Fields Galois-equivalent to a local field of positive characteristic

by

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Introduction

A celebrated theorem of Artin and Schreier [AS] characterizes the fields $K$ whose absolute Galois group $G_K$ is isomorphic to that of $\mathbb{R}$ as the real closed fields. In the present paper we consider the analogous problem for non-archimedean local fields of positive characteristic $F = \mathbb{F}_{p^n}((t))$. We show that a field $K$ with absolute Galois group isomorphic to $G_F$ possesses a Henselian valuation $v$ such that:

1. the value group $\Gamma$ of $v$ satisfies $\Gamma/l \cong \mathbb{Z}/l$ for all prime numbers $l \neq p$;
2. the residue field $\bar{K}$ of $v$ has characteristic $p$;
3. the maximal prime to $p$ Galois group $G_{K}(p')$ of $\bar{K}$ is $\hat{\mathbb{Z}}/\mathbb{Z}_p$;
4. if $\text{char} K = 0$ then $\Gamma = p\Gamma$ and $\bar{K}$ is perfect.

For every positive integer $r$ we construct such fields $K$ of characteristic $p$ with $\Gamma/p \cong (\mathbb{Z}/p)^r$. Likewise we construct examples with $\Gamma \cong \mathbb{Z}$, $G_{\bar{K}} \not\cong \hat{\mathbb{Z}}$ and $\bar{K}$ imperfect.

The similar problem for $p$-adic fields was answered by Koenigsmann [Kn] and the first named author [E1] (for $p \neq 2$), extending earlier results by Neukirch [N2] and Pop [P1]: the fields $K$ such that $G_{K} \cong G_{F}$ for some finite extension $F$ of $\mathbb{Q}_p$ are precisely the $p$-adically closed fields in the sense of [PR].

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Notation

We denote the algebraic, separable, and inseparable closures of a field $K$ by $\bar{K}$, $K_{\text{sep}}$, and $K_{\text{ins}}$, respectively. For a positive integer $m$ with $\text{char } K \nmid m$ let $\mu_m$ be the group of roots of unity of order dividing $m$ in $\bar{K}$. For a prime $l$ with $l \nmid \text{char } K$ let $\mu_{l^{\infty}} = \lim_{r \to \infty} \mu_{l^r}$. Given a profinite group $G$ and a prime number $l$, denote the quotient $\lim_{r \to \infty} G/N$, where $N$ ranges over all open normal subgroups of $G$ with $G/N$ abelian (resp., of $l$-power order, of order prime to $l$) by $G(\text{ab})$ (resp., $G(l)$, $G(l')$). We define $G(\text{ab}, l)$, $G(ab, l')$ similarly. For a (Krull) valuation $v$ on $K$ let $\mathcal{O}_v$, and $K_v$ be the corresponding value group, valuation ring, and residue field, respectively.

1. Galois groups of Henselian fields

We first recall several basic facts about the structure of the decomposition group of $(K, v)$ relative to $K_{\text{sep}}$ (see e.g. [Ed], [P2, §1] or [E1, §1] for more details and proofs). For simplicity we assume here that $v$ is Henselian, i.e., the decomposition group is $G_K$. Let $K_{\text{ur}}$ and $K_{\text{tr}}$ be the maximal unramified and maximal tamely ramified Galois extensions of $(K, v)$, respectively. If $p = \text{char } K_{\bar{v}} > 0$ then $G_{K_{\text{tr}}}$ is the unique $p$-Sylow subgroup of $G_{K_{\text{ur}}}$. If $\text{char } K_{\bar{v}} = 0$ then $G_{K_{\text{tr}}} = 1$. There are natural short exact sequences

$$1 \to \text{Gal}(K_{\text{tr}}/K_{\text{ur}}) \to \text{Gal}(K_{\text{tr}}/K) \to G_{K_{\bar{v}}} \to 1$$

$$1 \to G_{K_{\text{tr}}} \to G_K \to \text{Gal}(K_{\text{tr}}/K) \to 1$$

which are split, by [N1] and [KPR], respectively. For a prime number $l$ let $\delta_l = \dim_{\mathbb{F}_l} \Gamma_v/l$. Then $\text{Gal}(K_{\text{tr}}/K_{\text{ur}}) \cong \prod_{l \nmid \text{char } K_{\bar{v}}} (\lim_{r \to \infty} \mu_{l^r})^{\delta_l}$ as $G_{K_{\bar{v}}}$-modules. In particular, if $\Gamma_v \cong \mathbb{Z}$ and $\text{char } K_{\bar{v}} = p > 0$ then $\delta_l = 1$ for all primes $l$, so the $G_{K_{\bar{v}}}$-module $\text{Gal}(K_{\text{tr}}/K_{\text{ur}})$ is $\hat{\mu} = \lim_{(m, p) = 1} \mu_m$, and $\text{Gal}(K_{\text{tr}}/K) \cong \hat{\mu} \times G_{K_{\bar{v}}}$ with the Galois action.

The analogous result for the maximal pro-$l$ Galois group $G_K(l)$ of $K$ is the following: If $l \neq \text{char } K_{\bar{v}}$ is prime and $\mu_l \subseteq K$ then $G_K(l) \cong \mathbb{Z}_l^{\delta_l} \times G_{K_{\bar{v}}}(l)$, where $\sigma \in G_{K_{\bar{v}}}(l)$ acts on $\tau \in \mathbb{Z}_l^{\delta_l}$ according to $\sigma \tau \sigma^{-1} = \chi_{K_{\bar{v}}, l}(\sigma) \tau$; here $\chi_{K_{\bar{v}}, l}: G_{K_{\bar{v}}}(l) \to 1 + l\mathbb{Z}_l$ is the pro-$l$ cyclotomic character of $K_{\bar{v}}$, induced by the restriction homomorphism $G_{K_{\bar{v}}}(l) \to \text{Aut}(\mu_{l^{\infty}}) \cong \mathbb{Z}_l^\times$.  

