ON DEEPLY RAMIFIED EXTENSIONS

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Recently J. Coates and R. Greenberg have introduced a new important class of extensions of local number fields with finite residue field which they call deeply ramified fields. These extensions play an essential role in their study of the arithmetics of abelian varieties over local fields with finite residue fields [**1**].

The first aim of this paper is to provide an 'elementary' treatment of deeply ramified extensions of local fields with arbitrary perfect residue fields using a method different from the original approach of Coates and Greenberg. Equivalent properties-definitions (1), (3)–(8) of deeply ramified extensions in the first section are due to them, and for their proofs a presentation of the different as an integral was involved. We translate the most important constructions into the language of the Hasse–Herbrand function (Section 1) and then apply methods of the third Chapter of [**3**], where the Hasse-Herbrand function is defined in terms of the norm map. Some of properties of deeply ramified extensions (two implications in the language of this paper) have been already studied by M. Matignon [**7**] and J. Fresnel, M. Matignon [**5**] for different purposes (see Remark (1.6) and the beginning of the second section).

The second aim of this text is to expose relations among classes of deeply ramified extensions, that of arithmetically profinite extensions (subsections 2.1-2.4) and that of p-adic Lie extensions (subsection 2.5-2.7). For local fields with finite residue field we give in (2.2) an example of a Galois deeply ramified extension with infinite residue extension in which every Galois deeply ramified subextension is not arithmetically profinite; and in (2.4) – an example of a Galois deeply ramified extension with finite residue field extension and a nondiscrete set of breaks (that means that this extension is not arithmetically profinite). The main result is that for local fields with finite residue field the class of Galois deeply ramified extensions with finite residue extension and a discrete set of breaks coincides with the class of infinite Galois arithmetically profinite extensions (Proposition 2.3). However, in the case of the fields with infinite residue fields Proposition (2.3) doesn't hold (subsection 2.1). In (2.5) we construct an example that shows that the class of infinite Galois totally ramified arithmetically profinite extensions is strictly larger than the class of the most natural arithmetic origin – the class of totally ramified p-adic Lie extensions. Subsection (2.6) contains an example of a Galois deeply

J. London Math. Soc. 57(1998) 325–335

and totally ramified extension L of a local field F with finite residue field such that the norm group of L/E is of finite index in E^*.

1. *Equivalent properties of deeply ramified extensions*

Let F be a local field (in this paper — a complete discrete valuation field with perfect residue field of characteristic $p > 0$). We will assume that extensions of fields are separable. Let $\mathcal{F}/F$ be an extension (possibly infinite). Let $\mathcal{M}_{\mathcal{F}}$ denote the maximal ideal of $\mathcal{F}$ with respect to the extension of the discrete valuation v_F from F to $\mathcal{F}$. For a finite extension E/F we denote by $e(E|F)$ the ramification index of E/F and by $h_{E/F}$ the Hasse–Herbrand function of E/F, see for example Section 3 Chapter III of [**3**]. For a cyclic ramified extension L/F of a prime degree put $s(L|F) = v_L(\pi_L^{-1}\sigma\pi_L - 1)$ with a prime element π_L of L and a generator σ of $\mathrm{Gal}(L/F)$. Then the invariant s is well defined and $s = 0, > 0$ for tamely totally ramified and wildly ramified extensions resp. (see for instance Section 1 Chapter III of [**3**]).

1.1. Theorem. *The following properties of an extension $\mathcal{F}/F$ are equivalent*:

(1) *for every $m \geqslant -1$ and every $\varepsilon > 0$ there exists a finite subextension E/F in $\mathcal{F}/F$ such that $h_{E/F}(m)/e(E|F) < \varepsilon$;*
(2) *for every cyclic ramified extension $\mathcal{F}'/\mathcal{F}$ of prime degree and every $\varepsilon > 0$ there exists a finite subextension E/F in $\mathcal{F}/F$ such that $\mathcal{F}'/\mathcal{F}$ is defined over E (i.e. $\mathcal{F}' = \mathcal{F}E'$ for a cyclic extension E'/E of the same degree) and $s(E'|E)/e(E|F) < \varepsilon$;*
(3) $e(\mathcal{F}|F) = +\infty$ *and* $H^1(\mathrm{Gal}(\mathcal{F}'/\mathcal{F}), \mathcal{M}_{\mathcal{F}'}) = 0$ *for every cyclic extension $\mathcal{F}'/\mathcal{F}$ of prime degree*;
(4) $H^1(\mathrm{Gal}(\mathcal{F}'/\mathcal{F}), \mathcal{M}_{\mathcal{F}'}) = 0$ *for every finite extension $\mathcal{F}'/\mathcal{F}$*;
(5) $\mathrm{Tr}_{\mathcal{F}'/\mathcal{F}}\, \mathcal{M}_{\mathcal{F}'} = \mathcal{M}_{\mathcal{F}}$ *for every finite extension $\mathcal{F}'/\mathcal{F}$.*

1.2. Remark. It follows immediately from the properties of the Hasse–Herbrand function ($h_{L/F} = h_{L/E} \circ h_{E/F}$, $h_{L/F}(x) \leqslant e(L|F)x$) that $h_{L/F} \leqslant e(L|E)h_{E/F}$ for a finite extension L/E. From this we deduce that if $h_{E/F}(m)/e(E|F) < \varepsilon$, then $h_{L/F}(m)/e(L|F) < \varepsilon$ for a finite extension L/E. Note also that for a finite extension M/F and $m' = h_{M/F}(m)$

$$h_{ME/M}(m')/e(ME|M) = e(M|F)h_{ME/F}(m)/e(ME|F).$$

We conclude that if property (1) holds for $\mathcal{F}/F$, then it holds for $\mathcal{F}'/F$ and $\mathcal{F}/M$, where $\mathcal{F}'/\mathcal{F}$ is an extension and M/F is a finite subextension in $\mathcal{F}/F$. In addition, if (1) holds for $\mathcal{F}/F$ and $\mathcal{F}/\mathcal{F}_0$ is finite, $F \subseteq \mathcal{F}_0$, then (1) holds for $\mathcal{F}_0/F$.

1.3. Remark. [1] Recall the well known elementary fact: if R/S and T/S are totally ramified cyclic extensions of degree p, linearly disjoint with each other, then

[1] three changes, marked by footnotes, provide an improved or more detailed version; they are due to questions from V. Edenfeld

$s(R|S) < s(T|S)$ implies $s(RT|R) = ps(T|S) - (p-1)s(R|S)$, and $s(R|S) = s(T|S)$ implies $s(RT|R) \leqslant s(T|S)$.

