Measure, integration and elements of harmonic analysis on generalized loop spaces

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In this work we extend the first part of the previous paper [F4] to higher dimensional local fields. We introduce a nontrivial translation invariant measure on the additive group of higher dimensional local fields, and then develop elements of integration and harmonic analysis. We also discuss its relation with several other measure theories.

For an n-dimensional local field F a translation invariant measure μ is defined on a certain ring $\mathcal{A}$ of measurable sets and takes values in $\mathbb{R}((X_1))\dots((X_{n-1}))$ (which itself is an n-dimensional local field whose last residue field is the archimedean field $\mathbb{R}$). The ring $\mathcal{A}$ in the case of higher dimensional fields with finite last residue field is the ring generated by characteristic functions of shifts of fractional ideals, i.e. $a + bO$ with $a, b \in F$ and O the ring of integers of F with respect to the n-dimensional structure. The measure is countably additive in a refined sense (see sections 7 and 8). Elements of integration theory are introduced in sections 9–13.

The additive group of a higher dimensional field has a certain topology on it with respect to which it is not locally compact for $n > 1$. This topology plays a key role in higher class field theory [F1–F3]. The additive group of F is self dual, which together with the measure and integration leads to analogs of many classical results in Fourier analysis (section 14). In particular, for functions in certain space we introduce their transform

$$\mathcal{F}(f)(\alpha) = \int \psi(\alpha\beta)\, f(\beta)\, d\mu(\beta)$$

and show that $\mathcal{F}^2(f)(\alpha) = f(-\alpha)$.

In section 16 we discuss the theory in the case where the last residue field is $\mathbb{R}$ or $\mathbb{C}$.

In sections 17–21 we extend the previous theory to the case of generalized algebraic loop and path spaces (including in particular the complexified space of smooth loops), and indicate how the measure of this work can be used to extend p-adic zeta integrals associated to schemes over complete discrete valuation fields, and how it is related to nonarchimedean measures on spaces of arcs, which have recently found applications in algebraic geometry.

Theory of this paper can be viewed as a part of yet unknown general theory of harmonic analysis on certain classes of non locally compact groups. It is expected to find further applications, in particular towards integration on path spaces, which is important for mathematical explanation of quantum physics (see section 18).

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Analysis on higher dimensional local fields

For various results about higher local fields see sections of [IHLF].

1. Let F be an n-dimensional local field, i.e. a complete discrete valuation field whose residue field F_{n-1} is an $(n-1)$-dimensional local field; we include $\mathbb{R}$ and $\mathbb{C}$ in the class of one dimensional local fields. Assume $n > 1$. Fix a local parameter t_n of the discrete valuation field F. Choosing lifts of local parameters of the residue fields of F gives a system of local parameters $t_n, \dots, t_1$ of F ($t_n, \dots, t_2$ if the 1-dimensional residue field K of F is archimedean). Fix a lift of elements of the last finite residue field to F (e.g. as multiplicative representatives); or fix a coefficient field for the archimedean 1-dimensional residue field K in F. In the latter case F is isomorphic to the field of formal power series $K((t_2))\dots((t_n))$. In all other cases F is isomorphic to $E((t_{m+1}))\dots((t_n))$ where either $m = 0$ and E is a finite field, or $m \geqslant 1$, E is of characteristic 0 with residue field of positive characteristic; E is a finite extension of a field of type $\mathbb{Q}_p\{\{t_1\}\}\dots\{\{t_{m-1}\}\}$ (for the definition see [F1, sect. 17], [Z2]). Note that if if $m > 1$ then E is not a power series type field.

2. The previous choices supply a set S of representatives of the $(n-1)$-dimensional local field F_{n-1} in F and every element of F can be uniquely written as $\sum s_i t_n^i$, $s_i \in S$, with finitely many non-zero coefficients for negative i.

Define a shift invariant topology on the additive group of F by induction on dimension in two steps (the topology on 1-dimensional local fields is the usual one). First, a sequence $\alpha^{(m)} = \sum s_i^{(m)} t_n^i$, $m \in \mathbb{N}$, converges to 0 iff the residue of $s_i^{(m)}$ tends to 0 in the topology of F_{n-1} for every i, and there is i_0 such that $s_i^{(m)} = 0$ for all $i < i_0$ and all m. Similarly one defines fundamental sequences. It is easy to see that every fundamental sequence converges. Now consider on F the so called sequential saturation topology (sequential topology for short): a subset of F is open iff every sequence which converges to any element of it has almost all its elements in the set. For more details on the topology see [F1–F3], [IHLF].

3. From now on assume that the 1-dimensional residue field F_1 of F is non-archimedean (for the remaining case see section 15). Denote by q the cardinality of the last finite residue field of F. Denote the ring of integers of F by O_F or even O, the maximal ideal by M_F or M, and the group of units by U; denote by R the set of multiplicative representatives of the last finite residue field. The multiplicative group $F^\times$ is the product of infinite cyclic groups generated by $t_n, \dots, t_1$ and the group of units U.

Denote by $\mathcal{O}$ the ring of integers of F with respect to the discrete valuation of rank one (so t_n generates the maximal ideal $\mathcal{M}$ of $\mathcal{O}$). There is a projection $p\colon \mathcal{O} \longrightarrow \mathcal{O}/\mathcal{M} = F_{n-1}$.

Fractional ideals of F are of two types: principal $t_n^{i_n}\dots t_1^{i_1} O$ and nonprincipal

$$t_n^{i_n}\dots t_{m+1}^{i_{m+1}} p^{-1}(F_m) = \cup_i\, t_n^{i_n}\dots t_{m+1}^{i_{m+1}} t_m^i O = \cap_j\, t_n^{i_n}\dots t_{m+1}^{i_{m+1}+1} t_m^j O.$$

4. For a field $L((t))$ denote by $\mathrm{res}_i = \mathrm{res}_{t^i}\colon L((t)) \longrightarrow L$ the linear map $\sum a_j t^j \to a_i$. Similarly define $\mathrm{res} = \mathrm{res}_{-1}\colon L\{\{t\}\} \longrightarrow L$ in the case where L is a complete discrete valuation field of characteristic zero (for the definition of $L\{\{t\}\}$ see section 17 or [F1], [Z2]).

Let ψ_L be a complex valued character of the additive group L (L is a multi-dimensional local field) with conductor O_L (here and below the conductor of a character is by definition the largest fractional ideal on which the character is trivial). Introduce

$$\psi' = \psi_L \circ \mathrm{res}_0\colon L((t)) \longrightarrow \mathbb{C}^\times, \quad \psi' = \psi_L \circ (\pi_L^{-1}\,\mathrm{res})\colon L\{\{t\}\} \longrightarrow \mathbb{C}^\times,$$

where π_L is a prime element of L. The conductor of ψ' is the ring O of the corresponding field: i.e. $\psi'(O) = 1 \neq \psi'(t_1^{-1}O)$.