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Now fix a prime number \( p \). Given a pro-\( p \) group \( H \) and a cardinal number \( c \) let 
\( F_p(H; c) \) be the free \( H \)-operator pro-\( p \) group on \( c \) generators, in the sense of [K1], [MSH].

The following is a modest generalization of the main result of [MSH] (which treats Laurent series fields).

**Theorem 1.1:** Let \((K, v)\) be a Henselian discretely valued field of characteristic \( p \) and let 
\( c = \max\{\mathfrak{m}_0, |\bar{K}_v|\} \).
Then 
\( G_{K_{tr}} \cong F_p(\text{Gal}(K_{tr}/K); c) \) as \( \text{Gal}(K_{tr}/K) \)-operator pro-\( p \) groups; in particular, 
\( G_K \cong F_p(\text{Gal}(K_{tr}/K); c) \times \text{Gal}(K_{tr}/K) \) with the canonical action.

When \( k = \bar{K}_v \) is perfect, this theorem can be proven using precisely the same argument as in [MSH, Th. 1]. We therefore omit the details. When \( k \) is not perfect, one can prove it as follows: Let \( u \) be the unique prolongation of \( v \) to \( L = k_{\text{ins}}K \). The restriction \( G_L \to G_K \) is an isomorphism mapping \( G_{L_{ur}}, G_{L_{tr}} \) onto \( G_{K_{ur}}, G_{K_{tr}} \), respectively. By the result for perfect residue fields, 
\( G_{L_{tr}} \cong F_p(\text{Gal}(L_{tr}/L); c) \) as \( \text{Gal}(L_{tr}/L) \)-operator groups. It follows that 
\( G_{K_{tr}} \cong F_p(\text{Gal}(K_{tr}/K); c) \) as \( \text{Gal}(K_{tr}/K) \)-operator groups, as desired.

**Corollary 1.2:** Let \((K_1, v_1), (K_2, v_2)\) be Henselian discretely valued fields of characteristic \( p \), and let \( \bar{K}_1, \bar{K}_2 \) be the corresponding residue fields. Suppose that 
\( G_{\bar{K}_1} \cong G_{\bar{K}_2} \), that this isomorphism is compatible with the Galois actions on the roots of unity, and that
\[ \max\{\mathfrak{m}_0, |\bar{K}_1|\} = \max\{\mathfrak{m}_0, |\bar{K}_2|\} \]. Then 
\( G_{K_1} \cong G_{K_2} \).

**Proof:** We have 
\( \text{Gal}(K_{1, tr}/K_1) \cong \hat{\mu} \times G_{\bar{K}_1} \cong \hat{\mu} \times G_{\bar{K}_2} \cong \text{Gal}(K_{2, tr}/K_2) \) with the Galois actions. Now apply Theorem 1.1. \( \square \)

**Proposition 1.3:** Let \((K, v)\) be a Henselian discretely valued field of characteristic \( p \). Suppose that 
\( |K| = \max\{\mathfrak{m}_0, |\bar{K}_v|\} \).

Let \( L \) be a maximal totally tamely ramified extension of \((K, v)\). Let \((E, u)\) be a Henselian discretely valued field of characteristic \( p \) with \( E_u = L \). Then 
\( G_E \cong G_K \).

**Proof:** Let 
\( c = |K| = \max\{\mathfrak{m}_0, |\bar{K}_v|\} \), let 
\( H = \hat{\mu} \times G_{\bar{K}_v} \) with the Galois action, and let 
\( V = F_p(H; c) \). By Theorem 1.1, 
\( V \cong G_{K_{tr}} \) and 
\( G_K \cong V \rtimes H \). Since \( \bar{F}_p \subseteq K_{tr} \), the Galois action of \( V \) on \( \hat{\mu} \) is trivial. Further, the unique prolongation of \( v \) to \( L \) has residue field 
\( \bar{K}_v \). Hence 
\( G_L \cong V \rtimes G_{\bar{K}_v} \), and this isomorphism is compatible with the Galois action on
\[ \hat{\mu}. \] Therefore
\[ \Gal(E_{tr}/E) \cong \hat{\mu} \times G_L \cong \hat{\mu} \times (V \rtimes G_{\hat{K}_u}) \cong V \rtimes (\hat{\mu} \times G_{\hat{K}_u}) = V \rtimes H. \]

Now from [MSh, §1, Prop. 1] we deduce that
\[ F_p(V \rtimes H; c) \rtimes (V \rtimes H) \cong (F_p(H; c) \rtimes V) \rtimes H \]
\[ = (F_p(H; c) \rtimes F_p(H; c)) \rtimes H \cong F_p(H; c) \rtimes H = V \rtimes H, \]
where \( * \) denotes free pro-\( p \) product. Since \( |L| = |K| \), from Theorem 1.1 we deduce
\[ G_E \cong F_p(\Gal(E_{tr}/E); c) \rtimes \Gal(E_{tr}/E) \cong F_p(V \rtimes H; c) \rtimes (V \rtimes H) \cong V \rtimes H \cong G_K. \]

### 2. Existence of Henselian valuations

Let \( K_2^M(E) \) be the second Milnor \( K \)-group of the field \( E \), and let \( \{ , \} : E^\times \times E^\times \to K_2^M(E) \) be the natural symbolic map. The following theorem combines powerful constructions of Ware [Wr], Arason–Elman–Jacob [AEJ] (for \( l = 2 \)), and Hwang–Jacob [HJ] (for \( l \neq 2 \)); see also [E3] and [Kn].