Observation: $s(QN|N)/e(N|F) \leqslant s(Q|K)/e(K|F)$ for linearly disjoint totally ramified Q/K and N/K. Indeed, $s(QN|N) \leqslant h_{QN/Q}(s(Q|K)) \leqslant e(N|K)s(Q|K)$.

Clearly, property (2) does not depend on the finite base field F change. We will show in the rest of this Remark that if property (2) holds for $\mathcal{F}/F$ and $\mathcal{L}/\mathcal{F}$ is a finite extension, then property (2) holds for $\mathcal{L}/F$.

It suffices to check this when $\mathcal{L}/\mathcal{F}$ is cyclic of degree p and not unramified. Abbreviate $e(K) := e(K|F)$, $\mathfrak{s}(Q|N) := s(Q|N)/e(N|F)$.

By the assumption, for every $\delta \in (0, (2p(p-1))^{-1})$ there is a finite subextension E_0/F of $\mathcal{F}/F$ such that $\mathcal{L}/\mathcal{F}$ is defined over E_0 (with the same meaning as in Theorem, so in particular $\mathcal{L} = \mathcal{F}E_0'$, E_0'/E_0 is cyclic of degree p) such that $\mathfrak{s}(E_0'|E_0) < \delta$. Due to the observation, the same inequality holds for any finite extension E of E_0 inside $\mathcal{F}$. Since $\mathcal{L}/\mathcal{F}$ is assumed not to satisfy (2), there is a totally ramified cyclic extension $\mathcal{M}/\mathcal{L}$ of degree p and $\varepsilon' > 0$ such that for every finite subextension P/F of $\mathcal{L}/F$ such that $\mathcal{M}/\mathcal{L}$ is defined over P (in particular, $\mathcal{M} = \mathcal{L}P'$ for a cyclic extension P'/P of degree p) we have $\mathfrak{s}(P'|P) \geqslant \varepsilon'$. Using the observation, denote by ε the infimum of the set of real numbers $\mathfrak{s}(QP'|Q)$ where Q/P are finite subextensions of $\mathcal{L}/P$. Increasing P if necessary, we can assume that for all such Q/P we have $\varepsilon \leqslant \mathfrak{s}(QP'|P') < c\varepsilon$ where $1 < c < p$ and $(c-1)\varepsilon < p^{-1} - p^{-2}$. We can also assume that P is a totally ramified cyclic extension of E as above, disjoint with $\mathcal{F}/E$, and that $e(E) > 4p$.

Fix a totally ramified cyclic extension R/E, linearly disjoint with P/E, such that $s(R|E)+2 > pe(E)/(p-1)$, $s(R|E) \neq s(P|E)$ (so then $s(RP|P) \geqslant s(R|E)$). Suppose R/E were a subextension of $\mathcal{F}/E$. Then $s(RP|P) \geqslant s(P'|P)$ implies $\mathfrak{s}(RP'|RP) \leqslant p^{-1}\mathfrak{s}(P'|P)$ which contradicts $c < p$, and $s(RP|P) < s(P'|P)$ implies $(c-1)\varepsilon > \mathfrak{s}(P'|P) - \mathfrak{s}(RP'|RP) = (1-p^{-1})\,\mathfrak{s}(RP|P) > 4 - 2p^{-1}$ which again contradicts the choice of c. Hence R/E is linearly disjoint with $\mathcal{F}/E$.

Using the observation, property (2), the extension $\mathcal{F}R/\mathcal{F}$ defined over E and P/E, we can find a finite subextension V/P of $\mathcal{L}/P$ such that $\mathfrak{s}(VPR|V) < \delta$. We can assume that $V = P_n$, $P = P_1$, P_{i+1}/P_i for $1 \leqslant i \leqslant n-1$ is totally ramified cyclic of degree p. Denote $T_i = RP_i$, $s_i = s(P_{i+1}|P_i)$ and $s_i' = s(T_i|P_i)$. If $s_1 < s_1'$ then either for some $2 \leqslant m < n$ we have $s_i < s_i'$ for $1 \leqslant i \leqslant m-1$ and $s_m \geqslant s_m' = p^{m-1}s_1' - (p-1)\sum_{i=0}^{m-2} p^i s_{m-1-i}$, or $s_n' = p^{n-1}s_1' - (p-1)\sum_{i=0}^{n-2} p^i s_{n-1-i} < e(P_n)\delta$. Therefore $(1-p^{-1})\sum_{i=1}^{n-1} s_i/e(P_i) + \Delta \geqslant s_1'/e(P) \geqslant p^{-1}\mathfrak{s}(R|E)$ where $\Delta = \max(\delta, s_n/e(P_n|F))$. Thus, $\sum_{i=1}^{n} s_i/e(P_i) > p^{-1}$.

Now denote $P_i' = P_iP'$, $r_i = s(P_i'|P_i)$. If $s_i \geqslant r_i$ then $r_{i+1} \leqslant r_i$ and so $r_{i+1}/e(P_{i+1}) \leqslant p^{-1}r_i/e(P_i)$ which contradicts $c < p$. So $s_i < r_i$ for all i and $r_{i+1}/e(P_{i+1}) = r_i/e(P_i) - (1-p^{-1})s_i/e(P_i)$ for all i. Hence $(c-1)\varepsilon > \mathfrak{s}(P'|P) - \mathfrak{s}(P_{n+1}'|P_{n+1}) = (1-p^{-1})\sum_{i=1}^{n} s_i/e(P_i) > (p-1)p^{-2}$, a contradiction.

1.4. *Proof of the theorem.*

(1)⇒(2): (1) implies that $e(\mathcal{F}|F) = +\infty$. Assume that $\mathcal{F}'/\mathcal{F}$ is defined over E_0.

Let, according to Remark 1.2, the extension E/E_0 for $m = s(E_0'|E_0)$ be such that the inequality $h_{E/E_0}(m)/e(E|E_0) < \varepsilon$ holds. Then for $E' = EE_0'$

$$s(E'|E)/e(E|F) \leqslant h_{E/E_0}(s(E_0'|E_0))/e(E|F) < \varepsilon/e(E_0|F).$$

(2)⇒(3): (2) implies that $e(\mathcal{F}|F) = +\infty$. For a tamely ramified extension $\mathcal{F}'/\mathcal{F}$ (3) obviously holds. Assume that $\mathcal{F}'/\mathcal{F}$ is wildly ramified.