The field E of section 1 has a finite extension of type $M\{\{t_1\}\}\dots\{\{t_{m-1}\}\}$ which is the compositum of E and a finite extension of $\mathbb{Q}_p$ [sect. 3 of Z2], so the restriction of $\psi'_{M\{\{t_1\}\}\dots\{\{t_{m-1}\}\}}$ on E has conductor O_E.

Thus by induction on dimension for an n-dimensional local field F there is a character (not uniquely determined, of course)

$$\psi: F \longrightarrow \mathbb{C}^\times$$

with conductor O_F.

5. The additive group F is self-dual:

LEMMA. *The group X_F of continuous characters on F is isomorphic to F:*

$$X_F = \{\alpha \mapsto \psi(a\alpha) : a \in F\}.$$

Proof. Given a continuous character ψ', there are $i_1, \dots, i_n$ such that $\psi'(t_n^{i_n} \dots t_1^{i_1} O) = 1$, and so we may assume that the conductor of ψ' is O. If F has the same characteristic as F_{n-1} (equal characteristic case) then the restriction of ψ' on $\mathcal{O}$ induces a continuous character on F_{n-1} which by the $(n-1)$-dimensional theory is a shift of $\psi_{F_{n-1}}$. Hence there is $a_0 \in \mathcal{O}$ such that $\psi_1(\alpha) = \psi'(\alpha) - \psi(a_0\alpha)$ is trivial on $\mathcal{O}$. Similarly by induction, there is $a_i \in t_n^i\mathcal{O}$ such that $\psi_{i+1}(\alpha) = \psi_i(\alpha) - \psi(a_i\alpha)$ is trivial on $t_n^{-i}\mathcal{O}$. Then $\psi'(\alpha) = \psi(a\alpha)$ with $a = \sum_0^{+\infty} a_i$.

In the mixed characteristic case it suffices to consider the case of $L\{\{t_{n-1}\}\}$. The restriction of ψ' on $t_{n-1}^i L$ is of the form $\alpha \mapsto \psi(a_i\alpha)$ with $a_i \in t_{n-1}^{-i-1} M_L$, and $a_i \to 0$ when $i \to +\infty$; hence $\psi'(\alpha) = \psi(a\alpha)$ for $a = \sum_{-\infty}^{+\infty} a_i$.

REMARK. Equip the group of characters of F with the topology of uniform convergence on compact subsets (with respect to the sequential topology) of F. It is easy to verify that the map $a \mapsto (\alpha \mapsto \psi(a\alpha))$ is a homeomorphism between F and the group of its characters.

6. To introduce a measure on F we first specify a nice class of measurable sets.

DEFINITION. A distinguished subset is empty or a shift of a principal fractional ideal of O, i.e. is of the form $\alpha + t_n^{i_n} \dots t_1^{i_1} O$. Denote by $\mathcal{A}$ the minimal ring (i.e. closed with respect to finite union and difference) containing all distinguished sets.

The following properties are easy to check, and we indicate proofs of only some of them.

It is straightforward to see that if the intersection of two distinguished sets is nonempty, then it equals to one of them.

Call a set of type $A = \bigcup_i A_i \setminus (\bigcup_j B_j)$ with distinguished disjoint sets A_i and distinguished disjoint sets B_j such that $\bigcup B_j \subset \bigcup A_i$ a *dd-set*. One can arrange that each B_j is a subset of some A_i: $A_i \setminus B_j = A_i \setminus (A_i \cap B_j)$ and $A_i \cap B_j$, if nonempty, is a distinguished set.

One easily checks that if A is also of the similar form $\bigcup_l C_l \setminus (\bigcup_k D_k)$ then $A = \bigcup (A_i \bigcap C_l) \setminus \big(\bigcup (A_i \bigcap B_j \bigcap C_l \bigcap D_k)\big)$.

Call a disjoint union of dd-sets a ddd-set.

LEMMA. *The set of dd-sets is closed with respect to intersection, but not with respect to difference.*

The set of ddd-sets is closed with respect to intersection, difference and union, and thus coincides with $\mathcal{A}$.

A nonempty ddd-set B can be written as a disjoint union of matreshkas $B(A_i)$ with disjoint distinguished sets A_i. Here "matreshka" $B(A_i)$ is defined as

$$B(A_i) = (A_i \setminus \cup_j A_{ij}) \cup_j (\cup_k A_{ijk} \setminus \cup_l A_{ijkl}) \cup \dots$$

where $A_{i\dots yz}$ are some disjoint distinguished subsets of $A_{i\dots y}$.

Notice the misprint in the published version of [F4, sect. 4] where in the description of $\mathcal{A}$ the words "disjoint union of" sets of type A are omitted.

Every element of $\mathcal{A}$ is a disjoint (maybe infinite countable) union of some distinguished subsets. For $n > 1$ distinguished sets are closed but not open.

Alternatively, $\mathcal{A}$ is the minimal ring which contains sets $\alpha + t_n^i p^{-1}(S)$, $i \in \mathbb{Z}$, where S is in the ring of sets of F_{n-1}.

DEFINITION–LEMMA. There is a unique measure μ with values in $\mathbb{R}((X_1))\dots((X_{n-1}))$ which is a shift invariant finitely additive measure on $\mathcal{A}$ such that $\mu(\emptyset) = 0$,

$$\mu(t_n^{i_n}\dots t_1^{i_1} O) = q^{-i_1} X_{n-1}^{i_n}\dots X_1^{i_2}.$$

The proof immediately follows from the properties of the distinguished sets. The measure does depend on the choice of $t_2, \dots, t_n$.

We get $\mu(t_n^{i_n} p^{-1}(S)) = X_{n-1}^{i_n}\mu_{F_{n-1}}(S)$ where $\mu_{F_{n-1}}$ is the normalized Haar measure on F_{n-1} such that $\mu_{F_{n-1}}(O_{F_{n-1}}) = 1$.

DEFINITION. Let $v\colon \mathbb{R}((X_1))\dots((X_{n-1})) \to \mathbb{Z}^{n-1}$ be the discrete valuation of rank $n-1$ associated to the local parameters $X_{n-1}, \dots, X_1$. For a distinguished set A call $I = v(\mu(A))$ its level.