**Theorem 2.1:** Let \( l \) be a prime number, let \( E \) be a field of characteristic \( \neq l \), let \( T \) be a subgroup of \( E^\times \) containing \( (E^\times)^l \) and \(-1\). Suppose that:

(i) For every \( x, y \in E^\times \) which are \( \mathbb{F}_l \)-linearly independent in \( E^\times / T \) one has \( \{ x, y \} \neq 0 \) in \( K_2^M(E) \);

(ii) For every \( x \in E^\times \setminus T \) and \( y \in T \setminus (E^\times)^l \) one has \( \{ x, y \} \neq 0 \) in \( K_2^M(E) \).

Then there exists a valuation \( u \) on \( E \) such that \( (\Gamma_u : l\Gamma_u) \geq (E^\times : T)/l \) and \( u(l) \neq 0 \). Furthermore, if \( E_u = \bar{E}_u \) then \( (\Gamma_u : l\Gamma_u) \geq (E^\times : T) \).

The **rank** of a profinite group is the minimal number (possibly \( \infty \)) of topological generators of it.

**Proposition 2.2** ([E1, Prop. 2.1]): Let \( l \) be a prime number and let \( (E, u) \) be a valued field such that \( \text{char } E_u \neq l \) and \( G_{\hat{E}_u}(l) \) is infinite. Suppose that
\[ \sup_{M} \text{rank } G_M(l) < \infty, \]
where \( M \) ranges over all finite separable extensions of \( E \). Then \((E, u)\) is Henselian.

Combining the previous two facts, we obtain the following result (which is essentially proven in [E1] for \( l = 2 \)).
Proposition 2.3: Let $l, p$ be distinct prime numbers and let $K$ be a field of characteristic $\neq l$. Let $E_0$ be a finite extension of $K$ containing $\mu_l$ and containing $\sqrt{-1}$ if $l = 2$. Suppose that for every finite separable extension $E$ of $E_0$ one has

$$G_E(l) \cong \langle \sigma, \tau \mid \sigma \tau \sigma^{-1} = \tau^p \rangle_{\text{pro}-l}$$

for some $s = s(E) \geq 1$ such that $p^s \equiv 1 \mod l$. Then there exists a Henselian valuation $v$ on $K$ such that $\Gamma_v/l \cong \mathbb{Z}/l$, char $\bar{K}_v \neq l$, and $\bar{K}_v$ is not algebraically closed.

Proof: For $E$ as above denote $H^1(E) = H^1(G_E(l), \mathbb{Z}/l)$. We consider the cup product $H^1(E) \times H^1(E) \to H^2(E)$. Let $\varphi_1, \varphi_2$ be an $\mathbb{F}_l$-linear basis of $H^1(E)$ which is dual to the basis of $G_E(l)/G_E(l)[G_E(l), G_E(l)]$ consisting of the images of $\sigma$ and $\tau$. From the defining relation $\tau^{p^s-1}[\tau, \sigma] = 1$ of $G_E(l)$ we deduce that $\varphi_1 \cup \varphi_2 \neq 0$ [K2, §7.8]. Furthermore, when $l \neq 2$ one has $\varphi_1 \cup \varphi_1 = \varphi_2 \cup \varphi_2 = 0$ by the anti-commutativity of $\cup$. When $l = 2$ we may identify $\varphi_i \cup \varphi_i$ with the class of a quaternion algebra $(a_i, a_i/E)$ in the Brauer group $\text{Br}(E)$; here $a_i(E^\times)^2$ corresponds to $\varphi_i$ under the Kummer isomorphism $E^\times/(E^\times)^2 \cong H^1(E)$. Since $(a_i, a_i/E) = (a_i, -1/E)$ in $\text{Br}(E)$ and $\sqrt{-1} \in E$ we obtain that $\varphi_i \cup \varphi_i = 0$, $i = 1, 2$, in this case as well. Consequently, $H^2(E) \cong \wedge^2 H^1(E)$.

By the Kummer theory and the Merkur’ev–Suslin theorem [MSu], this implies that $K^M_2(E)/l \cong \wedge^2(E^\times/l)$ naturally. Hence (i) and (ii) of Theorem 2.1 hold for $T = (E^\times)^l$. Since $\dim_{\mathbb{F}_l}(E^\times/(E^\times)^l) = \text{rank } G_E(l) = 2$, Theorem 2.1 therefore gives rise to a valuation $u$ on $E$ such that $\dim_{\mathbb{F}_l}(\Gamma_u/l) \geq 1$ and char $\bar{E}_u \neq l$.

Furthermore, if $\tilde{E}_u = \bar{E}_u$ then $\delta_l = \dim_{\mathbb{F}_l}(\Gamma_u/l) \geq 2$ by the last statement of Theorem 2.1. From the discussion in §1 this would imply $\mathbb{Z}_l^2 \leq G_E(l)$, which is not the case [E1, Lemma 4.1]. We conclude that $G_{E_u}(l) \neq 1$. This implies that the latter group is in fact infinite ([B]; note that when $l = 2$, $\sqrt{-1} \in \bar{E}_u$). Proposition 2.2 therefore shows that $(E, u)$ is Henselian.

