_1,\dots,\alpha_{n+1}$ of $F^\times$ define $\mathfrak{c}(\alpha_1,\alpha_2,\dots,\alpha_{n+1})$ as the $(q-1)$th root of unity whose residue is equal to the residue of $\alpha_1^{b_1}\dots\alpha_{n+1}^{b_{n+1}}(-1)^b$ in the last residue field k_0, where $b=\sum_{s,i<j}v^{(s)}(\alpha_i)v^{(s)}(\alpha_j)b_{i,j}^s$ and b_j is the determinant of the matrix obtained by omitting the jth column with the sign $(-1)^{j-1}$ from the matrix $A=(v^{(i)}(\alpha_j))$, and $b_{i,j}^s$ is the determinant of the matrix obtained by omitting the ith and jth columns and sth line from A. The map $\mathfrak{c}$ is well defined and is sequentially continuous and symbolic. Therefore, it induces the *tame symbol* — the pairing

$$K_n^{\mathrm{M}}(F)/(q-1)\times F^\times/(q-1)\to\mu.$$

The tame symbol is the composition

$$K_{n+1}^{\mathrm{M}}(F)\xrightarrow{\ \partial\ }K_n^{\mathrm{M}}(k_{n-1})\xrightarrow{\ \partial\ }\dots\xrightarrow{\ \partial\ }K_1(k_0)=\mathbb{F}_q^\times,$$

where ∂ are border homomorphism in K-theory, c.f. [FV, Ch. IX].

4. Topological K-groups.

Let λ_m be the finest topology on $K_m(F)$ for which the map $\phi\colon F^{\times\oplus m} \to K_m(F)$, $\phi(\alpha_1,\dots,\alpha_m) = \{\alpha_1,\dots,\alpha_m\}$ is sequentially continuous with respect to the product of the introduced above topology λ on $F^\times$ and for which the subtraction in $K_m(F)$ is sequentially continuous. Put

$$K_m^{\mathrm{top}}(F) = K_m(F)/\Lambda_m(F)$$

with the quotient topology where $\Lambda_m(F)$ is the intersection of all neighborhoods of 0 with respect to λ_m (hence is a subgroup). From the definition it follows that $\lambda = \lambda_1$ and $K_1^{\mathrm{top}}(F)$ coincides with the quotient of $F^\times$ by the maximal divisible subgroup of it. If the first residue field k_{n-1} is of characteristic p, then $K_1^{\mathrm{top}}(F) = K_1(F)$.

Denote by $VK_m(F)$ the subgroup of $K_m(F)$ generated by V_F. Using the tame symbol defined in 3°, one can show that $K_m(F)$ splits into the direct sum of $VK_m(F)$ and several copies of $\mathbb{Z}$ and the group of roots of unity of order prime to p in F. The induced topology on $VK_m(F)$ from λ_m coincides with the the finest topology on $VK_m(F)$ for which the restriction of ϕ $V_F \oplus F^{\times\oplus m-1} \to VK_m(F)$ is sequentially continuous with respect to the product of the introduced above topology λ on $F^\times$ and for which the subtraction in $VK_m(F)$ is sequentially continuous.

In what follows "open and "close" for subsets of $K_m(F)$ means with respect to λ_m.

The topology λ_m coincides with the finest topology on $K_m(F)$ for which the map ϕ is sequentially continuous with respect to the product of the topology τ on $F^\times$ and for which the subtraction in $K_m(F)$ is sequentially continuous. Every sequentially open (closed) subset with respect to λ_m is open (closed resp.).

A symbolic sequentially continuous homomorphism from the tensor product of m copies of $F^\times$ to a topological Hausdorff group G induces a continuous homomorphism from $K_m^{\mathrm{top}}(F)$ to G. Therefore, the Artin–Schreier–Parshin, Vostokov pairings and tame symbol defined in 3° are factorized through topological K-groups.

The structure of the topological Milnor K-groups of multidimensional local fields is almost completely known (in contrast to the Milnor K-groups). The simplest way to describe it is to use the Artin–Schreier–Witt–Parshin, Vostokov and higher tame pairings. The role of the Artin–Schreier–Witt pairings in one-dimensional case is known from the theory of class formation in characteristic p [KS]. Recall that the Vostokov pairing [V2] has appeared as a multidimensional variant of his explicit formulas [V1] for the Hilbert norm residue symbol in number local fields in case the residue field being of odd characteristic. A general philosophy due to Shafarevich [Sh] as reflection of similarities between Riemann surfaces and algebraic number fields is to find an explicit formula for the p^rth Hilbert symbol, then forgetting about class field theory, and using the pairing correctly defined by the explicit formula develop independently and explicitly class field theory for Kummer extensions. For higher local fields both Artin–Schreier–Witt–Parshin and Vostokov pairings are first applied to determine structure of the quotient group $K_n^{\mathrm{top}}(F)/p^r$ (in characteristic p for arbitrary r). Then they serve as implements to construct Artin–Schreier–Witt and Kummer extensions which correspond via these pairings to open subgroups of finite index in $K_n^{\mathrm{top}}(F)$. Coincidence of both pairings with the corresponding pairings induced from class field theory enables one to deduce existence theorem.

5. Structure of topological K-groups.

For two principal units $\epsilon, \eta \in F^\times$ in $K_2(F)$:

$$\{\epsilon,\eta\} = \{1-\epsilon, 1-(1-\epsilon^{-1})(1-\eta)\} + \{\eta, 1-(1-\epsilon^{-1})(1-\eta)\}$$

with the principal unit $1-(1-\epsilon^{-1})(1-\eta)$ of higher order than that of ϵ,η. So we get

$$\begin{aligned}\{1-\theta t_n^{i_n}\dots t_1^{i_1}, 1-\beta\} = \{&\theta t_n^{i_n}\dots t_1^{i_1}, 1+\theta t_n^{i_n}\dots t_1^{i_1}\beta/(1-\theta t_n^{i_n}\dots t_1^{i_1})\}\\ &-\{1+\theta t_n^{i_n}\dots t_1^{i_1}\beta/(1-\theta t_n^{i_n}\dots t_1^{i_1}), 1-\beta\}.\end{aligned}$$

Continue these transformations and write $\{1-\theta t_n^{i_n}\dots t_1^{i_1}, 1-\beta\}$ in $K_2^{\mathrm{top}}(F)$ as a sum of symbols in the form $\{\rho_i, t_i\}$ with principal units ρ_i and local parameters t_i.

Elements ρ_i sequentially continuously depend on ϵ and η. Note that $\{\theta,\theta'\}=\{\theta,\epsilon\}=0$ for $(q-1)$th roots of unity θ,θ' and a principal unit ϵ. Therefore, every element x of $K_m(F)$ can be written as a sum of a fixed number of terms in the form $\{\alpha_i\}\cdot\{\text{some local parameters}\}$ with $\alpha_i\in F^\times$ plus an element of $\Lambda_m(F)$.

The group $K_m^{\mathrm{top}}(F)$ is topologically generated (with admissible sets playing the same role as in the case of $F^\times$) by the symbols

(1) $\{t_{j_1},\dots,t_{j_m}\}$,
(2) $\{\theta, t_{j_1},\dots,t_{j_{m-1}}\}$ with $\theta\in\mathcal{R}^\times$,
(3) $\{1+\theta t_n^{i_n}\dots t_1^{i_1}, t_{j_1},\dots,t_{j_{m-1}}\}$ with $\theta\in\mathcal{R}$.

Topological relations among these generators (modulo p^r for each r in the case of $\mathrm{char}(F)=p$, modulo p^r in the case $\mathrm{char}(F)=0$ and a primitive p^rth root of unity belongs to F, modulo p if $\mathrm{char}(F)=0$ and a primitive pth root of unity doesn't belong to F) are revealed using the Artin–Schreier–Witt–Parshin, Vostokov and tame pairings, see [P1,P3,P4], [F1–F2]. Simultaneously one verifies that all the pairings are non-degenerate.