Let $\operatorname{Tr}_{\mathcal{F}'/\mathcal{F}} \alpha = 0$ for $\alpha \in \mathcal{M}_{\mathcal{F}'}$. Take E/F for $\varepsilon = v_F(\alpha)$ as in (2). Then $v_{E'}(\alpha) > s(E'|E)$, $\operatorname{Tr}_{E'/E} \alpha = 0$. Thanks to standard properties of $s(E'|E)$, see for instance (1.4) Chapter III of [**3**], we deduce that $\alpha \in (\sigma - 1)\mathcal{M}_{E'}$ for a generator σ of $\operatorname{Gal}(E'/E)$. Thence $\alpha \in (\sigma - 1)\mathcal{M}_{\mathcal{F}'}$.

(3)⇒(2): Assume that (2) doesn't hold. Then there exists $\varepsilon > 0$ such that for every finite extension E/F, $E \subseteq \mathcal{F}$ with $\mathcal{F}'/\mathcal{F}$ being defined over E the inequality $s(E'|E)/e(E|F) \geqslant \varepsilon$ holds. Let E_0/F be a finite subextension in $\mathcal{F}/F$ such that $\varepsilon e(E_0|F) > 2$ and $\mathcal{F}'/\mathcal{F}$ is defined over E_0.

Let σ be a generator of $\operatorname{Gal}(E_0'/E_0)$, $s = s(E_0'|E_0) > 1$, and let π be a prime element of E_0'. Then $\alpha = (\sigma-1)\pi^{i-s} \in \mathcal{M}_{E_0'}$ for $i-s$ prime to p, $1 \leqslant i \leqslant 2$ and $\operatorname{Tr}_{E_0'/E_0} \alpha = 0$, $v_{E_0'}(\alpha) = i$. We claim that $\alpha \notin (\sigma - 1)\mathcal{M}_{\mathcal{F}'}$. Indeed, otherwise there would exist a finite extension E/E_0 in $\mathcal{F}/F$ such that $\alpha \in (\sigma - 1)\mathcal{M}_{E'}$. This is equivalent to $v_{E'}(\alpha) = ie(E|E_0) > s(E'|E)$. Then $s(E'|E)/e(E|F) < \varepsilon$, a contradiction.

(2)⇒(1): Assume that (1) doesn't hold. Then there exist m and $\varepsilon > 0$ such that $h_{E/F}(m)/e(E|F) \geqslant \varepsilon$ for every finite subextension E/F in $\mathcal{F}/F$. Let M/F be a finite extension in $\mathcal{F}/F$ such that $h'_{E/M}(x) = e(E|M)$ for every E/M, $E \subseteq \mathcal{F}$, and $x \geqslant h_{M/F}(m)$.

Let $M = L_0 - \cdots - L_{n-1} - L_n$, $n \geqslant 1$, be an extension in which L_i/L_{i-1} are cyclic ramified of degree p and $s(L_n|L_{n-1}) > h_{L_{n-1}/F}(m)$. Such an extension exists. Indeed, in characteristic p one can take $n = 1$ and L_1/L_0 as a suitable Artin–Schreier extension. In characteristic 0 one can take L_i/L_{i-1} as a suitable Artin–Schreier extension with $s(L_i|L_{i-1}) \geqslant pe(L_{i-1}|\mathbb{Q}_p)/(2p-2)$ (see Section 2 Chapter III of [**3**]). Now, if $h_{L_{i-1}/F}(m) \geqslant s(L_i|L_{i-1})$, then

$$\begin{aligned} h_{L_i/F}(m)/e(L_i|\mathbb{Q}_p) &= \big(s(L_i|L_{i-1}) + p(h_{L_{i-1}/F}(m) - s(L_i|L_{i-1}))\big)/e(L_i|\mathbb{Q}_p) \\ &\leqslant h_{L_{i-1}/F}(m)/e(L_{i-1}|\mathbb{Q}_p) - 1/2. \end{aligned}$$

Therefore, $h_{L_i/F}(m)/e(L_i|\mathbb{Q}_p) \leqslant h_{M/F}(m)/e(M|\mathbb{Q}_p) - i/2$ and

$$pe(L_{n-1}|\mathbb{Q}_p)/(p-1) > s(L_n|L_{n-1}) > h_{L_{n-1}/F}(m)$$

for sufficiently large n.

$$\begin{array}{cccc} E_1 & \dots & E_1L_{n-1} & E_1L_n \\ \\ E & \dots & EL_{n-1} & EL_n \end{array}$$

$$M = L_0 \qquad \dots \qquad L_{n-1} \qquad L_n$$

$$F$$

For a finite extension E/M in $\mathcal{F}/F$ with $EL_n \neq EL_{n-1}$ we get

$$s(EL_n|EL_{n-1}) > h_{EL_{n-1}/L_{n-1}}(h_{L_{n-1}/F}(m)) = h_{EL_{n-1}/F}(m) \geqslant h_{E/F}(m),$$

since $s(EL_n|EL_{n-1}) = h_{EL_{n-1}/L_{n-1}}(s(L_n|L_{n-1}))$.

If E_1/E is a cyclic ramified extension of degree p with $E_1 \subseteq \mathcal{F}$ and $E_1 \not\subseteq EL_{n-1}$, then the choice of M/F implies that $h'_{E_1/E}(h_{E/M}(x)) = p$ for $x \geqslant h_{M/F}(m)$ and $s(E_1|E) \leqslant h_{E/F}(m)$. Therefore,

$$s(E_1L_{n-1}|EL_{n-1}) \leqslant h_{EL_{n-1}/E}(s(E_1|E)) \leqslant h_{EL_{n-1}/F}(m),$$

and hence $E_1L_{n-1} \neq EL_n$, $E_1L_{n-1} \neq E_1L_n$.

Thus, $\mathcal{L}' = \mathcal{F}L_n$ is a ramified extension of degree p over $\mathcal{L} = \mathcal{F}L_{n-1}$. Since $s(EL_n|EL_{n-1})/e(EL_{n-1}|F) \geqslant \varepsilon/e(EL_{n-1}|E)$, (2) doesn't hold for $\mathcal{L}/F$ by Remark 1.3. The same Remark shows that (2) doesn't hold for $\mathcal{F}/F$.

(1)+(2)+(3)⇒(5)+(4); (5)⇒(2); (4)⇒(3): Recall that

$$\operatorname{Tr}_{E'/E}\mathcal{M}^i_{E'} = \mathcal{M}_E^{s(E'|E)+1+[(i-1-s(E'|E))/p]}$$

for $|E' : E| = p$, see for instance Proposition (1.4) Chapter III of [**3**]. Then property (5) for a cyclic ramified extension $\mathcal{F}'/\mathcal{F}$ of prime degree is equivalent to (2). Using Remark 1.2 we deduce now that property (5) holds for arbitrary finite extension. Then (5) and (3) imply (4).