Now take $E = E_0$, let $u$ be as above, and let $v = \text{Res}_K u$. Then $\dim_{\mathbb{F}_l}(\Gamma_v/l) = \dim_{\mathbb{F}_l}(\Gamma_u/l) \geq 1$ [E1, Lemma 1.2] and char $\bar{K}_v \neq l$. Since the finite extension $\bar{E}_u$ of $\bar{K}_v$ is not algebraically closed, neither is $\bar{K}_v$. Also, Henselianity goes down in finite extensions, provided that the upper residue field is not separably closed [Eg, Cor. 3.5]. Therefore $(K, v)$ is Henselian. \[\Box\]
Valuations $v, v'$ are called comparable if one of $O_v, O_{v'}$ contains the other.

**Proposition 2.4** (Endler-Engler [EE, Prop.]): Let $v, v'$ be valuations on a field $K$. Suppose that $v$ is Henselian and that $K_{v'}$ is not algebraically closed. Then $v, v'$ are comparable.

**Lemma 2.5:** Let $(L, w)/(K, v)$ be a Galois extension of Henselian valued fields of degree $n$. Suppose that the norm homomorphism $N_{L/K}: L^\times \to K^\times$ is surjective. Then $(\Gamma_v : n\Gamma_v) = (\Gamma_w : \Gamma_v)$.

**Proof:** By the Henselianity, one has a well-defined commutative square

$$
\begin{array}{ccc}
L^\times & \xrightarrow{w} & \Gamma_w \\
\downarrow{N_{L/K}} & & \downarrow{n} \\
K^\times & \xrightarrow{v} & \Gamma_v
\end{array}
$$

By assumption, the left vertical map is surjective. Hence so is the right vertical map; i.e., $\Gamma_v = n\Gamma_w$. By [E1, Lemma 1.2] again,

$$(\Gamma_v : n\Gamma_v) = (\Gamma_w : n\Gamma_w) = (\Gamma_w : \Gamma_v) . \quad \square$$

3. The absolute Galois group of a local field of characteristic $p$

From now on we fix a local field $F = \mathbb{F}_q((t))$ of characteristic $p > 0$. Let $G = G_F, T = G_{F_{ur}},$ and $V = G_{F_{tr}},$ taken with respect to the canonical discrete valuation on $F$. The group structure of $G$ was described by Koch [K1] (and follows from the general considerations in §1); namely:

(i) $G = V \times (G/V)$;

(ii) $T/V \cong \hat{\mathbb{Z}}/\mathbb{Z}_p$;

(iii) $G/T \cong \hat{\mathbb{Z}}$;

(iv) $G/V \cong (T/V) \times (G/T)$, where a generator $\sigma$ of $G/T$ acts on a generator $\tau$ of $T/V$ according to the Hasse–Iwasawa relation $\sigma \tau \sigma^{-1} = \tau^q$;

(v) $V \cong F_p(G/V; \mathfrak{N}_0)$; in particular, $V$ is a free pro-$p$ group on countably many generators [MSh, §1, Lemma 4].
Proposition 3.1: (a) $T/V$ intersects non-trivially every non-trivial normal closed subgroup of $G/V$.

(b) $T$ intersects non-trivially every non-trivial normal closed subgroup of $G$.

Proof: (a) We need to show that if $L$ is a Galois extension of $F$ such that $LF_{ur} = F_{tr}$ then $L = F_{tr}$. To this end, denote $L' = L \cap F_{ur}$. Then $\text{Gal}(F_{tr}/L') \cong \text{Gal}(F_{tr}/F_{ur}) \times \text{Gal}(F_{ur}/L')$. In particular, $\text{Gal}(F_{tr}/L')$ is abelian, by (ii) and (iii) above. For each positive integer $m$ which is prime to $p$ choose $t_m \in F_{tr}$ such that $t_m^m = t$. The abelianity implies that $L'(t_m)/L'$ is normal. Since $L'/F$ is unramified it follows that $\mu_m \subseteq L'$. Conclude that $F_{ur} = \bigcup_{(m, p) = 1} F(\mu_m) \subseteq L'$, whence $L = LF_{ur} = F_{tr}$.

(b) follows from (a). \qed

Lemma 3.2: Let $E$ be a totally ramified extension of $F$ of prime degree $l$ and let $\sigma$ be a generator of $\text{Gal}(E/F)$. Let $v$ be the canonical valuation on $E$ and let $\pi$ be a prime element of $E$. Let $s$ be the maximal integer such that the $s$th ramification group of $\text{Gal}(E/F)$ is non-trivial. Then:

(a) $v((\sigma - 1)(\pi^n)) = s + n$ for every integer $n$ relatively prime to $pl$;

(b) $E/(F + \wp(E))$ is infinite.

Proof: (a) When $l = p$ this is proven in [FV, Ch. III, (1.4)]. Suppose $l \neq p$. Then $s = 0$ [FV, Ch. II, §4.4, Cor. 1] and $\sigma(\pi) = \zeta \pi$ for a primitive $l$th root of unity $\zeta$. Hence $(\sigma - 1)(\pi^n) = (\zeta^n - 1)\pi^n$. It remains to observe that $v(\zeta^n - 1) = 0$.