Every set in $\mathcal{A}$ is a disjoint union of *elementary dd-sets* A of the form $B \setminus C$ with $B \supset C$, where B is a distinguished set and C is a disjoint union of distinguished sets C_i. If the level of A equals the level of every distinquished C_i, then

$$A = A' \setminus A''$$

where $A'' = \emptyset$ and A' is a finite disjoint union of distinguished sets of the same level. Otherwise $A = A' \setminus A''$ where A' is a finite disjoint union of distinguished sets of the same level as A, and $A'' \subset A'$ is a finite disjoint union of distinguished sets of higher level. In both cases for every I we have

$$0 \leqslant \operatorname{res}_I \mu(A') \leqslant |\operatorname{res}_I \mu(A)|, \qquad 0 \leqslant \operatorname{res}_I \mu(A'') \leqslant |\operatorname{res}_I \mu(A)|.$$

7. REMARKS.

1. This measure can be viewed as induced (in appropriate sense) from a measure which takes values in hyperreals ${}^*\mathbb{R}$. If one fixes a set of positive infinitesimals $\omega_1, \dots, \omega_{n-1} \in {}^*\mathbb{R}$, each next infinitesimally smaller than the previous one, then a surjective homomorphism from the fraction field of approachable polynomials $\mathbb{R}[X_1, \dots, X_{n-1}]^{ap}$ to $\mathbb{R}((X_1))\dots(X_{n-1}))$, $\omega_i \mapsto X_i$, determines an isomorphism of a subquotient of ${}^*\mathbb{R}$ onto $\mathbb{R}((X_1))\dots((X_{n-1}))$.

2. The measure μ takes values in $\mathbb{R}((X_1))\dots((X_{n-1}))$ which has the following total ordering: $\sum_{n\geqslant n_0} a_n X^n > 0$, $a_n \in \mathbb{R}((X_1))\dots((X_{n-2}))$, $a_{n_0} \neq 0$, iff $a_{n_0} > 0$ in $\mathbb{R}((X_1))\dots((X_{n-2}))$. For

every non-empty $A \in \mathcal{A}$ we have $\mu(A) > 0$; this property can be viewed a generalization of positive real measures.

Notice that for $n > 1$ not every subset bounded from below has an infimum. Thus, the standard concepts in real valued (or Banach spaces valued) measure theory, e.g. the outer measure, do not seem to be useful here.

3. The choice of distinguished sets reflects in a way the structure of valued field. These sets are different from cylinder sets which are used in analysis on infinite dimensional Hilbert spaces.

4. The set $O \in \mathcal{A}$ of measure 1 is the disjoint union of $t_n O \in \mathcal{A}$ of measure X_{n-1}, $t_n t_{n-1}^{-j} O \setminus t_n t_{n-1}^{-j+1} O$ of measure $q^j(1-q^{-1})X_{n-1}$ for $j > 0$, and $t_{n-1}^l O \setminus t_{n-1}^{l+1} O$ of measure $q^{-l}(1-q^{-1})$ for $l \geqslant 0$. Since $\sum_{j>0} q^j$ diverges, the measure μ is not countably additive.

A very important property of the measure μ is its *countably additivity* in the following *refined* sense (which takes into consideration the topology on $\mathbb{R}((X_1))\dots((X_{n-2}))$): for countably many disjoint sets A_i in the ring of sets $\mathcal{A}$ such that their union $\cup A_i$ is in $\mathcal{A}$ *and* $\sum \mu(A_i)$ absolutely converges we have $\mu(A) = \sum \mu(A_i)$ (see section 9).

5. The measure μ on elements of $\mathcal{A}$ takes values in $\mathbb{R}[X_1][X_1^{-1}]\dots[X_{n-1}][X_{n-1}^{-1}]$. So, at the current stage (but not for the following material) one can choose positive real numbers $r_1, \dots, r_{n-1}$ and using the ring homomorphism

$$\mathbb{R}[X_1][X_1^{-1}]\dots[X_{n-1}][X_{n-1}^{-1}] \longrightarrow \mathbb{R}, \quad X_i \mapsto r_i$$

define a real valued finitely additive measure $\mu_{r_1,\dots,r_{n-1}}$ on $\mathcal{A}$. This measure is not positive and is not countably additive; and the measure μ might be viewed as a range extension/lift of the measure $\mu_{r_1,\dots,r_{n-1}}$ to a countably additive measure in the sense of the previous remark.

The following theory cannot be directly applied to the measure $\mu_{r_1,\dots,r_{n-1}}$, since starting from section 8 the integrals will take values in $\mathbb{R}((X_1))\dots((X_{n-1}))$, and then the substitution in such power series is not defined in general.

8. For $A \in \mathcal{A}$, $\alpha \in F^\times$ one has $\alpha A \in \mathcal{A}$ and $\mu(\alpha A) = |\alpha|\mu(A)$, where $|\ |$ is an n-dimensional module: $|0| = 0$, $\quad |t_n^{i_n}\dots t_1^{i_1}u| = q^{-j}X_1^{i_2}\dots X_{n-1}^{i_n}$ $\quad$ for $u \in U$. The module is a generalization of the usual module on locally compact fields. For example, every $\alpha \in F$ can be written as a convergent series $\sum \alpha_{i_1,\dots,i_n}$ with $\alpha_{i_1,\dots,i_n} \in t_n^{i_n}\dots t_1^{i_1}O$ and $|\alpha_{i_1,\dots,i_n}| \to 0$.

9. Introduce a space R_F of complex valued functions on F and their integrals against the measure μ.

DEFINITION. Call a series $\sum c_i$, $c_i \in \mathbb{C}((X_1))\dots((X_{n-1}))$, absolutely convergent if it converges and if the series $\sum \operatorname{res}_{X_{n-1}^j}(c_i)$ absolutely converges in $\mathbb{C}((X_1))\dots((X_{n-2}))$ for every j.

For an absolutely convergent series $\sum c_i$ and every subsequence i_j the series $\sum c_{i_j}$ absolutely converges and the limit does not depend on the terms order.

If A_i are disjoint elementary dd-sets of the form $A_i' \setminus A_i''$ as in section 6, and if $\sum_i \mu(A_i)$ absolutely converges, then $\sum_i \mu(A_i')$, $\sum_i \mu(A_i'')$ absolutely converge.

From this it is easy to deduce that if A is a similar disjoint union of $B_j = B_j' \setminus B_j''$ and $\sum_j \mu(B_j)$ absolutely converges, then so does $\sum \mu(A_i \cap B_j)$.

LEMMA. *Suppose that a function $f\colon F \longrightarrow \mathbb{C}$ can be written as $\sum c_i\, char_{A_i}$ with countably many disjoint dd-sets A_i, $c_i \in \mathbb{C}$, where $char_C$ is the characteristic function of C, and suppose that the* *series* $\sum c_i\mu(A_i)$ *absolutely converges. If* f *has a second presentation of the same type* $\sum d_j\, char_{B_j}$*, then* $\sum c_i\mu(A_i) = \sum d_j\mu(B_j)$.

