NOMINATION OF CAUCHER BIRKAR FOR THE FIELDS MEDAL

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Caucher Birkar (abbreviated as CB) has made outstanding fundamental contributions, at the leadership level, to most of the key developments of *birational algebraic geometry* in the last 15 years.

He was born in July 1978 in the remote city of Marivan in Kurdistan (Iran) where he spent his school years during a difficult period of war, in a region severely disadvantaged educationally, economically, and politically. He did not have access to any proper mathematical education apart from school textbooks and a handful of books in the local library, before entering Tehran University to do an undergraduate degree. CB was a PhD student at Nottingham University, his supervisor was Ivan Fesenko, and his second informal supervisor was Vyacheslav Shokurov (John Hopkins Univ.). He defended his PhD thesis in 2004. He was awarded an EPSRC postdoctoral fellowship in mathematics and started it at Warwick University and then moved it to Cambridge University in 2006. In 2008 CB was appointed a lecturer in Cambridge University, where he is currently a professor (`https://www.dpmms.cam.ac.uk/~cb496/`).

CB was awarded prizes of three countries: the Philip Leverhulme Prize (2010), the Prize of the Mathematics Foundation of Paris (2010), and the Moore Prize of the American Mathematical Society (2016).

Background

Birational geometry, with the so-called minimal model programme (MMP) at its core, aims to classify algebraic varieties up to birational isomorphism by identifying "nice" elements in each birational class and then classifying such elements, e.g study their moduli spaces. The two-dimensional case of the MMP is classical: one starts with a surface and successively removes curves along which the surface has "positive curvature". The three-dimensional case was developed in the 70's-90's through work of many people notably Iitaka, Iskovskikh, Kawamata, Kollár, Mori, Reid, Shokurov, Ueno, which was recognised in the mathematical community by awarding to Mori a Fields Medal.

Consider a smooth and projective variety X over an algebraically closed field of characteristic zero, e.g $\mathbb{C}$. In general, the curvature of X is not uniformly positive, negative, or zero, that is, it may be positive at some points yet negative at some other points. The curvature is encoded in the canonical divisor K_X of X. In simple terms, the aim of the MMP is to show that we can modify X birationally so that at the end we have a uniform sign of curvature at least in some direction. More precisely, we want to show X is birational to a projective variety Y with "good singularities" so that K_Y is essentially positive everywhere, or that it is negative or trivial in some direction, that is, Y admits a fibration $Y \to Z$ by Fano or Calabi-Yau varieties.

The development of this programme in higher dimension has proved enormously difficult. It involves many fundamental problems of local and global nature many of which have been solved in the last decade. One of the key ideas has been to consider a more general form of the programme, the so-called log MMP, in which one studies pairs (X,B) consisting of a variety X and a boundary divisor B. Understanding singularities of such pairs is in many situations crucial.

Date: September 2016.

The minimal model and abundance conjectures

The following two problems, formulated in technical terms, are among the most important unsolved problems in algebraic geometry.

Minimal model conjecture. Let (X,B) be a projective pair with log canonical singularities. Then the pair has a log minimal model or a Mori fibre space.

Abundance conjecture. Let (X,B) be a projective pair with log canonical singularities. Then the Kodaira dimension and the numerical dimension of $K_X + B$ are equal.

The two conjectures basically say that running the MMP on (X,B) produces a model which is log canonically polarised or that it admits a fibration by log Fano or log Calabi-Yau varieties.

The theory has seen amazing progress in higher dimension in the last two decades. This progress can be divided into *two major breakthroughs*.

The *first breakthrough* was the 2010 paper of four mathematicians [22] in which they proved, building on ideas of Shokurov, the conjecture for *varieties of general type* along with other long-standing problems such as *existence of flips* and *finite generation of canonical rings*. The paper consists of three key steps: inductive construction of minimal models, termination with scaling via finiteness of minimal models, and non-vanishing. Before joining efforts, CB had figured out the main ideas of the first two steps, independently, while the other authors had figured out the main ideas of the three steps. The paper transformed the subject and opened up territories not accessible before. However, it should be emphasized that while further key developments (discussed below) have relied on the results and methods of this paper but they have all required completely new ideas and techniques.

Its further developments included [21,16], where CB derived the conjecture from the *abundance conjecture*. CB's other results on the conjecture include: a special case proved in [7] using derived categories and analytic geometry; [15,10] relate the conjecture to existence of *Zariski decompositions*, and [24] where CB establishes links with the *ACC for lc thresholds*. The minimal model conjecture is proved in dimension ≤ 4 in full generality and in any dimension for varieties of general type while the abundance conjecture is proved in dimension ≤ 3 and in any dimension for varieties of general type.

Singularities and boundness for Fano varieties

The *second breakthrough* in birational geometry in the last 15 years is in the study of singularities and bounding them for Fano varieties.

A feature of modern birational geometry is the presence of singularities, in particular, *Kawamata log terminal* (eg, cone over a rational curve) and *log canonical* singularities (eg, cone over an elliptic curve). Most fundamental problems have local aspects which are expressed in terms of behaviour of singularities. Understanding differences between the two types of singularities mentioned is crucial for tackling problems such as *finite generation* for lc pairs and *abundance*. In [14], a new powerful method is developed for reducing problems concerning log canonical singularities to Kawamata log terminal ones. In particular, existence of flips and key cases of the minimal model and abundance conjectures are proved for log canonical pairs. Hacon and Xu independently and at the same time developed the same technique and applied it to a crucial step of the construction of moduli spaces of stable pairs (higher dimensional analogues of Deligne-Mumford stable curves).