(b) In all cases except $l = p = 2$ let $I$ be the set of all integers $n$ such that $n < -s$ and $(pl, n(s + n)) = 1$. When $l = p = 2$ let $I$ be the set of all integers $n$ such that $2 \nmid n$ and $4 \nmid s + n < 0$. Using again [FV, Ch. II, §4.4, Cor. 1] we see that $I$ is always infinite.

We claim that the elements $\pi^n$, where $n \in I$, are distinct modulo $F + \wp(E)$. Indeed, suppose that $\pi^n - \pi^n' = y + \wp(x)$, with $y \in F$, $x \in E$, $n, n' \in I$, and $n < n'$. By (a),

$$0 > s + n = v((\sigma - 1)(\pi^n - \pi^n')) = v(\wp((\sigma - 1)(x))) \quad .$$

However, negative elements of $v(\wp(E))$ are divisible by $p$. Thus, we get a contradiction in all cases except $l = p = 2$. 

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In the remaining case \( l = p = 2 \) we obtain \( v(x) < 0 \) and hence \( v((\sigma - 1)(\varphi(x))) = 2v((\sigma - 1)(x)) \). Since \( \pi \) is a primitive element for the extension \( E/F \), we can write \( x = c_0 + c_1 \pi \) with \( c_0, c_1 \in F \). Then

\[
v((\sigma - 1)(x)) = v(c_1) + v((\sigma - 1)(\pi)) = v(c_1) + s + 1,
\]

by (a). But \( 2|v(c_1) \) and \( 2 \nmid s \) [FV, Ch. III, Prop. 2.3], so \( v((\sigma - 1)(x)) \) is even. We conclude that \( 4|s + n \), a contradiction. \( \square \)

**Proposition 3.3:** \( V \) intersects non-trivially every non-trivial normal closed subgroup of \( G \).

**Proof:** (Compare [P1, Satz 1.4].) Let \( H \) be a non-trivial normal closed subgroup of \( G \) and let \( L \) be its fixed field. It follows from Proposition 3.1(b) that \( LF_{ur} \neq F_{sep} \). Hence we can take a finite Galois extension \( N \) of \( F \) such that \( N \not\subseteq LF_{ur} \). Denote the maximal elementary \( p \)-abelian Galois extension of \( N \) by \( N[p] \). It is a Galois extension of \( F \). Set \( K = L \cap N \) and \( M = L \cap N[p] \). Then \( N \not\subseteq KF_{ur} \), i.e., the extension \( N/K \) has a non-trivial inertia group. Since \( \text{Gal}(N/K) \) is solvable, we may therefore find an intermediate field \( K \subseteq N_0 \subset N \) such that \( N/N_0 \) is a totally ramified extension of prime degree. By Lemma 3.2(b), \( N/(N_0 + \varphi(N)) \) is infinite. Hence so is \( N/(K + \varphi(N)) \).

By the Artin–Schreier theory, the dual of the natural homomorphism \( K/\varphi(K) \to N/\varphi(N) \) may be canonically identified with the restriction homomorphism \( \text{Gal}(N[p]/N) \to \text{Gal}(K[p]/K) \). Since the cokernel \( N/(K + \varphi(N)) \) of the former homomorphism is infinite, so is the kernel \( \text{Gal}(N[p]/K[p]N) \) of the latter homomorphism.

Now the group \( \text{Gal}(M/K) \cong \text{Gal}(MN/N) \) is an epimorphic image of \( \text{Gal}(N[p]/N) \), hence \( M \subseteq K[p] \). It follows that \( \text{Gal}(N[p]/MN) \) is infinite. Since it is an elementary abelian \( p \)-group, it is not cyclic. Therefore the \( p \)-Sylow subgroups of \( \text{Gal}(LN[p]/L) \cong \text{Gal}(N[p]/M) \) are not cyclic (note that as \( N[p]/F \) is Galois, so are \( LN[p]/L \) and \( N[p]/M \)). It follows that \( \text{Syl}_p(G_L) \) is not cyclic. On the other hand, \( G_L/(G_L \cap V) \) embeds in \( G/V \cong (\widehat{Z}/\mathbb{Z}_p) \times \mathbb{Z} \), hence its \( p \)-Sylow subgroups are cyclic. Conclude that \( H \cap V = G_L \cap V \neq 1 \), as required. \( \square \)
4. The main results

We still fix a local field $F = \mathbb{F}_q((t))$ of characteristic $p > 0$.

**Theorem 4.1:** Let $K$ be a field with $G_K \cong G_F$. There exists a Henselian valuation $v$ on $K$ such that:

(a) $(\Gamma_v : l\Gamma_v) = l$ for all primes $l \neq p$;

(b) char $K_v = p$.

**Proof:** Fix an isomorphism $\sigma: G_K \to G_F$. For a separable extension $E$ of $K$ let $E'$ denote the separable extension of $F$ such that $E, E_0 \subset E'$ contain $\mu_l$, and contain $\sqrt{-1}$ if $l = 2$. Then for every finite separable extension $E$ of $E_l$ one has

$$G_E(l) \cong G_{E'}(l) \cong \langle \sigma, \tau \mid \sigma \tau \sigma^{-1} = \tau^{p^s} \rangle_{\text{pro-}l}$$

for some $s = s(E) \geq 1$ such that $p^s \equiv 1 \mod l$ (namely, $p^s$ is the cardinality of the residue field of $E'$; see §1). Proposition 2.3 gives rise to a Henselian valuation $v_l$ on $K$ such that $\Gamma_{v_l}/l \cong \mathbb{Z}/l$, char $K_{v_l} \neq l$, and $\bar{K}_{v_l}$ is not algebraically closed.