Proof. We have the following property: if a distinguished set C is a coset of $t_n^k\mathcal{O}$ and is represented as the disjoint union of distinguished sets C_i such that $\sum\mu(C_i)$ absolutely converges, then each C_i contains a coset of $t_n^k\mathcal{O}$. By reducing the proof to the previous dimension, we then obtain $\mu(C) = \sum\mu(C_i)$. Also notice that if $\bigcup A_i = \bigcup B_j$ for distinguished disjoint sets, then for every i either A_i equals to the union of some of B_j, or the union of A_i and possibly several other A's equals to one of B_j.

Let $e_{i,j} = c_i$ if $A_i\cap B_j\neq\emptyset$ and $e_{i,j} = 0$ otherwise. Then similarly to the paragraph preceding the Lemma one deduces that $\sum e_{i,j}\mu(A_i\cap B_j)$ absolutely converges. Hence we can assume that as soon as $A_i\cap B_j\neq\emptyset$ then $B_j\subset A_i$ and $d_j = c_i$. Concentrating on one A_i it is sufficient to show that if O is the disjoint union of elementary dd-sets $E_i = E_i'\setminus E_i''$ and $\mu(E_i)$ absolutely converges, then $1 = \sum\mu(E_i)$. Then O is the disjoint union of E_i' of level 0, and the disjoint union of E_i'' of level 0 equals the disjoint union of E_i of positive level. To complete the proof use the first paragraph, induction of the level and absolute convergence.

Definition. Define R_F as the vector space generated by functions f as in the previous lemma and by functions which are zero outside finitely many points. For an f as in the previous lemma define its integral

$$\int f\,d\mu = \sum c_i\,\mu(A_i),$$

and for f which are zero outside finitely many points define its integral as zero.

The previous definition implies that the integral is an additive function on R_F.

Remark. Denote by $\mathcal{A}'$ the ring of subsets A of F which are disjoint unions of countably many $A_i\in\mathcal{A}$ such that $\sum\mu(A_i)$ absolutely converges. Extend the measure μ on $\mathcal{A}'$ by $\mu(A) = \sum\mu(A_i)$; this measure is well defined by the previous lemma and one can show it is countably additive in the sense of Remark 4 in section 7.

10. Remark. One can generalize the previous measure theory. Let C be a complete discrete valuation ring with field of fractions K, and let t be its local parameter. Suppose that there is a shift invariant $\mathbb{R}((X_1))\dots((X_{n-1}))$-valued measure on the residue field $B = C/tC$. Similar to the theory of the previous sections one defines a shift invariant $\mathbb{R}((X_1))\dots((X_{n-1}))((X_n))$-valued measure and integration on K. For example, the analog of the ring $\mathcal{A}$ is the minimal ring which contains sets $\alpha + t^ip^{-1}(S)$, where $p\colon C\to B$ is the residue map, S is from a class of measurable subsets of B; the space of integrable functions is generated as a vector space by functions $\alpha\to g\circ p(t^{-i}\alpha)$ extended by zero outside t^iC, where g is an integrable function on B.

11. Some important for harmonic analysis functions do not belong to R_F, for example, the function $\alpha\mapsto\psi(a\alpha)\,char_A(\alpha)$ for, say, $A = O\in\mathcal{A}$, $a\notin\mathcal{O}$. In the one dimensional case all such functions do belong to the analog of R_F.

We now extend the space R_F.

Definition. Suppose that a function $f\colon F\to\mathbb{C}$ is zero outside a distinguished subgroup A of F. Suppose that there are finitely many $a_1,\dots,a_m\in A$ such that the function $g(x) = \sum_i f(a_i + x)$ belongs to R_F. Denote the space of such functions by R_F'. For $f\in R_F'$ define

$$\int f\,d\mu = \frac{1}{m}\int g\,d\mu.$$

LEMMA. *The integral is well defined. If $f \in R_F$ then the integral coincides with the one defined in the previous section. The integral is additive. For $f \in R'_F$*

$$\int f(\alpha + a)\, d\mu(\alpha) = \int f(\alpha)\, d\mu(\alpha)$$

and

$$\int f(\alpha)\, d\mu(\alpha) = |a| \int f(a\alpha)\, d\mu(\alpha).$$

Proof. The integral is well defined: if $h(x) = \sum_j f(b_j+x)$ belongs to R_F for $b_j \in A$, $j = 1, \dots, m'$, then $\sum h(a_i + x) = \sum g(b_j + x)$. So from $h(a_i + x) \in R_F$ and the shift invariant property and the similar property for g one gets $m \int h\, d\mu = m' \int g\, d\mu$.

The integral is additive: for two functions $f_1, f_2 \colon A \to \mathbb{C}$ if

$$\sum_i f_1(a_i + x), \sum_j f_2(b_j + x) \in R_F,$$

then $\sum_{i,j} f_l(a_i + b_j + x) \in R_F$ for $l = 1, 2$ and so

$$\int (f_1 + f_2)\, d\mu = \int f_1\, d\mu + \int f_2\, d\mu.$$

The rest is clear.

For a subset S of F put

$$\int_S f d\mu = \int f\, char_S d\mu.$$

12. REMARK. Alternatively, one can define the integral by analogy with the one dimensional property: the integral of a nontrivial character over an open compact subgroup is zero.

Put $\psi(C) = 0$ if ψ takes more than one value on a distinguished set C and = the value of ψ if it is constant on C. For a distinguished set A and $a \in F^\times$ define

$$\int \psi(a\alpha)\, char_A(\alpha)\, d\mu(\alpha) = \mu(A)\, \psi(aA).$$

Extend the definition of the integral of $\psi(a\alpha)\, char_A(\alpha)$ by linearity to dd-sets A.

We claim that for a function $f = \sum c_n\, \psi(a_n\alpha)\, char_{A_n}(\alpha)$, with finite set $\{a_n\}$ and with countably many disjoint dd-sets A_n such that the series $\sum c_n \mu(A_n)$ absolutely converges, the sum $\sum c_n \int \psi(a_n\alpha)\, char_{A_n}(\alpha)\, d\mu(\alpha)$ does not depend on the choice of c_n, a_n, A_n.

Indeed, one can reduce to sets on which $|f|$ is constant, then use the following property: if a distinguished set C is the disjoint union of distinguished sets C_i, and

$$\psi(a\alpha)\, char_C(\alpha) = \sum d_i \psi(b_i\alpha)\, char_{C_i}(\alpha)$$

with $|d_i| = 1$, absolutely convergent series $\sum d_i\, \mu(C_i)$ and finitely many distinct b_i, then

$$\psi(aC)\, \mu(C) = \sum d_i\, \psi(b_i C_i)\, \mu(C_i).$$

To deduce this property, one can assume $b_i = a$, $d_i = 1$, $C = O$, $a \in \mathcal{O}$, and then as in the proof of Lemma in section 9 all C_i are cosets of $t_n O$, and therefore everything is reduced to the previous dimension $n - 1$.