It remains to show that assertion (4) implies $e(\mathcal{F}|F) = +\infty$. Indeed, if $e(\mathcal{F}) > 1$ then one can find a cyclic totally ramified extension $\mathcal{F}'/\mathcal{F}$ of degree p such that $1 < s(\mathcal{F}'|\mathcal{F}) < pe(\mathcal{F})/(p-1)$. Then in the same way as in the proof of (3)⇒(2) one obtains that (4) doesn't hold for $\mathcal{F}'/\mathcal{F}$. If $e(\mathcal{F}) = 1$, then (4) doesn't hold for $\mathcal{F}(\zeta)/\mathcal{F}$ where ζ is a primitive p^2th root of unity.

1.5. Denote by $\delta_{E/F}$ the different of a finite extension E/F. Recall that an extension $\mathcal{F}/F$ has infinite conductor if and only if for every $m \geqslant 0$ there exists a finite subextension E/F in $\mathcal{F}/F$ such that $h'_{E/F}(m) \neq h'_{E/F}(m+1)$. An equivalent condition is that G_F^m acts nontrivially on $\mathcal{F}$ for every $m \geqslant 0$, where G_F^m is the mth ramification subgroup of the absolute group G_F of F with respect to the upper numbering.

COROLLARY. *The following properties of an extension $\mathcal{F}/F$ are equivalent to the properties* (1) – (5) *of the extension $\mathcal{F}/F$*:

(6) *the extension $\mathcal{F}/F$ has infinite conductor*;

(7) *for every finite extension $\mathcal{F}'/\mathcal{F}$ and any $\varepsilon > 0$ there exists a finite subextension E/F in $\mathcal{F}/F$ such that $\mathcal{F}'/\mathcal{F}$ is defined over E and $v_F(\delta_{E'/E}) < \varepsilon$*;

(8) *the different $\delta_{\mathcal{F}/F} = \cap \delta_{E/F}$, where the intersection is taken for all finite subextensions E/F in $\mathcal{F}/F$, is equal to zero*; *in other words, for every $\varepsilon > 0$ there exists a finite subextension E/F in $\mathcal{F}/F$ such that $v_F(\delta_{E/F}) > \varepsilon$.*

Proof. It is plain that (1) is equivalent to (6). In order to show that (2) is equivalent to (7) one can use relations between the different and the invariant s for a cyclic extension of prime degree, multiplicativity of the different and Remark 1.3. Finally, similar observations imply equivalence of (6) and (8).

1.6. REMARK. The implication (8)⇒(7) for the fields of characteristic 0 has been proved by J. Fresnel and M. Matignon as Theorem 3 of [**5**]. For the fields of positive characteristic this can be deduced from Theorem 3 and Proposition 10 of [**7**].

1.7. REMARK. Remark 1.2 and Theorem imply that if $H^1(\mathrm{Gal}(\mathcal{F}^{\mathrm{sep}}/\mathcal{F}), \mathcal{M}_{\mathcal{F}^{\mathrm{sep}}}) = 1$, then $H^1(\mathrm{Gal}(\mathcal{L}^{\mathrm{sep}}/\mathcal{L}), \mathcal{M}_{\mathcal{L}^{\mathrm{sep}}}) = 1$ for an extension $\mathcal{L}/\mathcal{F}$ or a finite extension $\mathcal{F}/\mathcal{L}$.

1.8. An extension $\mathcal{F}/F$ satisfying one of the equivalent properties (1)–(8) is called *deeply ramified* (after J. Coates and R. Greenberg). In particular $e(\mathcal{F}|F) = +\infty$. According to Remark 1.2, if $\mathcal{F}/F$ is a deeply ramified extension, then $\mathcal{F}'/F$, $\mathcal{F}/M$, $\mathcal{F}''/F$ are deeply ramified for an extension $\mathcal{F}'/\mathcal{F}$ and finite subextensions M/F, $\mathcal{F}/\mathcal{F}''$. An extension $\mathcal{F}/F$ is deeply ramified if and only if $\mathcal{F}/\mathcal{F}_0$ is deeply ramified, where $\mathcal{F}_0/F$ is the maximal unramified subextension of $\mathcal{F}/F$.

In the case of characteristic 0 the field $\mathcal{F}$ is called *deeply ramified* if it is a deeply ramified extension over $\mathbb{Q}_p$.

2. *Deeply ramified and arithmetically profinite extensions*

From now on we will discuss relations of deeply ramified extensions with arithmetically profinite (APF) extensions introduced by J.-M. Fontaine and J.-P. Wintenberger which play a central role in their theory of fields of norms ([**4**], [**8**], see also Chapter III of [**3**]). For an infinite Galois extension $\mathcal{F}/F$ denote by B the set of breaks of the Galois group with respect to the upper numbering, i.e. the set of real numbers $a \geqslant -1$ with the property $G_F^{a+\varepsilon} G_{\mathcal{F}} \neq G_F^a G_{\mathcal{F}}$ for every $\varepsilon > 0$. The extension $\mathcal{F}/F$ is deeply ramified if and only if B is not bounded. The extension $\mathcal{F}/F$ is APF if and only if B is a countable sequence (a_n) with $a_n \to +\infty$ and the fixed subfield of $G_F^{a_n}$ in $\mathcal{F}$ is of finite degree over F for all n. In particular, an infinite arithmetically profinite extension has infinite conductor. Now Corollary (1.6) implies that every infinite arithmetically profinite extension is deeply ramified. That every Galois extension with a p-adic Lie group (which is arithmetically profinite, see (2.5)) is deeply ramified has been noted by M. Matignon for the fields of positive characteristic in his thesis in 1979.

2.1. Using a correspondence between abelian totally ramified extension and normic subgroups of the group of principal units established in p-class field theory [**2**] one can show that for local fields F with infinite residue field there exist abelian totally ramified p-extensions L/F which are deeply ramified and at the same time are not arithmetically profinite. Indeed, if the residue field $\overline{F}$ of F is infinite, then $f(\overline{F}) \neq 0$ for any additive polynomial $f(X) \in \overline{F}[X]$, and then by p-class field theory the abelian totally ramified p-extension F_1/F with the norm group $1 + \mathcal{M}_F^2$ in the group of principal units is of infinite degree. Then the compositum of F_1/F with an APF extension isn't APF and is deeply ramified.

In addition, every deeply ramified abelian extension of a local field with infinite residue field contains an arithmetically profinite subextension.

From now on we will study deeply ramified extensions of classical local number fields with finite residue field.