In [11], a new technique is introduced to study singularities in Fano fibrations attacking a conjecture of M^{c}Kernan and Shokurov on boundedness of singularities (a special case of which was conjectured by Iskovskikh and proved by Prokhorov and Mori). Combining this technique with recent results (see below) almost certainly solves the conjecture.

The birational geometry of a projective variety X of nonnegative Kodaira dimension is largely determined by the linear systems $|mK_X|$ and the associated Iitaka fibration. The Kodaira dimension is a non-negative integer which is less than or equal to dimension of X. An old fundamental question of Iitaka (from the 70's) asks whether there is m depending only on dimension of X such that $|mK_X|$ defines an Iitaka fibration. When X is of general type, that is, when its Kodaira dimension is maximal, the question was answered by Hacon-M^{c}Kernan (and in more recent work with Xu) and Takayama after work of Kawamata, Kollár, Siu, Tsuji, etc. Substantial progress is made on the question, in [5], for varieties of intermediate Kodaira dimension by reducing it to bounding certain invariants of the very general fibres of the Iitaka fibration which means one only needs to understand the case of Kodaira dimension zero. This is done via developing a new theory of pairs and singularities, the *theory of generalised polarised pairs*. This theory has also been very important for more recent progress [2].

As pointed out above, Fano varieties are one of the building blocks of algebraic varieties. They behave well in many respects, eg they are rationally connected as shown by Kollár, Mori, Miyaoka, and Campana. However, sometimes they are very hard to treat because unlike varieties of general type they lose many of their useful properties after a birational transformation, eg after a resolution of singularities. In his recent work, CB investigates singularities of linear systems in general and boundedness properties of Fano varieties tackling the difficulties mentioned. He proves Shokurov's conjecture on *boundedness of complements* (from mid 90's) by developing effective birationality techniques in the context of Fano varieties and by developing the theory of complements for generalised polarised pairs [2]. He then proves the famous Borisov-Alexeev-Borisov conjecture on *boundedness of Fano varieties* by developing new techniques to study singularities of linear systems [1]. The list of mathematicians who had tried to prove that BAB conjecture includes names of many famous geometers. CB's breakthrough also implies solutions of open problems in other areas of geometry such as solution of a problem of Tian on α*-invariants* (from early 90's), answer to a question of Serre on *Jordan property of Cremona groups* through work of Prokhorov and Shramov. This is a major transformative event in birational geometry. The ideas developed in [2,1] will very likely lead, in due course, to solution of other central problems in birational geometry, including the minimal model conjecture.

Positive characteristic

Birational geometry over fields of characteristic zero is well-developed as can be seen from the results mentioned above. In contrast, progress in the positive characteristic case is very recent. The main reasons are that in positive characteristic resolution of singularities is not known and Kodaira vanishing fails. However, these have been replaced by tools originating in commutative algebra in positive characteristic, eg use of the Frobenius endomorphism. Building on work by Keel, Schwede, etc, Hacon and Xu in [25] established the existence of flips (for pairs with Kawamata log terminal singularities) and existence of minimal models (for pairs with canonical singularities) in dimension 3 and characteristic $p > 5$ for pairs whose boundary coefficients belong to a special set. The restriction on the coefficients meant their results could not be applied to many problems. By generalising their result to general coefficients and by solving a host of open problems including *existence of flips*, *the minimal model conjecture*, *cone and contraction*, *base point freeness*, *termination with* *scaling*, *Kollár-Shokurov connectedness*, *ACC for lc thresholds*, etc, an almost complete picture of the theory is given in [8,4] in dimension 3 and characteristic $p > 5$.

Conferences and workshops in 2017 on CB's work

Taipei, April 2017 (`http://www.ncts.ntu.edu.tw/events_2_detail.php?nid=128`) with more than 20 hours of talks on CB's work

Edinburgh, June 2017 (`http://www.maths.ed.ac.uk/cheltsov/edge2017/index.html`)

New York, August 2017 (`http://www.maths.ed.ac.uk/cheltsov/simons/index.html`)

Beijing, August-September 2017 (`http://bicmr.pku.edu.cn/meeting/index?id=41?catid=KiQhKyYs`)

Beijing, Tsinghua Univ., December 2017 (Yau)

Pohang, December 2017 (Park)

Support of the research work

CB was supported by an EPSRC postdoc fellowship from 2004 to 2006, in particular, for the work [22] on flips, minimal models, and finite generation. He was supported by a prize of the Mathematics Foundation of Paris for a one year visit (2010-2011) to the Jussieu Mathematics Institute where the work [14] on lc singularities and special cases of the minimal model and abundance conjectures was carried out. He was supported by a grant of the Leverhulme trust from 2011 to 2016 and from 2017 during which he worked on birational geometry in positive characteristic [8,4], Iitaka fibrations [5], and Fano varieties [2]. CB was a scholar of the National Center for Theoretical Sciences (NCTS) of Taiwan from 2013 to 2015. He did parts of the works [8,4,5,2] on his multiple visits to the National Taiwan University with the support of the NCTS in September 2013, August-September 2014, and May and August 2015 (four months in total).

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