Thus we can define

$$\int f\,d\mu = \sum c_n \int \psi(a_n\alpha)\,char_{A_n}(\alpha)\,d\mu(\alpha).$$

This definition is compatible with the previous one: of course, for nontrivial characters on a distinguished subgroup one has $g = 0$.

13. Examples.

1. We have $\int_{t_n^{i_n}\dots t_1^{i_1}O}\psi(a\alpha)\,d\mu(\alpha) = q^{-i_1}X_1^{i_2}\dots X_{n-1}^{i_n}$ if $a \in t_n^{-i_n}\dots t_1^{-i_1}O$ and $= 0$ otherwise (since then $\psi(a\alpha)$ is a nontrivial character on $t_n^{i_n}\dots t_1^{i_1}O$). Hence

$$\int_{t_n^{i_n}\dots t_1^{i_1}U}\psi(a\alpha)\,d\mu(\alpha) = \begin{cases} 0 & \text{if } a \notin t_n^{-i_n}\dots t_1^{-i_1-1}O,\\ -q^{-1-i_1}X_1^{i_2}\dots X_{n-1}^{i_n} & \text{if } a \in t_n^{-i_n}\dots t_1^{-i_1-1}U,\\ q^{-i_1}(1-q^{-1})X_1^{i_2}\dots X_{n-1}^{i_n} & \text{if } a \in t_n^{-i_n}\dots t_1^{-i_1}O. \end{cases}$$

2. We get $\int_{t_n^i\mathcal{O}} d\mu = 0$ for every i. Either one can argue as in [F4, sect.8] or use $\int char_{\{0\}}\,d\mu_{F_{n-1}} = 0$ and Lemma below. Of course, the sets $t_n^i\mathcal{O}$ are not in $\mathcal{A}$ for $n > 1$.

If two functions $f, h\colon \mathcal{O} \longrightarrow \mathbb{C}$ are constant on $t_n\mathcal{O}\setminus\{0\}$ and their restriction to $\mathcal{O}\setminus t_n\mathcal{O}$ coincide, then

$$\int_{\mathcal{O}} f\,d\mu = \int_{\mathcal{O}} h\,d\mu.$$

Indeed, if $(f-h)|_{t_n\mathcal{O}\setminus\{0\}} = c$, then

$$\int_{\mathcal{O}}(f-h)d\mu = \int_O (f-h)d\mu = \int_{\mathcal{O}\setminus t_n\mathcal{O}}(f-h)\,d\mu + \int_{t_n\mathcal{O}} c\,d\mu = 0.$$

From the definitions and Lemma in [F4, sect. 8] we immediately get

Lemma. *Suppose that a function $g\colon F_{n-1} \longrightarrow \mathbb{C}$ is in $R'_{F_{n-1}}$ if $n > 2$ and is absolutely integrable if $n = 2$ with respect to the normalized measure $\mu_{F_{n-1}}$ as in section 6. Then the function $g \circ p$ extended by zero outside $\mathcal{O}$ belongs to R'_F and*

$$\int_{\mathcal{O}} g\circ p\,d\mu = \int_{F_{n-1}} g\,d\mu_{F_{n-1}}.$$

14. It is more natural to extend the class of functions to $\mathbb{C}((X_1))\dots((X_{n-1}))$-valued functions on F.

Definition. Denote by $\mathcal{R}_F$ the space of $\mathbb{C}((X_1))\dots((X_{n-1}))$-valued function

$$\left\{f = \sum f_{i_1,\dots,i_{n-1}}X_1^{i_1}\dots X_{n-1}^{i_{n-1}},\quad f_{i_1,\dots,i_{n-1}} \in R'_F\right\}.$$

For $f \in \mathcal{R}_F$ define

$$\int_F f d\mu = \sum X_1^{i_1}\dots X_{n-1}^{i_{n-1}}\int_F f_{i_1,\dots,i_{n-1}}\,d\mu.$$

Similarly to the previous text one checks the correctness of the definition and the properties of the integral.

DEFINITION. Denote by $\mathcal{Q}_F$ the subspace $\mathcal{R}_F$ of $\mathbb{C}((X_1))\dots((X_{n-1}))$-valued functions $f = \sum f_{i_1,\dots,i_{n-1}} X_1^{i_1}\dots X_{n-1}^{i_{n-1}}$ with $f_{i_1,\dots,i_{n-1}} \in R_F$ generated by the characteristic functions of shifts of fractional ideas of F.

For $f \in \mathcal{Q}_F$ introduce the transform function

$$\mathcal{F}(f)(\beta) = \int_F f(\alpha)\,\psi(\alpha\beta)\,d\mu(\alpha).$$

In particular,

$$\mathcal{F}(char_{t_n^{i_n}\dots t_1^{i_1}O}) = q^{-i_1} X_1^{i_2}\dots X_{n-1}^{i_n}\, char_{t_n^{-i_n}\dots t_1^{-i_1}O}.$$

THEOREM. *Given $f \in \mathcal{Q}_F$, the function $\mathcal{F}(f)$ belongs to $\mathcal{Q}_F$ and we have a double transform formula*

$$\mathcal{F}^2(f)(\alpha) = f(-\alpha).$$

Proof. First note that $\psi(\alpha) = \psi_{F_{n-1}} \circ p(\alpha)$ for all $\alpha \in \mathcal{O}$ where $\psi_{F_{n-1}}$ is an appropriate character on F_{n-1} with conductor $O_{F_{n-1}}$.

It is sufficient to check the theorem for a complex valued function f. Furthermore, we can assume that f has support in $\mathcal{O}$, so $f|_{\mathcal{O}} = g \circ p$ for a function $g \in \mathcal{Q}_{F_{n-1}}$.

We shall verify that

if $\beta \notin \mathcal{O}$ then $\mathcal{F}(f)(\beta) = 0$;

if $\beta \in \mathcal{O}$ then $\mathcal{F}(f)(\beta) = \mathcal{F}_{n-1}(g) \circ p(\beta)$ (where $\mathcal{F}_{n-1}(g)$ denotes the transform of g with respect to $\psi_{F_{n-1}}$ and $\mu_{F_{n-1}}$).