2.2.[2] Consider the following example which demonstrates that Galois deeply ramified extensions of $\mathbb{Q}_p$ with infinite residue field extensions are very far from being related with arithmetically profinite extensions.

Example. A Galois deeply ramified extension of $\mathbb{Q}_p$, $p > 2$, in which every Galois subextension with infinite conductor has infinite residue field extension and therefore is not arithmetically profinite.

For a local field K we denote by $U_{i,K}$ the ith group of principal units, i.e. $U_{i,K} = 1 + \mathcal{M}_K^i$ where $\mathcal{M}_K$ is the maximal ideal of the ring of integers $\mathcal{O}_K$ of K. Denote by φ the absolute Frobenius map.

Let F_i be the unramified extension of $\mathbb{Q}_p$ of degree p^i with the ring of integers $\mathcal{O}_i$. Put $F_\infty = \cup F_i$. Let Tr_i be the trace map from $p\mathcal{O}_i$ to $p\mathcal{O}_{i-1}$.

[2] this is an improved and simplified version

Let V_0 be the $\mathbb{Z}_p$-module $p\mathbb{Z}_p = p\mathcal{O}_0$. Let V_1 be a free $\mathbb{Z}_p$-submodule of $p\mathcal{O}_1$ generated by $p\omega_1, \dots, p\omega_{p-1}, p^2$ where $1, \omega_1, \dots, \omega_{p-1}$ is a $\mathbb{Z}_p$-basis of $\mathcal{O}_1$ such that $\omega_i = \varphi^{i-1}\omega_1$ for $1 \leqslant i \leqslant p-1$. Since the trace map on the residue fields is surjective, $\mathrm{Tr}_1 V_1 = V_0$. We also have $\varphi V_1 = V_1$ and V_1 has a 'break' at the level of the first power of p. Continue in the similar way to define V_i as $p^{i+1}\mathcal{O}_i + W_i$, where $\mathrm{Tr}_i^{-1}(V_{i-1})$ is the direct sum of its submodules W_i and $p^i\mathbb{Z}_p$, $\mathrm{Tr}_i W_i = V_{i-1}$ and $\varphi(W_i) = W_i$. Then $\mathrm{Tr}_i V_i = V_{i-1}$, $\varphi V_i = V_i$ and V_i has a 'break' at the level of p^i.

Define a subgroup $N_i = (p^{\mathbb{Z}}) \times R_i \times \exp(V_i)$ of F_i^* with R_i being the group of Teichmüller representatives in F_i. Let L_i/F_i be the abelian totally ramified extension corresponding to N_i. Put $F = F_0$ and let L be the compositum of L_i. Translating the conditions on V_i we deduce: $N_{L_i/F_i}F_i^*$ is φ-stable, $U_{1,F_i} \cap N_{L_{i+1}/F_i}L_{i+1}^* = U_{1,F_i} \cap N_{L_i/F_i}L_i^*$, $U_{i,F_i}N_{L_i/F_i}L_i^* \neq U_{i+1,F_i}N_{L_i/F_i}L_i^*$.

Class field theory and the properties of V_i imply that L_i/F is Galois, $L_iF_{i+1} \subset L_{i+1}$, for $i \geqslant 1$ we get $\mathrm{Gal}(L_i/F_i)^i \neq \{1\}$, $\mathrm{Gal}(L_i/F_i)^{i+\varepsilon} = \{1\}$ for $\varepsilon > 0$. The field $L = LF_\infty = \cup L_iF_\infty$ is abelian over F_∞, and upper ramification breaks of L/F are the set of positive integers. The Galois extension L/F has infinite conductor.

By class field theory, the norm group of any abelian totally ramified field extension F_i inside the field L includes the intersection of U_{1,F_i} with the intersection of all $N_{L_j/F_i}L_j^*$ where j runs over all integers $\geqslant i$. This intersection equals $N_{L_i/F_i}U_{1,L_i}$. Hence the norm group of the maximal abelian totally ramified extension R_i of F_i inside L intersected with U_{1,F_i} is an open subgroup of U_{1,F_i}, hence R_i/F_i is finite.

Now for every Galois subextension E/F of L/F, if E/F has infinite conductor then it has infinite residue field extension. Indeed, if E/F is a Galois subextension of L/F with finite residue field extension, then the field E is abelian over its inertia subfield $F_m = E \cap F_\infty$ (since the Galois group of E/F_m is isomorphic to that of EF_∞/F_∞ which is a subextension of the abelian extension L/F_∞) and hence by the previous paragraph E/F_m is finite.

2.3. Now for local fields with finite residue fields we have the following

Proposition. *Let F be a local field with finite residue field. Let L/F be a Galois extension with finite residue field extension and with a discrete set of breaks $0 \leqslant u_1 < u_2 < \dots$ with respect to the upper numbering. Then $|\mathrm{Gal}(L/F)^{u_j} : \mathrm{Gal}(L/F)^{u_{j+1}}| \leqslant p^f$ for all $j \geqslant 1$, where p^f is the cardinality of the residue field of L.*

Proof. First, let M/F be a finite Galois extension with two successive breaks $u_1 < u_2$ with respect to the upper numbering. Let K_1, K_2 be the fixed fields of $\mathrm{Gal}(M/F)^{u_1}$ and $\mathrm{Gal}(M/F)^{u_2}$ corr. Then, if $u_j = h_{M/F}(v_j)$ for appropriate integers v_j, the ramification group $\mathrm{Gal}(M/F)_{v_2}$ coincides with $\mathrm{Gal}(M/F)_{v_1+1}$ and $\mathrm{Gal}(M/F)_{v_1}/\mathrm{Gal}(M/F)_{v_1+1}$ injects into either the additive or the multiplicative group of the residue field of M. We deduce that $|K_2 : K_1|$ isn't greater than the cardinality of the residue field of M.

Now denote by E_j the fixed field in L of $\mathrm{Gal}(L/F)^{u_j}$, put $E_0 = F$. Note that since the set of breaks is discrete, every break is a break of a finite Galois subextension of L/F, c.f. [11].

We shall prove the inequality $|E_{j+1} : E_j| \leqslant p^f$ by induction on j. For $j=0$ this is easy. Assume that the inequality holds for all integer smaller than j, and deduce it for j. From the inductive assumption E_j is of finite degree over E_0. Choose an infinite tower of finite Galois extensions M_n/F, $M_n \subset M_{n+1}$ such that $L = \cup M_n$. It is clear that for a sufficiently large n the field M_n contains E_j and u_j, u_{j+1} are two succesive breaks with respect to the upper numbering of $\mathrm{Gal}(M_n/F)$. Then $E_j, E_{j+1} \cap M_n$ are the fixed fields of $\mathrm{Gal}(M_n/F)^{u_j}$, $\mathrm{Gal}(M_n/F)^{u_{j+1}}$ respectively. Therefore, by the first part of the proof, we get $|E_{j+1} \cap M_n : E_j| \leqslant p^f$. Thus, $|E_{j+1} : E_j| \leqslant p^f$.