In particular, $K_n^{\mathrm{top}}(F)$ is isomorphic to the direct sum of the cyclic group generated by $\{t_1,\dots,t_n\}$, n copies of the cyclic groups of order $q-1$, and the subgroup $VK_n^{\mathrm{top}}(F)$ generated by principal units V_F.

For $m\leqslant n+1$, $d=\binom{n}{m-1}$ define the homomorphism $g\colon V_F^{\oplus d}\to VK_m(F)$,

$$g(\beta_J)=\sum\{\beta_J, t_{j_1},\dots,t_{j_{m-1}}\},$$

where $J=\{j_1,\dots,j_{m-1}\}$ runs over all $m-1$ elements subsets of $\{1,\dots,n\}$.

Theorem 1.
(i) $\mathrm{im}(g)+\Lambda_m(F)=VK_m(F)$ and $g_0\colon V_F^{\oplus d}/g^{-1}(\Lambda_m(F))\to VK_m^{\mathrm{top}}(F)$ is a homeomorphism of $V_F^{\oplus d}/g^{-1}(\Lambda_m(F))$ with the sequential saturation of the quotient topology of the product topology of the λ topologies on $V_F^{\oplus d}$ and of $VK_m^{\mathrm{top}}(F)$ with the topology λ_m.
(ii) $\Lambda_m(F)$ is the intersection of all open subgroups of finite index in $K_m(F)$.
(iii) $p^rVK_m(F)$ is a closed subgroup for every r, $K_m(F)/p^r\simeq K_m^{\mathrm{top}}/p^r$, and $\Lambda_m(F)$ coincides with $\cap_{l\geqslant 1} lK_m^{\mathrm{M}}(F)$.

Proof.
Let Y be a subgroup of $VK_m(F)$ such that $\mathrm{im}(g)+Y=VK_m(F)$ and there is a sequentially continuous map $f_Y\colon V_F\times F^{\times\oplus m-1}\to V_F^{\oplus d}$ such that its composition with g coincides with the restriction of the map ϕ on $V_F\oplus F^{\times\oplus m-1}$ modulo Y.

Let $U+Y$ be an open subset in $VK_m(F)$. Then $g^{-1}(U+Y)$ is open in the sequential saturation of the product topology of the topology λ on $V_F^{\oplus d}$. Indeed, otherwise there were a sequence $\alpha_J^{(i)}\notin g^{-1}(U+Y)$ which converges to $\alpha_J\in g^{-1}(U+Y)$. Then $\alpha_J^{(i)}$ converges to α_J, so by the properties of the map ϕ the sequence $\phi(\alpha_J^{(i)})\notin U+Y$ converges to $\phi(\alpha_J)\in U+Y$ which contradicts the openness of $U+Y$.

On the other hand, the sequential saturation of the quotient topology of λ_m on $VK_m(F)/Y$ is $\geqslant$ the sequential saturation of the product topology of λ on $V_F^{\oplus d}$ via g. Indeed, the sum of two convergent sequences x_i, y_i in $V_F^{\oplus d}/g^{-1}(Y)$ converges to the sum of their limits by 3°.

Thus, $g_Y\colon V_F^{\oplus d}/g^{-1}(Y)\to VK_m(F)/Y$ is a homeomorphism of $V_F^{\oplus d}/g^{-1}(Y)$ with the sequential saturation of the quotient of the product topology of the λ topologies on $V_F^{\oplus d}$ and of $VK_m(F)/Y$ with the sequential saturation of the quotient topology of λ_m.

To deduce (i) apply the previous part to $Y=\Lambda_m(F)$: from the definitions and the beginning of this section it follows that for $\alpha_1\in V_F$, $\alpha_2,\dots,\alpha_m\in F^\times$ there exist elements $\beta_J\in V_F$,

$J = \{j_1, \dots, j_{m-1}\}$, which sequentially continuously depend on $\alpha_1, \dots, \alpha_m$ such that the symbol $\{\alpha_1, \dots, \alpha_m\}$ can be written as $\sum\{\beta_J, t_{j_1}, \dots, t_{j_{m-1}}\} \mod \Lambda_m(F)$.

Note that $\Lambda_m(F)$ is closed: if $x_i \in \Lambda_m(F)$ tends to x, then $x = x_i + y_i$ with x_i, y_i convergent to 0, so x converges to 0 and hence belongs to $\Lambda_m(F)$. To deduce (ii) notice that every closed subgroup in $V_F^{\oplus d}$ is the intersection of certain open subgroups of finite index, hence every closed subgroup in $VK_m^{\mathrm{top}}(F)$ is the intersection of certain open subgroups of finite index.

To deduce (iii) first note that

$$\cap_{l\geqslant 1} lK_m(F) \subset \Lambda_m(F),$$

since $\cap_{l\geqslant 1} lK_m(F) \subset VK_m(F)$; lx for $x \in K_m(F)$ can be written as the sum of symbols $\{\alpha^l\} \cdot \{\text{some local parameters}\}$ and an element of $\Lambda_m(F)$; for any $\alpha_v \in V_F$ the sequence $\alpha_v^{p^v}$, $v \to +\infty$, converges to 1.

The subgroup $p^r VK_m(F) + \Lambda_m(F)$ is closed, since $g^{-1}(p^r VK_m(F) + \Lambda_m(F))$ being the product of $g^{-1}(\Lambda_m(F))$ and $(V_F^{p^r})^{\oplus d}$ is closed according to section 2.

Hence to complete the proof it remains to show that $p^r VK_m(F)$ contains $\Lambda_m(F)$.

Let $\{U\}$ be the set of all open subgroups in $V_F^{\oplus d}$. From the definition of λ_m it follows that the topology on $VK_m(F)$ with the base of neighbourhoods $\{g(U)\}$ is $\leqslant \lambda_m$. Therefore $g(U)$ is open; in particular, $g(U) \supset \Lambda_m(F)$. The closed subgroup $(V_F^{p^r})^{\oplus d}$ is the intersection of open subgroups according to section 2, hence $g((V_F^{p^r})^{\oplus d}) \supset \Lambda_m(F)$. Thus, $p^r VK_m(F) \supset \Lambda_m(F)$, as required.

Remark 1. Using the completeness property of V_F stated in 3° and the previous theorem we deduce that every decreasing sequence of closed subsets in $VK_m^{\mathrm{top}}(F)$ has nonempty intersection.

Remark 2. A sequence in $VK_m^{\mathrm{top}}(F)$ converges if and only if it converges with respect to the topology with open sets $g_{\Lambda_m(F)}(\oplus U)$ where U running over open sugroups with respect to the topology τ described in 3°.

Remark 3. From the proof we deduce that λ_m coincides with the finest topology on $K_m(F)$ for which the map from $(F^{\times\oplus m})^{\oplus d}$ to $K_m(F)$, $(\beta_{j,i}) \to \sum_{1\leqslant i\leqslant d}\{\beta_{1,i}, \dots, \beta_{m,i}\}$, is sequentially continuous. In addition, λ_m coincides with the finest topology on $K_m(F)$ for which for all $\alpha_i \in F^\times$, $1 \leqslant i \leqslant m-1$, the map from $F^\times$ to $K_m(F)$, $\alpha \to \{\alpha, \alpha_1, \dots, \alpha_{m-1}\}$, is sequentially continuous and the subtraction in $K_m(F)$ is sequentially continuous.

Remark 4. Let ν_m be the finest topology on $K_m(F)$ for which the map from $F^{\times\oplus m}$ to $K_m(F)$ is sequentially continuous and the intersection of all neighbourhoods of zero in $K_m(F)$ contains $\cap_{l\geqslant 1} lK_m(F)$ (the topology ν_m was used in [F1,F2]). Then ν_m is $\geqslant \lambda_m$ and the intersection of all neighbourhoods of zero in $K_m(F)$ with respect to ν_m coincides with $\cap_{l\geqslant 1} lK_m(F)$. On the level of subgroups λ_m and ν_m coincide.