For $\beta \notin \mathcal{O}$ the definitions imply

$$\mathcal{F}(f)(\beta) = \int_{\mathcal{O}} f(\alpha)\,\psi(\alpha\beta)\,d\mu(\alpha) = 0.$$

For $\beta \in \mathcal{O} \setminus \mathcal{O}^\times$

$$\begin{aligned}\mathcal{F}(f)(\beta) &= \int_{\mathcal{O}} f(\alpha)\,d\mu(\alpha) = \int_{\mathcal{O}} g \circ p(\alpha)\,d\mu(\alpha)\\ &= \int_{F_{n-1}} g(\overline{\alpha})\,d\mu_{F_{n-1}}(\overline{\alpha}) = \mathcal{F}_{n-1}(g)(0) = \mathcal{F}(f)(0).\end{aligned}$$

For $\beta \in \mathcal{O}^\times$

$$\begin{aligned}\mathcal{F}(f)(\beta) &= \int_F f(\alpha)\,\psi(\alpha\beta)\,d\mu(\alpha) = \int_{\mathcal{O}} f(\alpha)\,\psi(\alpha\beta)\,d\mu(\alpha)\\ &= \int_{\mathcal{O}} g \circ p(\alpha)\,\psi_E \circ p(\alpha\beta)\,d\mu(\alpha) = \int_{F_{n-1}} g(\overline{\alpha})\,\psi_{F_{n-1}}(\overline{\alpha}\overline{\beta})\,d\mu_{F_{n-1}}(\overline{\alpha}) = \mathcal{F}_{n-1}(g)(p(\beta)).\end{aligned}$$

It remains to use the $(n-1)$-dimensional double transform formula.

15. REMARKS.

1. In general we need to work with normalized measures corresponding to shifted characters. Similar to the one dimensional case, if we start with a character ψ with conductor aO then the dual to μ measure μ' on F is normalized such that $|a|\mu(O)\mu'(O) = 1$. The double transform formula of section 14 holds.

2. Of course, one has an appropriate change of variable property of the integral. For example, suppose that for a map $h: F \to F$ there are distinguished sets $A_i = c_i + t_n^{i_n} \dots t_1^{i_1} O$ ($i_n, \dots, i_1$ depend on i) such that a) F is the disjoint union of A_i and finitely many points; b) $h(A_i)$ is a distinguished set or an element of F; c) $h|_{A_i}$ is a map of type $a \mapsto \sum_{j \geqslant 0} a_j (a - c_i)^j$, $a \in A_i$, $a_j \in F$ such that $h' = \sum_{j \geqslant 1} a_j j (a - c_i)^{j-1}$ is defined on A_i and its module $|h'(a)|$ is constant on A_i. Then it is straightforward, similar to the one dimensional case, to show that for $f \in R_F$

$$\int_{h(F)} f(\beta)\, d\mu(\beta) = \int_F f(h(\alpha))\, |h'(\alpha)|\, d\mu(\alpha).$$

3. By the well known theorem of Weil one can restore the topology of a group endowed with a shift invariant nonzero separated real valued measure: it is induced from a locally compact group in which the original group is thick (for more precise statements see e.g. [H, §62]). Relations between a shift invariant $\mathbb{R}((X_1)) \dots ((X_{n-1}))$-measure on groups which are iterated inductive projective limits of locally compact groups and their (higher dimensional) topology remains unclear, and would be interesting to study.

Analysis on fields over archimedean local fields

Now assume that the 1-dimensional residue field $K = F_1$ of F is archimedean.

16. Define a character $\psi: K((t_2)) \dots ((t_n)) \longrightarrow \mathbb{C}^\times$ as the composite of several res_0:

$$K((t_2)) \dots ((t_n)) \longrightarrow K((t_2)) \dots ((t_{n-1})) \longrightarrow \dots \longrightarrow K$$

and the archimedean character $\psi_K(\alpha) = \exp(2\pi i \operatorname{Tr}_{K/\mathbb{R}}(\alpha))$ on K. The role of distinguished sets is played by $A = a + t_n^{i_n} \dots t_2^{i_2} D + t_n^{i_n} \dots t_2^{i_2+1} K[[t_2]]$ where D is an open ball in K. In this case if the intersection of two distinguished sets is nonempty then it equals either to one of them, or to a smaller distinguished set. The measure is the shift invariant additive measure μ on the ring $\mathcal{A}$ generated by distinguished sets with values in $\mathbb{R}((X_1)) \dots ((X_{n-1}))$ such that $\mu(A) = \mu_K(D) X_{n-1}^{i_n} \dots X_1^{i_2}$. It can be extended to

$$\mu\bigl(a + t_n^{i_n} \dots t_2^{i_2} C + t_n^{i_n} \dots t_2^{i_2+1} K[[t_2]]\bigr) = \mu_K(C) X_n^{i_n} \dots X_2^{i_2},$$

where $C \subset K$ is a Lebesgue measurable set.

The module is

$$\Bigl| \sum_{(i_2, \dots, i_n) \geqslant (j_2, \dots, j_n)} a_{i_2, \dots, i_n} t_2^{i_2} \dots t_n^{i_n} \Bigr| = |a_{j_2, \dots, j_n}|_K X_1^{j_2} \dots X_{n-1}^{j_n}, \quad a_{j_2, \dots, j_n} \neq 0$$

where the module on the real K is the absolute value and on the complex K is the square of the absolute value.

The definitions of spaces R_F and R'_F follow the general pattern of sections 9 and 10: for disjoint dd-sets A_i such that $\sum c_i\, \mu(A_i)$ absolutely converges put

$$\int \sum c_i\, char_{A_i}\, d\mu = \sum c_i\, \mu(A_i).$$

For a function $f = \sum c_i\, \psi(a_i \alpha)\, char_{A_i}(\alpha)$ with finite set $\{a_i\}$ and with countably many disjoint dd-sets A_i such that the series $\sum c_i\, \mu(A_i)$ absolutely converges put

$$\int f\, d\mu = \sum c_i \int_{A_i} \psi(a_i \alpha)\, d\mu(\alpha),$$

where

$$\int_A \psi(c\alpha)\, d\mu(\alpha) = \psi(ca)\, X_{n-1}^{i_n} \ldots X_1^{i_2} \int_D \psi_K(c_{-(i_2,\ldots,i_n)}\beta)\, d\mu_K(\beta),$$
$$c_{-(i_2,\ldots,i_n)} = \operatorname{res}_{t_n^{-i_n}} \ldots \operatorname{res}_{t_2^{-i_2}} c,$$

$A, a, D, i_2, \ldots, i_n$ are as above, and then we extend the definition to dd-sets by linearity.

We have analogs of the definitions and results of sections 9, 10.

Extend the definitions of spaces $\mathcal{R}_F$ and $\mathcal{Q}_F$ of section 14; for example $\mathcal{Q}_F$ is the subspace $\mathcal{R}_F$ of $\mathbb{C}((X_1))\ldots((X_{n-1}))$-valued functions $f = \sum f_{i_1,\ldots,i_{n-1}} X_1^{i_1} \ldots X_{n-1}^{i_{n-1}}$ with $f_{i_1,\ldots,i_{n-1}} \in R_F$ generated by functions $\alpha \mapsto g \circ \operatorname{res}_{t_n^0}(t_n^{i_n}\alpha)$ for a complex valued function $g \in \mathcal{Q}_{F_{n-1}}$; and $\mathcal{Q}_K$ is the Schwarz space on K.

