# ON ASYMPTOTIC EQUIVALENCE OF ELLIPTIC CURVES OVER $\mathbb{Q}$

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ABSTRACT. This short paper asks a question about a new asymptotic symmetry of the moduli space of Frey–Hellegouarch elliptic curves over rational numbers. If the answer to the question is positive then this allows to deduce an effective $(1+\varepsilon)$ abc-inequality from effective abc-inequalities established in [3].

**1**. Let $a,b$ be coprime non-zero integers. The affine equation of elliptic curve (this curve is sometimes called the Frey or Frey–Hellegouarch curve) $E_{a,b}$

$$y^2 = x(x+a)(x-b)$$

can be written in the Weierstrass form as

$$Y^2 = X^3 - 27c_4X - 54c_6, \quad c_4 = 16(a^2+ab+b^2), \quad c_6 = 32(b-a)(2a+b)(a+2b).$$

The discriminant $\Delta = (c_4^3 - c_6^2)/1728 = 16(ab(a+b))^2$. The minimal discriminant of $E_{a,b}$ is the same if 16 does not divide $abc$ or if $a \equiv -1 \mod 4$ and $b \equiv 0 \mod 16$, and $16^{-2}(ab(a+b))^2$ if $a \equiv 1 \mod 4$ and $b \equiv 0 \mod 16$.

Every elliptic curve over $\mathbb{Q}$ all of whose 2-torsion points are in $\mathbb{Q}$ is isomorphic over the algebraic closure of $\mathbb{Q}$ to such a curve.

In particular,

$$(a^2+ab+b^2)^3 = ((b-a)(2a+b)(a+2b)/2)^2 + 3^3(ab(a+b)/2)^2. \tag{†}$$

The $j$-invariant of the Weierstrass equation is

$$j_{a,b} = 2^8 \cdot \frac{(a^2+ab+b^2)^3}{(ab(a+b))^2} = 2^6 \cdot \frac{((b-a)(2a+b)(a+2b))^2}{(ab(a+b))^2} + 2^6 \cdot 3^3.$$

For a non-zero integer its radical rad is the product of its prime divisors taken each with multiplicity one and its odd radical rad$'$ is the product of its odd prime divisors taken each with multiplicity one.

If $16 \nmid ab(a+b)$ then $\mathrm{cond}(E_{a,b}) < 2^{10}\mathrm{rad}'(ab(a+b))$. If $16|ab(a+b)$ and say $4|(a-1)$, $16|b$ then $\mathrm{cond}(E_{a,b}) = \mathrm{rad}(2^{-4}ab(a+b)) \leqslant \mathrm{rad}(ab(a+b))$. If $16|ab(a+b)$ and say $4|(a+1)$, $16|b$ then $\mathrm{cond}(E_{a,b}) \leqslant 2^{4+2l}\mathrm{rad}'(ab(a+b))$ where $l$ is the maximal power of 2 dividing $b$.

All this is very well known. See e.g. sect. 12.5 of [1]. Note that the statement "Since $E$ has multiplicative reduction all all primes $p|\Delta$" in the top line of its p.434 is incorrect as the example of $E_{1,16}$ shows, but that does not affect the LHS and RHS of the displayed inequality in its next line on that page.

Now let in addition $0 < a < b$, $a,b$ are still coprime. Put $c = a+b$. Define

$$A = (b-a)/d, B = (2a+b)/d, C = A+B = (a+2b)/d,$$

where $d = \gcd(b-a, 2a+b)$ $(= 1$ or $3)$. Then $0 < A < B$, and $A,B$ are coprime.

We have

$$a^2+ab+b^2 = d^2(A^2+AB+B^2)/3,$$
$$ab(a+b) = d^3(B-A)(A+2B)(2A+B)/3^3,$$
$$(b-a)(2a+b)(a+2b) = d^3AB(A+B).$$

The map $\phi\colon (a,b)\mapsto (A,B)$ is an involution: $\phi^2=\mathrm{id}$. It is a special map relating the two terms on the RHS of (†). Thus we have an involution map on the moduli space of Frey–Hellegouarch elliptic curves: $E_{a,b}\mapsto E_{A,B}$.

From (†) one gets

$$(a^2+ab+b^2)^3 = ((b-a)(2a+b)(a+2b)/2)^2 + 3^3\,(ab(a+b)/2)^2.$$

and

$$(A^2+AB+B^2)^3 = 3^3\,(AB(A+B)/2)^2 + ((B-A)(2A+B)(A+2B)/2)^2.$$

We also have $j_{A,B} = 12^3 j_{a,b}/(j_{a,b}-12^3) = (12^{-3}-j_{a,b}^{-1})^{-1}$.