Theorem 2.
(i) If a multidimensional local field F of characteristic zero contains a primitive pth root ζ_p of unity and $px = 0$ for $x \in K_m^{\mathrm{top}}(F)$, then $x = \{\zeta_p\} \cdot y$ for some $y \in K_{m-1}^{\mathrm{top}}(F)$.
(ii) For every multidimensional local field F the group $\Lambda_m(F)$ is divisible.
(iii) The sequence $0 \to \Lambda_m(F) \to K_m^{\mathrm{M}}(F) \to K_m^{\mathrm{top}}(F) \to 0$ splits.

Proof. In the case of fields of characteristic 0 it is sufficient to consider the case of a field F of characteristic 0 with residue field of characteristic p.

Let F contain a primitive pth root of unity ζ_p. We will use the theorem of Bloch–Kato [BK] (a shorter proof for higher local fields see in [F1]), which implies that for a higher local field F the norm residue homomorphism

$$K_m(F)/p^s \to H^m(F, \mu_{p^s}^{\otimes m})$$

is an isomorphism for all $m \geqslant 1, s \geqslant 1$.

The exact sequence

$$0 \to \mu_{p^s}^{\otimes m} \to \mu_{p^{s+1}}^{\otimes m} \to \mu_p^{\otimes m} \to 0$$

induces the commutative diagram

$$\begin{array}{ccccc}
\mu_p \otimes K^{\mathrm{top}}_{m-1}(F)/p & \longrightarrow & K^{\mathrm{top}}_m(F)/p^s & \xrightarrow{p} & K^{\mathrm{top}}_m(F)/p^{s+1} \\
\downarrow & & h\downarrow & & \downarrow \\
H^{m-1}(F, \mu_p^{\otimes m}) & \longrightarrow & H^m(F, \mu_{p^s}^{\otimes m}) & \longrightarrow & H^m(F, \mu_{p^{s+1}}^{\otimes m})
\end{array}$$

and the bottom horizontal sequence is exact, the left and the right vertical homomorphisms are isomorphisms. Hence if $pz \in p^r K_m(F)$, then $z = \{\zeta_p\} \cdot a_{r-1} + p^{r-1} c_{r-1}$. Denote by D_r the preimage of the closed subgroup $p^r K_m^{\mathrm{top}}(F)$ with respect to the continuous homomorphism $VK^{\mathrm{top}}_{m-1}(F) \to VK^{\mathrm{top}}_m(F)$, $z \to \{\zeta_p\} \cdot z$. Then $a_r + D_r$ form a decreasing sequence of closed subsets in $VK_m^{\mathrm{top}}(F)$ and according to Remark 1 of Theorem 1 they have a nonempty intersection. For any y in the intersection we have $z - \{\zeta_p\} \cdot y \in p^r K_m^{\mathrm{top}}(F)$ for every r, hence, $z - \{\zeta_p\} \cdot y \in \Lambda_m(F)$.

Let F be of characteristic 0. Let $z \in \Lambda_m(F)$. Then $z = px$ for some $x \in VK_m(F)$. Hence $x = \{\zeta_p\} \cdot y + w$ with $w \in \Lambda_m(F)$. Hence $z = pw$ and $\Lambda_m(F)$ is a divisible group. Then $\Lambda_m(F) = \cap_{l \geqslant 1} l K_m(F)$. If F doesn't contain a primitive p-th root, then pass to $F(\zeta_p)$ and then back using the norm map and the equality $(|F(\zeta_p) : F|, p) = 1$.

In the case of $\mathrm{char}(F) = p$ let J consist of $j_1, \dots, j_{m-1}$ and run all $(m-1)$-elements subsets of $\{1, \dots, n\}$, $m \leqslant n+1$. Let $\mathcal{E}_J$ be the subgroup of V_F generated by $1 + \theta t_n^{i_n} \cdots t_1^{i_1}$, $\theta \in \mathcal{O}_0$, with restrictions that p doesn't divide $\gcd(i_1, \dots, i_n)$ and the smallest index l for which i_l is prime to p doesn't belong to J. Consider the induced topology from λ on $\mathcal{E}_J$. From Theorem 1 and applications of the Artin–Schreier–Witt–Parshin pairing we deduce that there exists an isomorphism and homeomorphism ψ from the group $\oplus_J \mathcal{E}_J$ endowed with the sequential saturation of the product topology onto $VK_m^{\mathrm{top}}(F)$. This provides an explicit and satisfactory description of the topology on $K_m^{\mathrm{top}}(F)$ in the positive characteristic case. Since the group $K_m(F)/\Lambda_m(F) = \psi(\prod \mathcal{E}_J)$ does not have nontrivial p-torsion, we deduce that $\Lambda_m(F)$ is divisible.

Remark. From Theorem 2 it follows that for l running over primes different from characteristic of F the homomorphism $K_m(F) \to \prod H^m(F, \mathbb{Z}_l(m))$ induces the monomorphism $K_m^{\mathrm{top}}(F) \to \prod H^m(F, \mathbb{Z}_l(m))$.

For $\mathrm{char}(F) = 0$ I. B. Zhukov found (applying higher class field theory) a complete algebraic description of $K_n^{\mathrm{top}}(F)$ in several cases (see [Zh2]). In particular, if $T_p K_n^{\mathrm{top}}(F)$ is the topological closure of the p-torsion in $K_n^{\mathrm{top}}(F)$ and F has a local parameter t_n algebraic over $\mathbb{Q}_p$, then $VK_n^{\mathrm{top}}(F)/T_p K_n^{\mathrm{top}}(F)$ possesses a topological basis of the form $\{\varepsilon, t_{n-1}, \dots, t_1\}$ with ε running free $\mathbb{Z}_p$-generators of the group of principal units of the algebraic closure of $\mathbb{Q}_p$ in F modulo its p-primary torsion.

For another approach to topologies on K_m see [Ka6].

6. *Norm.*

It follows easily from 5° that for a cyclic extension L/F of a prime degree $K_m^{\mathrm{top}}(L)$ is generated by $L^{\times}$ over the image of $K_{m-1}^{\mathrm{top}}(F)$.

In characteristic p there is a very simple way to define the norm mapping on topological K-groups:

(1) for a cyclic extension L/F of a prime degree introduce $N_{L/F}: K_m^{\mathrm{top}}(L) \to K_m^{\mathrm{top}}(F)$ as induced by the norm on K_1;

(2) for an arbitrary abelian extension L/F define the norm decomposing L/F in cyclic extension of prime degree.

Correctness of this definition follows easily using the Artin–Schreier–Witt–Parshin and tame pairings. The norm on $K_n^{\mathrm{top}}(L)$ is dual to the map induced by the fields embedding $F \to L$, for details (one should replace K_2 by K_1 there) see [P4].

For an arbitrary multidimensional local field define the norm on $K^{\mathrm{top}}(F)$ as induced from the norm on Milnor K-groups. Compatibility of the just defined norm with induced from the Milnor K-groups follows then from 5°. It is easy to check that the image of a closed subgroup with respect to the norm map is a closed subgroup, the preimage of an open subgroup is an open subgroup.

Hilbert Satz 90 plays a very significant role in K-theory. For a general type of fields and arbitrary cyclic extension it is still only known for K_2. If F is a higher local field, then Hilbert Satz 90 holds for K_n^{top}. The proof uses the description of the torsion in K_n^{top} in 5° and small Hilbert Satz 90:
if L/F is of a prime degree l with a generator σ, then the sequence

$$K_n^{\mathrm{top}}(F)/l \oplus K_n^{\mathrm{top}}(L)/l \xrightarrow{i_{F/L}\oplus(1-\sigma)} K_n^{\mathrm{top}}(L)/l \xrightarrow{N_{L/F}} K_n^{\mathrm{top}}(F)/l$$

is exact, where $i_{F/L}$ is induced by the fields embedding. The latter theorem is verified by explicit calculations in K_n^{top}/l-groups whose structure is completely known due to the tame and Vostokov pairings (adjoin if necessary a primitive lth root of unity ζ and then return without problems as $|F(\zeta):F|$ is prime to l). Similar calculations show that the index of the norm group $N_{L/F}K_n^{\mathrm{top}}(L)$ in $K_n^{\mathrm{top}}(F)$ is finite of order $|L:F|$ if the latter is prime [F1, F2].