Corollary 1. *Let F be a local field with finite residue field. Let L/F be a Galois deeply ramified extension with a discrete set of breaks and with finite residue field extension. Then $\mathrm{Gal}(L/F)^u$ is of finite index in $\mathrm{Gal}(L/F)$ for all u.*

Corollary 2. *For local fields with finite residue field the class of Galois deeply ramified extensions with finite residue field extensions and a discrete set of breaks coincides with the class of Galois arithmetically profinite extension.*

The last statement doesn't hold for the fields with infinite residue field in view of (2.1).

2.4.[3] Note that there exist infinite Galois totally ramified extensions with countable and bounded set of breaks. We will construct such an extension $\mathcal{F}/F_0$ with $\mathcal{F} = \cup F_n$, where F_0 has finite residue field of p^p elements, $p > 3$, and F_{n+1} is the normal closure over F_0 of a certain cyclic totally ramified extension F'_n/F_n of degree p defined as follows.

Denote $s_1 = s(F'_0|F_0) = 1$ and $F_1 = F'_0$. For $n \geqslant 1$, denote $|F_n : F_0| = e(F_n|F_0) = M_n$, $m_n = \log_p M_n$. Denote by l_n the smallest integer $\geqslant pm_n/(p-1)$. By induction on $n \geqslant 1$ we will construct the F_{n+1} such that for the maximal upper ramification break s_{n+1} of the extension F_{n+1}/F_n the inequality $(*)_n$

$$s_{n+1} \leqslant (p/(p-1) - \delta_{n+1})M_n \quad \text{with } \delta_{n+1} = (1 + 2pm_{n+1}/(p-1))/M_{n+1}$$

holds for $n \geqslant 1$. Assume that this inequality is known for s_j with $j < n+1$.

Put $r_n = h_{F_n/F_{n-1}}(s_n)$. Note that $r_n + l_n < pe(F_n)/(p-1)$ for $n = 1$ and odd p. For $n \geqslant 2$ the inequality $r_n + l_n < pe(F_n)/(p-1)$ follows from the induction hypothesis inequality for s_n and the fact $r_n \leqslant e(F_n|F_{n-1})s_n$.

Therefore the dimension of the $\mathbb{F}_p$-space $U_{r_n+1,F_n}/(U_{r_n+l_n+1,F_n}(U_{r_n+1,F_n} \cap U^p_{1,F_n}))$ is at least $M_n = (\dim_{\mathbb{F}_p} \mathbb{F}_{p^p})^{m_n}$ (use the well known description of $U_{1,F_n}/U^p_{1,F_n}$, e.g. as given in **[3, Ch. I, sect. 5]**).

Let π be a prime element of F_n, and let $U_{i,F_n} = 1 + \mathcal{M}^i_{F_n}$. Let A be the subgroup of U_{1,F_n} generated by U^p_{1,F_n} and all $\sigma(\pi)/\pi$, where σ runs through $\mathrm{Gal}(F_n/F_0)$. The group $A/U^p_{1,F_n}$ is an $\mathbb{F}_p$-vector subspace of $U_{1,F_n}/U^p_{1,F_n}$ and it has $M_n - 1$ generators,

[3] this is a more detailed version

therefore there is an integer $i < pe(F_n)/(p-1)$, i prime to p, $r_n + 1 \leqslant i \leqslant r_n + l_n$, and such that the group $(A \cap U_{i,F_n})U_{i+1,F_n}$ doesn't coincide with U_{i,F_n}.

Put $s_{n+1} = i$. Using class field theory, choose a totally ramified Galois extension F'_n/F_n of degree p such that $r_n < s_{n+1} = s(F'_n|F_n)$ and the norm group of the extension F'_n/F_n contains π and the group A. Let F_{n+1} be the normal closure of F'_n over F_0. The extension F_{n+1}/F_n is the compositum of $\sigma F'_n/F_n$ and its norm group is the intersection of the σ-images of the norm groups of F'_n/F_n. Since the norm group of F'_{n+1}/F_n contains $U_{s_{n+1}+1,F_n}$, the norm group of F_{n+1}/F_n contains $U_{s_{n+1}+1,F_n}$. Due to the definition of A, $\pi \in (\sigma\pi)A$, $A \subset (\sigma A)U_{s_{n+1}+1,F_n}$; hence, the prime element π belongs to the norm group of F_{n+1}/F_n. Thus, F_{n+1}/F_0 is totally ramified and the maximal upper ramification break of F_{n+1}/F_n is $s_{n+1} \leqslant r_n + l_n$.

From the construction we get $s_2 \leqslant 1 + l_1 = 3 \leqslant p \leqslant (p/(p-1) - \delta_2)p$, since $\delta_2 \leqslant 1/p^2 + 2p/(p-1) \cdot 2/p^2 \leqslant 1/(p-1)$ for $p > 3$. This proves the inequality $(*)_1$ for s_2. The inequality $(*)_n$ for s_{n+1} with $n > 1$ follows from $s_{n+1} - l_n \leqslant r_n \leqslant e(F_n|F_{n-1})s_n \leqslant (p/(p-1) - \delta_n)M_n \leqslant (p/(p-1) - \delta_{n+1})M_n - l_n$, since $(\delta_n - \delta_{n+1})M_n \geqslant l_n$ due to $pm_n/(p-1) \geqslant (1 + 2pm_{n+1}/(p-1))M_{n+1}^{-1}$.

Finally,

$$\begin{aligned} h^{-1}_{F_{n+1}/F_0}(h_{F_{n+1}/F_n}(s_{n+1})) - h^{-1}_{F_n/F_0}(h_{F_n/F_{n-1}}(s_n)) \\ = (s_{n+1} - r_n)/M_n \leqslant l_n/M_n \leqslant {M_n}^{-1/2}, \end{aligned}$$

and $\sum M_n^{-1/2} < \infty$. Hence the set of upper ramification breaks of the extension $\mathcal{F}/F_0$ contains $(h^{-1}_{F_n/F_0}(s_{n+1}))$ and bounded from above by $\sup h^{-1}_{F_n/F_0}(s_{n+1}) < \infty$.