By Proposition 2.4, the valuations $v_l, l \neq p$, are pairwise comparable. It follows that

$$\bigcap_{l \neq p} O_{v_l}$$

is a Henselian valuation ring on $K$. Let $v$ be the corresponding valuation on $K$. For every prime number $l \neq p$ the fact that $O_v \subseteq O_{v_l}$ implies that $\Gamma_{v_l}$ is an epimorphic image of $\Gamma_v$; hence dim$_{F_l}(\Gamma_v/l) \geq$ dim$_{F_l}(\Gamma_{v_l}/l) = 1$. Moreover, $\mathbb{Z}_l^2 \not\subseteq G_{E_l}(l)$ [E1, Lemma 4.1]. We conclude as before using [E1, Lemma 1.2] and the considerations of §1 that dim$_{F_l}(\Gamma_v/l) = 1$, proving (a).

To prove (b), let $T_v, V_v$ be the inertia and ramification groups, respectively, of $v$ in $K_{\text{sep}}/K$. For every prime number $l \neq p$, char $\bar{K}_v$, part (a) gives Syl$_p(T_v/V_v) \cong \mathbb{Z}_l$. In particular, $T_v/V_v$ is non-trivial. Now the closed normal subgroup $\sigma^{-1}(V)$ of $G_K$ is free pro-$p$ of infinite rank. According to Proposition 3.3 it intersects every non-trivial closed normal subgroup of $G_K$. Thus $T_v \cap \sigma^{-1}(V) \neq 1$. Since a free pro-$p$ group of rank $\geq 2$ does not have non-trivial abelian closed normal subgroups, $T_v \cap \sigma^{-1}(V)$ is non-abelian. Therefore Syl$_p(T_v)$ is non-abelian, which can happen only when char $\bar{K}_v = p$. \qed
**Lemma 4.2:** Let $H$ be a profinite group such that $\text{cd}_p(H) \leq 1$ and such that $\text{Syl}_l(H) \cong \mathbb{Z}_l$ for all primes $l \neq p$. Then $H(p') \cong \hat{\mathbb{Z}}/\mathbb{Z}_p$.

**Proof:** By [S, I-23, Prop. 16] and [FJ, Cor. 20.14], $H$ embeds as a closed subgroup of a free profinite group $\hat{F}$. Now any closed subgroup of $\hat{F}$ isomorphic to $\mathbb{Z}_l$, $l \neq p$, is mapped bijectively by the canonical projection $\hat{F} \to \hat{F}(ab, p')$. Since the induced homomorphism $H \to \hat{F}(ab, p')$ breaks through $H(ab, p')$, any $l$-Sylow subgroup of $H$ is mapped bijectively onto an $l$-Sylow subgroup of $H(ab, p')$. It follows that $H(ab, p') \cong \prod_{l \neq p} \mathbb{Z}_l \cong \hat{\mathbb{Z}}/\mathbb{Z}_p$. Since $\text{cd}(\hat{\mathbb{Z}}/\mathbb{Z}_p) \leq 1$, the projection $H(p') \to H(ab, p')$ has a continuous homomorphic section. Then $H(p')$ and the image of this section have the same $l$-Sylow subgroups, hence they coincide. Thus $H(p') \cong \hat{\mathbb{Z}}/\mathbb{Z}_p$. □

**Proposition 4.3:** Let $K$ and $v$ be as in Theorem 4.1 and let $l \neq p$ be a prime number. Then:

(a) $G_{K_v}(p') \cong \hat{\mathbb{Z}}/\mathbb{Z}_p$.

(b) For $s \geq 0$, $\mu_{l^s} \subseteq K_v(\mu_l)$ if and only if $\mu_{l^s} \subseteq \mathbb{F}_q(\mu_l)$.

(c) If $\mu_l \subseteq K_v$ then $\mu_l \subseteq \mathbb{F}_q$.

(d) $\text{Syl}_p(G_{K_v})$ is a non-trivial free pro-$p$ group.

**Proof:** Fix an $l$-Sylow extension $(E_l, v_l)$ of $(K, v)$ relative to $K_{\text{sep}}$. Denote its residue field by $\bar{E}_l$. Then $G_{E_l} \cong \text{Syl}_l(G_{K_v})$. One has $\mu_l \subseteq E_l$ and $\mu_l \subseteq \bar{E}_l$. Also, the $l$-primary component of $\Gamma_{v_l}/\Gamma_v$ is trivial. Hence [E2, Lemma 2.4(b)] and Theorem 4.1(a) give $(\Gamma_{v_l} : l\Gamma_{v_l}) = (\Gamma_v : l\Gamma_v) = l$. Take $1 \leq s \leq \infty$ such that $\text{Im}(\chi_{E_l,l}) = 1 + l^s\mathbb{Z}_l$ (where we make the convention $l^{-\infty} = 0$). Then $G_{E_l} \cong \mathbb{Z}_l \times G_{E_l}$, where any $\sigma \in G_{E_l}$ acts on the generator $\tau$ of $\mathbb{Z}_l$ according to $\sigma \tau \sigma^{-1} = \chi_{E_l,l}(\sigma)\tau$ (see §1). It follows that $G_{E_l}(ab) \cong (\mathbb{Z}_l/l^s) \times G_{E_l}(ab)$.