Now we review 4 approaches to higher class field theory: of K. Kato [Ka1-Ka7], Y. Koya [Ko1-Ko2], A. N. Parshin [P1-P5] and the author [F1,F2,F4].

7. *Kato's approach.* For a field F K. Kato introduced remarkable groups $H^m(F)$ as follows.

(1) $H^m(F) = \varinjlim H^m(F, \mu_l^{\otimes(m-1)})$ for a field of characteristic 0, where μ_l is the group of all lth roots of unity in F^{sep}, $\mu_l^{\otimes(m-1)}$ is the $(m-1)$th tensor power, $l \geqslant 1$ and the homomorphisms of the inductive system are induced by the canonical injections $\mu_l^{\otimes(m-1)} \to \mu_{l'}^{\otimes(m-1)}$ when l divides l';

(2) $H^m(F) = \varinjlim H^m(F, \mu_l^{\otimes(m-1)}) \oplus \varinjlim H^m_{p^r}(F)$ for $\mathrm{char}(F) = p > 0$, where l runs all positive integers prime to p, r runs all positive integers.

Here $H^m_{p^r}(F) = W_r(F) \otimes \underbrace{(F^\times \otimes \dots \otimes F^\times)}_{m-1 \text{ times}}/J$, where J is the subgroup generated by the following three types of elements:

a) $(\mathrm{Frob}(y) - y) \otimes \beta_1 \otimes \dots \otimes \beta_{m-1}$ with $y \in W_r(F)$, $\beta_i \in F^\times$;

b) $(\underbrace{0, \dots, 0}_{i \text{ times}}, \beta_1, 0, \dots, 0)(\beta_1) \otimes \beta_1 \otimes \dots \otimes \beta_{m-1}$ with $0 \leqslant i < r$;

c) $y \otimes \beta_1 \otimes \dots \otimes \beta_{m-1}$ with $\beta_i = \beta_j$ for some $i \neq j$.

Equivalently one can put $H^m_{p^r}(F)$ to be $H^1(F, W_r\Omega^{m-1}_{F,\log})$ where $W_r\Omega^{m-1}_{F,\log}$ is the logarithmic part of the De Rham–Witt complex.

For every field F the group $H^1(F)$ is isomorphic to the group of all continuous homomorphisms $\mathrm{Gal}(F^{\mathrm{ab}}/F) \to \mathbb{Q}/\mathbb{Z}$ and $H^2(F)$ is isomorphic to $\mathrm{Br}(F)$.

For an n-dimensional local field F a celebrated theorem of Kato claims that there exists a canonical homomorphism $H^{n+1}(F) \simeq \mathbb{Q}/\mathbb{Z}$. This is an analog of the classical theorem describing the Brauer group of a local field with finite residue field. The proof of Kato's theorem is easy for the prime to p part where it follows from typical arguments involving the Hochshild–Serre spectral sequence. The proof is more difficult for the p part and relies in particular on relations among quotients of Milnor K-groups, Galois cohomology groups and subquotient modules in the module of differentials of fields of positive characteristic. Some ingredients are the theorem of Kato on the residue symbol $K_n(F)/p \to H^n(F, \mathbb{Z}/p)$ mentioned in 5° and the study of the cohomological residue $H^{n+1}(F, \mu_l^{\otimes(n)}) \to H^n(k_{n-1}, \mu_l^{\otimes(n-1)})$, see [Ka3–Ka4], [R]. In fact, many results established by Kato in [Ka3–Ka4] hold for arbitrary complete discrete valuation fields.

Using the canonical pairing

$$H^1(F)\times K_n(F)\to H^1(F)\times H^n(F)\to H^{n+1}(F)\simeq \mathbb{Q}/\mathbb{Z}$$

one obtains the *higher local reciprocity map*

$$\Psi_F\colon K_n(F)\to \mathrm{Gal}(F^{\mathrm{ab}}/F).$$

It describes finite abelian extensions L/F in a sense that Ψ_F induces an isomorphism of the group $K_n(F)/N_{L/F}K_n(L)$ onto $\mathrm{Gal}(L/F)$.

Kato has also proved existence theorem which describes norm subgroups [Ka6]. His approach is different from that of 10°.

8. Koya's approach.

The previous theory can be treated as a generalization of Tate's approach in classical class field theory. For one-dimensional fields the notion of formation of classes seems to be the most standard way to define the reciprocity map. There is an important obstruction for its generalization to higher-dimensional fields — already for (>1)-dimensional fields the Galois descent for K_n^{top} fails: if L/F is a finite Galois extension, then $i_{F/L}\colon K_n^{\mathrm{top}}(F)\to K_n^{\mathrm{top}}(L)$ induced by $F\to L$ isn't in general injective, and $i_{F/L}K_n^{\mathrm{top}}(F)$ doesn't in general coincide with the $\mathrm{Gal}(L/F)$-invariant elements of $K_n^{\mathrm{top}}(L)$ (the same is true for Milnor K-groups).

For instance, consider $K=\mathbb{Q}_p(\zeta)$ with a primitive pth root of unity ζ. Let ω be a p-primary element in K, i.e. a principal unit such that $K_1=K(\sqrt[p]{\omega})$ is the unramified extension of degree p over K. Let t be a transcendental element over K and let $F=K\{\{t\}\}$. Consider the totally ramified with respect to the 2-dimensional structure extension $L=F(\sqrt[p]{t})$ of degree p. Put $F_1=FK_1$, $L_1=LF_1$. Then, according to properties of K_2 of a local field (see, e.g., [FV, Ch. IX]) for a prime element $\pi=1-\zeta$ of K the symbol $i_{K/K_1}\{\omega,\pi\}$ is a divisible element in $K_2(K_1)$, since $i_{K/K_1}\{\omega,\pi\}$ belongs to $pK_2(K_1)$. Hence $i_{F/F_1}(i_{K/F}\{\omega,\pi\})=0$ in $K_2^{\mathrm{top}}(F_1)$, but from explicit calculations it follows that $i_{K/F}\{\omega,\pi\}\notin N_{L/F}K_2^{\mathrm{top}}(L)$. Since $\sigma\{\sqrt[p]{t},\pi\}-\{\sqrt[p]{t},\pi\}$ is a multiple of $\{\zeta,1-\zeta\}=0$ for a generator σ of L/F, the symbol $\{\sqrt[p]{t},\pi\}$ is σ-invariant but it doesn't belong to $i_{F/L}K_2^{\mathrm{top}}(F)$.

Y. Koya found a class formation approach to higher class field theory using bounded complexes of Galois modules and their modified hypercohomology groups $\widehat{\mathbb{H}}$ instead of respectively Galois modules and their modified (Tate) cohomology groups [Ko1-Ko2]. His generalized Tate–Nakayama theorem claims that if for a finite group G and a bounded complex $\mathbb{A}$ of G-modules there is an element $a\in\widehat{\mathbb{H}}^2(G,\mathbb{A})$ such that for every prime l and Sylow l-group G_l of G the group $\widehat{\mathbb{H}}^1(G_l,\mathbb{A})$ is trivial and the group $\widehat{\mathbb{H}}^2(G_l,\mathbb{A})$ is generated by the symbol $\mathrm{res}_{G/G_l}(a)$ of order $|G_l|$, then for every subgroup H of G and $i\in\mathbb{Z}$ the cup-product with $\mathrm{res}_{G/H}(a)$ induces an isomorphism of the Tate cohomology group $\widehat{H}^{i-2}(H,\mathbb{Z})$ onto $\widehat{\mathbb{H}}^i(H,\mathbb{A})$.