Now the compositum of $\mathcal{F}/F$ with an infinite APF Galois extension furnishes an example of a Galois deeply ramified extension with finite residue field extension and nondiscrete set of breaks which is not arithmetically profinite.

Thus the class of Galois deeply ramified totally ramified extensions is larger than that of Galois totally ramified APF extensions.

2.5. From Sen's and Wintenberger's theorems ([**8**], [**9**]) it follows that every Galois p-adic Lie extension of a local field (with infinite Galois group being a p-adic Lie group of positive dimension and with finite residue field extension) is strictly APF.

We consider

Example. An infinite Galois totally ramified arithmetically profinite extension of a local number field which is not a p-adic Lie extension.

Let $K/\mathbb{Q}_p$ be the abelian extension corresponding to $p^{\mathbb{Z}}\mu_{p-1} \subset \mathbb{Q}_p^*$. Then the Galois group of $K/\mathbb{Q}_p$ has breaks $1 < 2 < 3 < \dots$ with respect to the upper numbering, and $h_{K/\mathbb{Q}_p}(x) = 1 + p + \dots + p^{i-1} + p^i(x - i)$ for $i \leqslant x \leqslant i + 1$, $i > 0$. Let (π_i), $\pi_0 = p$, be a sequence of prime elements compatible with the norm map in the cyclic subextensions of $K/\mathbb{Q}_p$.

Let F be the unramified extension of $\mathbb{Q}_p$ of degree 2, and $M = FK$. Then M/F is an arithmetically profinite extension, and the field of norms $E = N(M|F)$ can be identified with $\mathbb{F}_{p^2}((X))$, where the prime element X corresponds to the sequence (π_i). If τ is a generator of $\mathrm{Gal}(M/F)$, then $\pi_1^{-1}\tau\pi_1 = 1+\theta\pi_1$ and the residue $\overline{\theta}$ is not zero. Denote by T the automorphism of E which corresponds to τ. We will use properties of arithmetically profinite extensions, see for example Section 5 Chapter III of [**3**]. From the previous considerations it follows that $TX = XA_0$ with $A_0 = 1 + aX + \dots$, $a \in \mathbb{F}_{p^2}^*$. Put $A_i = A_{i-1}^{-1}TA_{i-1}$. Then the order of $A_i - 1$ is higher than that of $A_{i-1} - 1$. Hence the group $N \in E^*$ generated by X, $\mathbb{F}_{p^2}^*$ and all A_i is invariant under the action of T; and

$$(N \cap (1+\mathcal{M}_E^j))(1+\mathcal{M}_E^{j+1})$$

is of index $\geqslant p$ in $1+\mathcal{M}_E^j$, since the residue field of E is of cardinality $> p$ and $N \cap (1+\mathcal{M}_E) \subset \mathbb{F}_p((X))$.

Let Q/E be the abelian extension corresponding to N. By class field theory we deduce that $T(Q) = Q$ and $\mathrm{Gal}(Q/E)$ has breaks $1 < 2 < 3 < \dots$. Let L/M be the Galois extension corresponding to Q/E via the theory of fields of norms. Note that $\tau L = L$, therefore L/F is a Galois totally ramified extension. It is also arithmetically profinite as an arithmetically profinite extension of an arithmetically profinite extension; and $h_{L/F} = h_{Q/E} \circ h_{M/F}$.

The set B of breaks of $\mathrm{Gal}(L/F)$ is the set of breaks of the derivative of $h_{L/F}$, and it is straightforward that $B = \{n + p^{-n}m : n \geqslant 1, 0 \leqslant m \leqslant p^n - 1\}$. We assert that L/F is not a p-adic Lie extension. Indeed, one of the properties of a Galois extension K''/K' with a p-adic Lie group G is that $(G^i)^p = G^{i+e}$ for all sufficiently large i, where e is the absolute ramification index of K' (c.f. Proposition 4.5 of [**8**]). From this it is easy to deduce that there exists a c such that $u_{i+jc} = u_i + je, j \geqslant 1$ for all sufficiently large i, where $u_1 < u_2 < \dots$ are all breaks of G (c.f. Proposition 2 of [**6**]). The extension L/F obviously doesn't satisfy this property.

2.6. We present now an example of a Galois p-extension L of a local field E (not necessarily of characteristic 0) with finite residue field such that L/E has infinite conductor and the norm group $N_{L/E}L^*$ which is the intersection of all $N_{M/E}M^*$ where M/E runs finite subextensions in L/E is of finite index in E^*.

Example. A Galois deeply ramified p-extension L of a local field E with finite residue field such that $N_{L/E}L^*$ is of finite index in E^*.

Let E be a a local field with finite fesidue field such that a primitive pth root of unity isn't contained in E for $p \neq 2$. Suppose that E has an abelian noncyclic totally ramified extension M/E of degree p^2. Assume that Q/E is a subextension of degree p in M/E such that $s_2 = s(M|Q) \geqslant s_1 = s(Q|E)$, s_1, s_2 are prime to p.

We will show by induction that there is a tower of totally ramified extensions $M_n - M_{n-1} - \dots - M_1 - M_0 = E$ such that (1) M_i/M_{i-1} is abelian of degree a power of p, (2) M_i/M_0 is Galois, (3) $N_{M_i/M_0}M_i^* = N_{M_{i-1}/M_0}M_{i-1}^*$ for $i \geqslant 3$, (4) $M_0^* \subset N_{M_i/M_1}M_i^*$, (5) $(M_{i-1}^*)^{\sigma-1}\ker N_{M_{i-1}/M_1} \subset N_{M_i/M_{i-1}}M_i^*$ where $\sigma \in \mathrm{Gal}(M_i/M_0)$ is a lifting of a generator of $\mathrm{Gal}(M_1/M_0)$, (6) for $i \geqslant 3$ the maximal ramification break of M_i/M_1 with respect to the upper numbering is at least by 1 greater than the maximal ramification break of M_{i-1}/M_1 with respect to the upper numbering.

Put $M_0 = E$, $M_1 = Q$, $M_2 = M$. Since M_2/M_0 is not cyclic, $M_0^* \subset N_{M_2/M_1} M_2^*$. Since M_2/M_0 is abelian, $M_1^{*\sigma-1} \subset N_{M_2/M_1} M_2^*$. Hence M_2/M_0 satisfies the properties (1) – (6) for $i = 2$.