The same analysis holds for $F$, so we obtain that $G_{F_l}(ab) \cong (\mathbb{Z}_l/l^{s'}) \times \mathbb{Z}_l$, where $F_l$ and $s'$ are defined in a similar manner. Since the residue field $\bar{F}_l$ of $F_l$ is the $l$-Sylow extension of $\mathbb{F}_q$, it does not contain $\mu_{l^{\infty}}$. Hence $s' < \infty$. If $s = \infty$ then we would obtain that $G_{E_l}(ab) \cong \mathbb{Z}_l/l^{s'}$, which is impossible at positive characteristic. We conclude that $s = s' < \infty$ and $G_{E_l}(ab) \cong \mathbb{Z}_l$. It follows that $\bar{E}_l, \bar{F}_l$ contain the same roots of unity of $l$-power order, and $G_{E_l} \cong \mathbb{Z}_l$. As $\text{cd}_p(G_{K_v}) \leq 1$ [S, II-4, Prop. 3], (a) follows from Lemma
4.2.

To prove (b) it remains to observe that $\mu_{l^v} \subseteq \bar{K}_v(\mu_l)$ if and only if $\mu_{l^v} \subseteq \bar{E}_l$, and likewise for $\mathbb{F}_q$ and $\bar{F}_l$.

To prove (c) assume that $\mu_l \subseteq \bar{K}_v$ and $\mu_l \not\subseteq \mathbb{F}_q$. Then $G_K(l) \cong \mathbb{Z}_l \times \mathbb{Z}_l \not\cong \mathbb{Z}_l$ (§1). On the other hand, $G_F(l) \cong G_{\mathbb{F}_q}(l) \cong \mathbb{Z}_l$ [E2, Lemma 2.1], a contradiction.

Finally, we prove (d). By [S, I-37, Cor. 2], Syl$_p(G_{\bar{K}_v})$ is indeed a free pro-$p$ group. Suppose that it is trivial. Then the maximal pro-$p$ Galois extension of $\mathbb{F}_p$ is contained in $\mathbb{F}_p \cap \bar{K}_v$. However, (b) and (c) imply that $\mathbb{F}_p \cap \bar{K}_v \subseteq \mathbb{F}_q$, a contradiction. □

**Theorem 4.4:** Let $K$ and $v$ be as in Theorem 4.1 and suppose that char $K = 0$. Then:

(a) $\Gamma_v = p\Gamma_v$;

(b) $K_v$ is perfect.

**Proof:** For any algebraic extension $E$ of $K(\mu_p)$ the $p$-torsion part of Br($E$) is isomorphic to $H^2(G_E, \mathbb{Z}/p) = H^2(G_{E'}, \mathbb{Z}/p) = 0$ ([S, II-4, Prop. 3]; here $E'$ is as before the extension of $F$ corresponding to $E$ with respect to a fixed isomorphism $\sigma: G_K \to G_F$). It follows that for every Galois extension $M$ of $E$ of degree $p$, the norm homomorphism $N_{M/E}: M^\times \to E^\times$ is surjective (see e.g. [M, Th. 15.7]).

To prove (a), let $E = K(\mu_p)$ and let $u$ be the unique extension of $v$ to $E$. By Proposition 4.3, $\bar{K}_v$ contains only finitely many roots of unity. Hence so does its finite extension $\bar{E}_u$. It follows that $\text{Gal}(\bar{\mathbb{F}}_p \bar{E}_u/\bar{E}_u) \cong \hat{\mathbb{Z}}$. Therefore there is an unramified extension $(L, w)$ of $(E, u)$ of degree $p$; thus $\Gamma_w = \Gamma_u$. By [E1, Lemma 1.2] and by Lemma 2.5 (for the extension $L/E$), $(\Gamma_v : p\Gamma_v) = (\Gamma_u : p\Gamma_u) = 1$, as required.

To prove (b), let $T_v, V_v$ be again the inertia and ramification groups, respectively, of $v$ in $G_K$. By Proposition 3.3, $T_v \cap \sigma^{-1}(V) \neq 1$. From Theorem 4.1(b) we get $p \not| (T_v : V_v)$. Since $V$ is pro-$p$, these two facts imply that the pro-$p$ group $V_v$ is non-trivial. Therefore we can take a tower of finite extensions $K(\mu_p) \subseteq E \subseteq M$ such that $M/E$ is a wildly ramified extension of degree $p$. Then the residue field extension $\bar{M}/\bar{E}$ is trivial. The surjectivity of $N_{M/E}: M^\times \to E^\times$ established above implies that $\bar{E} = \bar{M}^p = \bar{E}^p$; i.e., $\bar{E}$ is perfect. Hence so is $\bar{K}_v$. □
5. Constructions

We conclude by showing that various restrictions made in our main results in §4 are indeed necessary.

**Example 5.1:** For every positive integer \( r \) we construct a Henselian valued field \((K_r, u_r)\) of characteristic \( p \) such that \( G_{K_r} \cong G_F \) and \( \Gamma_{u_r}/p \cong (\mathbb{Z}/p)^r \).

We first construct inductively countable Henselian discretely valued fields \((K_r, v_r)\) as follows: Let \((K_1, v_1)\) be a Henselization of \( \mathbb{F}_q(t_1) \) with respect to the discrete valuation with uniformizer \( t_1 \). Assuming that \((K_r, v_r)\) has already been defined, let \( L_r \) be a maximal totally tamely ramified extension of it. Then the (supernatural) degree \([L_r : K_r]\) is prime to \( p \). Let \((K_{r+1}, v_{r+1})\) be a Henselization of \( L_r(t_{r+1}) \) with respect to its discrete valuation with uniformizer \( t_{r+1} \). Since both \( L_r \) and \( K_r \) are countable, Proposition 1.3 implies that \( G_{K_{r+1}} \cong G_{K_r} \).