Koya's generalized axioms of class formation for a profinite group G and a bounded complex $\mathbb{A}$ of G-modules are the following:

(1) $\widehat{\mathbb{H}}^i(U,\mathbb{A})=0$ for every open subgroup U in G and $i=1$;
(2) for every open subgroup U in G there is a canonical isomorphism

$$\mathrm{inv}_U\colon\widehat{\mathbb{H}}^2(U,\mathbb{A})\to\mathbb{Q}/\mathbb{Z}$$

such that $\mathrm{inv}_V\circ\mathrm{res}_{U/V}=|U:V|\,\mathrm{inv}_U$ for every pair of open subgroups $V\subset U$ in G.

For a 2-dimensional local field F it follows from results of S. Saito [S] and Koya [Ko1] that the shifted Lichtenbaum complex $\mathbb{Z}(2)_F[2]$ satisfies the generalized axioms of class formation. Then for every open normal subgroup $U=\mathrm{Gal}(F^{\mathrm{sep}}/L)$ of $G_F=\mathrm{Gal}(F^{\mathrm{sep}}/F)$ the finite group $\mathrm{Gal}(L/F)$ and the complex $\mathbb{A}=\tau_{\leqslant 0}(\mathbb{R}\Gamma(U,\mathbb{Z}(2)_F[2]))$ satisfy the assumption of the generalized

Tate–Nakayama theorem. The isomorphism of this theorem for $i = 0$ implies the isomorphism $K_2(F)/N_{L/F}K_2(L) \to \mathrm{Gal}(L/F)^{\mathrm{ab}}$. This is a more general result than that in 7°, because it provides certain information about nonabelian extensions.

Since a Lichtenbaum complex $\mathbb{Z}(n)$ for $n > 2$ has not yet been constructed, one can't extend this approach for higher dimensions directly. Recently M. Spieß [Sp] proved that for an n-dimensional local field F the shifted complex $\mathbb{Z}'(n)_F[n]$, where $\mathbb{Z}'(n)$ either is the decomposable part of motivic cohomology $\mathbb{G}_m \otimes^{\mathbb{L}_n}[-n]$ studied by B. Kahn in [Kn] or is $\tau_{\geqslant 0}\mathcal{Z}^n(F^{\mathrm{sep}}, 2n-.)$ with $\mathcal{Z}^{\cdot}(F^{\mathrm{sep}}, \cdot)$ being the Bloch complex [B], satisfies Koya's class formation axioms. Thus there is an isomorphism

$$K_n(F)/N_{L/F}K_n(L) \to \mathrm{Gal}(L/F)^{\mathrm{ab}}.$$

9. Parshin's approach.

In fact, we need to say only few things in addition to those in 2°–4°, 6° in order to describe the Kawada–Satake–Parshin method to construct the reciprocity map in characteristic p.

Recall that for an n-dimensional local field F of characteristic p the torsion subgroup $TK_n^{\mathrm{top}}(F)$ in $K_n^{\mathrm{top}}(F)$ is isomorphic to $(k_0^{\times})^n$, $k_0 = \mathbb{F}_q$ (see 5°). Put $\tilde{K}(F) = K_n^{\mathrm{top}}(F)/TK_n^{\mathrm{top}}(F)$.

Non-degeneracy of the tame pairing of 4° and the Kummer theory provide the prime to p part map $K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F(\sqrt[q-1]{F^{\times}})/F)$, that of the Artin–Schreier–Witt–Parshin pairing — the p part map $K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F^{\mathrm{abp}}/F)$ where F^{abp} is the maximal abelian p-extension of F. There is the third map which transforms the symbol $\{t_1, \dots, t_n\}$ for a system of local parameters $t_n, \dots, t_1$ in F to the lifting of the Frobenius automorphism on $F \otimes_{\mathbb{F}_q} \mathbb{F}_q^{\mathrm{sep}}$. All three maps are compatible, and their stitching is the *reciprocity map*

$$K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F).$$

Thus, the whole construction of the reciprocity map in the Parshin theory is cohomology free. In contrast to the Milnor K-groups used in the previous theories, the group $K_n^{\mathrm{top}}(F)$ describing abelian extensions is completely known, see 5°– 6°.

10. Explicit higher local class field theory.

Finally, we describe main ideas of the approach to higher local class field theories in [F1,F2,F4]. This approach is a nontrivial generalization of two explicit constructions of the one dimensional reciprocity map and its inverse one for classical local fields due to M. Hazewinkel [H1–H3] and J. Neukirch [N1–N2], essential ingredients are the tame, Artin–Schreier–Witt–Parshin, Vostokov pairings and topological K-groups.

For an n-dimensional local field F denote by $v_F\colon K_n^{\mathrm{top}}(F) \to \mathbb{Z}$ the composition

$$K_n^{\mathrm{top}}(F) \xrightarrow{\ \partial\ } K_{n-1}^{\mathrm{top}}(k_{n-1}) \xrightarrow{\ \partial\ } \dots \xrightarrow{\ \partial\ } K_0(k_0) = \mathbb{Z}\ ,$$

where ∂ are the border homomorphism in K-theory. An element Π_F of $K_n^{\mathrm{top}}(F)$ which is mapped to 1 is called prime. Its role in higher class field theory is in many respects similar to the role of a prime element of a classical local field. Given a system of local parameters $t_n, \dots, t_1$ a prime element can be written as $\{t_1, \dots, t_n\} + \epsilon$ with $\epsilon \in \ker v_F$. Let $\widetilde{F}$ be the maximal purely unramified extension of F i.e. the unramified extension with respect to n structure corresponding to k_0^{sep}/k. The Galois group of $\widetilde{F}/F$ has a canonical generator — the lifting of the Frobenius automorphism from G_{k_0} which is called by the same name.

The inverse map to the reciprocity map can be explicitly described as follows: let L/F be a finite Galois extension, attach to an automorphism σ the element $N_{\Sigma/F}\Pi_{\Sigma} \mod N_{L/F}K_n^{\mathrm{top}}(L)$ where Π_{Σ} is any prime element of $K_n^{\mathrm{top}}(\Sigma)$ and Σ is the fixed field of a lifting of the σ on $\mathrm{Gal}(\tilde{L}/F)$ such that its restriction on $\mathrm{Gal}(\tilde{F}/F)$ is a positive integer power of the Frobenius automorphism. This is a direct generalization of the Neukirch definition in the classical case. The main result is that the map just defined doesn't depend on the choice of lifting of σ and the choice of a prime element, and induces an isomorphism of groups

$$\mathcal{Y}_{L/F}\colon \mathrm{Gal}(L/F)^{\mathrm{ab}} \to K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L)$$

[F1,F2,F4].

The proof is essentially based on Hilbert Satz 90 and the norm index calculation for extensions of a prime degree in 6°. It is convenient to introduce the second map acting in inverse direction from topological K-groups to the Galois group as a generalization of the Hazewinkel description of the reciprocity map for classical local fields. However, in complete extent this can be done only in characteristic p.

Put $K_n^{\mathrm{top}}(\widetilde{F}) = \varinjlim K_n^{\mathrm{top}}(F')$ with F' running finite subextensions of F' in $\widetilde{F}$.