**Question (abc-ABC question)**. *Are the following equivalent statements true?*
*1.* $\mathrm{rad}(abc)$ *and* $\mathrm{rad}(ABC)$ *are effectively asymptotically equal, i.e. for every* $\varepsilon>0$ *there are constants* $\mathfrak{c}_\varepsilon, \mathfrak{c}'_\varepsilon$, *effectively depending on* $\varepsilon$, *such that for all relatively prime positive* $a<b$

$$\mathrm{rad}(abc) < \mathfrak{c}_\varepsilon\cdot \mathrm{rad}(ABC)^{1+\varepsilon},\quad \mathrm{rad}(ABC) < \mathfrak{c}'_\varepsilon\cdot \mathrm{rad}(abc)^{1+\varepsilon}.$$

*2. For every* $\varepsilon>0$ *there is a positive constant* $\kappa_\varepsilon$ *such that for all positive coprime integers* $a<b$

$$\mathrm{rad}((b-a)(2a+b)(a+2b)) < \kappa_\varepsilon\cdot \mathrm{rad}(ab(a+b))^{1+\varepsilon}$$

*with* $\kappa_\varepsilon$ *effectively dependent on* $\varepsilon$.
*3.* $\mathrm{rad}(\Delta(E_{a,b}))$ *and* $\mathrm{rad}(\Delta(E_{A,B}))$ *are effectively asymptotically equivalent.*
*4.* $\mathrm{rad}(c_6(E_{a,b}))$ *and* $\mathrm{rad}(\Delta(E_{a,b}))$ *are effectively asymptotically equivalent.*

The proof of the equivalences is immediate.

The involution $\phi$ corresponds to $x\mapsto (1-x)/(2x+1)$ on $\mathbb{P}^1$ sending the divisor $[0]+[1]+[\infty]$ to $[0]+[1]+[-1/2]$. We have $\mathrm{rad}(abc) = \mathrm{cond}_{[0]+[1]+[\infty]}(a:b) = \mathrm{cond}_{[0]+[1]+[-1/2]}(A:B)$ and $\mathrm{rad}(ABC) = \mathrm{cond}_{[0]+[1]+[\infty]}(A:B)$.

The positive answer to the Question signifies a new asymptotic symmetry of the moduli space of elliptic curves over $\mathbb{Q}$ all of whose 2-torsion points are $\mathbb{Q}$-rational.

**2**. A recent paper [3] slightly extends the IUT theory of S. Mochizuki [2] and establishes two effective abc inequalities.

One of the established effective abc inequalities is:

*for every* $\varepsilon>0$ *there is an effectively described constant* $C'_\varepsilon$ *such that for all relatively prime positive integer numbers* $a,b$, *the inequality*

$$\log(a+b) < 1.5(1+\varepsilon)\cdot\log\mathrm{rad}(ab(a+b)) + C'_\varepsilon$$

*holds.* The constant $C'_1$ is slightly larger than $8.5\cdot 10^{29}$. A version of this inequality is also established over quadratic imaginary fields.

Another established effective abc inequality is:

*for every* $\varepsilon > 0$ *there is an effectively described constant* $C_\varepsilon$ *such that for all relatively prime positive integer numbers* $a, b$, *the inequality*

$$\log(ab(a+b)) < 3(1+\varepsilon)\cdot\log\operatorname{rad}(ab(a+b)) + C_\varepsilon$$

*holds.* The constant $C_1$ is slightly larger than $1.7\cdot 10^{30}$.

The second abc inequality implies the first one. The second inequality was stated as a conjecture by Szpiro in [4] in 1990.

Among several motivations for the Question in the previous section, one motivation comes from the study of an issue of how to deduce an effective $(1+\varepsilon)$-abc inequality from the effective abc inequalities in [3] and mentioned above. Let's see how a potential positive answer to the abc-ABC Question helps in this direction.

Fix a positive integer $m$. The second abc inequality above implies that *for every positive* $\varepsilon$ *for all non-zero integers* $a, b, c$ *such that* $a+b+c=0$ *and* $\gcd(a,b,c)$ *divides* $m$ *we have*

$$\log|abc| < 3(1+\varepsilon)\cdot\log\operatorname{rad}(abc) + C_\varepsilon + 3\log m. \tag{$\sharp$}$$

In view of (†), consider the equation

$$x^3 = y^2 + 3^3 z^2, \qquad x,y,z>0, \quad \gcd(x,y,z)|3.$$

The following is a variation of arguments presented in sect. 12.5 of [1].