For example, we have

$$\mu\big(t^i\mathbb{R}[[t]]\big) = 0 \quad \text{for every } i,$$
$$\mu\big((-a,a)t^i + t^{i+1}\mathbb{R}[[t]]\big) = 2aX^i, \quad a \geqslant 0;$$

$(-a,a)t^i + t^{i+1}\mathbb{R}[[t]] \to t^i\mathbb{R}[[t]]$, $2aX^i \not\to 0$ when $a \to \infty$, which is compatible with the above restriction on absolute convergence. The equality

$$\int char_{\mathbb{R}[[t]]}\, d\mu = 0$$

can be viewed, in physical terminology, as a renormalization of the divergent integral $\int_{-\infty}^{+\infty} dx$ with respect to Lebesque measure dx. One can further extend this to radiative corrections with finitely many parameters $X_2, \ldots, X_n$ playing the role of infinitesimally small elements: after an integral is calculated, one is allowed to substitute concrete real values for X_i (see Remark 5 in section 8).

The transform of a function $f \in \mathcal{Q}_F$ is defined by the same formula

$$\mathcal{F}(f)(\beta) = \int_F f(\alpha)\, \psi(\alpha\beta)\, d\mu(\alpha).$$

The double transform formula is $\mathcal{F}^2(f)(\alpha) = f(-\alpha)$ for $f \in \mathcal{Q}_F$.

There is an appropriate change of variable formula for the integral.

Further extensions and relations to other measures

17. Generalized higher dimensional loop spaces. For a topological group H define the topology on (algebraic) loop space $H((t))$ similarly to section 2; extend this to the case of power series in several variables.

If V is a discrete valuation space such that there is a prime p which is topologically nilpotent, then introduce

$$V\{\{t\}\} = \{\sum_{-\infty}^{+\infty} v_i t^i : v_i \in V, v_i \to 0 \text{ when } i \to -\infty, (v_i) \text{ is bounded above}\}$$

with topology defined similarly to the topology on $L\{\{t\}\}$ in [F3].

Recall that for a finite dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ the formal loop algebra $\mathfrak{g}((t))$ is defined as $\mathfrak{g} \otimes \mathbb{C}((t))$ [FZ, Ch. II].

For a finite dimensional vector space g over a one-dimensional local field F (non-archimedean and archimedean) the space $G = g((t_2))\dots((t_n))$ and for nonarchimedean g the space $g\{\{t_1\}\}\dots\{\{t_{m-1}\}\}((t_{m+1}))\dots((t_n))$ with the defined above topology on them, and finite degree vector spaces over them are natural to call *generalized (higher dimensional, or iterated) loop spaces*. The additive group of an n-dimensional local field is a generalized loop space.

Similarly to section 5 one shows that a generalized loop space is self-dual. Similarly to sections 6–16 one defines a shift invariant measure

$$\mu: \mathcal{A} \to \mathbb{R}((X_1))\dots((X_{n-1}))$$

on the appropriate ring $\mathcal{A}$ of subsets of G, integration, transform and proves the double transform property.

One immediately gets the extension of the measure from the algebraic loop space to the algebraic path space, as well as to the space of an affine Lie algebra, and more generally to the space of a toroidal Lie algebra.

18. Links to Feynman path integral. The loop space L of continuous complex valued functions f from $[0,1]$, satisfying $f(0) = f(1) = 0$, contains a subspace of functions meromorphic in a neighbourhood of the unit ball with a pole of finite order at the origin, which in turn is a subspace of $\mathbb{C}((t))$ by considering formal Laurent expansion. The space of paths P is the direct sum of L and $\mathbb{C}$.

The measure (resp. transform) on algebraic loop space $\mathbb{C}((t))$ over $\mathbb{C}$ introduced in this paper can in some sense be viewed as an arithmetic analogue of the physical Feynman measure "$\mathcal{D}x$" on the path space P, which is used in the not yet fully mathematically justified Feynman functional integral

$$\int_P f(x)\exp(-iS(x)/\hbar)\,\mathcal{D}x$$

for appropriate functions f, S on P.

Here is the list of several analogies:

(a) The (Gaussian measure if a is real) normalization for almost-eigenfunction $\exp(-a||x||^2)$, $a \in \mathbb{R}_{>0} \cup i\mathbb{R}_{>0}$, of Fourier transform is

$$(\pi/a)^{-1/2}\int \exp(-a||x||^2) = 1.$$

Its analog for the theory of this work is

$$(\pi/a)^{-1/2}\int_{\mathbb{C}((t))} \exp(-a|p(x)|^2)\,d\mu(x) = 1$$

where $p = \mathrm{res}_0: \mathbb{C}[[t]] \to \mathbb{C}$ is evaluation at 0 extended by zero outside $\mathbb{C}[[t]]$.

(b) Example 1 in section 13 reminds of the property of the Feynman integral to vanish on nonclassical paths due to oscillation of the integrand.

(c) Example 8 in [F4] and the discussion in section 16 remind of renormalization issues in quantum physics.

(d) The Wiener measure is often used in relation to the path integral, and there was an attempt to use it for integration on two dimensional local fields [Sa]. An abstract approach [CD] to the path integral based on prodistributions is reminiscent of an abstract approach to the Haar type measure on two dimensional local fields in [Ka2].

(e) Feynman's works used a translation invariant property of $\mathcal{D}x$, and there is no non trivial translation invariant countably additive real valued measure on the path space of continuous functions.

The $\mathbb{R}((X))$-valued measure μ is translation invariant and sigma additive in the refined sense (see section 7).

For various mathematical approaches (functional analysis, stochastic, white noise, continuous quantum measurement methods) to the path integral see [AHS], [JL], [Ko2] and references therein. Methods involving analytic continuation can handle a wider range of potentials, but don't normally include the method of stationary phase and semiclassical limit when $\hbar \to 0$. The latter aspect of Feynman integration is of utmost importance for physical applications, in particular in string theory. It is not a part of most of mathematical approaches to the Feynman integral, with the notable exception of the approach of Maslov–Chebotarev and its recent development by Kolokoltsov, and the approach of Albeverio–Hoegh-Rohn–Truman and many of its extensions. However, in those approaches the class of potentials which can be handled is smaller. None of known mathematical approaches to the Feynman path integral goes far enough to justify numerous renormalization recipes in physics. Recall that yet in 1930ies the problem of infinities was seen as "a gap in the understanding of relativistic quantum field theory on the most fundamental level" [W, p. 33].