Assume we have constructed M_n and construct then M_{n+1}. Let L_n/M_{n-1} be a subsextension of M_n/M_{n-1} such that $|M_n : L_n| = p$ and $s_n = s(M_n|L_n)$ is such that the ramification group $\operatorname{Gal}(M_n/M_{n-1})_{s_n+1}$ is trivial (see [**3, Chap. III, sect. 3**].) Let h_{M_n/M_0} be the Hasse–Herbrand function of the extension M_n/M_0. Then $h^{-1}_{M_n/M_0}(s_n)$ is the maximal ramification break of M_n/M_0 with respect to the upper numbering. Take an element $\alpha \in 1 + \mathcal{M}_{L_n}^{s_n}$ such that $\alpha \notin N_{M_n/L_n} M_n^*$. Then $\alpha^{\sigma-1} = N_{M_n/L_n}\beta$ for $\beta \in 1 + \mathcal{M}_{M_n}^{s_n+ps_1}$.

We claim that β doesn't belong to the group $M_n^{*\sigma-1} N^{-1}_{M_n/M_1}(M_0^*)$. Indeed, otherwise $N_{L_n/M_1}\alpha^{\sigma-1} = \mu N_{M_n/M_1}(\lambda^{\sigma-1})$ with $\mu \in M_0^*$. Then $N_{M_1/M_0}\mu = \mu^p = 1$, hence $\mu = 1$. Therefore, $N_{L_n/M_1}(\alpha) N_{M_n/M_1}(\lambda^{-1})$ belongs to M_0^* which is $\subset N_{M_n/M_1} M_n^*$ by the induction assumption. We get $N_{L_n/M_{n-1}}\alpha \in N_{M_n/M_{n-1}} M_n^*$ because $\ker N_{M_{n-1}/M_1} \subset N_{M_n/M_{n-1}} M_n^*$ by the induction assumption. Since M_n/M_{n-1} is abelian, it follows that $\alpha \in N_{M_n/L_n} M_n^*$, a contradiction.

Take now a subgroup $\mathcal{N}_n$ of index a power of p in M_n^* such that (1) it contains some prime element of M_n and the subgroup $M_n^{*\sigma-1} N^{-1}_{M_n/M_1}(M_0^*)$, (2) it doesn't contain β, (3) M_n^* is generated by β and $\mathcal{N}_n$. Let $\mathcal{N}_n = N_{M_{n+1}/M_n} M_{n+1}^*$ for an abelian totally ramified extension M_{n+1}/M_n of degree a power of p. Note that $M_n^{*\tau-1} \subset \mathcal{N}_n$ and then $\tau\mathcal{N}_n = \mathcal{N}_n$ for every $\tau \in \operatorname{Gal}(M_n/M_0)$. Hence M_{n+1}/M_0 is a Galois extension and the properties (1) – (5) hold for M_{n+1}/M_0.

Since M_n/M_{n-1} is abelian, it follows from properties of the Hasse–Herbrand function [**3, Ch. III, sect. 3**] that $U_{j,M_n} \ker N_{M_n/M_{n-1}} = U_{j+1,M_n} \ker N_{M_n/M_{n-1}}$ for $j > s_n$, $j \notin h_{M_n/M_{n-1}}(\mathbb{N})$. Then induction on n shows that $U_{j,M_n} \ker N_{M_n/M_1} = U_{j+1,M_n} \ker N_{M_n/M_1}$ for $j > s_n$, $j \notin h_{M_n/M_1}(\mathbb{N})$. Thus, we deduce that the maximal ramification break of M_{n+1}/M_n with respect to its upper numbering which is $\geqslant s_n + ps_1$ is at least $s_n + |M_n : M_1|$. Property (6) follows.

In characteristic 0 for $p = 2$ take $M_0 = E = \mathbb{Q}_2(\sqrt[4]{-1})$ and $R = M_0(\sqrt[8]{-1})$. Then $s(R|M_0) = 7$ and the unit $\omega = N_{R/M_0}(1 + (\sqrt[8]{-1} - 1)^3)$ belongs exactly to $1 + \mathcal{M}_{M_0}^3$. Take $M_1 = Q = M_0(\sqrt{\omega})$, then $s(M_1|M_0) = 5$. The extension $M_2 = M_1R/M_0$ is noncyclic and since $\omega \in N_{M_2/M_0} M_2^*$ we get $\sqrt{\omega} \in N_{M_2/M_1} M_2^*$. Now one can construct a tower of fields M_n satisfying the properties (1) – (6) and property (7) $\sqrt{\omega} \in N_{M_i/M_1} M_i^*$ for $i \geqslant 2$. Taking the same β as above for $p > 2$ one needs only to check that β doesn't belong to $M_n^{*\sigma-1} N^{-1}_{M_n/M_1}(M_0^*) N^{-1}_{M_n/M_1}(\sqrt{\omega})$, too. But from the equality $N_{L_n/M_1}\alpha^{\sigma-1} = \mu\sqrt{\omega} N_{M_n/M_1}(\lambda^{\sigma-1})$ with $\mu \in M_0^*$ one gets $\omega = \mu^{-2} \in M_0^{*2}$, a contradiction.

Put $L = \cup M_n$. The extension L/E is a Galois p-extension and $N_{L/E}L^* = N_{M_2/E} M_2^*$ is of index p^2 in E^*. The sequence of the breaks of L/E with respect to the upper numbering tends to infinity due to property (6), therefore L/E has infinite conductor.

2.7. In the example (2.5) if Q_i/E is the abelian extension corresponding to $N(1+\mathcal{M}_E^{i+1})$, and L_i/M is the Galois extension corresponding to Q_i/E, then L_i/F is a Galois p-adic Lie extension and $L = \cup L_i$. In the example of (2.6) the extension L/E doesn't contain a $\mathbb{Z}_p$-extension M/E, but L/M_1 is an infinite abelian totally ramified p-extension, therefore it is a Galois p-adic Lie extension.

The following basic problem has been raised by J. Coates and R. Greenberg [**1**]: given a finite extension F of $\mathbb{Q}_p$ is there a Galois deeply ramified extension K of F which does not contain a subfield M which is a p-adic Lie extension of a finite extension of $\mathbb{Q}_p$? This problem is solved in [12] using special closed subgroups of the group of wild automorphisms of a local field of positive characteristic.

Finally we state a problem a positive answer on which will provide another solution of Coates-Greenberg's problem.

Problem. Let E be a finite totally ramified extension of $\mathbb{Q}_p$ with the absolute ramification index $e(E)$ being odd. Is there a Galois deeply ramified extension L of E having a subset of odd positive integers as the set of breaks with respect to the upper numbering?

Then L is a Galois deeply ramified extension of E which does not contain a subfield M which is a p-adic Lie extension of E, since the set of breaks u_i of M/E doesn't satisfy the property $u_{i+c} = u_i + e(E)$, for all sufficiently large i (see the last paragraph of (2.5)).

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