Applying ($\sharp$), we obtain $x^3y^2z^2 \ll_\varepsilon \operatorname{rad}(xyz)^{3(1+\varepsilon)}$. Since $y^2\cdot 3^3z^2 \leqslant x^6/4$, we deduce $yz \ll_\varepsilon \operatorname{rad}(xyz)^{1+\varepsilon}$. Assume that $y^2 \leqslant 3^3z^2$, then we deduce $y \ll_\varepsilon \operatorname{rad}(xyz)^{(1+\varepsilon)/2}$, and since $x^3 \leqslant 2\cdot 3^3z^2$, we get $x^6y^2 \leqslant x^3\cdot 2\cdot 3^3z^2\cdot y^2 \ll_\varepsilon \operatorname{rad}(xyz)^{3(1+\varepsilon)}$ and $x^6y^6 \ll_\varepsilon \operatorname{rad}(xyz)^{5(1+\varepsilon)}$, so $xy \ll_\varepsilon \operatorname{rad}(z)^{5(1+\varepsilon)}$. Substituting the latter in the RHS of $y \ll_\varepsilon \operatorname{rad}(xyz)^{(1+\varepsilon)/2}$, we obtain $y \ll_\varepsilon \operatorname{rad}(z)^{3(1+\varepsilon)}$. From $x^6y^2 \ll_\varepsilon \operatorname{rad}(xyz)^{3(1+\varepsilon)}$ we deduce $x^3 \ll_\varepsilon y^{1+\varepsilon}\cdot\operatorname{rad}(z)^{3(1+\varepsilon)}$ so $x^3 \ll_\varepsilon \operatorname{rad}(z)^{6(1+\varepsilon)}$, hence $x \ll_\varepsilon \operatorname{rad}(z)^{2(1+\varepsilon)}$. Thus, ($\sharp$) implies: if $y^2 \leqslant 3^3z^2$ then $x \ll_\varepsilon \operatorname{rad}(z)^{2(1+\varepsilon)}$. We obtain similarly that if $y^2 \geqslant 3^3z^2$ then $x \ll_\varepsilon \operatorname{rad}(y)^{2(1+\varepsilon)}$. All the implied constants are explicit functions of $C_\varepsilon$.

Now, for positive coprime $a<b$ denote $x = a^2+ab+b^2$, $y=(b-a)(2a+b)(a+2b)/2$, $z=ab(a+b)/2$. Then $x^3 = y^2+3^3z^2$. Note that since $a$ and $b$ are coprime, $\gcd(x,y,z)$ divides 3, so we can apply the previous paragraph to $x,y,z$. We deduce from the previous paragraph: if $((b-a)(2a+b)(a+2b))^2 \leqslant 3^3(ab(a+b))^2$ then $3c^2/4 \leqslant a^2+ab+b^2 \ll_\varepsilon \operatorname{rad}(abc)^{2+\varepsilon}$ and hence $c \ll_\varepsilon \operatorname{rad}(abc)^{1+\varepsilon}$; if $((b-a)(2a+b)(a+2b))^2 \geqslant 3^3(ab(a+b))^2$, i.e. $((B-A)(2A+B)(A+2B))^2 \leqslant 3^3(AB(A+B))^2$, then $A^2+AB+B^2 \ll_\varepsilon \operatorname{rad}(ABC)^{2+\varepsilon}$ and hence $c \ll_\varepsilon \operatorname{rad}(ABC)^{1+\varepsilon}$. All the implied constants are explicit functions of $C_\varepsilon$.

Therefore, the inequality ($\sharp$) implies:

**Theorem 1**. *For every positive* $\varepsilon$ *there is an effectively described constant* $K_\varepsilon$ *such that for all coprime positive integers* $a, b$ *and their sum* $c = a+b$ *and* $A, B, C$ *defined for* $a, b$ *as above*

$$\log c < (1+\varepsilon)\cdot\log\max\{\operatorname{rad}(abc),\operatorname{rad}(ABC)\} + K_\varepsilon. \tag{$\diamond$}$$

Using Theorem 1 we obtain

**Theorem 2**. *Assume that the* abc-ABC Question *has positive answer. Then for every positive* $\varepsilon$ *there is an effectively described constant* $L_\varepsilon$ *such that for all coprime positive integers* $a, b$ *and their sum* $c = a + b$ *the inequality*

$$\log c < (1+\varepsilon) \cdot \log \mathrm{rad}(abc) + L_\varepsilon$$

*holds.*

## REFERENCES

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