For the fields of positive characteristic the Galois descent for the topological K_n-groups holds (see, for example, [F2]). Given a finite Galois extension L/F linearly disjoint with $\widetilde{F}/F$ denote by $V(L|F)$ the subgroup in $K_n^{\mathrm{top}}(\widetilde{L})$ generated by elements $\sigma\alpha-\alpha$ with $\sigma \in \mathrm{Gal}(\widetilde{L}/\widetilde{F})$, $\alpha \in VK_n^{\mathrm{top}}(\widetilde{L})$. Then the sequence

$$1 \longrightarrow \mathrm{Gal}(\widetilde{L}/\widetilde{F}) \xrightarrow{\ c\ } K_n^{\mathrm{top}}(\widetilde{L})/V(L|F) \xrightarrow{N_{\widetilde{L}/\widetilde{F}}} K_n^{\mathrm{top}}(\widetilde{F}) \longrightarrow 0$$

is exact where $c(\sigma) = \sigma\Pi_L - \Pi_L$ modulo $V(L|F)$ doesn't depend on the choice of Π_L [F4].

This allows one to define for a finite Galois extension L/F linearly disjoint with $\widetilde{F}/F$ a generalization of the Hazewinkel homomorphism

$$\Psi_{L/F}\colon K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L) \to \mathrm{Gal}(L/F)^{\mathrm{ab}}$$

as follows. Given an element $\epsilon \in \ker v_F$ write it as $N_{\widetilde{L}/\widetilde{F}}\eta$ with $\eta \in K_n^{\mathrm{top}}(\widetilde{L})$. Then for a lifting $\varphi \in \mathrm{Gal}(\widetilde{L}/F)$ of the Frobenius automorphism the element $\varphi\eta - \eta$ belongs to the kernel of $N_{\widetilde{L}/\widetilde{F}}$ and according to the description of this kernel can be written as $\tilde{\sigma}\Pi_L - \Pi_L$ modulo $V(L|F)$ with $\tilde{\sigma} \in \mathrm{Gal}(\widetilde{L}/\widetilde{F})$. The generalized Hazewinkel map attaches the automorphism $\tilde{\sigma}^{-1}|_{L\cap F^{\mathrm{ab}}}$ to $\epsilon \bmod N_{L/F}K_n^{\mathrm{top}}(L)$. It is a well defined homomorphism. The generalized Neukirch and Hazewinkel maps are inverse to each other and thus are isomorphisms. The whole theory here is cohomology free similar to the Parshin theory.

In the case of characteristic zero all essential problems are concentrated in p-extensions. There is a class of p-extensions which are very close to extensions in positive characteristic — so-called $\wp$-extensions (or Artin–Schreier towers) which are towers of subsequent cyclic extensions of degree p generated at each step by roots of an Artin–Schreier polynomial $\wp(X) - a = X^p - X - a$. One can prove that for a cyclic $\wp$-extension L/F linearly disjoint with $\widetilde{F}/F$ a weak Galois descent holds: the homomorphism v_F maps the $\mathrm{Gal}(\widetilde{L}/\widetilde{F})$-invariant elements of $K_n^{\mathrm{top}}(\widetilde{L})$ onto $|L:F|\mathbb{Z}$ [F4, sect. 3]. A generalized Hazewinkel map $\Psi_{L/F}$ for an arbitrary extension in the case of characteristic zero doesn't exist, see 8°. However, it can be defined by the same rule as in the positive characteristic case above for a finite Galois p-extension L/F linearly disjoint with $\widetilde{F}/F$ which is an Artin–Schreier tree (AST) that means that every cyclic intermediate subextension in L/F is a $\wp$-extension. Then for an AST-extension L/F the composition $\Psi_{L/F} \circ \mathcal{Y}_{L\cap F^{\mathrm{ab}}/F}$ is identity. AST-extensions are "dense" in the class of all p-extensions: for a finite Galois p-extension linearly disjoint with $\widetilde{F}/F$ there exists a p-extension Q/F linearly disjoint with $\widetilde{F}/F$ such that $\widetilde{Q} \cap \widetilde{L} = \widetilde{F}$ and every intermediate cyclic extension in LQ/Q is an AST-extension. This allows one to prove that $\mathcal{Y}_{L/F}$ is an isomorphism [F4]. For another proof when three assertions: Hilbert Satz 90, $|K_n^{\mathrm{top}}(F) : N_{L/F}K_n^{\mathrm{top}}(L)| = |L:F|$, $\mathcal{Y}_{L/F}$ is an isomorphism are verified for a cyclic extension L/F by simultaneous induction on degree see [F1].

Now for an n-dimensional local field F passing to the projective limit for $\Psi_{L/F}$ when L/F runs all abelian subextensions in F^{ab}/F we obtain the *reciprocity map*

$$\Psi_F^{\mathrm{top}}\colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F).$$

It is compatible with the reciprocity maps defined in 7°–10°.

The reciprocity map Ψ_F^{top} is injective and its image is dense in $\mathrm{Gal}(F^{\mathrm{ab}}/F)$. The maximal divisible subgroup $\Lambda_n(F)$ of $K_n(F)$ coincides with the intersection of all open subgroups of finite index in $K_n(F)$ by 5°; the latter is the kernel of Ψ_F due to existence theorem: the lattice of open subgroups of finite index in $K_n^{\mathrm{top}}(F)$ is in an order reversing bijection with the lattice of the finite abelian extensions L/F, $L \to N_{L/F}K_n^{\mathrm{top}}(L)$ [F1,F2].

Using the description of the topology on the Milnor K-groups one can verify that for a finite Galois extension M/F the preimage of an open subgroup of finite index in $K_n^{\mathrm{top}}(F)$ is an open subgroup of finite index in $K_n^{\mathrm{top}}(M)$ and $N_{M/F}K_n^{\mathrm{top}}(M)$ is an open subgroup of finite index in $K_n^{\mathrm{top}}(F)$. Then it is sufficient to prove existence theorem for a prime index. The abelian extension attached to an open subgroup is constructed then as corresponding to the annihilator of the open subgroup via the Artin–Schreier–Witt–Parshin, Vostokov and tame pairings (again, if necessarily, adjoining a root of unity and then descending).

For another approach to existence theorem see [Ka6].

There are several works on class field theory of a local field with a global residue field in terms of K_2-idele groups: [Ka5], [Ko3], [Kuc]. In the next part we describe totally ramified abelian p-extensions of a complete discrete valuation field with arbitrary residue field of characteristic p in terms of the group of principal units.

Totally ramified abelian p-extensions and the group of principal units

Let F be a complete discrete valuation field with residue field $\overline{F}$ of characteristic p. In this part we deal with reciprocity maps describing abelian totally ramified p-extensions of F in terms of subquotients of the group of principal units of F [F3,F5]. We indicate then their relations with Miki's [M2] and Kurihara's results [Kur2].

11. Perfect residue fields.

Let F be a local field with a perfect residue field $\overline{F}$ of characteristic $p > 0$. Let $\wp(X)$ denote as above the polynomial $X^p - X$. Put $\kappa = \dim_{\mathbb{F}_p} \overline{F}/\wp(\overline{F})$. We will assume that $\kappa \neq 0$, the case $\kappa = 0$ when the field $\overline{F}$ is algebraically p-closed may be treated as a limit of class field theories of local fields with nonalgebraically p-closed residue field tending to $\overline{F}$.

To describe the maximal abelian extension F^{ab}/F one must study abelian prime to p-extensions and abelian p-extensions. Totally tamely ramified abelian extensions over F are easily described by the Kummer theory, since every such extension L/F is generated by adjoining a root $\sqrt[l]{\pi}$ for a suitable prime element π in F and in this case a primitive lth root of unity belongs to F.

The description of the maximal unramified abelian p-extension follows from the Witt theory. Thus, the nontrivial part is study of abelian totally ramified p-extensions of F. A variant of their description in terms of constant pro-quasi-algebraic groups as a generalization of geometric Serre's class field theory was furnished by M. Hazewinkel ([H1–H2]). We describe another approach to abelian totally ramified p-extensions which is cohomology free and of explicit nature [F3].

Let $\widetilde{F}$ denote the maximal abelian unramified p-extension of F. The Witt theory shows that $\mathrm{Gal}(\widetilde{F}/F) \simeq \prod_\kappa \mathbb{Z}_p$. Denote by $\widehat{F}$ the maximal unramified p-extension of F. The Galois group of $\widehat{F}/F$ is a free pro-p-group and the group $\mathrm{Gal}(\widetilde{F}/F)$ is its maximal abelian quotient. The residue field of $\widehat{F}$ does not have nontrivial separable p-extensions.