The measure μ on $\mathbb{C}((t))$ takes into account structures different from those of a Hilbert or Banach space. Still, it seems very interesting to ask the following question.

Is the translation invariant formal power series valued measure μ or its modification useful for a mathematical description of the integration over path spaces in quantum physics, satisfactory both from mathematical and physical point of view?

For example, the analytic duality supplied by the double Fourier transform property on loop spaces could be related to some of dualities in string theory.

In accordance with Remark 5 in section 7 the real valued measure on the algebra of distinguished sets will correspond to substitution $X \to \hbar$; it is not positive and not countably additive, which again agrees with similar phenomena in many other approaches to the path integral.

One of applications of the measure μ is to the study of zeta functions of elliptic curves over number fields [F5], and on the other hand Feynman path integral via semiclassical limit and Gutzwiller trace formula can be used as an alternative method to random matrix theory in explanation of properties of chaotic systems [St, Ch. 7–8]. Recall also that calculation of some of path integrals involves zeta-regularized determinants. All this might potentially be of use in explanation of why the distribution of the neighbour spacings of the zeros of ζ and L-functions of elliptic curves has the same statistics as the distribution of eigenvalues of random matrices in the Gaussian unitary ensemble [LRMT].

19. Measure on generalized loop groups associated to algebraic groups. Let G be a connected reductive split algebraic group over an n-dimensional local field F. Using the measure μ and integration on F Kim and Lee define and study a spherical Hecke algebra of $SL_2(F)$ and prove its isomorphism with a spherical Hecke algebra of a maximal torus [KL]. It is expected that using the measure μ on F one can define a translation invariant measure on $G(F)$. The analysis on $G(F)$ (and its central extension – a generalized Kac–Moody group) is useful in the study of their representation theory. In particular, it could lead to a refinement and extension of results of [Ka1] and [KG].

20. Higher dimensional local zeta integrals. Let F be of characteristic zero (in particular, a formal power series field over $\mathbb{R}$ or $\mathbb{C}$). In line with the second part of [F4] one can generalize the p-adic zeta integral (often called Igusa zeta function) ([I], [D]) to an integral

$$\int_A \chi(f(x))\, d\mu(x)$$

for an algebraic set $A \subset F^m$, a polynomial f and a quasi-character $\chi\colon F \to \mathbb{C}^\times$ (usually for this theory it is assumed to be at most tamely ramified): extending the measure on F to finite dimensional affine spaces over it and using Hironaka's desingularization as in [I], [D]. If F is entirely nonarchimedean then one can do that even for semi-algebraic sets (for their definition in one dimensional case see [DL3]).

What analogs of the known properties of the one-dimensional zeta function, stated e.g. in [D], *will hold for this higher dimensional zeta integral?*

21. Links with the nonarchimedean measure. For a field k of characteristic zero and an appropriate k-scheme S (e.g. a k-variety of pure dimension) the scheme of arcs of germs $\mathcal{S}$ is the projective limit of truncated schemes $\mathcal{S}_n$, which represents the functor of k-algebras $R \to \mathrm{Mor}_k(\mathrm{Spec}\ R[t]/t^n, S)$. A nonarchimedean measure μ_{DL} on the algebra of $k[t]$-semi-algebraic subsets [DL3] of $\mathcal{S}$ and its applications in algebraic geometry are studied in many works, see [B], [DL2–DL4]. The measure takes values in the separated completion M of $\mathrm{G}_k[L^{-1}]$ with respect to filtration subgroups generated by $[Z]L^{-r}$, $r - \dim Z - r > i$, $i \to +\infty$; here G_k is the Grothendieck ring of reduced separated schemes over k of finite type, and L is the class of $\mathbb{A}^1_k$. The group M is a quotient of formal power series $\mathrm{G}_k[[L^{-1}]]$; it is endowed with the nonarchimedean norm corresponding to the one dimensional filtration [DL3]. Similarly to the Gaussian and Wiener measures in analysis, μ_{DL} is first defined on cylinder sets and then extended to a countably additive measure. Sometimes the measure μ_{DL} is called "motivic", which unfortunately is a misleading terminology.

Now let's follow the material of section 10 and start with the discrete counting measure on k adding to the ring of measurable sets the set k (which gets measure X). Then we obtain the measure on $k((t))$ and then, as in the previous section, a measure μ_0 on affine spaces and algebraic sets, this measure takes values in $\mathbb{Z}((X))$. This measure is a weaker version of μ_{DL} (when X gets replaced by L^{-1}).

The measure μ of this work does take into account a non-trivial measure (and analytic topology) of k if k is a locally compact field, because of this it has to take values in higher dimensional spaces like $\mathbb{R}((X))$. The measure μ_{DL} takes into account the weaker Zariski topology, and its range is essentially one dimensional.

A modification of the range of values of μ_{DL} from G_k to the group K_0 of Chow motives over k with coefficients in $\mathbb{Q}^{\mathrm{ab}}$ in [DL1] leads to the definition of corresponding zeta integral such that its specialization (L goes to q) for varieties over a one-dimensional nonarchimedean local field of characteristic zero with good reduction gives the p-adic zeta integral of section 20, see [DL1].

A modification of the range of values of μ_{DL} from G to the group K_0 of Chow motives over k with coefficients in $\mathbb{Q}^{\mathrm{ab}}$ in [DL1] leads to the definition of corresponding zeta integral such that its specialization (L goes to q) for varieties over a one-dimensional nonarchimedean local field of characteristic zero with good reduction gives the p-adic zeta integral of section 20, see [DL1]. A refinement of the measure μ_{DL} and of the zeta integral, which uses in particular elimination of quantifiers for finite and Henselian valuation fields and Galois stratification theory is suggested in [DL5–DL6]. It is proved that from a refined zeta function one can get uniformly p-adic zeta functions for almost all p [DL5].

In the case of one-dimensional local fields k (archimedean or nonarchimedean) of characteristic zero is there a kind of measure on $\mathcal{S}(k)$ which generalizes the measure μ of this work and simultaneously generalizes the nonarchimedean measure μ_{DL} or its refined version by taking into account the analytic topology and nontrivial Haar measure on k?

Such a theory is supposed to include harmonic analysis, as does the theory of this paper. It then can be used for analytic dualities studies, and applications, both in algebraic geometry and higher arithmetic (for some of which see [F5]). It is clear that the values of such unknown measure should be taken in a refined completed version M' of M (in particular, not integral power series type object but its quotient) endowed with a refined topology (not one-dimensional as in the case of the measure μ_{DL}, but of the type similar to the sequential topology on formal power series over $\mathbb{R}$), so that M' is self dual in appropriate sense.

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