Next we construct the valuations \( u_r \) on \( K_r \) inductively as follows: Take \( u_1 = v_1 \). Assuming that \( u_r \) has already been defined, let \( w_r \) be its unique prolongation to \( L_r \). Let \( u_{r+1} \) be the refinement of \( v_{r+1} \) such that the residue valuation \( u_{r+1}/v_{r+1} \) on \( L_r \) is \( w_r \) \([R]\). Since both \( w_r \) and \( v_{r+1} \) are Henselian, so is \( u_{r+1} \) \([R, \text{pp. } 210-211]\). One has an exact sequence

\[
0 \to \Gamma_{w_r} \to \Gamma_{u_{r+1}} \to \Gamma_{v_{r+1}} \to 0
\]

of ordered abelian groups, and \( \Gamma_{w_r} \) is convex in \( \Gamma_{u_{r+1}} \). We obtain an exact sequence of abelian groups

\[
0 \to \Gamma_{w_r}/p \to \Gamma_{u_{r+1}}/p \to \Gamma_{v_{r+1}}/p \to 0.
\]

Since the \( p \)-primary part of \( \Gamma_{w_r}/\Gamma_{u_r} \) is trivial, \( \Gamma_{w_r}/p \cong \Gamma_{u_r}/p \cong (\mathbb{Z}/p)^r \) \([E2, \text{Lemma } 2.4(b)]\). Combining this with \( \Gamma_{v_{r+1}}/p \cong \mathbb{Z}/p \), we conclude that \( \Gamma_{u_{r+1}}/p \cong (\mathbb{Z}/p)^{r+1} \), as desired.

In fact, \( K_r, L_r \) embed in a maximal totally tamely ramified extension \((M_r, w_r)\) of the \( r \)-dimensional local field \( \mathbb{F}_q((t_1)) \cdots ((t_r)) \) with its canonical discrete valuation of rank \( r \) \([FV, \text{Appendix B}]\). By considering the restrictions of \( w_r \) to these fields one can obtain an alternative proof that \( \Gamma_{u_r}/p \cong (\mathbb{Z}/p)^r \).
Example 5.2: There exists a Henselian discretely valued field \((K, v)\) of characteristic \(p\) such that \(G_K \cong G_F\), \(\bar{K}_v\) is imperfect, and \(G_{\bar{K}_v} \not\cong \hat{\mathbb{Z}}\). Indeed, take \((K, v) = (K_2, v_2)\) (with terminology as in Example 5.1). Then \(\bar{K}_v = L_1\). Since \(K_1\) is imperfect, so is its separable extension \(L_1\). According to \(\S 1\), \(G_{L_1} \cong F_p(\mu_{\infty} \times G_{\mathbb{F}_q}; \mathbb{R}_0) \times G_{\mathbb{F}_q}\). In particular, \(\text{Syl}_p(G_{L_1})\) has infinite rank. Conclude that \(G_{\bar{K}_v} = G_{L_1} \not\cong \hat{\mathbb{Z}}\).

Example 5.3: Let \((K, v)\) be a complete discretely valued field. Suppose that \(\text{char } K = p\), \(|\bar{K}_v| \leq \mathbb{R}_0\), \(G_{\bar{K}_v} \cong \hat{\mathbb{Z}}\), and \(\bar{K}_v\) has the same group of roots of unity as \(\mathbb{F}_q\) (e.g., this happens when \(K\) is a finite extension of \(\mathbb{Q}_p\) with residue field \(\mathbb{F}_q\)). Let \(L/K\) be an arithmetically profinite totally ramified extension (for the definitions see [Wi] or [FV, Ch. III, \S 5]). In particular, if \([L : K] = \prod l^{n(l)}\), then \(n(p) = \infty\) and \(\sum_{l \neq p} n(l) < \infty\). The theory of fields of norms of Fontaine–Wintenberger [Wi, 3.2.3] implies that \(G_L \cong G_{\bar{K}_v((X))}\). By Corollary 1.2, the latter group is isomorphic to \(G_F\). If \(u\) is the extension of \(v\) to \(L\), then \(\Gamma_u = p\Gamma_u\) and \(\Gamma_u/l \cong \mathbb{Z}/l\) for \(l \neq p\) prime.

Remark 5.4: Let \(M\) be an \(n\)-dimensional local field such that its canonical valuation of rank \(n\) has residue characteristic \(p\) (cf. [FV, Appendix B]). From the discussion in \(\S 1\) it follows that for every prime number \(l \neq p\) one has

\[
G_M(l) \cong \langle \sigma, \tau_1, \ldots, \tau_n \mid \sigma \tau_i \sigma^{-1} = \tau_i^q, \tau_i \tau_j = \tau_j \tau_i \rangle_{\text{pro-}l}.
\]

Now let \(K\) be a field such that \(G_K \cong G_M\). Similarly to the proof of Theorem 4.1 one can show that there is a Henselian valuation \(v\) on \(K\) such that \((\Gamma_v : l\Gamma_v) = l^n\) for all primes \(l \neq p\) and such that \(\text{char } \bar{K}_v = p\) or \(0\).

References


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