Let L/F be a Galois totally ramified p-extension, then $\mathrm{Gal}(L/F)$ can be identified with $\mathrm{Gal}(\widetilde{L}/\widetilde{F})$ and with $\mathrm{Gal}(\widehat{L}/\widehat{F})$. Let

$$\mathrm{Gal}(L/F)^{\sim} = \mathrm{Hom}_{\mathbb{Z}_p}\big(\mathrm{Gal}(\widetilde{F}/F), \mathrm{Gal}(L/F)\big)$$

denote the group of continuous homomorphisms from profinite group $\mathrm{Gal}(\widetilde{F}/F)$, which is a $\mathbb{Z}_p$-module ($a \cdot \sigma = \sigma^a$, $a \in \mathbb{Z}_p$), to the discrete $\mathbb{Z}_p$-module $\mathrm{Gal}(L/F)$. This group is isomorphic (non-canonically) with $\oplus_\kappa \operatorname{Gal}(L/F)$. Denote by

$$\operatorname{Gal}(L/F)^\frown = \operatorname{Hom}\big(\operatorname{Gal}(\widehat{F}/F), \operatorname{Gal}(L/F)\big)$$

the group of continuous homomorphisms from the profinite group $\operatorname{Gal}(\widehat{F}/F)$ to the discrete group $\operatorname{Gal}(L/F)$. If L/F is abelian then $\operatorname{Gal}(L/F)^\frown = \operatorname{Gal}(L/F)^\smile$.

Now let L/F be of finite degree. Let $\chi \in \operatorname{Gal}(L/F)^\frown$ and Σ_χ be the fixed field of all $\tau_\varphi \in \operatorname{Gal}(\widehat{L}/F)$, where $\tau_\varphi|_{\widehat{F}} = \varphi, \tau_\varphi|_L = \chi(\varphi)$ and φ runs a topological basis of $\operatorname{Gal}(\widehat{F}/F)$. Then $\Sigma_\chi \cap \widehat{F} = F$, i.e., Σ_χ/F is a totally ramified p-extension. Let U_F and $U_{1,F}$ be the groups of units and the group of principal units of F respectively. Let π_χ be a prime element of Σ_χ. Put

$$\Upsilon_{L/F}(\chi) = N_{\Sigma_\chi/F}\pi_\chi N_{L/F}\pi_L^{-1} \mod N_{L/F}U_L,$$

where π_L is a prime element in L. The group $U_F/N_{L/F}U_L$ is mapped isomorphically onto the group $U_{1,F}/N_{L/F}U_{1,L}$ (multiplicative representatives are mapped to 1). So we obtain the map

$$\operatorname{Gal}(L/F)^\frown \to \operatorname{Gal}(L \cap F^{\mathrm{ab}}/F)^\frown = \operatorname{Gal}(L \cap F^{\mathrm{ab}}/F)^\smile \to U_{1,F}/N_{L/F}U_{1,L}$$

which is well defined and is another generalization of the Neukirch map of [N1–N2]. Denote

$$\Upsilon_{L/F}\colon \operatorname{Gal}(L \cap F^{\mathrm{ab}}/F)^\frown \to U_{1,F}/N_{L/F}U_{1,L}.$$

There is an analogue of the exact sequence in 10°:

$$1 \longrightarrow \operatorname{Gal}(\widehat{L}/\widehat{F})^{\mathrm{ab}} \longrightarrow U_{1,\widehat{L}}/V(L|F) \xrightarrow{N_{\widehat{L}/\widehat{F}}} U_{1,\widehat{F}} \longrightarrow 1,$$

where $V(L|F)$ is generated by $\epsilon^{\sigma-1}$ with $\epsilon \in U_{1,\widehat{L}}$. Now define the generalized Hazewinkel map $\Psi_{L/F}$ as follows. Let $\epsilon \in U_{1,F}$ and $\phi \in \operatorname{Gal}(\widehat{F}/F)$. Write $\epsilon = N_{\widehat{L}/\widehat{F}}\eta$ with $\eta \in U_{1,\widehat{L}}$. Let $\varphi \in \operatorname{Gal}(\widehat{L}/F)$ be a lifting of ϕ. Then $\eta^{-1}\varphi(\eta) \equiv \pi_L\sigma(\pi_L^{-1}) \mod V(L|F)$ for a suitable $\sigma \in \operatorname{Gal}(\widehat{L}/\widehat{F})^{\mathrm{ab}}$ where π_L is a prime element in L. Set $\chi(\phi) = \sigma|_{L\cap F^{\mathrm{ab}}}$. Then $\chi \in \operatorname{Gal}(L \cap F^{\mathrm{ab}}/F)^\frown$. Put $\Psi_{L/F}(\epsilon) = \chi$. The main result is that $\Upsilon_{L/F}$ and $\Psi_{L/F}$ are inverse isomorphisms with natural functorial properties [F3]. Thus, the quotients $U_{1,F}/N_{L/F}U_{1,L}$ still as in classical theories describe abelian extensions. However they are roughly κ times larger than the Galois group of L/F.

Passing to the projective limit one obtains the *reciprocity map*

$$\Psi_F\colon U_{1,F} \to \operatorname{Hom}_{\mathbb{Z}_p}\big(\operatorname{Gal}(\widetilde{F}/F), \operatorname{Gal}(F^{\mathrm{abp}}/\widetilde{F})\big),$$

where $U_{1,F}$ is the group of principal units, F^{abp} is the maximal abelian p-extension of F.

By using extended theory of additive polynomials one can describe for a fixed prime element π of F those open subgroups of finite index in $U_{1,F}$ (normic subgroups) which are norm groups $N_{L/F}U_{1,L}$ for finite abelian totally ramified p-extensions L/F such that π belongs to $N_{L/F}L^\times$. Existence theorem in the perfect residue field case claims that for a fixed prime π in F the lattice of abelian extensions L/F such that $\pi \in N_{L/F}L^\times$ is in order reversing bijection with the lattice of normic subgroups in $U_{1,F}$ [F3].

We note that there is a synthesis of the theories of 10° and 12°: a description of totally ramified with respect to n-dimensional structure abelian p-extensions of an n-dimensional local field with last residue field being perfect of characteristic p [F4].

12. General residue field case.

Let F be a complete discrete valuation field with arbitrary residue field $\overline{F}$ which isn't separably p-closed.

Denote again by $\widetilde{F}$ the maximal unramified abelian p-extension of F, i.e. the unramified extension corresponding to the maximal abelian p-extension $\overline{F}^{\mathrm{abp}}$ of the residue field $\overline{F}$. Denote by $\widehat{F}$ the maximal unramified p-extension of F.

Let L/F be a totally ramified finite Galois p-extension.

In the same way as in the perfect residue field case introduce the generalized Neukirch map

$$\Upsilon_{L/F}\colon \mathrm{Gal}(L\cap F^{\mathrm{ab}}/F)^{\widehat{}} \to (U_{1,F}\cap N_{\widehat{L}/\widehat{F}}U_{1,\widehat{L}})/N_{L/F}U_{1,L}.$$

Assume that the residue field of F is not perfect. Denote by $\mathcal{F}=\mathrm{P}(F)$ a complete discrete valuation field which is an extension of F such that $e(\mathcal{F}|F)=1$ and the residue field of $\mathcal{F}$ is the perfection $\overline{F}^{\mathrm{perf}}=\cup_{i\geqslant 1}\overline{F}^{p^{-i}}$ of the residue field of F ($\mathcal{F}$ isn't uniquely defined). In the same way define $\mathfrak{F}=\mathrm{P}(\widehat{F})$.

For $\sigma\in\mathrm{Gal}(L/F)$ put $c(\sigma)={\pi_L}^{-1}\sigma\pi_L \mod V(L|F)$, where π_L is a prime element in L, and $V(L|F)$ is the subgroup of $U_{1,\widehat{L}}$ generated by the elements $\epsilon^{-1}\sigma(\epsilon)$ with $\epsilon\in U_{1,\mathfrak{L}}$, $\sigma\in\mathrm{Gal}(L/F)$, $\mathfrak{L}=L\mathfrak{F}$. Then the sequence

$$1\to\mathrm{Gal}(L/F)^{\mathrm{ab}}\xrightarrow{c} U_{1,\widehat{L}}/V(L|F)\xrightarrow{N_{\widehat{L}/\widehat{F}}} N_{\widehat{L}/\widehat{F}}U_{1,\widehat{L}}\to 1$$

analogous to 10° and 11° is exact.

Similarly to the above one can introduce a reciprocity map $\Psi_{L/F}$ acting in the inverse direction with respect to $\Upsilon_{L/F}$. The generalized Hazewinkel map $\Psi_{L/F}\colon (U_{1,F}\cap N_{\widehat{L}/\widehat{F}}U_{1,\widehat{L}})/N_{L/F}U_{1,L}\to \mathrm{Gal}(L\cap F^{\mathrm{ab}}/F)^{\widehat{}}$ is well defined and a homomorphism [F5]. The composition $\Psi_{L/F}\circ\Upsilon_{L/F}$ is the identity map.

The previous maps induce an isomorphism

$$(U_{1,F}\cap N_{\widehat{L}/\widehat{F}}U_{1,\widehat{L}})/N_*(L/F)\simeq\mathrm{Gal}(L\cap F^{\mathrm{ab}}/F)^{\widetilde{}}$$

where

$$N_*(L/F)=U_{1,F}\cap N_{\widehat{L}/\widehat{F}}U_{1,\widehat{L}}\cap N_{\mathcal{L}/\mathcal{F}}U_{1,\mathcal{L}}$$

is the group of elements of $U_{1,F}$ which are norms at the level of the maximal unramified p-extension (where the residue field is separably p-closed) and at the level of $\mathcal{F}$ (where the residue field is perfect).

In contrast to all previous class field theories a new problem comes on to the stage. The objects which describe abelian extensions in this case are not very simple especially because of the term $N_{\widehat{L}/\widehat{F}}U_{1,\widehat{L}}$. And for a finite abelian totally ramified p-extension L/F there is no a priori as in other class field theories induction on degree

$$(*)\qquad N_{M/F}U_{1,M}\cap N_{\widehat{E}/\widehat{F}}U_{1,\widehat{E}}=N_{M/F}(U_{1,M}\cap N_{\widehat{E}/\widehat{M}}U_{1,\widehat{E}})$$

for every subextension $M/F\subset E/F$ in L/F. One can show that the property (*) holds if and only if $\Psi_{L/F}$ and $\Upsilon_{L/F}$ are isomorphisms [F5].

If L/F is a cyclic extension, then the property (*) holds, thus we obtain that $\Psi_{L/F}\colon (U_{1,F}\cap N_{\widehat{L}/\widehat{F}}U_{1,\widehat{L}})/N_{L/F}U_{1,L}\to\mathrm{Gal}(L/F)^{\widetilde{}}$ is an isomorphism [F5]. Moreover, the left hand side is isomorphic to $(U_{1,F}\cap N_{\widetilde{L}/\widetilde{F}}U_{1,\widetilde{L}})/N_{L/F}U_{1,L}$.

If the residue field of F is imperfect, one can show that $\Psi_{L/F}$ is an isomorphism in the following cases: (1) L is the compositum of cyclic extensions M_i over F, $1\leqslant i\leqslant m$, such that all the breaks of $\mathrm{Gal}(M_i/F)$ with respect to the upper numbering are not greater than every break of $\mathrm{Gal}(M_{i+1}/F)$ for all $1\leqslant i\leqslant m-1$; (2) $\mathrm{Gal}(L/F)$ is the product of cyclic groups of order p and a cyclic group.

Merits of the theory just exposed with respect to higher local class field theories are more simple structure of the objects in comparison to K-groups and more independence of a concrete type of the residue field. The main demerit is that only totally ramified abelian extensions are covered, and not abelian extensions with inseparable residue field extension.

Miki in [M2] has shown without explicit introduction of reciprocity maps that for a totally ramified cyclic extension L/F of degree m and for a finite abelian unramified extension E/F of exponent m the group $(F \cap N_{EL/E}U_{EL})/N_{L/F}U_L$ is canonically isomorphic to the character group of $\mathrm{Gal}(E/F)$.

13. Some applications.

The description of the kernel of $\Psi_{L/F}$ for cyclic extensions has numerous applications. First, one can show that for an abelian totally ramified p-extension E/F the norm groups $N_{L/F}U_{1,L}$ are in bijection with subextensions L/F of the extension E/F.

A deeper result is that for a complete discrete valuation field F with non-separable-p-closed residue field the norm group $N_{L/F}L^\times$ is uniquely determined by an abelian totally ramified p-extension L/F [F5]: for abelian totally ramified p-extensions L_1, L_2 over F the equality of their norm groups $N_{L_1/F}L_1^\times = N_{L_2/F}L_2^\times$ holds if and only if $L_1 = L_2$. This generalizes the classical assertion to the most possible extent.

In the case of imperfect residue field, one needs additional information in comparison with the perfect residue field case about the structure of norm subgroups. Existence theorem seems to be very difficult even to formulate. This is natural in view of the description of the norm groups in multidimensional class field theory where one uses all power of topological K-groups. However, for cyclic extensions of the fields with the absolute ramification index 1 there is a satisfactory description of norm groups.

For a complete discrete valuation field F of characteristic 0 with residue field $\overline{F}$ of characteristic $p > 2$, absolute ramification index 1 and a fixed prime element π introduce the function

$$\mathcal{E}_{n,\pi}\colon W_n(\overline{F}) \to U_{1,F}/U_{1,F}^{p^n}$$

by the formula

$$\mathcal{E}_{n,\pi}\left((a_0,\dots,a_{n-1})\right) = \prod_{0\leqslant i\leqslant n-1} E(\tilde{a}_i^{p^{n-1-i}}\pi)^{p^i} \mod U_{1,F}^{p^n}.$$

Here $E(X) = \exp(X + X^p/p + X^{p^2}/p^2 + \dots)$ is the Artin–Hasse function, and $\tilde{a}_i$ is a lifting of $a_i \in \overline{F}$ in the ring of integers of F. Then cyclic totally ramified extensions L/F of degree p^n, such that a fixed prime element π of F belongs to $N_{L/F}L^\times$, are in one-to-one correspondence with subgroups

$$\mathcal{E}_{n,\pi}(\wp W_n(\overline{F})\mathrm{Frob}(a_0,\dots,a_{n-1}))U_{1,F}^{p^n}$$

in $U_{1,F}$, where $(a_0,\dots,a_{n-1})$ is invertible in $W_n(\overline{F})$, $\wp = \mathrm{Frob} - 1$, and Frob is the Frobenius map [F5]. This was first discovered by Kurihara [Kur2] for $\pi = p$. He proved that there is an exact sequence

$$1 \to H^1(F, \mathbb{Z}/p^n)_{nr} \to H^1(F, \mathbb{Z}/p^n) \to W_n(\overline{F}) \to 1$$

with nice functorial properties, i.e. there is a canonical connection between Witt vectors and cyclic p-extensions (in the case of $e < p-1$ every cyclic p-extension has separable residue field extension, see [M1]). The approach of Kurihara is based on the study of the sheaf of the etale vanishing cycles on the special fiber of a smooth scheme over the ring of integers of F and of filtrations on Milnor's K-groups of local rings.

The class field theories described above demonstrate "a vivid and lively picture of the great and beautiful edifice of class field theories".

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